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Indeterminacy and Imperfect Information*

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Abstract

We study equilibrium determination in an environment where two kinds of agents have different information sets: The fully informed agents know the structure of the model and observe histories of all exogenous and endogenous variables. The less informed agents observe only a strict subset of the full information set. All types of agents form expectations rationally, but agents with limited information need to solve a dynamic signal extraction problem to gather information about the variables they do not observe. We show that for parameter values that imply a unique equilibrium under full information, the limited information rational expectations equilibrium can be indeterminate. We illustrate our framework with a monetary policy problem where an imperfectly informed central bank follows an interest rate rule.

JEL Classification: C11; C32; E52
Keywords: Limited information; rational expectations; Kalman filter; belief shocks

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1 Introduction

Asymmetric information is a pervasive feature of economic environments. Even when agents are fully rational their expectation formation and decision-making process are constrained by the fact that information may be imperfectly distributed in the economy for reasons such as costs of information acquisition. Asymmetric information is also a central issue for the conduct of monetary policy as policymakers regularly face uncertainty about the true state of the economy, either because they are uncertain about the structure of the economy or because they receive data in real time that are subject to measurement error. In environments where information is perfect and symmetrically shared, the literature has shown that ill-designed policy rules can cause indeterminacy. We study equilibrium determinacy in an asymmetric information setting, where policy is conducted based on estimates of the true state of the economy.

We consider an economic environment with two types of agents, one who has full information about the state of economy while the other agent is imperfectly informed. More specifically, the less informed agent's information set is nested within the fully informed agent's. We think of the two agents as a fully informed public, or alternatively, the private sector as a data-generating process for aggregate outcomes, and a less informed policymaker. Respectively, we model the private sector as a homogenously informed representative agent who is perfectly informed about the aggregate state whereas the policymaker operates under imperfect imperfection, which, for instance, can take the form of aggregate data subject to measurement error.[1]

A key assumption of our modelling framework is that both types of agents, the policymaker and the private sector, employ rational expectations, but use different information sets. Private-sector behavior is characterized by a set of linear, expectational difference equations. On the other hand, the policymaker's behavior is characterized by the use of a policy instrument. It is set according to a rule that responds to the policymaker's optimal estimates of economic conditions.[2] Formally, we consider linear, stochastic equilibria with time-invariant decision rules and Gaussian shocks. In this case, the rational inference efforts of the policymaker are represented by a dynamic signal extraction problem as captured by the Kalman filter. The interaction of the two sets of expectation formation processes represents the fundamental mechanism underlying equilibrium determination.

The central result of our paper is that equilibrium indeterminacy is generic in this imperfect information environment for a broad class of linear models that have unique equilibria under full information. Mechanically, optimal information processing of the less informed agent introduces additional stable dynamics into the equation system that then lead to self-fulling expectations. Intuitively, the interaction of the two expectation processes generates an endogenous feedback mechanism in a similar vein to strategic complimentarities or the application of ad-hoc behavior in

[1] Such a dichotomy is well-established in the learning literature. Where our work differs is that both agents have rational expectations and know the structure of the economy, although not necessarily its state.

[2] As a specific example, we consider a Taylor-type interest-rate rule that responds to the policymaker's projection of inflation.

the standard indeterminacy literature. Moreover, the interplay of expectations based on different information sets results in equilibrium outcomes that are not certainty equivalent even though we only consider environments that are linear. While the rationality of expectations under both information sets places non-trivial restrictions on outcomes, they are not sufficient to rule out multiple equilibria.

We characterize the outcomes of different equilibria as the result of non-fundamental disturbances, similar to the perfect-information literature on equilibrium determinacy in linear rational expectations models. Such belief shocks are unrelated to fundamental shocks in the original economic setup and can be interpreted as self-fulfilling shifts in expectations, or beliefs, that cause fluctuations consistent with the concept of a linear, stationary equilibrium. When there is indeterminacy in the perfect-information case, there are no restrictions on the scale of effects caused by belief shocks. In contrast, the potential effects of belief shocks are tightly bounded in our imperfect information environment. The bounds arise from the required consistency of expectations of the public and the policymaker and the assumption that we consider only environments that have a unique equilibrium under full information.

Our paper touches upon three strands in the literature. In a broad sense, our paper contributes to the burgeoning literature on imperfect information in macroeconomic models. The main thrust of the existing literature has been on the implications of dispersed information among different members of the public and the resulting effects on their strategic interactions. Key contributions by Angeletos and La'O (2013), Nimark (2008a, 2008b, 2014), and Acharya, Benhabib and Huo (2017) demonstrate that imperfect information has important implications for the amplification and propagation of economic shocks. However, while the literature has been aware of the potential for multiple equilibria, this has usually not been a key issue of the analysis. In that vein, our paper is much closer to Benhabib, Wang and Wen (2015) who consider sentiment shocks in their New Keynesian model with imperfect information in the private sector. In contrast to our setup, which involves the interaction between a policymaker and the public under different information sets, sunspot or belief shocks do not arise endogenously in their setting. In that respect, our framework is closer to Rondina and Walker (2017).

Our research also makes a contribution to the literature on indeterminacy in linear rational expectations models by expanding the set of plausible economic mechanisms that can lead to multiple equilibria. A key element of the indeterminacy literature is the presence of a mechanism that validates self-fulfilling expectations. In the standard literature, these could arise from what is often termed strategic complimentarities, such as increasing returns to scale in production that are not internalized, as in the seminal contributions of Benhabib and Farmer (1994), Farmer and Jang-Ting (1994), and Schmitt-Grohe (1997). An alternative mechanism is the interplay between economic agents' forward-looking behavior and the reaction function of a policymaker, which Clarida, Gali and Gertler (2000) and Lubik and Schorfheide (2004) show to be a key feature of macroeconomic fluctuations.[3]

In contrast, our framework does not rely on these previously identified sources of indeterminacy but rather on the interaction of different expectation formation processes under asymmetric information sets. This also sets our framework apart from the general imperfect information literature, which is largely concerned with the strategic interaction between agents in the private sector. Although our framework utilizes the formalism of the indeterminacy literature, where we build on the contributions of Lubik and Schorfheide (2003, 2004) and Farmer, Khramov and Nicolò (2015), the mechanism to get there is novel. More specifically, we show that in an imperfect information environment the standard root-counting approach in the literature is inadequate in identifying the set of multiple equilibria. At the same time, we show that the set of multiple equilibria, despite the generic pervasiveness of indeterminacy, is tightly circumscribed by internal consistency requirements for the interaction between the two expectation processes. Our paper thereby puts some caveats on the notion that sunspot shocks are unrestricted in their effects on macroeconomic outcomes.

It is a well-known result from the monetary policy literature that the Taylor principle, namely that interest rate rules need to respond to endogenous variables with sufficient strength, is required for determinacy and to avoid multiple equilibria. Clarida et al. (2000) and Lubik and Schorfheide (2004) have pointed to a neglect of the Taylor rule as a possible factor behind the Great Inflation. However, their evidence is based on a full-information perspective that does not account for the uncertainties faced by the Federal Reserve in assessing the state of the economy in real time, as discussed by Orphanides (2001). The framework that we develop in this paper sheds new light on this issue.

A key paper in this literature is Orphanides (2003) who models the consequences of an imperfectly informed central bank for economic outcomes. Similar to our framework, his model considers a policy rule that responds to estimates of economic conditions generated from optimal signal extraction efforts. But in a fundamental difference to our framework, his model is purely backward-looking so that the issue of indeterminacy does not arise. Our paper also relates to Svensson and Woodford (2004) and Aoki (2006) who derive conditions for optimal policy when the policymaker is less informed than the public in forward-looking linear rational expectations models, but take determinacy as given.[4]

Our paper is structured as follows. In the next section we introduce our framework by means of a simple example in which we can derive analytical results. The section proceeds by developing the various model components sequentially so as to build up the full set of equilibrium relationships.

[3] Ascari, Bonomolo and Lopes (2019) also point to sunspot-driven equilibria as a key source of fluctuations during the high-inflation era of the 1970s.

[4] Applications following Svensson and Woodford to various economic issues are Carboni and Ellison (2011), Dotsey and Hornstein (2003), and Nimark (2008b). Evans and Honkapohja (2001) and Orphanides and Williams (2006, 2007) revisit the question of policymaking under imperfect information in an environment with learning. Faust and Svensson (2002) and Mertens (2016) study the implications for optimal policy of the opposite informational asymmetry, where the public does not perfectly share the policymaker's information set.

We also discuss various extensions and some additional findings that connect our framework to the literature. Section 3 contains the main body of the paper. We present a general linear rational expectations framework with heterogenous information sets and use results from general linear systems theory to prove existence of a variety of equilibria. It is here that we establish our central result that equilibrium indeterminacy is generic in this framework. We conduct some quantitative exercises in section 4. We first solve the simple example of section 2 numerically in order to provide additional insights. In the next step, we solve a New Keynesian model under our informational assumptions. We show that while indeterminacy is, in fact, generic in this policy-relevant model, the quantitative implications appear relatively limited. In section 5, we consider a set of alternative policy rules that could lead to determinate outcomes. Section 6 concludes and discusses further extensions of our framework.

2 A Simple Example Model

We develop the basic concepts and ideas underlying our modelling framework by means of a simple example. First, we describe the basic structural relationships before introducing two types of information sets. For exposition purposes we distinguish information sets where the observed signals reflect solely exogenous variables or where the signal also reflects endogenous variables. We then introduce the key component of the framework, namely optimal information extraction by the less informed agent via Kalman-filtering, and a projection condition that rational expectations equilibria in our framework need to obey. We conclude this section by discussing the underlying intuition and some special results from the simple framework.

2.1 Economic Framework

We consider a simple textbook model of inflation determination in a frictionless economy. The economy is described by a Fisher equation that links the nominal interest rate i_t to the real rate r_t via expected inflation $E_t\pi_{t+1}$, where E_t is an expectation operator. The nominal rate is set according to a monetary policy rule where it responds to current inflation π_t.[5] Reflecting the central role of the Fisher equation here, we refer to the small model also as a Fisher economy. We assume that the real rate is characterized by an exogenous AR(1) process with a Gaussian shock. The equation system is thus given by:

$$i_t = r_t + E_t\pi_{t+1}, \quad (1)$$

$$i_t = \phi\pi_t, \quad (2)$$

$$r_t = \rho r_{t-1} + \varepsilon_t, \quad (3)$$

where $\varepsilon_t \sim iid\ N(0, \sigma_\varepsilon^2)$ and $|\rho| < 1$. Throughout this paper we assume that the monetary policy parameter ϕ is outside the unit circle, $|\phi| > 1$.

[5] In the Supplementary Appendix, we also consider a policy rule of the type: $i_t = r_t + \phi\pi_t$, with a time-varying intercept given by the real rate of interest.

There are two agents in this economy: a representative private-sector agent whose behavior is characterized by the Fisher equation (1), and a central bank whose behavior is given by a monetary policy rule such as (2). We assume that the agents know the structure of the economy, including the structural parameters, and that they observe the history of their respective information sets. Crucially, both agents form expectations rationally. The central assumption of our framework is that the two agents have different, but nested information sets. The full information set S^t contains realizations of all shocks through time t, where E_t is the rational expectations operator under full information, so that for some variable x_t, $E_t x_{t+h} = E\left(x_{t+h}|S^t\right)$, for all h, and $E_t x_t = x_t$. We also define a limited information set Z^t which is nested in S^t, $Z^t \subset S^t$.[6] The expectations, or projections, of the less informed agent for any variable x_t are denoted as $x_{t|t} = E\left(x_t|Z^t\right)$ and $x_{t+h|t} = E\left(x_{t+h}|Z^t\right)$. Since Z^t is spanned by S^t we can apply the law of iterated expectations to obtain: $E\left(E(x_{t+h}|Z^t)|S^t\right) = x_{t+h|t}$.

We consider two informational environments: full and limited information. Under full information rational expectations (FIRE), both agents are assumed to know S^t.[7] This means that they observe all variables in the model without error, that they know the history of all shocks, and that they understand the structure of the economy and the solution concepts. Under limited information rational expectations (LIRE), we assume that one agent has access to the full information set S^t, while the other observes the limited information set Z^t only. For the purposes of this simple example, we assume that the private sector is fully informed whereas the central bank has limited information.

2.2 Rational Expectations Equilibria with Full Information

The equation system (1) - (3) forms a linear rational expectations model that can be solved under FIRE with standard methods. Substituting the policy rule into the Fisher equation yields a relationship in inflation with driving process r_t:

$$E_t \pi_{t+1} = \phi \pi_t + r_t. \tag{4}$$

The type of solution depends on the value of the policy coefficient ϕ. It is well known that the solution is unique if and only if $|\phi| > 1$. In this case, the determinate rational expectations (RE) solution is: $\pi_t = \frac{1}{\phi-\rho} r_t$ and $i_t = \frac{\phi}{\phi-\rho} r_t$. Inflation and the nominal rate inherit the properties of the exogenous process r_t and are thus first-order autoregressive processes.

When $|\phi| < 1$, the full-information solution is indeterminate, and there are infinitely many solutions to equation (4). Although the remainder of the paper considers the case $|\phi| > 1$, it is instructive to review the implications of equilibrium indeterminacy when ϕ is inside the unit circle, since we utilize these concepts later. We follow the approach developed by Lubik and Schorfheide (2003), which extends the Sims (2002) solution method to the case of indeterminacy.

[6] Allowing Z^t to be only weakly nested in S^t, $Z^t \subseteq S^t$, would also encompass the case when both agents are fully informed. We will typically consider the limited-information case as such that Z^t is strictly less informative than S^t.

[7] In terms of notation, the full-information case corresponds to the situation where $Z^t = S^t$.

We define the rational expectations forecast error $\eta_t = \pi_t - E_{t-1}\pi_t$, whereby $E_{t-1}\eta_t = 0$ by construction. This allows us to substitute out inflation expectations $E_t\pi_{t+1}$ in (4), so that we can write:

$$\pi_t = \phi\pi_{t-1} + r_{t-1} + \eta_t. \tag{5}$$

It is easily verifiable that this representation is a solution to the expectational difference equation (4). Inflation is a stationary process with autoregressive parameter $|\phi| < 1$ and driving process r_{t-1}. What makes this equilibrium indeterminate is the fact that the solution imposes no restriction on the evolution of η_t other than that it is a martingale difference sequence with $E_{t-1}\eta_t = 0$. Consequently, there can be infinitely many solutions.

Without loss of generality, we can, however, put some structure on the solution.[8] Following Farmer, Khramov, and Nicolò (2015) we decompose the RE forecast error η_t into a fundamental component, namely the policy innovation ε_t, and a non-fundamental component, the belief shock b_t. More specifically, we can write $\eta_t = \gamma_\varepsilon\varepsilon_t + \gamma_b b_t$, where $E_{t-1}b_t = 0$.[9] The loadings $-\infty < \gamma_\varepsilon, \gamma_b < \infty$ on the two sources of uncertainty are unrestricted and their choice is arbitrary. They can be used to index specific equilibria within the set of indeterminate equilibria. A specific solution to (4) when $|\phi| < 1$ can therefore be written as:

$$\pi_t = \phi\pi_{t-1} + r_{t-1} + \gamma_\varepsilon\varepsilon_t + \gamma_b b_t. \tag{6}$$

Returning to the case of $|\phi| > 1$, we can also compute an RE equilibrium for the case when the dynamic system is conditioned down onto the information set Z^t. Since the content of this information set is known to both agents, we can apply the law of iterated expectations: $E\left(E(x_{t+h}|Z^t)|S^t\right) = x_{t+h|t}$. Except for changing the information set on which the expectations operator is conditioned, the structure of the system remains unchanged. The policy rule now becomes:

$$i_{t|t} = \phi\pi_{t|t}. \tag{7}$$

Following the same steps as before, we find that:

$$\pi_{t+1|t} = \phi\pi_{t|t} + r_{t|t}, \tag{8}$$

which is a first-order difference equation in projected inflation $\pi_{t|t}$. Under the maintained assumption that $|\phi| > 1$, the RE equilibrium is $\pi_{t|t} = \frac{1}{\phi-\rho}r_{t|t}$ and $i_{t|t} = \frac{\phi}{\phi-\rho}r_{t|t}$. The form of the solution is isomorphic to the FIRE solution above. That is, central bank projections of inflation and the evolution of the policy rate obey the same functional form as the actual variables in the full information model. Moreover, in our setup central bank decisions are always based on Z^t such that

[8] Strictly speaking, this is without loss of generality within the set of equilibria that are time-invariant and linear. There are other non-linear equilibria that can be constructed in this linear model. See Evans and McGough (2005) for further discussion.

[9] The interpretation of a belief shock in the terminology of Lubik and Schorfheide (2003) and Farmer, Khramov, and Nicolò (2015) emerges when the inflation equation is rewritten in terms of expectations only. Define $\xi_t = E_t\pi_{t+1}$ and rewrite equation (4) as $\xi_t = \phi\xi_{t-1} + r_t + \phi\eta_t$. In this representation, the forecast error η_t is akin to an innovation to the conditional expectation ξ_t.

$i_t = i_{t|t}$. The key insight is that under Z^t, the central bank has less information than under S^t, but it still forms expectations rationally under its own information set, given its real rate projections $r_{t|t}$.[10]

2.3 Rational Expectations Equilibria with Limited Information

The key aspect of our limited information framework is that there are two expectation formation processes interacting with each other. The nature of this interaction, and how it affects equilibrium determination, depends on how the limited-information agent extracts and updates information. Our framework has four building blocks: first, the relationships describing the fully-informed agent; second, those of the limited-information agent; third, the filtering and updating mechanism used by the latter to gain additional information; and fourth, restrictions on agents' projections to ensure consistency of expectation formation in an RE equilibrium.

In the simple example considered thus far, the first element is given by the Fisher equation (1) and the law of motion of the real rate (3). Considering the second building block, we assume that the central bank is the less informed agent and therefore has access to the information set Z^t. As in Svensson and Woodford (2004), policy is set based on target variable projections. Specifically, the behavior of the central bank is given by a limited information policy rule where the policy rate responds to the inflation projection $\pi_{t|t}$:

$$i_t = \phi\,\pi_{t|t} \tag{9}$$

The third element is the specification of the central bank's information extraction and updating problem. The policymaker is aware of the limited information set and solves a signal extraction problem to conduct inference about unobserved variables.[11] The model is linear and the exogenous shocks are Gaussian; in addition, we assume that the belief shocks are Gaussian. Without loss of generality, the variance of belief shocks is normalized to one:[12]

$$b_t \sim N(0,1). \tag{10}$$

As a result, the Kalman filter is the optimal filter in this environment. The gain in the optimal projection equation is endogenous and depends on the second moments of the model variables in an equilibrium. In turn, existence and uniqueness of an equilibrium depends on the endogenous Kalman gain. This feature of our framework implies a non-trivial fixed-point problem. Finally, the

[10] This is a key difference to the framework in Lubik and Matthes (2016) who assume that the central bank engages in least-squares learning to gain information about private-sector outcomes. In our setup, the deviation from the standard RE benchmark is only minor in the sense that the central bank does not observe everything that the private sector does, but employs fully rational expectations in its inferences about current and future conditions.

[11] Conceptually, this is an environment where the central bank receives noisy measurements of incoming data but makes decisions in real time based on its best projections of the true underlying data.

[12] Similar to the full-information case shown in (6), belief shocks will enter the system only via the endogenous forecast error, which linearly depends on the belief shock b_t with sensitivity γ_b, This allows us to normalize the variance of belief shocks.

fourth building block of our framework is an additional restriction on equilibrium determination. We posit that rational expectations formation across all information sets has to be mutually and internally consistent. Specifically, the central bank's behavior is constrained by its own projections, namely $\pi_{t|t} = \frac{1}{\phi-\rho} r_{t|t}$ and $i_t = \frac{\phi}{\phi-\rho} r_{t|t}$, and a projection for $r_{t|t}$. These projections imply a restriction on the joint behavior of the model variables' second moments so as to validate the RE of the fully informed and the limited-information agents.

We now discuss the solution of our simple framework in two steps. We specify simple information sets that make the central bank's projection equations analytically tractable, whereby we distinguish between exogenous and endogenous information sets. The former contains only exogenous variables where there is no feedback between projection and model evolution. Specifically, we assume that the central bank receives a noisy measurement of the real rate of interest. In the second step, we assume endogenous information where the central bank observes inflation with a measurement error.

2.3.1 Equilibrium with an Exogenous Signal

We assume that the central bank observes the real rate with measurement error $\nu_t \sim iid\ N(0, \sigma_\nu^2)$, so that its information set is $Z^t = \{Z_t, Z_{t-1}, \ldots\}$, with $Z_t = r_t + \nu_t$.[13] The signal Z_t is exogenous in that the real rate is an exogenous process that does not depend on other endogenous variables.[14] The Kalman projection equation for the real rate is:

$$r_{t|t} = r_{t|t-1} + \kappa_r \left(r_t - r_{t|t-1} + \nu_t \right), \tag{11}$$

where the Kalman gain κ_r is an endogenous coefficient, and the one-step-ahead projection of the real rate is $r_{t|t-1} = \rho r_{t-1|t-1}$.

We now combine the private sector Fisher equation (1) with the policy rule (9):

$$\phi \pi_{t|t} = r_t + E_t \pi_{t+1}. \tag{12}$$

The evolution of inflation depends on two expectation processes: the central bank's projection of inflation $\pi_{t|t}$ and the private sector's expectation $E_t \pi_{t+1}$. Using the formalism described above, we introduce the RE forecast error η_t and rewrite this equation as:

$$\pi_t = \phi \pi_{t-1|t-1} - r_{t-1} + \eta_t. \tag{13}$$

In addition, recall that after conditioning down all equations of the model onto Z^t and solving for an RE equilibrium conditional on Z^t, we obtain:

$$\pi_{t|t} = \frac{1}{\phi - \rho} r_{t|t}\,. \tag{14}$$

[13] In addition, the central bank knows the structure of the economy and all parameters of the model, which are common knowledge. For brevity, these elements of the information set are omitted in our notation.

[14] However, the process of making projections of the real rate, that is, of gaining information about its true value can depend on endogenous outcomes.

Equation (14) is a consistency condition for the RE equilibrium that we will also refer to as "projection condition" since it restricts central bank projections to be consistent with predictions from the full-information model.

We can now combine these equations into a linear RE system:

$$\begin{aligned} \pi_t &= \frac{\phi}{\phi-\rho} r_{t-1|t-1} - r_{t-1} + \eta_t, \\ r_{t|t} &= (1-\kappa_r)\,\rho r_{t-1|t-1} + \kappa_r \rho r_{t-1} + \kappa_r \varepsilon_t + \kappa_r \nu_t, \\ r_t &= \rho r_{t-1} + \varepsilon_t. \end{aligned} \tag{15}$$

The first equation in (15) is derived from the Fisher equation, where we substituted out the central bank's lagged inflation projection by using the projection condition (14). The second equation is derived from the Kalman projection equation for the real rate, while the third equation is the law of motion of the actual real rate. The set of equations in (15) is a well-specified equation system in three unknowns: inflation π_t, the exogenous real rate r_t, and the central bank projection of the real rate $r_{t|t}$. In principle, it can be solved using standard methods for linear RE models that allow for indeterminacy such as Lubik and Schorfheide (2003). However, there are two key differences to the standard framework. First, the gain coefficient κ_r is endogenous and has to be computed from the second moments of the model solution. The second difference is that the central bank's projection $r_{t|t}$ has to be consistent with the solution of the full system it determines. This projection condition necessitates an additional computational step in the solution of the model.

We solve the model in three steps. First, in the exogenous signal case, the Kalman filtering problem can be solved independently of the solution for inflation dynamics. Second, as shown below, the Kalman gain lies between zero and one. As a result, $r_{t|t}$ is stationary and standard root-counting implies that the system (15) is indeterminate. Third, we impose the projection condition. Following Farmer, Khramov, and Nicolò (2015), we find it convenient to express the endogenous forecast error as a linear combination of fundamental and belief shocks:[15]

$$\eta_t = \gamma_\varepsilon \varepsilon_t + \gamma_\nu \nu_t + \gamma_b b_t. \tag{16}$$

The solution is determinate if $\gamma_b = 0$ and γ_ε and γ_ν are uniquely determined. An RE equilibrium may not exist when there are no loadings that fulfill the restrictions imposed by and on the model, specifically, the projection condition (14).

We find it convenient to define innovations of any variable x_t as its unexpected component relative to the limited information set Z^{t-1}: $\widetilde{x}_t = x_t - x_{t|t-1}$, that is, the projection innovations. We can then define the projection error variance $\Sigma = var\left(\widetilde{r}_t - \widetilde{r}_{t|t}\right) = var\left(\widetilde{r}_t\right) - var\left(\widetilde{r}_{t|t}\right)$, whereby $cov(\widetilde{r}_t, \widetilde{r}_{t|t}) = var(\widetilde{r}_{t|t})$. The steady-state Kalman gain is given by:

$$\kappa_r = \frac{cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)}{var(\widetilde{Z}_t)}, \tag{17}$$

[15] Alongside ε_t, we refer to the measurement error ν_t as a fundamental shock, too.

where $\widetilde{Z}_t = \widetilde{r}_t + \nu_t$. It is straightforward to verify that $var\left(\widetilde{r}_t\right) = \rho^2\Sigma + \sigma_\varepsilon^2$ and that $var(\widetilde{Z}_t) = var\left(\widetilde{r}_t\right) + \sigma_\nu^2$. Similarly, we have that $cov(\widetilde{r}_t, \widetilde{Z}_t) = var\left(\widetilde{r}_t\right)$. This leads to the following expression:

$$\kappa_r = \frac{\rho^2\Sigma + \sigma_\varepsilon^2}{\rho^2\Sigma + \sigma_\varepsilon^2 + \sigma_\nu^2}, \tag{18}$$

whereby clearly $0 < \kappa_r < 1$. We can now find an expression for the projection error variance Σ by noting that $cov(\widetilde{r}_t, \widetilde{r}_{t|t}) = var(\widetilde{r}_{t|t})$ and $var\left(\widetilde{r}_{t|t}\right) = \kappa_r \ cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$, given the projection equation $\widetilde{r}_{t|t} = \kappa_r \widetilde{Z}_t$. Substituting these expressions into the definition of Σ results in a quadratic equation, commonly known as a Riccati equation:

$$\Sigma = \frac{\rho^2\Sigma + \sigma_\varepsilon^2}{\rho^2\Sigma + \sigma_\varepsilon^2 + \sigma_\nu^2}\sigma_\nu^2. \tag{19}$$

The (positive) solution to this equation is given by:

$$\Sigma = \frac{1}{2\rho^2}\left[-\left(\sigma_\varepsilon^2 + \left(1-\rho^2\right)\sigma_\nu^2\right) + \sqrt{\left(\sigma_\varepsilon^2 + \left(1-\rho^2\right)\sigma_\nu^2\right)^2 + 4\sigma_\varepsilon^2\sigma_\nu^2\rho^2}\right]. \tag{20}$$

We can now establish that the LIRE model with an exogenous signal vector always has multiple equilibria. We have $0 < \kappa_r < 1$, from which it follows that $\left|\left(1-\kappa_r\right)\rho\right| < 1$, so that the law of motion for $r_{t|t}$ in the full equation system is a stable difference equation. In (13), inflation depends only on lags of the real rate r_t and lags of the real-rate projection $r_{t|t}$, which are both stationary. Without any dependence of inflation on lags of its own, inflation is stationary for any specification of η_t, and we can conclude that the equilibrium cannot be determinate. That is, the structure of the model does not impose restrictions that would uniquely pin down the endogenous forecast error η_t and which would typically derive from the set of explosive roots in the system. One such restriction could be $|\kappa_r| > 1$, which we can rule out in this case. In other words, the representation (15) is already a candidate solution to the model.

In the final step, we need to ensure that central bank projections for inflation and the real rate are mutually consistent. Specifically, $\pi_{t|t} = \frac{1}{\phi-\rho} r_{t|t}$ needs to hold along any equilibrium path. This projection condition imposes a second-moment restriction on innovations with respect to the central bank's information set $cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right) = \frac{1}{\phi-\rho} cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$. Since $cov\left(\widetilde{r}_t, \widetilde{Z}_t\right) = \rho^2\Sigma + \sigma_\varepsilon^2$, we can write $cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right) = cov\left(\widetilde{\pi}_t, \widetilde{r}_t\right) + cov\left(\widetilde{\pi}_t, \nu_t\right)$. Using the innovation representation of the projection equation for π_t, we have:

$$\widetilde{\pi}_t = -\left(\widetilde{r}_{t-1} - \widetilde{r}_{t-1|t-1}\right) + \eta_t, \tag{21}$$

where after some substitution we find that $cov\left(\widetilde{\pi}_t, \widetilde{r}_t\right) = -\rho\Sigma + \gamma_\varepsilon\sigma_\varepsilon^2$. Similarly, we have that $cov\left(\widetilde{\pi}_t, \nu_t\right) = \gamma_\nu\sigma_\nu^2$. Combining all expressions results in the following linear restriction on the shock loadings of the forecast error $\eta_t = \gamma_\varepsilon\varepsilon_t + \gamma_b b_t + \gamma_\nu\nu_t$:

$$\gamma_\nu = \frac{\phi}{\phi-\rho}\frac{\Sigma}{\sigma_\nu^2} + \frac{1}{\phi-\rho}\frac{\sigma_\varepsilon^2}{\sigma_\nu^2} - \frac{\sigma_\varepsilon^2}{\sigma_\nu^2}\gamma_\varepsilon. \tag{22}$$

This condition places a linear restriction on γ_ε and γ_ν to guarantee that central bank projections for inflation and the real rate co-vary as they would in the full information case. However, this projection condition does not uniquely determine γ_ε and γ_ν. Moreover, γ_b is left unrestricted. We can now summarize the solution in the following proposition.

PROPOSITION 1 (LIRE Equilibrium in the Fisher Economy with Exogenous Signal) *The set of stationary RE equilibria in the model (15) under LIRE with exogenous signal $Z_t = r_t + \nu_t$ is characterized by:*

$$\pi_t = \frac{\phi}{\phi - \rho} r_{t-1|t-1} - r_{t-1} + \gamma_\varepsilon \varepsilon_t + \gamma_\nu \nu_t + \gamma_b b_t, \tag{23}$$

$$r_{t|t} = (1 - \kappa_r) \rho r_{t-1|t-1} + \kappa_r \rho r_{t-1} + \kappa_r \varepsilon_t + \kappa_r \nu_t, \tag{24}$$

$$r_t = \rho r_{t-1} + \varepsilon_t, \tag{25}$$

where:

$$\kappa_r = \frac{\rho^2 \Sigma + \sigma_\varepsilon^2}{\rho^2 \Sigma + \sigma_\varepsilon^2 + \sigma_\nu^2}, \tag{26}$$

$$\Sigma = \frac{1}{2\rho^2} \left[- \left(\sigma_\varepsilon^2 + \left(1 - \rho^2\right) \sigma_\nu^2 \right) + \sqrt{\left(\sigma_\varepsilon^2 + (1 - \rho^2) \sigma_\nu^2\right)^2 + 4 \sigma_\varepsilon^2 \sigma_\nu^2 \rho^2} \right], \tag{27}$$

$$-\infty < \gamma_b < \infty, -\infty < \gamma_\varepsilon < \infty, \ \gamma_\nu = \frac{\phi}{\phi - \rho} \frac{\Sigma}{\sigma_\nu^2} + \left(\frac{1}{\phi - \rho} - \gamma_\varepsilon \right) \frac{\sigma_\varepsilon^2}{\sigma_\nu^2}. \tag{28}$$

Proof. *The result follows directly from the positive solution to the Riccati equation (19) and (20) as well as the projection condition (22).* ■

We can draw the following conclusion at this point. Equilibrium indeterminacy is generic in this setting in that the endogenous forecast error is not uniquely determined and that *any* stationary RE equilibrium allows for the presence of sunspot shocks. Mechanically, the optimal filter employed by the central bank introduces a stable root into the system, associated with the Kalman gain κ_r, and thereby leaves the endogenous forecast error undetermined. Although policy obeys the Taylor principle with $|\phi| > 1$, and there is a unique mapping from central bank projections to endogenous outcomes, equilibrium is generically indeterminate in the full model, in particular, the component that is orthogonal to the central bank's information set.[16]

A second observation is that the projection condition imposes restrictions on the set of multiple equilibria which stands in contrast to the typical indeterminacy case under full information. Optimal filtering restricts how private agents coordinate on an equilibrium, that is, which equilibrium is admissible and consistent with central bank projections. Although the effects of belief shocks with exogenous information are still unrestricted, the relationship between the fundamental real-rate shock and the measurement error is subject to a second moment restriction on their comovement.[17]

[16] By the logic of the root-counting approach to solving linear RE models, the system needs an 'unstable' root outside the unit circle to pin down the endogenous forecast error when there is one 'jump variable', namely inflation. In the FIRE case, this is provided by the policy parameter $|\phi| > 1$, while the Kalman filter introduces a stable root.

[17] From an empirical perspective, the FIRE solution results in a reduced-form representation for inflation that is first-order autoregressive. The LIRE solution on the other hand exhibits much richer dynamics. In particular, the resulting inflation process can be quite persistent when the signal-to-noise ratio is small since a large σ_ν^2 translates into a small Kalman gain.

However, this simple example is restrictive in that the central bank only observes an exogenous process with error. In the next step, we therefore analyze an endogenous signal which creates additional feedback within the model.

2.3.2 Equilibrium with an Endogenous Signal

We now assume that the central bank observes the inflation rate with measurement error ν_t such that $Z_t = \pi_t + \nu_t$. We present the analysis in terms of the projection equation for the real rate to facilitate comparison with the previous case:

$$r_{t|t} = r_{t|t-1} + \kappa_r \left(\pi_t - \pi_{t|t-1} + \nu_t\right). \tag{29}$$

This leads to the full equation system:

$$\begin{aligned}
\pi_t &= \frac{\phi}{\phi - \rho} r_{t-1|t-1} - r_{t-1} + \eta_t, \\
r_{t|t} &= \left(\rho + \kappa_r\right) r_{t-1|t-1} + \kappa_r r_{t-1} + \kappa_r \nu_t + \kappa_r \eta_t, \\
r_t &= \rho r_{t-1} + \varepsilon_t.
\end{aligned} \tag{30}$$

While the structure of the system is the same as before under exogenous information, the key difference is the coefficient $(\rho + \kappa_r)$ on the lagged real rate projection. In addition, real rate projections depend on the endogenous forecast error η_t. As a result, the solution for κ_r depends on the equilibrium law of motion for π_t.

To solve the model, we first derive the endogenous Kalman gain and the associated forecast error variance. We then derive the projection condition and assess consistency with the proposed equilibrium paths. The steady-state Kalman gain is $\kappa_r = cov\left(\widetilde{r}_t, \widetilde{Z}_t\right) \diagup var(\widetilde{Z}_t)$, where $\widetilde{r}_t = r_t - r_{t|t-1}$ and $\widetilde{Z}_t = \widetilde{\pi}_t + \nu_t$. As before, we decompose the endogenous forecast error $\eta_t = \gamma_\varepsilon \varepsilon_t + \gamma_\nu \nu_t + \gamma_b b_t$. It can be quickly verified that $cov\left(\widetilde{r}_t, \widetilde{Z}_t\right) = -\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2$. The negative sign in this expression reflects the inverse relationship between inflation and the real rate when the signal is endogenous. Using $\widetilde{\pi}_t = -\left(\widetilde{r}_{t-1} - \widetilde{r}_{t-1|t-1}\right) + \eta_t$, we find that $var(\widetilde{Z}_t)$ can be expressed as $var(\widetilde{Z}_t) = \Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 \sigma_b^2 + \left(1 + \gamma_\nu\right)^2 \sigma_\nu^2$.

We can now derive the following expression for the Kalman gain:

$$\kappa_r = \frac{-\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2}{\Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 \sigma_b^2 + \left(1 + \gamma_\nu\right)^2 \sigma_\nu^2}. \tag{31}$$

Although the forecast error variance Σ still needs to be determined as a function of the structural parameters, we can make two observations already. First, in contrast with the exogenous signal case the gain κ_r can be negative for small enough γ_ε, that is, $\kappa_r < 0$ if $\gamma_\varepsilon < \rho\Sigma/\sigma_\varepsilon^2$. Second, existence of a steady-state Kalman filter implies that $|\rho + \kappa_r| < 1$ as long as $\Sigma > 0$, as shown in Proposition 2 below. We return to a discussion of the case where $r_t = r_{t|t}$, so that $\Sigma = 0$, later in this section. In the next step, we compute the projection error variance $\Sigma = var\left(\widetilde{r}_t\right) - var\left(\widetilde{r}_{t|t}\right)$. Using

$var\left(\widetilde{r}_t\right) = \rho^2\Sigma + \sigma_\varepsilon^2$ and $var\left(\widetilde{r}_{t|t}\right) = \kappa_r cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$ we can derive the following Riccati equation, which is quadratic in Σ:

$$\Sigma = \rho^2\Sigma + \sigma_\varepsilon^2 - \frac{\left(-\rho\Sigma + \gamma_\varepsilon\sigma_\varepsilon^2\right)^2}{\Sigma + \gamma_\varepsilon^2\sigma_\varepsilon^2 + \gamma_b^2\sigma_b^2 + \left(1+\gamma_\nu\right)^2\sigma_\nu^2}. \tag{32}$$

Finally, an equilibrium has to obey the restrictions imposed by central bank projections, namely $\pi_{t|t} = \frac{1}{\phi-\rho} r_{t|t}$. This implies a covariance restriction of projection errors which differs from the exogenous signal case because of different information sets. Specifically, we have that $cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right) = \frac{1}{\phi-\rho} cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$ or alternatively $\left(\phi-\rho\right) cov\left(\widetilde{\pi}_t, \widetilde{\pi}_t + \nu_t\right) = cov\left(\widetilde{r}_t, \widetilde{\pi}_t + \nu_t\right)$. After some rearranging we can write this expression as:

$$\gamma_\nu\left(1+\gamma_\nu\right) = -\frac{\phi}{\phi-\rho}\frac{\Sigma}{\sigma_\nu^2} - \frac{\gamma_b^2}{\phi-\rho}\frac{\sigma_b^2}{\sigma_\nu^2} + \frac{\left[1-\left(\phi-\rho\right)\gamma_\varepsilon\right]\gamma_\varepsilon}{\phi-\rho}\frac{\sigma_\varepsilon^2}{\sigma_\nu^2}. \tag{33}$$

The condition places a quadratic restriction on all three innovation loadings γ in contrast to the linear restriction on γ_ε and γ_n and unrestricted γ_b in the exogenous signal case. This can imply that there are no or multiple solution to this equation, and thus for the overall equilibrium, for a given parameterization of the model. We summarize our findings in the following proposition.

PROPOSITION 2 (LIRE Equilibrium in the Fisher Economy with Endogenous Signal) *The set of stationary RE equilibria in the model (30) under LIRE with endogenous signal $Z_t = \pi_t + \nu_t$ is characterized by the following dynamic equations:*

$$\begin{aligned}
\pi_t &= \frac{\phi}{\phi-\rho} r_{t-1|t-1} - r_{t-1} + \gamma_\varepsilon\varepsilon_t + \gamma_\nu\nu_t + \gamma_b b_t, && (34)\\
r_{t|t} &= \left(\rho+\kappa_r\right) r_{t-1|t-1} - \kappa_r r_{t-1} + \kappa_r\gamma_\varepsilon\varepsilon_t + \kappa_r\left(1+\gamma_\nu\right)\nu_t + \kappa_r\gamma_b b_t, && (35)\\
r_t &= \rho r_{t-1} + \varepsilon_t,
\end{aligned}$$

With $\Sigma > 0$, we have:

$$\begin{aligned}
&\left|\rho+\kappa_r\right| < 1 && (36)\\
&\Sigma = \frac{1}{2}\left(\alpha + \sqrt{\alpha^2 + 4\beta}\right), && (37)\\
&\alpha = \left(1+2\rho\gamma_\varepsilon\right)\sigma_\varepsilon^2 - \left(1-\rho\right)^2\left(\gamma_\varepsilon^2\sigma_\varepsilon^2 + \gamma_b^2 + \left(1+\gamma_\nu\right)^2\sigma_\nu^2\right), && (38)\\
&\beta = \left(\gamma_b^2 + \left(1+\gamma_\nu\right)^2\sigma_\nu^2\right)\sigma_\varepsilon^2, && (39)\\
&\kappa_r = \frac{-\rho\Sigma + \gamma_\varepsilon\sigma_\varepsilon^2}{\Sigma + \gamma_\varepsilon^2\sigma_\varepsilon^2 + \gamma_b^2 + \left(1+\gamma_\nu\right)^2\sigma_\nu^2}, && (31)\\
&\gamma_\nu\left(1+\gamma_\nu\right) = -\frac{\phi}{\phi-\rho}\frac{\Sigma}{\sigma_\nu^2} - \frac{1}{\phi-\rho}\frac{\sigma_b^2}{\sigma_\nu^2} + \left(\frac{1}{\phi-\rho} - \gamma_\varepsilon\right)\gamma_\varepsilon\frac{\sigma_\varepsilon^2}{\sigma_\nu^2}. && (33)
\end{aligned}$$

Proof. *Equations* (37), (38) *and* (39) *follow directly from solving the quadratic equation for $\Sigma > 0$ in* (32). *The expression for the Kalman gain κ_r in* (31) *and the restrictions from the projection* *condition in* (33) *restate earlier results. The requirement that with* $\Sigma > 0$ *we must have* $|\rho + \kappa_r| < 1$ *is an application of Theorem 3 in Appendix A. In this specific example, the result that* $|\rho + \kappa_r| < 1$ *can be derived as follows: Consider the candidate value* $\kappa_r = 0$ *for the Kalman gain; in this case we would have* $\Sigma = \text{Var}(r_t)$*. The optimal Kalman gain seeks to minimize* Σ *and the optimal value of* Σ *must thus be (weakly) smaller than* $\text{Var}(r_t) = \sigma_\varepsilon^2/(1-\rho^2)$ *and finite. With the optimal Kalman gain, projections are given by* (35) *and the process for the projection errors* $r_t^* = r_t - r_{t|t}$ *is* $r_t^* = (\rho + \kappa_r) r_{t-1}^* + \varepsilon_t - \kappa_r(\eta_t + \nu_t)$*. Recall that* $\Sigma \equiv \text{Var}(r_t^*)$*. We can thus conclude that for* $0 < \Sigma < \infty$ *the optimal Kalman gain must be such that* $|\rho + \kappa_r| < 1$*.* ■

Proposition 2 describes the set of solutions under indeterminacy. With $|\rho + \kappa_r| < 1$ the equation system has only stable roots and therefore lacks a restriction to determine the endogenous forecast error uniquely. As in the case of an exogenous signal, the projection condition that ensures internal consistency of central bank and private sector expectation formation restricts the set of multiple equilibria. Specifically, an equilibrium with $\Sigma > 0$ does not exist when no innovation loadings can be found to ensure existence of a steady-state Kalman filter that is consistent with the projection condition. Moreover, the set of solutions is restricted over the parameter space by the nonlinear Riccati equation for the forecast error variance, by non-negativity constraints on variances and by ruling out complex solutions.[18]

In contrast to the exogenous signal case, feedback between filtering and model solution is central to equilibrium determination. Filtering depends on the information set, the result of which affects equilibrium outcomes and the content of the information set. This fixed-point problem has been noted before, at least as early as Sargent (1991). We go beyond this insight by showing that equilibrium determination is substantially different from the standard linear RE case. A solution may not exist even when the root-counting criterion for existence of an equilibrium indicates a sufficient number of stable roots in standard FIRE settings. While the root-counting approach for given κ_r could indicate non-existence, uniqueness or indeterminacy, it is the second-moment restrictions due to the less informed agent's filtering problem that determine equilibrium. In that sense, indeterminacy is generic in a LIRE environment since existence of a stable Kalman filter introduces a stable root into the dynamic system. At the same time, the second-moment restrictions resulting from the projection condition restrain belief shock loadings, which stands in stark contrast to the case of indeterminacy in a FIRE scenario.

2.4 Additional Results

The remainder of this section provides further insights and intuition for key results arising from our framework. First, we consider a case where the RE equilibrium under LIRE can appear determinate in the sense that the Kalman gain implies an explosive root that pins down the forecast error. In a second exercise, we show that the simple model implies an upper variance bound for the dynamics

[18]Section 4 provides a full set of numerical solutions for this simple example.

of the model. We discuss additional results in the Supplementary Appendix. These include a comparison with the framework of Svensson and Woodford (2004) that shares similarities with our approach, and a derivation of the model solution for an alternative monetary policy rule.

2.4.1 Equilibrium with $\Sigma = 0$ and an Explosive Root

The determinacy properties of a full-information, linear RE model depend on the number of unstable eigenvalues in the dynamic system. In a standard root-counting approach (for instance, Blanchard and Kahn, 1980), the equilibrium is unique if the number of explosive roots matches the number of forward-looking, or jump, variables. With fewer explosive roots, the equilibrium is indeterminate and non-existent otherwise. In the simple example model there is one jump variable, inflation, as evidenced by the presence of the endogenous forecast error η_t; in order to achieve determinacy, this jump variable should be matched by an explosive root. We consider whether this possibility can arise in our simple model given the two types of information sets. The respective dynamic RE equation systems are given in Propositions 1 and 2.

In the exogenous-signal case, Proposition 1 establishes that the Kalman gain lies between zero and one, $0 < \kappa_r < 1$.[19] Consequently, the projection equation is a stable difference equation and the absence of an unstable root means that η_t is not pinned down uniquely. While the projection condition restricts the set of equilibria in terms of the loadings on the stochastic disturbances, there is a multiplicity of solutions to this problem and indeterminacy is generic in this setting.

The case of an endogenous information set is different. We can define $r_t^* = r_t - r_{t|t}$ as the error from the projection onto the current information set and rewrite the equation slightly:

$$r_t^* = (\rho + \kappa_r)\, r_{t-1}^* + \varepsilon_t - \kappa_r \nu_t - \kappa_r \eta_t. \tag{40}$$

This is a first-order difference equation driven by a linear combination of stochastic terms: the exogenous real-rate innovation ε_t, the exogenous measurement error ν_t, and the endogenous forecast error η_t. The stability of the difference equation for r_t^* hinges on $|\rho + \kappa_r| < 1$. As demonstrated in Proposition 2, for $\mathrm{Var}\,(r_t^*) = \Sigma > 0$, $|\rho + \kappa_r| < 1$ is assured by the existence of a solution to the Kalman filter.

Now suppose that $|\rho + \kappa_r| > 1$. In this case, the only stationary solution is $r_t^* = 0$ and thus $\Sigma = 0$, which is achieved by letting $\eta_t = \frac{1}{\kappa_r}\varepsilon_t - \nu_t$, so that the endogenous forecast error is determined as a function of fundamentals alone.[20] We can verify the proposed solution by substituting the expression into the projection equation (35), which yields $r_t = \rho r_{t-1} + \varepsilon_t$. Substituting the solution into the inflation equation (34) leads to $\pi_t = \frac{\rho}{\phi - \rho} r_{t-1} + \frac{1}{\kappa_r}\varepsilon_t - \nu_t$, so that the loadings in the forecast error decomposition $\eta_t = \gamma_\varepsilon \varepsilon_t + \gamma_b b_t + \gamma_\nu \nu_t$ are $\gamma_\varepsilon = 1/\left(\phi - \rho\right)$, $\gamma_\nu = -1$, and $\gamma_b = 0$. The latter

[19] When the central bank only observes linear combinations of exogenous variables, the Kalman filtering problem can be solved independently from the rest of the model since the measurement equation contains only exogenous variables.

[20] From the perspective of the private sector the measurement error is a fundamental innovation in that it is a primitive of the model and affects outcomes in any equilibrium.

simply restates that belief shocks do not affect equilibrium outcomes when the standard eigenvalue condition for an equilibrium holds, namely that the number of unstable roots equals the number of jump variables.

The proposal equilibrium perfectly reveals one of the exogenous drivers r_t, while providing no signal about other shocks. In contrast, the equilibrium inflation rate depends on the measurement error ν_t and is not perfectly revealed to the central bank. The full-information solution is $\pi_t^{FI} = \frac{\rho}{\phi-\rho} r_{t-1} + \frac{1}{\phi-\rho}\varepsilon_t$. Comparing the LIRE and FIRE solution we therefore find that $\pi_t^{LI} = \pi_t^{FI} - \nu_t$, which suggests that $\kappa_r = \operatorname{Cov}(\tilde{r}_t, \tilde{\pi}_t + \nu_t) / \operatorname{Var}(\tilde{\pi}_t + \nu_t) = \operatorname{Cov}(\varepsilon_t, \eta_t + \nu_t) / \operatorname{Var}(\eta_t + \nu_t) = \phi - \rho$.[21] The root of the projection equation is thus $\rho + \kappa_r = \phi > 1$, which validates our original conjecture.

This equilibrium in the LIRE model with an endogenous information set is special in the sense that it superficially appears like the unique equilibrium in a FIRE setting: the solution is not affected by sunspot shocks while the forecast error is pinned down by matching the numbers of explosive roots and jump variables. It is not, however, a unique equilibrium in the sense that there is only one solution to the dynamic equation system for a given set of parameters. This is because the gain κ_r is endogenous and, as such, there can be other equilibria with a different gain. This special equilibrium is thus one of many multiple equilibria.[22] However, as discussed in section I.2 of the Supplementary Appendix, the existence of the special equilibrium is germane to the simple structure of this example economy, and typically does not extend to more general models.

2.4.2 Variance Bounds

The projection condition ensures that expectation formation of the different types of agents in the model is mutually consistent. As it turns out, this condition also provides bounds on the variances of the model's endogenous variables. Specifically, we show that the special equilibrium discussed above has the highest inflation variance of all equilibria in the LIRE setting despite having the plausibly desirable property that it is not driven by belief or sunspot shocks.[23]

The RE solution in the space of central bank projections is $\pi_{t|t} = \frac{1}{\phi-\rho} r_{t|t}$. This implies the projection condition $cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right) = \frac{1}{\phi-\rho} cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$ for information set Z_t, whereby we focus on the case $Z_t = \pi_t + \nu_t$. Expanding terms we find:

$$var(\widetilde{\pi}_t) + cov\left(\widetilde{\pi}_t, \nu_t\right) = \frac{1}{\phi-\rho} cov\left(\widetilde{\pi}_t, \widetilde{r}_t\right), \tag{41}$$

[21] Formally, the proposed solution when $|\rho + \kappa_r| > 1$ corresponds to an unstable, non-positive solution of the Riccati equation in (32). In this particular case, the non-positive solution of the Riccati equation is exactly equal to zero.

[22] It is somewhat akin to the result described in Lubik and Schorfheide (2003), where an indeterminate equilibrium without sunspots is observationally equivalent to a corresponding determinate equilibrium. However, in their case, this sunspot equilibrium without sunspots belongs to a set of equilibria that are continuous in the parameter space whereas the special equilibrium is discretely different from the set of equilibria in Proposition 2.

[23] Following Taylor (1977), Blanchard (1979) proposed to select equilibria based on a minimum variance criterion. The variance bounds presented here, suggest that the minimum-variance equilibrium is not an equilibrium driven solely by fundamental shocks, but an equilibrium where fluctuations are at least in part due to non-fundamental belief shocks.

where we have made use of the fact that $cov\left(\widetilde{r}_t, \nu_t\right) = 0$. Collecting terms, we can write:

$$var(\widetilde{\pi}_t) = cov\left(\widetilde{\pi}_t, \frac{1}{\phi-\rho}\widetilde{r}_t - \nu_t\right). \tag{42}$$

Using the Cauchy-Schwarz inequality the upper bound on the inflation projection error variance is given by:

$$var(\widetilde{\pi}_t) \leq var\left(\frac{1}{\phi-\rho}\widetilde{r}_t - \nu_t\right) = \left(\frac{1}{\phi-\rho}\right)^2 var(\widetilde{r}_t) + \sigma_\nu^2. \tag{43}$$

Since $\pi_t = \widetilde{\pi}_t + \pi_{t|t-1}$ and $\pi_{t|t-1} = \frac{1}{\phi-\rho} r_{t|t-1}$, we can derive the expression:

$$var(\pi_t) = var(\widetilde{\pi}_t) + \left(\frac{1}{\phi-\rho}\right)^2 var(r_{t|t-1}) + 2cov\left(\widetilde{\pi}_t, r_{t|t-1}\right), \tag{44}$$

whereby the covariance is zero under optimal projections. Similarly, $var(r_t) = var(\widetilde{r}_t) + var(r_{t|t-1})$. Substituting these expressions and collecting terms results in the following upper bound for the inflation variance:

$$var(\pi_t) \leq \left(\frac{1}{\phi-\rho}\right)^2 var(r_t) + \sigma_\nu^2 = \sigma_\nu^2 + \frac{\sigma_\varepsilon^2}{(1-\rho^2)(\phi-\rho)^2}. \tag{45}$$

The first term in the expression is the measurement error variance σ_ν^2, while the second term is the variance under FIRE with solution $\pi_t = \frac{1}{\phi-\rho} r_t$. In the special equilibrium, the solution for inflation is $\pi_t^S = \frac{1}{\phi-\rho} r_t - \nu_t$, with variance $var(\pi_t^S) = \sigma_\nu^2 + \frac{\sigma_\varepsilon^2}{(1-\rho^2)(\phi-\rho)^2}$. This expression is equal to the upper bound above. This implies that the inflation variance in the special equilibrium is the highest inflation variance of any equilibria under LIRE with an endogenous information set, i.e., $var(\pi_t) \leq var(\pi_t^S)$. Moreover, the variance bound is a direct implication of the projection condition. It may seem counterintuitive that an equilibrium in which sunspots do not matter exhibits more volatility than a sunspot equilibrium. In fact, it also runs counter to the comparable scenario in a standard determinacy analysis where sunspot shocks under a multiple equilibrium add excess volatility. At the same time, it highlights the different nature of equilibrium determination in our framework.

2.4.3 Closed-form Solutions with an Alternative Information Set

Even in the simple Fisher economy presented above, closed-form solutions are difficult to obtain due to the intricate fixed-point problem of the imperfect-information equilibrium. We now present a variant of the Fisher economy with a particular information set that enables us to derive a number of results in closed form. In contrast to the case discussed above, the specification of the simple example shown here features only amplification but no additional persistence of inflation due to belief shocks.

As before, the model combines an exogenous AR(1) process for the real rate with a Fisher equation and a Taylor rule that responds to the central bank's inflation projection:

$$
\begin{aligned}
i_t &= r_t + E_t\pi_{t+1}\,, & i_t &= \phi\,\pi_{t|t}\,, & |\phi| &> 1\,, \\
r_t &= \rho\, r_{t-1} + \varepsilon_t & \varepsilon_t &\sim iidN(0,\sigma_\varepsilon^2)\,, & |\rho| &< 1\,,
\end{aligned}
$$

and the projection condition requires $\pi_{t|t} = r_{t|t}/(\phi-\rho)$. As before, we express the endogenous forecast error as a linear combination of fundamental and belief shocks. Collecting terms yields the following characterization of the inflation process:

$$
\pi_{t+1} = -(r_t - r_{t|t}) + \frac{\rho}{\phi-\rho} r_t + \eta_{t+1} \tag{46}
$$

$$
\begin{aligned}
\text{with} \quad \eta_{t+1} &\equiv \pi_{t+1} - E_t\pi_{t+1} \\
&= \gamma_\varepsilon \varepsilon_{t+1} + \gamma_\nu \nu_{t+1} + \gamma_b b_{t+1}\,, \qquad b_{t+1} \sim iidN(0,1)\,,
\end{aligned} \tag{47}
$$

We now assume that the central bank's information set is characterized by a bivariate signal, which includes a perfect reading of the real rate and a noisy signal of current inflation:

$$
\boldsymbol{Z}_t = \begin{bmatrix} r_t \\ \pi_t + \nu_t \end{bmatrix} \qquad \text{with} \quad \nu_t \sim iidN(0,\sigma_\nu^2) \tag{48}
$$

$$
\Rightarrow \quad r_{t|t} = r_t \quad \Rightarrow \quad \pi_{t|t} = \frac{1}{\phi-\rho} r_t\,, \tag{49}
$$

In light of (49), the inflation dynamics specified in (46) simplify to

$$
\pi_{t+1} = \frac{\rho}{\phi-\rho} r_t + \eta_{t+1}\,. \tag{50}
$$

Before determining the shock loadings γ_ε, γ_ν, and γ_b of the endogenous forecast error η_t, we can already note that one-step-ahead expectations of inflation, $E_t\pi_{t+1} = r_t/(\phi-\rho)$, are identical to the full-information case so that the effects of indeterminacy will be limited to changes in the amplification of shocks, without consequences for inflation persistence.

In light of (50), we can conclude that the history of $\boldsymbol{Z}_t$ spans the same information content as the history of

$$
\boldsymbol{W}_t = \begin{bmatrix} \varepsilon_t \\ (1+\gamma_\nu)\,\nu_t + \gamma_b\, b_t \end{bmatrix}\,. \tag{51}
$$

$\boldsymbol{W}^t$ spans $\boldsymbol{Z}^t$ since, with $|\rho| < 1$, ε^t spans r^t, and since observing $(1+\gamma_\nu)\,\nu_t + \gamma_b\, b_t$ adds the same information to the span of ε^t as observing $\pi_t + \nu_t$.

Conveniently, the signal vector $\boldsymbol{W}_t$ consists of two mutually orthogonal elements that are serially uncorrelated over time. As a result, projections onto $\boldsymbol{W}^t$ can be decomposed into the sum of projections onto its individual elements. The projection condition (49) then requires $\eta_{t|t} = \varepsilon_{t|t}/(\phi-\rho)$, and with $\varepsilon_{t|t} = \varepsilon_t$, we can conclude that

$$
\gamma_\varepsilon = \frac{1}{\phi-\rho}\,. \tag{52}
$$

In addition we have $\pi_t^* \equiv \pi_t - \pi_{t|t} = \gamma_\nu\, \nu_t + \gamma_b\, b_t$, and thus $E\left(\gamma_\nu\, \nu_t + \gamma_b\, b_t \,\middle|\, \boldsymbol{Z}^t\right) = 0$, which implies the following restriction on γ_ν and γ_b:

$$\operatorname{Cov}\left(\gamma_\nu\, \nu_t + \gamma_b\, b_t \,\middle|\, (1+\gamma_\nu)\, \nu_t + \gamma_b\, b_t\right) = \gamma_\nu (1+\gamma_\nu)\sigma_\nu^2 + \gamma_b^2 = 0\,. \tag{53}$$

$$\Rightarrow \quad \gamma_\nu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\gamma_b^2}{\sigma_\nu^2}} \tag{54}$$

Real solutions to (54) require $|\gamma_b| < 0.5\, \sigma_\nu$ leading to a continuum of solutions with $\gamma_\nu \in [-1, 0]$. While there is a unique solution for the shock loading γ_ε, as given in (52), there are multiple solutions for γ_ν and γ_b as characterized by (54).

Equilibrium dynamics of inflation are described by (46) and (47) together with (52) and (54). Collecting terms, inflation evolves according to:

$$\pi_t = \frac{1}{\phi - \rho} r_t + \gamma_\nu\, \nu_t + \gamma_b\, b_t \tag{55}$$

where γ_ν and γ_b are restricted by (54). In light of the projection condition (53), the variance of inflation is given by

$$\operatorname{Var}(\pi_t) = \left(\frac{1}{\phi - \rho}\right)^2 \operatorname{Var}(r_t) + |\gamma_\nu| \sigma_\nu^2\,. \tag{56}$$

Since γ_ν is bounded by one in absolute value, there is an upper bound for the inflation variance equal to $\operatorname{Var}(r_t)/(\phi-\rho)^2 + \sigma_\nu^2$.

A key feature of this example is that belief shock loadings are not generally zero, and non-fundamental belief shocks can affect equilibrium outcomes. In addition, there is an upper bound on belief shock loadings. The upper bound on belief shock loadings stems from the projection condition and is unique to our imperfect information framework. When indeterminacy arises in full-information models, there are no such bounds on the scale with which belief shocks can affect economic outcomes.

In the absence of measurement error on inflation, $\sigma_\nu = 0$, the outcomes in our simplified example collapse to the full-information solution $\pi_t = r_t/(\phi - \rho)$ and equilibria are continuous with respect to the full-information case as σ_ν approaches zero. For any measurement error variance, the range of possible equilibria includes the case where outcomes are identical to the full-information case, with $\gamma_\nu = \gamma_b = 0$.

2.5 Discussion

At the core of our simple model is that the Taylor principle is not satisfied under imperfect information even though it holds in the corresponding set-up with full information.[24] The Taylor principle

[24] In our simple example economy, the Taylor principle requires $|\phi| > 1$, as discussed, among others, by Woodford (2003). For more general interest rate rules, Bullard and Mitra (2002) study requirements on interest-rate rule coefficients to ensure determinacy in the New Keynesian model.

prescribes a sufficiently strong response of the nominal policy rate to actual inflation. Deviating from it leads to sunspot-driven movements in private sector expectations that the central bank cannot invalidate through its actions. Even though there is a unique mapping between central bank projections of outcomes and economic conditions, actual outcomes remain indeterminate in our framework. In standard models such as Clarida, Gali and Gertler (2000) or Lubik and Schorfheide (2004) indeterminacy arises because the central bank conducts a policy that does not satisfy the Taylor principle. In contrast, in our limited information setting the central bank applies the Taylor principle with respect to its reaction to projections derived from an optimal filter, which then leads to an insufficiently strong reaction of policy to actual inflation. The source of the indeterminacy thus lies in the interaction of expectations formed under the two information sets.

When outcomes are not uniquely determined by economic fundamentals, there is a role for belief shocks to drive economic fluctuations. The term "belief shocks" refers to a set of economic disturbances that matter since people believe that they do. In general, these disturbances are otherwise unrelated to economic fundamentals.[25] We can think of the implications of belief shocks in terms of the following thought experiment. Suppose that the realization of a sunspot leads the private sector to believe that inflation is higher than warranted by economic fundamentals. This implies a reassessment of the nominal interest-rate path and a higher i_t in compensation for higher expected inflation. At this point, the behavior of the central bank is crucial. If the Taylor principle holds under FIRE, it would raise the policy rate by proportionally more than the private-sector's sunspot-driven belief. If the Taylor principle does not hold, the central bank raises the policy rate by proportionally less and thereby validates the original belief. Consequently, next period's expected inflation is a fraction ϕ of this period's inflation rate, see equation (6), so that the resulting equilibrium is indeterminate and subject to belief shocks.

A similar intuition holds in the LIRE case, with a subtle but crucial wrinkle that captures the core of our framework. We assume that the central bank follows a projection-based policy rule as in Svensson and Woodford (2004). This is arguably common central bank practice as real-time data are generally noisy and an informative signal needs to be extracted. The policy rule is $i_t = \phi\pi_{t|t}$ with $|\phi| > 1$. The central bank's inflation projection is therefore $\pi_{t|t} = (\phi - \rho)^{-1} r_{t|t}$ so that $i_t = \phi (\phi - \rho)^{-1} r_{t|t}$. In the case of an exogenous signal with a real rate projection equation $r_{t|t} = r_{t|t-1} + \kappa_r \left(r_t - r_{t|t-1} + \nu_t\right)$, the implied policy rule is then:

$$i_t = \frac{\phi}{\phi - \rho} r_{t|t-1} + \kappa_r \frac{\phi}{\phi - \rho} \left(r_t - r_{t|t-1} + \nu_t\right). \tag{57}$$

The source of indeterminacy in this case is that policy responds only to movements in exogenous variables and the measurement error. The interest rate therefore evolves autonomously of the remainder of the model with no feedback from an endogenous variable. This stems, of course, from the fact that the central bank's information set only contains real-time real rate observations and

[25] The use of the term "beliefs" is conceptually distinct from the "projections" described as part of our imperfect information setup, where projections are the result of the policymaker's optimal signal extraction efforts.

is thus almost akin to an interest-rate peg, which even in a FIRE model implies indeterminacy. In contrast to the FIRE case, the central bank responds to *projected* inflation which, consistent with the projection condition, the central bank knows to be a function of real rate projections. However, when the signal is exogenous, the projections do not contain a signal from actual inflation; instead, they reflect the average comovement between inflation and the signal in equilibrium. Therefore, monetary policy cannot invalidate beliefs that arise along a particular inflation trajectory.

These basic insights also apply to the case of an endogenous information set as presented in Proposition 2. We can derive an implied policy rule as before:

$$i_t = \frac{\phi}{\phi-\rho} r_{t|t-1} + \kappa_r \frac{\phi}{\phi-\rho}\left(\pi_t - \pi_{t|t-1} + \nu_t\right), \tag{58}$$

where the central bank observes current inflation with error. The resulting feedback from inflation movements to real rate projections implies that current inflation matters for the interest rate path so that the effective policy coefficient is $\kappa_r\phi/(\phi-\rho)$ instead of ϕ. Since κ_r is likely small and also within the unit circle it implies that the Taylor principle in terms of feedback from actual inflation to the policy rate is not satisfied as the response is less than proportional.[26]

Continuing our thought experiment, a sunspot-driven increase in inflation also affects the central-bank's projection process. Signal extraction is imperfect in the sense that the central bank adjusts its inflation projection somewhat upward as it cannot fully distinguish between the signal and the sunspot noise. Because of the size of the Kalman gain the resulting effective interest rate increase is smaller than would be warranted so that the sunspot-driven belief is validated. At the same time, the central bank's projection is consistent with the Taylor principle as it observes data subject to measurement error in its limited information set; whereas the private sector is aware of the actual data and takes into account the effective policy feedback in setting expectations.[27]

Equilibrium determination in our framework is conceptually different from the standard linear RE model. In the latter, the parameter space can typically be divided in three distinct regions of determinacy, indeterminacy, and non-existence. Given a specific parameterization the model solution is thus placed in one of the regions so that a reduced-form representation can be obtained. The set of multiple equilibria can be parameterized using the approach of Lubik and Schofheide (2003) or Farmer, Khramov, and Nicolò (2015) which then can be used to describe adjustment dynamics. This set of equilibria is essentially unrestricted. Although our imperfect information model shares some similarities, the key difference is that there is no corresponding partition of the parameter space. Equilibrium indeterminacy is generic in the sense that a root-counting approach would generally imply indeterminacy and would not pin down the forecast forecast errors uniquely, with the exception of the special case as discussed above.

At the same time, an equilibrium for a given parameterization may not exist because it is inconsistent with the projection condition or conditions derived from the computation of the gain

[26] This qualitative observation is borne out quantitatively by the numerical exercises in section 4.

[27] In this sense, our framework is similar to the set-up in Lubik and Matthes (2015) where learning about the economy in a real-time environment with measurement error and an optimal policy choice can engender indeterminacy.

coefficients while fulfilling the criteria of the root-counting approach. Again, this insight reflects the fact that in this imperfect information environment the solution of a linear RE system depends on second moments of that system which in turn are endogenous to the model solution. However, the set of multiple equilibria under LIRE is restricted by the projection condition in stark contrast to the standard case. We leave it to the numerical analysis in section 4 to assess the quantitative implications.

3 General Framework

We now introduce the general modelling framework, of which the analysis in the previous section is an introductory example. We begin by laying out a general class of expectational linear difference systems that feature conditional expectations of two types of agents with possibly different information sets. After reviewing rational expectations outcomes under full information, we turn to the imperfect information case, where one of the agents ("the policymaker") is strictly less informed than the other ("the private sector").

3.1 An Expectational Difference Equation with Two Information Sets

We consider linear, time-invariant equilibria that solve a system of linear expectational difference equations of the following form:

$$E_t \boldsymbol{S}_{t+1} + \hat{\boldsymbol{J}} \boldsymbol{S}_{t+1|t} = \boldsymbol{A} \boldsymbol{S}_t + \hat{\boldsymbol{A}} \boldsymbol{S}_{t|t} + \boldsymbol{A}_i \, \boldsymbol{i}_t \tag{59}$$

$$\boldsymbol{i}_t = \boldsymbol{\Phi}_i \boldsymbol{i}_{t-1} + \boldsymbol{\Phi}_J \boldsymbol{S}_{t+1|t} + \boldsymbol{\Phi}_A \boldsymbol{S}_{t|t} \tag{60}$$

$$\boldsymbol{S}_t = \begin{bmatrix} \boldsymbol{X}_t \\ \boldsymbol{Y}_t \end{bmatrix} \tag{61}$$

where $\boldsymbol{i}_t$ denotes a vector of policy instruments (typically a scalar) and $\boldsymbol{X}_t$ and $\boldsymbol{Y}_t$ are vectors of backward- and forward-looking variables, respectively.[28] There are N_x backward- and N_y forward-looking variables as well as N_i policy instruments. As in Klein (2000) and Svensson and Woodford (2004), the backward-looking variables are characterized by *exogenous* forecast errors, $\boldsymbol{\varepsilon}_t$:

$$\boldsymbol{X}_t - E_{t-1} \boldsymbol{X}_t = \boldsymbol{B}_{x\varepsilon} \, \boldsymbol{\varepsilon}_t \qquad \boldsymbol{\varepsilon}_t \sim N(\boldsymbol{0}, \boldsymbol{I}) \tag{62}$$

where the number of independent, exogenous shocks N_ε may be smaller than the number of backward-looking variables, N_x, while $\boldsymbol{B}_{x\varepsilon}$ is assumed to have full rank (i.e. $\boldsymbol{B}_{x\varepsilon}$ has N_ε independent columns). As in Klein (2000), we also assume that the initial value of the backward-looking variables, $\boldsymbol{X}_0$, is exogenously given. In contrast, forecast errors for the forward-looking variables,

[28] Throughout, vectors and matrices will be denoted with bold letters; notice, however, that our use of lower- and uppercase letters does *not* distinguish between matrices and vectors. In most applications, $\boldsymbol{i}_t$ is likely to be a scalar, but nothing in our framework hinges on this assumption and so we use the generic vector notation, $\boldsymbol{i}_t$, throughout. In our context, keeping the policy instrument separate from $\boldsymbol{X}_t$ and $\boldsymbol{Y}_t$ will be useful since $\boldsymbol{i}_t$ will always be assumed to be perfectly known and observable to both public and central bank.

denoted

$$\boldsymbol{\eta}_t \equiv \boldsymbol{Y}_t - E_{t-1}\boldsymbol{Y}_t, \tag{63}$$

are endogenous and remain to be determined as part of the model's RE solution.[29]

The pair of linear difference equations (59) and (60) is intended to capture the interdependent decision making of two kinds of agents.[30] Both agents form rational expectations, but conditional on different information sets, that will be described further below: One agent has access to full information about the state of the economy; in the applications considered in our paper, this would be a representative agent for the private sector also referred to as "the public". Private sector decisions are represented by (59), which also depends on the setting of a policy instrument $\boldsymbol{i}_t$ chosen by the other agent. The second agent is an imperfectly informed policymaker. In light of our applications, we synonymously refer to the policymaker also as "central bank." The policymaker sets $\boldsymbol{i}_t$ according to the rule given in (60). By definition, the policymaker must know the current value and history of her instrument choices. Moreover, all variables entering the policy rule (60) are expressed as expectations conditional on the central bank's information set, denoted $\boldsymbol{S}_{t+1|t}$ and $\boldsymbol{S}_{t|t}$.

The policymaker is supposed to form rational expectations based on an information set that is characterized by the observed history of a signal, denoted $\boldsymbol{Z}_t$, as well as knowledge of all model parameters.[31] For any variable $\boldsymbol{V}_t$, and any lead or lag h, $E_t\boldsymbol{V}_{t+h}$ denotes expectations based on full information whereas

$$\boldsymbol{V}_{t+h|t} \equiv E(\boldsymbol{V}_{t+h}|\boldsymbol{Z}^t) \qquad\qquad \boldsymbol{Z}^t = \{\boldsymbol{Z}_t, \boldsymbol{Z}_{t-1}, \boldsymbol{Z}_{t-2}, \ldots\} \tag{64}$$

denotes conditional expectations under the central bank information set.[32] For further use, it will be helpful to introduce the following notation for innovations $\tilde{\boldsymbol{V}}_t$ and residuals $\boldsymbol{V}_t^*$:

$$\tilde{\boldsymbol{V}}_t \equiv \boldsymbol{V}_t - \boldsymbol{V}_{t|t-1}\ , \qquad\qquad \boldsymbol{V}_t^* \equiv \boldsymbol{V}_t - \boldsymbol{V}_{t|t} = \tilde{\boldsymbol{V}}_t - \tilde{\boldsymbol{V}}_{t|t}\ . \tag{65}$$

Henceforth we will use the term "shocks" in reference to martingale difference sequences defined relative to the full information set, and the term "innovations" when referring to martingale difference sequences with respect to the central bank's information set.

By construction, central bank actions, $\boldsymbol{i}_t$, are spanned by the history of observed signals, such that we always have $\boldsymbol{i}_t = \boldsymbol{i}_{t|t}$; note that $\boldsymbol{i}_t$ merely reflects information contained in $\boldsymbol{Z}^t$ and need not be added to the description of the measurement vector, even though the policy instrument will

[29]Note that there are in principle N_y endogenous forecast errors; though, as will be seen shortly, their variance-covariance matrix need not have full rank.

[30]In our setup, equations (59) and (60) serve as primitives. In principle, each of these equations could represent a mere behavioral characterization or, alternatively, a description of optimal decision making in the form of a (linearized) first-order condition as in Svensson and Woodford (2004), for example.

[31]There is common knowledge about the structure of the economy and all model parameters.

[32]For notational convenience, knowledge of model paramters is suppressed when describing the information sets underlying conditional expectations.

not be explicitly listed as part of the measurement vector $\boldsymbol{Z}_t$. The measurement vector has N_z elements, and each is a linear combination of backward- and forward-looking variables:

$$\boldsymbol{Z}_t = \boldsymbol{H}\boldsymbol{S}_t = \boldsymbol{H}_x\boldsymbol{X}_t + \boldsymbol{H}_y\boldsymbol{Y}_t\,. \tag{66}$$

The measurement vector may also be affected by "measurement errors" — disturbances to the measurement equation that would otherwise be absent from a full-information version of the model. Such measurement errors are assumed to have been lumped into the vector of backward-looking variables, $\boldsymbol{X}_t$.[33]

3.2 Full Information Equilibrium

Our setup nests the case of full information when $\boldsymbol{S}_{t+h|t} = E_t\boldsymbol{S}_{t+h}\ \forall h \geq 0$, which holds, for example, when $\boldsymbol{H} = \boldsymbol{I}$ such that $\boldsymbol{Z}_t = \boldsymbol{S}_t$. The full-information system can easily be solved using familiar methods like those of Sims (2002), Klein (2000), or King and Watson (1998). We stack all variables, including the policy control, into a vector $\boldsymbol{\mathcal{S}}_t$ that is partitioned into a vector of $N_i + N_x$ backward-looking variables, $\boldsymbol{\mathcal{X}}_t$, and a vector of $N_y + N_i$ forward-looking variables, $\boldsymbol{\mathcal{Y}}_t$:[34]

$$\boldsymbol{\mathcal{S}}_t = \begin{bmatrix}\boldsymbol{\mathcal{X}}_t\\ \boldsymbol{\mathcal{Y}}_t\end{bmatrix} \qquad \text{where} \quad \boldsymbol{\mathcal{X}}_t = \begin{bmatrix}\boldsymbol{i}_{t-1}\\ \boldsymbol{X}_t\end{bmatrix} \qquad \boldsymbol{\mathcal{Y}}_t = \begin{bmatrix}\boldsymbol{Y}_t\,,\\ \boldsymbol{i}_t\end{bmatrix} \tag{67}$$

Using $\boldsymbol{\mathcal{S}}_t' = \begin{bmatrix}\boldsymbol{i}_{t-1}' & \boldsymbol{S}_t' & \boldsymbol{i}_t'\end{bmatrix}$, the dynamics of the system under full information are then characterized by the following expectational difference equation:

$$\underbrace{\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}+\hat{\boldsymbol{J}} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{\Phi}_J & \boldsymbol{0}\end{bmatrix}}_{\boldsymbol{\mathcal{J}}} E_t\boldsymbol{\mathcal{S}}_{t+1} = \underbrace{\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}\\ \boldsymbol{0} & \boldsymbol{A}+\hat{\boldsymbol{A}} & \boldsymbol{A}_i\\ -\boldsymbol{\Phi}_i & -\boldsymbol{\Phi}_H & \boldsymbol{I}\end{bmatrix}}_{\boldsymbol{\mathcal{A}}} \boldsymbol{\mathcal{S}}_t \tag{68}$$

Throughout, we focus on environments where a unique full-information solution exists, and assume that the following assumption holds:

ASSUMPTION 1 (Unique full-information solution) *The pencil $|\boldsymbol{\mathcal{J}}\,z - \boldsymbol{\mathcal{A}}|$, with $\boldsymbol{\mathcal{J}}$ and $\boldsymbol{\mathcal{A}}$ as defined in* (68)*, is a regular pencil and has $N_i + N_x$ roots inside the unit circle and $N_y + N_i$ roots outside the unit circle.*[35]

As shown in Klein (2000) or King and Watson (1998), Assumption 1 ensures the existence of a unique equilibrium under full information.[36] The solution has the following form, and can be

[33] By construction, we have then $\boldsymbol{Z}_{t|t} = \boldsymbol{H}\boldsymbol{S}_{t|t} = \boldsymbol{Z}_t$ and thus $\boldsymbol{H}\,\mathrm{Var}\left(\boldsymbol{S}_t|\boldsymbol{Z}^t\right)\boldsymbol{H}' = \boldsymbol{0}$.

[34] The presence of the lagged policy control in $\boldsymbol{\mathcal{X}}_t$ serves to handle the case of interest-rate smoothing, $\boldsymbol{\Phi}_i \neq 0$, and can otherwise be omitted. In the case of interest rate smoothing, $\boldsymbol{i}_{t-1}$ enters the system as a backward-looking variables. In the setups of Klein (2000) or King and Watson (1998), it is required that all backward-looking variables be placed at the top of $\boldsymbol{\mathcal{S}}_t$.

[35] The pencil is regular if there is some complex number z such that $|\boldsymbol{\mathcal{J}}\,z - \boldsymbol{\mathcal{A}}| \neq 0$.

[36] In the case of the simple Fisher economy in section 2, the root-counting condition was satisfied by requiring that the central bank's interest-rate rule satisfied the Taylor principle, responding more than one-to-one to fluctuations in inflation.

computed, for example, using the numerical methods of Klein (2000):

$$E_t \boldsymbol{\mathcal{X}}_{t+1} = \boldsymbol{\mathcal{P}}\, \boldsymbol{\mathcal{X}}_t\,, \qquad \boldsymbol{\mathcal{Y}}_t = \boldsymbol{\mathcal{G}}\, \boldsymbol{\mathcal{X}}_t\,, \qquad \boldsymbol{\mathcal{G}} = \begin{bmatrix} \boldsymbol{\mathcal{G}}_{yi} & \boldsymbol{\mathcal{G}}_{yx} \\ \boldsymbol{\mathcal{G}}_{ii} & \boldsymbol{\mathcal{G}}_{ix} \end{bmatrix}. \tag{69}$$

where $\boldsymbol{\mathcal{P}}$ is a stable matrix.[37] Certainty equivalence holds, and the decision-rule coefficients $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{G}}$ do not depend on the shock variances encoded in $\boldsymbol{B}_{x\varepsilon}$ and, of course, not on the measurement loadings $\boldsymbol{H}$ either.[38] Equilibrium dynamics in the full-information case are then summarized by:

$$\boldsymbol{\mathcal{S}}_{t+1} = \bar{\boldsymbol{\mathcal{T}}}\boldsymbol{\mathcal{S}}_t + \bar{\boldsymbol{\mathcal{H}}}\boldsymbol{\varepsilon}_{t+1}\,, \qquad \bar{\boldsymbol{\mathcal{T}}} = \begin{bmatrix} \boldsymbol{\mathcal{P}} & \mathbf{0} \\ \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{P}} & \mathbf{0} \end{bmatrix}, \qquad \bar{\boldsymbol{\mathcal{H}}} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{I} \\ \boldsymbol{\mathcal{G}}_{yx} \\ \boldsymbol{\mathcal{G}}_{ix} \end{bmatrix} \boldsymbol{B}_{x\varepsilon}\,, \tag{70}$$

where $\bar{\boldsymbol{\mathcal{T}}}$ is stable because $\boldsymbol{\mathcal{P}}$ is, and the endogenous forecast errors are given by $\boldsymbol{\eta}_t = \boldsymbol{\mathcal{G}}_{yx}\, \boldsymbol{B}_{x\varepsilon}\, \boldsymbol{\varepsilon}_t$.

3.3 Expectation Formation in Imperfect Information Equilibria

In the imperfect-information case, we are interested in linear equilibria, driven by normally distributed disturbances, so that a Kalman filter delivers an exact representation of the true conditional expectations. Hence, we make the following assumption:

ASSUMPTION 2 (Jointly normal forecast errors) *The endogenous forecast errors are a linear combination of the N_ε exogenous errors, $\boldsymbol{\varepsilon}_t$, and N_y so-called belief shocks, $\boldsymbol{b}_t$, that are mean zero and uncorrelated with $\boldsymbol{\varepsilon}_t$:*

$$\boldsymbol{\eta}_t = \boldsymbol{\Gamma}_\varepsilon \boldsymbol{\varepsilon}_t + \boldsymbol{\Gamma}_b \boldsymbol{b}_t \tag{71}$$

Moreover, exogenous shocks and belief shocks are generated from a joint standard normal distribution,

$$\boldsymbol{w}_t \equiv \begin{bmatrix} \boldsymbol{\varepsilon}_t \\ \boldsymbol{b}_t \end{bmatrix} \sim N\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix} \right) \tag{72}$$

As a corollary, exogenous and endogenous forecast errors are joint normally distributed as well:

$$\begin{bmatrix} \boldsymbol{\varepsilon}_t \\ \boldsymbol{\eta}_t \end{bmatrix} \sim N\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \boldsymbol{I} & \boldsymbol{\Gamma}_\varepsilon' \\ \boldsymbol{\Gamma}_\varepsilon & \boldsymbol{\Omega}_\eta \end{bmatrix} \right), \qquad \textit{with} \quad \boldsymbol{\Omega}_\eta = \boldsymbol{\Gamma}_b \boldsymbol{\Gamma}_b' + \boldsymbol{\Gamma}_\varepsilon \boldsymbol{\Gamma}_\varepsilon'\,. \tag{73}$$

The matrix of belief shock loadings $\boldsymbol{\Gamma}_b$ need not have full rank so that linear combinations of endogenous and exogenous forecast errors might be perfectly correlated.

[37] A stable matrix has all eigenvalues inside the unit circle.

[38] As noted before, our imperfect information setup would include measurement errors as part of the vector or backward-looking variables, $\boldsymbol{X}_t$. The measurement errors would affect endogenous variables of the system only via $\boldsymbol{H}$, which does not play a role in the full information solution. But, also when computing a full-information solution, there is no harm including measurement errors in $\boldsymbol{X}_t$: The corresponding columns of $\boldsymbol{\mathcal{G}}_{yx}$ — as generated, for example, by the procedures of Klein (2000) or King and Watson (1998)— are set to zero in this case.

Assumption 2 nests properties of the full-information case, where - under the maintained assumption of a unique full-information equilibrium - endogenous forecast errors are a linear combination of the exogenous errors, and could be *perfectly* recovered by a regression of $\boldsymbol{\eta}_t$ on $\boldsymbol{\varepsilon}_t$. Joint normality of exogenous and endogenous errors then follows directly from the assumed normality of the exogenous errors. While Assumption 2 allows for part of $\boldsymbol{\eta}_t$ to be unrelated to the exogenous errors, this part takes the form of additive belief shocks that are normally distributed. In a linear equilibrium, where (71) holds, the belief shocks affect outcomes only via the product $\boldsymbol{\Gamma}_b \, \boldsymbol{b}_t$; thus, without loss of generality, $\boldsymbol{b}_t$ can be normalized to have a variance-covariance matrix equal to the identity matrix.[39]

For now, we treat the shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ as given and characterize a class of equilibria where central bank expectations are represented by a Kalman filter. Afterwards, we turn to solution methods that determine values for $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ consistent with these equilibria. Throughout this paper, we limit attention to a particular class of equilibria referred to as "stationary, linear and time-invariant equilibria," that are formally defined as follows:

DEFINITION 1 (Stationary, linear, time-invariant equilibrium) *In a stationary, linear, and time-invariant equilibrium, forward- and backward-looking variables, $\boldsymbol{Y}_t$ and $\boldsymbol{X}_t$, as well as the policy instrument, $\boldsymbol{i}_t$, are stationary and their equilibrium dynamics satisfy the expectational difference system described by* (59) *and* (60).[40] *All expectations are rational, and the imperfectly informed agent's information set is described by* (66). *In addition, Assumption 2 holds, which means that the forecast errors of the forward-looking variables are a linear combination of fundamental shocks and belief shocks, with time-invariant loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$, as in* (71)*, and normally distributed belief shocks $\boldsymbol{b}_t$.*

As argued next, in such an equilibrium, conditions are in place to ensure that the central bank's conditional expectations, as defined in (64) can be represented by a Kalman filter. The measurement equation of the central bank is given by (66). The state equation of the central bank's filtering problem is given by

$$\boldsymbol{S}_{t+1} + \hat{\boldsymbol{J}}\boldsymbol{S}_{t+1|t} = \boldsymbol{A}\boldsymbol{S}_t + \hat{\boldsymbol{A}}\boldsymbol{S}_{t|t} + \boldsymbol{A}_i \, \boldsymbol{i}_t + \boldsymbol{B}\boldsymbol{w}_{t+1} \, , \qquad \text{with} \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_{x\varepsilon} & \boldsymbol{0} \\ \boldsymbol{\Gamma}_\varepsilon & \boldsymbol{\Gamma}_b \end{bmatrix} , \tag{74}$$

which combines the expectational difference equation (59) with the implications of Assumption 2 for the endogenous forecast errors. The appearance of projections $\boldsymbol{S}_{t+1|t}$ and $\boldsymbol{S}_{t|t}$ in (74) lends this state equation a slightly non-standard format. However, when expressed in terms of innovations, the filtering problem can be cast in the canonical "ABCD" form, studied, among others, by Fernández-

[39] Note further that Assumption 2 could equivalently by restated by assuming that $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{\eta}_t$ are joint normally distributed zero-mean shocks. The linear relationship in (71) between endogenous and exogenous shocks then follows from a regression of $\boldsymbol{\eta}_t$ on $\boldsymbol{\varepsilon}_t$, which characterizes the distribution of $\boldsymbol{\eta}_t$ conditional on $\boldsymbol{\varepsilon}_t$. Viewed from this perspective, the belief shocks, $\boldsymbol{b}_t$, emerge as the standardized regression residual that is orthogonal to $\boldsymbol{\varepsilon}_t$.

[40] Note that all of the linear equilibria considered in this paper are driven by normally distributed shocks, leading to normally distributed outcomes, such that covariance stationarity also implies strict stationarity. Hence, we will not distinguish between both concepts and merely refer to stationarity.

Villaverde, Rubio-Ramírez, Sargent and Watson (2007):[41]

$$\tilde{\boldsymbol{S}}_{t+1} = \boldsymbol{A}\left(\tilde{\boldsymbol{S}}_t - \tilde{\boldsymbol{S}}_{t|t}\right) + \boldsymbol{B}\boldsymbol{w}_{t+1}\,, \tag{75}$$

$$\tilde{\boldsymbol{Z}}_{t+1} = \boldsymbol{C}\left(\tilde{\boldsymbol{S}}_t - \tilde{\boldsymbol{S}}_{t|t}\right) + \boldsymbol{D}\boldsymbol{w}_{t+1}\,, \tag{76}$$

$$\text{with} \quad \boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}\,, \quad \boldsymbol{D} = \boldsymbol{H}\boldsymbol{B}\,, \quad \text{since} \quad \tilde{\boldsymbol{Z}}_{t+1} = \boldsymbol{H}\tilde{\boldsymbol{S}}_{t+1}\,. \tag{77}$$

To ensure a well-behaved filtering problem, we impose the following assumption on the shocks to the central bank's measurement vector $\boldsymbol{D}\boldsymbol{w}_t = \boldsymbol{Z}_t - E_{t-1}\boldsymbol{Z}_t$.

ASSUMPTION 3 (Non-degenerate shocks to the signal equation) *Shocks to the central bank's measurement equation have a full-rank variance-covariance matrix; that is* $|\boldsymbol{D}\boldsymbol{D}'| \neq 0$.

A necessary condition for Assumption 3 to hold is that the signal vector has not more elements than the sum of endogenous and exogenous forecast errors: $N_z \leq N_\varepsilon + N_y \leq N_x + N_y$.

Together with Assumption 3, certain conditions on $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{H}$ known as "observability" and "unit-circle controllability" ensure the existence of a steady state Kalman filter; details are provided in the appendix. As shown in (74), $\boldsymbol{B}$ depends on the yet to be determined shock loadings $\boldsymbol{\Gamma}_\varepsilon$, $\boldsymbol{\Gamma}_b$. Further below, we will discuss how the conditions for the existence of a steady state filter impose only weak restrictions on these shock loadings.

ASSUMPTION 4 (Sufficient condition for existence of a steady-state Kalman filter) *The equilibrium shock-loadings* $\boldsymbol{\Gamma}_\varepsilon$, $\boldsymbol{\Gamma}_b$ *of the endogenous forecast errors* $\boldsymbol{\eta}_t$ *are such that* $\boldsymbol{A}$, $\boldsymbol{B}$ *and* $\boldsymbol{H}$ *are detectable and unit-circle controllable as stated in Definition 5 of the appendix.*

In general, central bank projections, $\boldsymbol{S}_{t|t}$ can be decomposed into central bank forecasts made in the previous period, $\boldsymbol{S}_{t|t-1}$, and an update reflecting the innovations in measurement vector. When a steady-state filter exists, the expectational update is linear and a constant Kalman gain matrix relates the projected innovations in the state vector, $\tilde{\boldsymbol{S}}_{t|t}$, to innovations in the measurement vector, $\tilde{\boldsymbol{Z}}_t$:

$$\boldsymbol{S}_{t|t} = \boldsymbol{S}_{t|t-1} + \tilde{\boldsymbol{S}}_{t|t} \qquad \text{with} \quad \tilde{\boldsymbol{S}}_{t|t} = \boldsymbol{K}\tilde{\boldsymbol{Z}}_t\,, \qquad \text{and} \quad \boldsymbol{K} = \operatorname{Cov}(\tilde{\boldsymbol{S}}_t, \tilde{\boldsymbol{Z}}_t)\left(\operatorname{Var}(\tilde{\boldsymbol{Z}}_t)\right)^{-1}. \tag{78}$$

As shown in the appendix, the Kalman gain matrix, $\boldsymbol{K}$, is given by the solution of a standard Riccati equation involving $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{H}$. To remain consistent with the equilibrium properties laid out in Definition 1, we limit attention to the case when a steady-state Kalman filter exists, which enables us to represent the central bank's conditional expectations as a recursive system of linear projections with time-invariant coefficients.

[41] The innovations form is obtained by projecting both sides of (74) onto $\boldsymbol{Z}^t$ and subtracting these projections from (74). When doing so, note that the policy instrument $\boldsymbol{i}_t$ is the central bank's decision variable and thus always in the central bank's information set. Notice that the innovations form given by (75) and (76) is identical to the innovations form of a state space system with $\boldsymbol{S}_{t+1} = \boldsymbol{A}\boldsymbol{S}_t + \boldsymbol{B}\boldsymbol{w}_{t+1}$ in place of (74) while maintaining (66) as measurement equation, as noted also by (Baxter, Graham and Wright 2011).

PROPOSITION 3 (Existence of steady-state Kalman filter) *When Assumptions 3 and 4 hold, a steady state Kalman filter exists that describes the projection of innovations in the state vector,* $\tilde{\boldsymbol{S}}_{t|t} = \boldsymbol{K}\tilde{\boldsymbol{Z}}_t$ *with a constant Kalman gain* $\boldsymbol{K}$ *as in* (78)*. Moreover, the variance-covariance matrix of projection residuals is constant,* $\mathrm{Var}\left(\boldsymbol{S}_t|\boldsymbol{Z}^t\right) = \mathrm{Var}\left(\boldsymbol{S}_t^*\right) = \boldsymbol{\Sigma}^*$*. Existence of a steady state Kalman filter ensures that innovations* $\tilde{\boldsymbol{S}}_t = \boldsymbol{S}_t - \boldsymbol{S}_{t|t-1}$ *and residuals* $\boldsymbol{S}_t^* = \boldsymbol{S}_t - \boldsymbol{S}_{t|t}$ *are stationary. Innovations to the measurement equation,* $\tilde{\boldsymbol{Z}}_t$*, are stationary as well.*

Proof. *See Theorem 3 of Appendix A. The stationarity of* $\tilde{\boldsymbol{Z}}_t = \boldsymbol{H}\tilde{\boldsymbol{S}}_t$ *follows then from the stationarity of* $\tilde{\boldsymbol{S}}_t$*.* ■

Equation (78) describes the update from policymaker forecasts, $\boldsymbol{S}_{t|t-1}$, to current projections, $\boldsymbol{S}_{t|t}$. What remains to be characterized is the transition equation from policymakers' projections $\boldsymbol{S}_{t|t}$ to their forecasts, $\boldsymbol{S}_{t+1|t}$, which is restricted by the linear difference equations (59) and the rule (60) for the policy instrument $\boldsymbol{i}_t$. Since the policy instrument may depend on its own lagged value via $\boldsymbol{\Phi}_i \neq \boldsymbol{0}$ in (60), we construct the transition from $\boldsymbol{S}_{t|t}$ to $\boldsymbol{S}_{t+1|t}$ based on the vector $\boldsymbol{\mathcal{S}}_t$, which includes $\boldsymbol{S}_t$ and $\boldsymbol{i}_{t-1}$, as defined in (67).

Conditioning down (59) and (60) onto the information set of the policymaker, $\boldsymbol{Z}^t$, yields a system of expectational linear difference equations in $\boldsymbol{\mathcal{S}}_t$ that is akin to the full-information system shown in (68), except for the use of policymaker projections in lieu of full-information expectations:

$$\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{S}}_{t+1|t} = \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{S}}_{t|t} \tag{79}$$

with $\boldsymbol{\mathcal{J}}$, and $\boldsymbol{\mathcal{A}}$ as defined in (68) above. In a stationary equilibrium consistent with Definition 1, $\boldsymbol{\mathcal{S}}_t$ is stationary, and so is its projection $\boldsymbol{\mathcal{S}}_{t|t}$. When the equilibrium is linear and time-invariant, the projections $\boldsymbol{\mathcal{S}}_{t|t}$ must follow

$$\boldsymbol{\mathcal{S}}_{t+1|t} = \boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{S}}_{t|t}\ , \qquad \text{with} \quad \left(\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{T}} - \boldsymbol{\mathcal{A}}\right)\boldsymbol{\mathcal{S}}_{t|t} = 0 \qquad \text{for some stable matrix } \boldsymbol{\mathcal{T}}. \tag{80}$$

$\boldsymbol{\mathcal{T}}$ spans a stable, invariant subspace of the matrix pencil $|\boldsymbol{\mathcal{J}}\,z - \boldsymbol{\mathcal{A}}|$. In principle, several choices of $\boldsymbol{\mathcal{T}}$ could satisfy this criterion. To see this, think of any $\boldsymbol{\mathcal{T}}$ whose columns include a (sub)set of stable eigenvectors of the pencil plus columns of zeros. The simple example of Section 2, is characterized by a dynamic system with only two non-zero eigenvalues, and a single, one-dimensional stable subspace of the associated matrix pencil. In this case, there is a single choice of $\boldsymbol{\mathcal{T}}$ consistent with the equilibrium, and it is identical to the full-information transition matrix $\bar{\boldsymbol{\mathcal{T}}}$.

In addition, $\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{T}} - \boldsymbol{\mathcal{A}}$ must be orthogonal to the space of projections $\boldsymbol{\mathcal{S}}_{t|t}$. Before discussing further the determination of $\boldsymbol{\mathcal{T}}$, note that in a linear equilibrium with normally distributed shocks with a given linear transformation from projections $\boldsymbol{\mathcal{S}}_{t|t}$ into forecasts $\boldsymbol{\mathcal{S}}_{t+1|t}$, the Kalman filter represents conditional expectations of $\boldsymbol{\mathcal{S}}_t$ (and thus also $\boldsymbol{S}_t$).

PROPOSITION 4 (Kalman filter represents conditional expectations) *When the conditions for Proposition 3 hold, and for a given stable transition matrix* $\boldsymbol{\mathcal{T}}$ *between policymaker projections and forecasts as in* (80)*, the steady state Kalman filter represents conditional expectations*

$\boldsymbol{\mathcal{S}}_{t|t} = E\left(\boldsymbol{\mathcal{S}}_t|\boldsymbol{Z}^t\right)$ *in a linear, time-invariant stationary equilibrium. For a given sequence of innovations in the measurement vector,* $\tilde{\boldsymbol{Z}}_t$*, the Kalman filter implies the following, stationary evolution of projections:*

$$\boldsymbol{\mathcal{S}}_{t+1|t+1} = \boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{S}}_{t|t} + \boldsymbol{\mathcal{K}}\tilde{\boldsymbol{Z}}_{t+1}\,, \qquad \textit{with} \quad \boldsymbol{\mathcal{K}} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{K}_x \\ \boldsymbol{K}_y \\ \boldsymbol{K}_i \end{bmatrix} \tag{81}$$

Proof. *In a linear, time-invariant stationary equilibrium shocks are jointly normal and propagate linearly so that the sequences of* $\boldsymbol{\mathcal{S}}_t$ *and* $\boldsymbol{Z}_t$ *are joint normally distributed, so that conditional expectations are identical to mean-squared-error optimal linear projections. By the law of iterated projections, we can decompose* $E\left(\boldsymbol{\mathcal{S}}_{t+1}|\boldsymbol{Z}^{t+1}\right) = E\left(\boldsymbol{\mathcal{S}}_{t+1}|\boldsymbol{Z}^t\right) + E\left(\tilde{\boldsymbol{\mathcal{S}}}_{t+1}|\tilde{\boldsymbol{Z}}_{t+1}\right)$ *and we have* $E\left(\boldsymbol{\mathcal{S}}_{t+1}|\boldsymbol{Z}^t\right) = \boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{S}}_{t|t}$ *based on* (80)*.* $E\left(\tilde{\boldsymbol{\mathcal{S}}}_{t+1}|\tilde{\boldsymbol{Z}}_{t+1}\right) = \boldsymbol{\mathcal{K}}\tilde{\boldsymbol{Z}}_{t+1}$ *follows from Proposition 3.* $\boldsymbol{K}_x$ *and* $\boldsymbol{K}_y$ *are appropriate partitions of* $\boldsymbol{K}$ *as defined in* (78) *and* $\boldsymbol{K}_i = \mathrm{Cov}\left(\tilde{\boldsymbol{i}}_t, \tilde{\boldsymbol{Z}}_t\right)\left(\mathrm{Var}\left(\tilde{\boldsymbol{Z}}_t\right)\right)^{-1}$*.*[42] *The upper block of* $\boldsymbol{\mathcal{K}}$*, corresponding to the Kalman gain coefficients for the lagged policy instrument, are zero since* $\boldsymbol{i}_{t-1} = \boldsymbol{i}_{t-1|t-1}$ *and thus* $\tilde{\boldsymbol{i}}_{t-1|t} = \boldsymbol{i}_{t-1|t} - \boldsymbol{i}_{t-1|t-1} = 0$. ■

3.4 A Class of Imperfect Information Equilibria

As noted above, there can be multiple solutions for $\boldsymbol{\mathcal{T}}$ in (80). As an application of certainty equivalence, a valid choice for $\boldsymbol{\mathcal{T}}$ is $\bar{\boldsymbol{\mathcal{T}}}$, known from the full-information solution given in (70).[43] In the full-information case, and under the maintained assumption that Assumption 1 holds, $\bar{\boldsymbol{\mathcal{T}}}$ characterizes the unique solution to the difference equation $\boldsymbol{\mathcal{J}}E_t\boldsymbol{\mathcal{S}}_{t+1} = \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{S}}_t$.

In the imperfect information case, $\boldsymbol{\mathcal{T}}$ need not be a unique solution. However, the multiplicity of equilibria highlighted in our paper does not stem from the implications of choosing different $\boldsymbol{\mathcal{T}}$. We rather focus solely on equilibria based on $\boldsymbol{\mathcal{T}} = \bar{\boldsymbol{\mathcal{T}}}$, which is consistent with the approach of Svensson and Woodford (2004) who assume that equilibrium is unique in a setup similar to ours.[44]. In order to ensure $\boldsymbol{\mathcal{T}} = \bar{\boldsymbol{\mathcal{T}}}$, we follow Svensson and Woodford (2004) and impose the following condition:[45]

DEFINITION 2 (Projection Condition) *The projection condition restricts the mapping between projected backward- and forward-looking variables to be identical to the full-information case:*

$$\boldsymbol{\mathcal{Y}}_{t|t} = \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{X}}_{t|t}\,, \qquad \textit{and} \quad \boldsymbol{\mathcal{X}}_{t+1|t} = \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{X}}_{t|t}\,. \tag{82}$$

where $\boldsymbol{\mathcal{G}}$ *and* $\boldsymbol{\mathcal{P}}$ *are the unique solution coefficients in the corresponding full-information case.*

[42] The value for $\boldsymbol{K}_i$ could be computed based on the policy rule (60) and the dynamics of $\boldsymbol{S}_t$; however, further below we utilize a more direct approach.

[43] To verify the validity of $\bar{\boldsymbol{\mathcal{T}}}$ as a solution to (79), note that $\bar{\boldsymbol{\mathcal{T}}}$ solves (68) and that (79) represents the same difference equation, when projected onto $\boldsymbol{Z}^t$.

[44] Applications that build on Svensson and Woodford (2004) are, for example, Dotsey and Hornstein (2003), Aoki (2006),Nimark (2008b), Carboni and Ellison (2011).

[45] Note that (80) corresponds to their equation (42), and that our projection condition (82) corresponds to their (42), (45), (46), and (47).

The projection condition is an equilibrium condition that imposes a linear mapping between projections of backward- and forward-looking variables. In particular, the projection condition imposes a *second-moment* restriction on the joint distribution of the innovations $\tilde{\boldsymbol{X}}_t$, $\tilde{\boldsymbol{Y}}_t$.[46] As a second-moment restriction, the projection condition restricts only co-movements of the innovations *on average* but not for any particular realization of $\tilde{\boldsymbol{X}}_t$ and $\tilde{\boldsymbol{Y}}_t$. The upshot of the projection condition (82) is the following restriction between Kalman gains of forward- and backward-looking variables:

$$\boldsymbol{Y}_{t|t} = \boldsymbol{\mathcal{G}}_{yx}\, \boldsymbol{X}_{t|t} + \boldsymbol{\mathcal{G}}_{yi}\, \boldsymbol{i}_{t-1|t} \quad \Longrightarrow \tilde{\boldsymbol{Y}}_{t|t} = \boldsymbol{\mathcal{G}}_{yx}\, \tilde{\boldsymbol{X}}_{t|t} \quad \Longleftrightarrow \quad (\boldsymbol{K}_y - \boldsymbol{\mathcal{G}}_{yx}\, \boldsymbol{K}_x)\, \tilde{\boldsymbol{Z}}_t = 0\,, \qquad (83)$$

where $\boldsymbol{K}_y$ and $\boldsymbol{K}_x$ denote the corresponding partitions of the Kalman gain, $\boldsymbol{K}$, defined in (78).[47] Since (83) must hold for every $\tilde{\boldsymbol{Z}}_t$, the projection condition implies a restriction on the Kalman gains, summarized in the following proposition.

PROPOSITION 5 (Projection Condition for Kalman gains) *The projection condition* (82) *holds only if the Kalman gains satisfy* $\boldsymbol{K}_y = \boldsymbol{\mathcal{G}}_{yx}\, \boldsymbol{K}_x$.
Proof. *As noted in* (83)*, a necessary condition for the projection to hold is* $(\boldsymbol{K}_y - \boldsymbol{\mathcal{G}}_{yx}\, \boldsymbol{K}_x)\, \tilde{\boldsymbol{Z}}_t = 0$ *for all realizations of* $\tilde{\boldsymbol{Z}}_t$*, which has a joint normal distribution. Assumption 3 implies that* $\mathrm{Var}\,(\tilde{\boldsymbol{Z}}_t) = \boldsymbol{C}\,\mathrm{Var}\,(\boldsymbol{S}_t^*)\boldsymbol{C}' + \boldsymbol{D}\boldsymbol{D}'$ *is strictly positive definite, so that the distribution of* $\tilde{\boldsymbol{Z}}_t$ *is non-degenerate. For* (83) *to hold, we must have* $\boldsymbol{K}_y = \boldsymbol{\mathcal{G}}_{yx}\, \boldsymbol{K}_x$. ■

Kalman gains are multivariate regression slopes.[48] As a result of Proposition 5, the projection condition imposes a linear restriction on covariances between $\tilde{\boldsymbol{Y}}_t$, $\tilde{\boldsymbol{X}}_t$, and $\tilde{\boldsymbol{Z}}_t$, i.e. $\mathrm{Cov}\,(\tilde{\boldsymbol{Y}}_t, \tilde{\boldsymbol{Z}}_t) = \boldsymbol{\mathcal{G}}_{yx}\ \mathrm{Cov}\,(\tilde{\boldsymbol{X}}_t, \tilde{\boldsymbol{Z}}_t)$.

Henceforth we only consider equilibria that are stationary, linear and time-invariant according to Definition 1 and that satisfy the projection condition laid out in Definition 2. In such equilibria, the dynamics of forward- and backward-looking variables, as well as the policy instrument, are characterized by a state vector that tracks both projections and actual values of the vector $\boldsymbol{\mathcal{S}}_t$, which contains backward- and forward-looking variables as well as the policy instrument. In fact, the joint vector of $\boldsymbol{\mathcal{S}}_t$ and $\boldsymbol{\mathcal{S}}_{t|t}$ does not need to tracked in its entirety: First, $\boldsymbol{\mathcal{S}}_t$ includes the policy instrument $\boldsymbol{i}_t$, which lies in the space of central bank projections, and need not be tracked twice. Thus, the state of the economy can be described by $\boldsymbol{S}_t$ — which differs from $\boldsymbol{\mathcal{S}}_t$ in omitting $\boldsymbol{i}_t$ — and $\boldsymbol{\mathcal{S}}_{t|t}$. Second, the state of the economy is equivalently described by $\boldsymbol{S}_t^* = \boldsymbol{S}_t - \boldsymbol{S}_{t|t}$ and $\boldsymbol{\mathcal{S}}_{t|t}$. Third, when the projection condition (82) is satisfied, we need only track $\boldsymbol{\mathcal{X}}_{t|t}$ rather than $\boldsymbol{\mathcal{S}}'_{t|t} = \begin{bmatrix} \boldsymbol{\mathcal{X}}'_{t|t} & \boldsymbol{\mathcal{Y}}'_{t|t} \end{bmatrix}$, since $\boldsymbol{\mathcal{Y}}_{t|t} = \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{X}}_{t|t}$. Properties of the resulting equilibria are summarized in the following theorem.

[46] In addition to $\boldsymbol{Y}_t$ and $\boldsymbol{X}_t$, $\boldsymbol{\mathcal{Y}}_t$ and $\boldsymbol{\mathcal{X}}_t$ also contain the current and lagged policy instrument, respectively. However, the projection condition does not impose a direct restriction on innovations in the policy instrument since $\boldsymbol{i}_t = \boldsymbol{i}_{t|t}$ and thus $\tilde{\boldsymbol{i}}_{t-1|t} = \boldsymbol{i}_{t-1|t} - \boldsymbol{i}_{t-1|t-1} = 0$.

[47] That means $\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}'_x & \boldsymbol{K}'_y \end{bmatrix}'$ so that $\boldsymbol{K}_x = \mathrm{Cov}\,(\tilde{\boldsymbol{X}}_t, \tilde{\boldsymbol{Z}}_t)\left(\mathrm{Var}\,(\tilde{\boldsymbol{Z}}_t)^{-1}\right)$, and $\boldsymbol{K}_y = \mathrm{Cov}\,(\tilde{\boldsymbol{Y}}_t, \tilde{\boldsymbol{Z}}_t)\left(\mathrm{Var}\,(\tilde{\boldsymbol{Z}}_t)^{-1}\right)$.

[48] Please recall that $\boldsymbol{K}_y = \mathrm{Cov}\,(\tilde{\boldsymbol{Y}}_t, \tilde{\boldsymbol{Z}}_t)\,\mathrm{Var}\,(\tilde{\boldsymbol{Z}}_t)^{-1}$ and $\boldsymbol{K}_x = \mathrm{Cov}\,(\tilde{\boldsymbol{X}}_t, \tilde{\boldsymbol{Z}}_t)\,\mathrm{Var}\,(\tilde{\boldsymbol{Z}}_t)^{-1}$.

THEOREM 1 (Difference System Under Imperfect Information) *Consider the model represented by the system of difference equations* (61) *and* (60) *in* $\boldsymbol{S}_t = \begin{bmatrix}\boldsymbol{X}_t' & \boldsymbol{Y}_t'\end{bmatrix}'$ *with a measurement vector that is linear in* $\boldsymbol{S}_t$ *as defined in* (66)*. In addition, let Assumptions 1, 2, 3, and 4 hold and consider stationary, linear, time-invariant equilibria that satisfy the projection condition, as stated in Definitions 1 and 2. In this case, equilibrium dynamics are characterized by the evolution of the following vector system:*

$$\overline{\boldsymbol{\mathcal{S}}}_{t+1} \equiv \begin{bmatrix}\boldsymbol{S}^*_{t+1} \\ \boldsymbol{\mathcal{X}}_{t+1|t+1}\end{bmatrix} = \underbrace{\begin{bmatrix}(\boldsymbol{A}-\boldsymbol{K}\boldsymbol{C}) & \boldsymbol{0} \\ \boldsymbol{\mathcal{K}}_x\boldsymbol{C} & \boldsymbol{\mathcal{P}}\end{bmatrix}}_{\overline{\boldsymbol{\mathcal{A}}}} \overline{\boldsymbol{\mathcal{S}}}_t + \begin{bmatrix}(\boldsymbol{I}-\boldsymbol{K}\boldsymbol{H}) \\ \boldsymbol{\mathcal{K}}_x\boldsymbol{H}\end{bmatrix} \begin{bmatrix}\boldsymbol{B}_{x\varepsilon} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix} \begin{bmatrix}\boldsymbol{\varepsilon}_{t+1} \\ \boldsymbol{\eta}_{t+1}\end{bmatrix} \tag{84}$$

where $\boldsymbol{\mathcal{K}}_x' = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{K}_x'\end{bmatrix}$, $\boldsymbol{C}$ *as defined in* (77)*, and* $\boldsymbol{\mathcal{P}}$ *known from the unique full-information solution in* (69)*.*

Proof. *Outcomes for* $\boldsymbol{S}_t$ *can be decomposed into* $\boldsymbol{S}_t = \boldsymbol{S}_t^* + \boldsymbol{S}_{t|t}$*. It remains to show that* $\boldsymbol{S}_{t|t}$ *can be constructed from* $\overline{\boldsymbol{\mathcal{S}}}_t$*. In addition, we need to show that the policy instrument* $\boldsymbol{i}_t = \boldsymbol{i}_{t|t}$ *can also be constructed from the proposed state vector* $\overline{\boldsymbol{\mathcal{S}}}_t$*. Recalling the definitions of* $\boldsymbol{S}_t$*,* $\boldsymbol{\mathcal{X}}_t$ *and* $\boldsymbol{\mathcal{Y}}_t$ *in* (61)*, and* (67)*, the projection condition then implies that*

$$\boldsymbol{S}_{t|t} = \begin{bmatrix}\boldsymbol{X}_{t|t} \\ \boldsymbol{Y}_{t|t}\end{bmatrix} = \underbrace{\begin{bmatrix}\boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{\mathcal{G}}_{yi} & \boldsymbol{\mathcal{G}}_{yx}\end{bmatrix}}_{\boldsymbol{\mathcal{G}}_S} \boldsymbol{\mathcal{X}}_{t|t}\,, \qquad \boldsymbol{i}_t = \underbrace{\begin{bmatrix}\boldsymbol{\mathcal{G}}_{ii} & \boldsymbol{\mathcal{G}}_{ix}\end{bmatrix}}_{\boldsymbol{\mathcal{G}}_i} \boldsymbol{\mathcal{X}}_{t|t}\,, \qquad \textit{with} \quad \boldsymbol{\mathcal{X}}_{t|t} = \begin{bmatrix}\boldsymbol{i}_{t-1} \\ \boldsymbol{X}_{t|t}\end{bmatrix},$$

where block matrices are partitioned along the lines of $\boldsymbol{\mathcal{X}}_{t|t}$ *above. The various cofficient matrices* $\boldsymbol{\mathcal{G}}_{..}$ *are known from the full-information solution given in* (69)*.*

The dynamics of $\boldsymbol{S}^*_{t+1}$*, as captured by the top rows of* $\overline{\boldsymbol{\mathcal{S}}}$*, follow from the innovation state space* (75)*,* (76) *as well as the steady state Kalman filter described in Appendix A. The dynamics of* $\boldsymbol{\mathcal{X}}_{t+1|t+1}$*, as captured by the bottom rows of* $\overline{\boldsymbol{\mathcal{S}}}$*, follow from* (81) *together with the projection condition* (82) *and the dynamics of* $\tilde{\boldsymbol{Z}}_t$ *given in* (76)*.* ■

The Kalman gain $\boldsymbol{K}_x$, defined in Proposition 4, depends on the equilibrium distribution of endogenous forecast errors $\boldsymbol{\eta}_t$. According to Assumption 2, $\boldsymbol{\eta}_t$ is a linear combination of exogenous shocks $\boldsymbol{\varepsilon}_t$ and belief shocks $\boldsymbol{b}_t$ with endogenous shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ that are yet to be determined so as to satisfy the projection condition stated in Definition 2. Before turning to the determination of the shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ of the endogenous forecast errors, a few critical results already emerge.

The state vector $\overline{\boldsymbol{\mathcal{S}}}_t$ follows a first-order linear difference system given in (84). The stability of the system depends on the eigenvalues of its transition matrix $\overline{\boldsymbol{\mathcal{A}}}$. The transition matrix $\overline{\boldsymbol{\mathcal{A}}}$ depends on the Kalman gain $\boldsymbol{K}$, which depends on the yet to be determined shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ of the endogenous forecast errors $\boldsymbol{\eta}_t$. Nevertheless, as argued next, the existence of a steady-state Kalman filter allows us to conclude that $\overline{\boldsymbol{\mathcal{A}}}$, is stable.

COROLLARY 1 (Stable Transition Matrix) *Provided that a steady-state Kalman filter exists, the transition matrix* $\overline{\boldsymbol{\mathcal{A}}}$ *in* (84) *is stable. The eigenvalues of* $\overline{\boldsymbol{\mathcal{A}}}$ *are given by the eigenvalues of* $\boldsymbol{\mathcal{P}}$*, which is stable and known from the full-information solution* (69)*, and* $\boldsymbol{A}-\boldsymbol{K}\boldsymbol{C}$*, whose stability is assured by the existence of the steady-state Kalman filter.*

Proof. *The stability of* $\boldsymbol{\mathcal{P}}$ *follows from Assumption 1 and the resulting solution of the full-information case in* (114)*. The stability of* $\boldsymbol{A}-\boldsymbol{K}\boldsymbol{C}$ *follows from Theorem 3 in Appendix A.* ■

The upshot of Corollary 1 is that the usual root-counting arguments do not pin down the shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ of the endogenous forecast errors $\boldsymbol{\eta}_{t+1}$ in (84), since $\overline{\boldsymbol{\mathcal{A}}}$ is a stable matrix for any choice of $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ consistent with the existence of a steady state Kalman filter. Moreover, the projection condition does typically not place sufficiently many restrictions on $\boldsymbol{\Gamma}_{\eta\varepsilon}$ and $\boldsymbol{\Gamma}_{\eta b}$ to uniquely identify the shock loadings:

COROLLARY 2 (Generic Indeterminacy) *With* $\overline{\boldsymbol{\mathcal{A}}}$ *stable, the endogenous forecast errors are only restricted by the projection condition given in Definition 2. The shocks loadings of the endogenous forecast errors,* $\boldsymbol{\Gamma}_\varepsilon$ *and* $\boldsymbol{\Gamma}_b$*, have* $N_y \times (N_\varepsilon + N_y)$ *unknown conditions. Stated as in* (83)*, the projection condition imposes only* $N_y \times N_z$ *restrictions. However, a necessary condition for Assumption 3 to hold is* $N_z \leq N_\varepsilon + N_y$*. As a result, the projection condition cannot uniquely identify the shock loadings* $\boldsymbol{\Gamma}_\varepsilon$ *and* $\boldsymbol{\Gamma}_b$*.*

Among others, Theorem 1 rests on the assumption of joint detectability and unit-circle controllability of $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{H})$ where $\boldsymbol{B}$ depends on the endogenous shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ while $\boldsymbol{A}$ and $\boldsymbol{H}$ are primitives of the model setup. The detectability condition can thus be verified independently from solving for $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$. An addition, as described in the appendix (see Proposition 8), a full rank of

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_{x\varepsilon} & \boldsymbol{0} \\ \boldsymbol{\Gamma}_\varepsilon & \boldsymbol{\Gamma}_b \end{bmatrix}$$

is sufficient to ensure unit-circle controllability . As part of the model setup, $\boldsymbol{B}_{x\varepsilon}$ is supposed to have full rank. Consequently, the criterion of a full rank of $\boldsymbol{B}$ is satisfied when the belief shock loadings $\boldsymbol{\Gamma}_b$ have full rank, and thus $\boldsymbol{\Gamma}_b \neq \boldsymbol{0}$. Non-zero belief shock loadings are a hallmark of equilibrium indeterminacy. While the projection condition places restrictions on $\boldsymbol{\Gamma}_b$, non-zero belief shock loadings are thus a sufficient condition for the existence of a steady state Kalman filter, which in turn assures the stability of $\overline{\boldsymbol{\mathcal{A}}}$.

Before turning to approaches to compute $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ that are consistent with the projection condition, we summarize the construction of an equilibrium for a given solution of the endogenous forecast errors.

THEOREM 2 (Equilibria under Imperfect Information) *Consider the difference system under characterized in Theorem 1 and let* $\boldsymbol{\eta}_t = \boldsymbol{\Gamma}_\varepsilon \boldsymbol{\varepsilon}_t + \boldsymbol{\Gamma}_b \boldsymbol{b}_t$ *with shock loadings* $\boldsymbol{\Gamma}_\varepsilon$ *and* $\boldsymbol{\Gamma}_b$ *such that* *the projection condition (82) is satisfied. Equilibrium outcomes are then characterized as follows:*

$$\boldsymbol{S}_t = \begin{bmatrix}\boldsymbol{I} & \boldsymbol{\mathcal{G}}_s\end{bmatrix}\ \overline{\boldsymbol{\mathcal{S}}}_t \qquad \boldsymbol{i}_t = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{\mathcal{G}}_i\end{bmatrix}\ \overline{\boldsymbol{\mathcal{S}}}_t \qquad \overline{\boldsymbol{\mathcal{S}}}_{t+1} = \overline{\boldsymbol{\mathcal{A}}}\ \overline{\boldsymbol{\mathcal{S}}}_t + \overline{\boldsymbol{\mathcal{B}}}\ \boldsymbol{w}_{t+1}\,, \tag{85}$$

$$\textit{with} \quad \overline{\boldsymbol{\mathcal{S}}}_t \equiv \begin{bmatrix}\boldsymbol{S}_t^* \\ \boldsymbol{\mathcal{X}}_t\end{bmatrix}, \qquad \boldsymbol{w}_{t+1} \equiv \begin{bmatrix}\boldsymbol{\varepsilon}_{t+1} \\ \boldsymbol{b}_{t+1}\end{bmatrix}, \qquad \textit{and} \quad \overline{\boldsymbol{\mathcal{B}}} \equiv \begin{bmatrix}(\boldsymbol{B} - \boldsymbol{K}\boldsymbol{D}) \\ \boldsymbol{\mathcal{K}}_{\mathcal{X}}\boldsymbol{D}\end{bmatrix}. \tag{86}$$

where $\boldsymbol{B}$ and $\boldsymbol{D}$ encode the shock loading $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ as stated in (77); $\boldsymbol{\mathcal{K}}_x$, $\boldsymbol{\mathcal{G}}_s$ and $\boldsymbol{\mathcal{G}}_i$ are defined in the proof of Theorem 1. Block matrices are partitioned along the lines of $\overline{\boldsymbol{\mathcal{S}}}_t$ as stated above.
Proof. *The proof follows straightforwardly from Theorem 1.* ■

3.5 Determination of the Endogenous Forecast Errors

As described by Sims (2002) and Lubik and Schorfheide (2003), restrictions for the endogenous forecast errors $\boldsymbol{\eta}_t$ emanate from explosive roots in the dynamic system. In our imperfect information case, further restrictions result from the projection condition stated in Definition 2. As noted in Corollary 1 above, the transition matrix $\overline{\boldsymbol{\mathcal{A}}}$ is always stable in a time-invariant equilibrium with a steady-state Kalman filter. Restrictions on $\boldsymbol{\eta}_t$ can only result from the projection condition; but as discussed in Corollary 2, the projection condition does generally not provide sufficiently many restrictions to pin down $\boldsymbol{\eta}_t$ uniquely.

The determination of shock loadings $\boldsymbol{\Gamma}_\varepsilon$, $\boldsymbol{\Gamma}_b$ for the endogenous forecast errors that are consistent with the projection condition poses an intricate fixed problem between shock loadings and Kalman gains. As noted already by Sargent (1991), the Kalman gains are endogenous equilibrium objects when the observable signals reflect information contained in endogenous variables. In contrast, the Kalman filtering problem can be solved independently of the equilibrium dynamics of the system, when the signal vector consists only of exogenous variables.

In (66), the central bank's measurement vector is generically described as a linear combination of backward- and forward-looking variables, $\boldsymbol{Z}_t = \boldsymbol{H}_x\boldsymbol{X}_t + \boldsymbol{H}_y\boldsymbol{Y}_t$. To facilitate the analysis, we now delineate two cases: one where the signal depends on endogenous variables (specifically, choosing $\boldsymbol{H}_y = \boldsymbol{I}$) as well as the case where the signal solely reflects exogenous variables ($\boldsymbol{H}_y = \boldsymbol{0}$ and $\boldsymbol{X}_t$ exogenous).

3.5.1 Endogenous Signal

In (66), the signal observed by the central bank involves a linear combination of forward- and backward-looking variables, such that the signal depends at least in part on endogenous variables. When considering this case, and to simplify some of the algebra, we limit ourselves to signal vectors that have the same length as the vector of forward-looking variables ($\boldsymbol{Y}_t$) and that have no rank-deficient loading on $\boldsymbol{Y}_t$. All told, we assume that $\boldsymbol{H}_y$ in (66) is square and invertible. In this case, $\boldsymbol{H}_y$ can be normalized to the identity matrix.[49] In the endogenous-signal case, we thus consider

[49] Consider the case of a signal $\hat{\boldsymbol{Z}}_t = \hat{\boldsymbol{H}}_x\boldsymbol{X}_t + \hat{\boldsymbol{H}}_y\boldsymbol{Y}_t$ where $\hat{\boldsymbol{H}}_y$ is square and nonsingular. The information content provided by $\hat{\boldsymbol{Z}}_t$ is equivalent to what is spanned by $\boldsymbol{Z}_t = \hat{\boldsymbol{H}}_y^{-1}\hat{\boldsymbol{Z}}_t$ with $\boldsymbol{H}_x = \hat{\boldsymbol{H}}_y^{-1}\hat{\boldsymbol{H}}_x$.

signal vectors of the form

$$\boldsymbol{Z}_t = \boldsymbol{H}_x \boldsymbol{X}_t + \boldsymbol{Y}_t \qquad \text{and thus} \quad \boldsymbol{H} = \begin{bmatrix} \boldsymbol{H}_x & \boldsymbol{I} \end{bmatrix}. \tag{87}$$

Note that the endogenous-signal setup also includes the case where each forward-looking variable is observed with error, as in $\boldsymbol{Z}_t = \boldsymbol{Y}_t + \boldsymbol{\nu}_t$ where $\boldsymbol{\nu}_t$ is an exogenous measurement error to be included among the set of backward-looking variables in $\boldsymbol{X}_t$.

In the context of the simple example described in section 2, we provided an analytical characterization of the fixed point problem posed by the endogenous-signal case. However, even in this stylized example the fixed point proved intractable to solve analytically. Instead, we have derived a fast numerical procedure to solve for shock loadings $\boldsymbol{\Gamma}_\varepsilon$, $\boldsymbol{\Gamma}_b$ that are consistent with the projection condition for this case.

Our numerical approach combines elements of standard techniques for solving linear RE models with a fast algorithm to solve the non-linear fixed-point problem for the Riccati equation embedded in the Kalman filter while ensuring consistency with the projection condition. For a given intial guess of $\boldsymbol{\Gamma}_\varepsilon$, $\boldsymbol{\Gamma}_b$, the procedure returns values that are consistent with the projection condition; different starting values then generate typically different result values. Details of our algorithm are described in Appendix B.

3.5.2 Exogenous Signal

For the case of an exogenous signal, we derive two general results: First, the projection condition does not restrict the belief shock loadings of the endogenous forecast errors, $\boldsymbol{\Gamma}_b$, when the signal is exogenous. Second, we derive an analytical expression for the restrictions on the loadings of the endogenous forecast errors on fundamental shocks (including the measurement errors) that result from the projection condition.

To consider the case of a purely exogenous signal, we need to distinguish between endogenous and exogenous components of the vector of backward-looking variables $\boldsymbol{X}_t$. Let $\boldsymbol{X}_t$ be partitioned into exogenous variables, denoted $\boldsymbol{X}_t^1$, and endogenous variables (like the lagged inflation rate in case of a Phillips Curve with indexation), denoted $\boldsymbol{X}_t^2$.

Exogeneity of $\boldsymbol{X}_t^1$ places zero restrictions on the system matrices in (59), and its dynamics are reduced to

$$\boldsymbol{X}_t^1 = \boldsymbol{A}_{xx}^{11}\, \boldsymbol{X}_{t-1}^1 + \boldsymbol{B}_{x\varepsilon}^1\, \boldsymbol{\varepsilon}_t \tag{88}$$

where $\boldsymbol{A}_{xx}^{11}$ and $\boldsymbol{B}_{x\varepsilon}^1$ are appropriate sub-blocks of $\boldsymbol{A}$ and $\boldsymbol{B}_{x\varepsilon}$.[50]. The signal is then given by

$$\boldsymbol{Z}_t = \boldsymbol{H}_x \boldsymbol{X}_t^1. \tag{89}$$

[50]Consistent with (59) , the endogenous component of $\boldsymbol{X}_t$ generally evolves according to

$$\boldsymbol{X}_t^2 = \boldsymbol{A}_{xx}^{21}\, \boldsymbol{X}_{t-1}^1 + \boldsymbol{A}_{xx}^{22}\, \boldsymbol{X}_{t-1}^2 + \boldsymbol{A}_{yx}^{2}\, \boldsymbol{Y}_{t-1} + \hat{\boldsymbol{A}}_{xx}^{21}\, \boldsymbol{X}_{t-1|t-1}^1 + \hat{\boldsymbol{A}}_{xx}^{22}\, \boldsymbol{X}_{t-1t-1}^2 + \hat{\boldsymbol{A}}_{yx}^{2}\, \boldsymbol{Y}_{t-1|t-1} + \boldsymbol{B}_{x\varepsilon}^2\, \boldsymbol{\varepsilon}_t\,,$$

where $\boldsymbol{A}_{xx}^{21}$, $\boldsymbol{A}_{xx}^{22}$, $\boldsymbol{A}_{yx}^{2}$, $\hat{\boldsymbol{A}}_{xx}^{21}$, $\hat{\boldsymbol{A}}_{xx}^{22}$, $\hat{\boldsymbol{A}}_{yx}^{2}$, and $\boldsymbol{B}_{x\varepsilon}^2$ are appropriate sub-blocks of $\boldsymbol{A}$, $\hat{\boldsymbol{A}}$ and $\boldsymbol{B}_{x\varepsilon}$, respectively.

For ease of notation, we consider henceforth the case where the entire vector of backward-looking variables is exogenous; that means $\boldsymbol{X}_t = \boldsymbol{X}_t^1$. (Our first result in this section, the lack of restrictions on $\boldsymbol{\Gamma}_b$, extends also to the case when $\boldsymbol{X}_t$ contains an endogenous component, that is, however, not reflected by the signal.) The signal extraction problem is then given by the following system:

$$\boldsymbol{X}_t = \boldsymbol{A}_{xx}\boldsymbol{X}_{t-1} + \boldsymbol{B}_{x\varepsilon}\boldsymbol{\varepsilon}_t\,, \tag{90}$$

$$\boldsymbol{Z}_t = \boldsymbol{H}_x\boldsymbol{X}_t\,, \tag{91}$$

where $\boldsymbol{A}_{xx}$ denotes the appropriate sub-block of $\boldsymbol{A}$ in (59).

Existence of steady state Kalman filter is assured by joint detectability and unit-circle controllability of $(\boldsymbol{A}_{xx}, \boldsymbol{B}_{x\varepsilon}, \boldsymbol{H})$, which does not depend on the equilibrium solution of the system and thus a weaker condition than what is required by Assumption 4. Henceforth, existence of a steady-state Kalman filter is assumed, resulting in a constant gain matrix $\boldsymbol{K}_x$ such that $\boldsymbol{A}_{xx}(\boldsymbol{I} - \boldsymbol{K}_x\boldsymbol{H}_x)$ is stable.

A defining feature of the exogenous-signal case is that the signal extraction problem can be solved independently from the dynamics of the forward-looking variables, $\boldsymbol{Y}_t$ and $\boldsymbol{K}_x$ and Var $(\boldsymbol{X}_t^*)$ are determined solely by (90) and (91). We have $\boldsymbol{X}_t^* = \boldsymbol{X}_t - \boldsymbol{K}_x\boldsymbol{Z}_t = (\boldsymbol{I} - \boldsymbol{K}_x\boldsymbol{H}_x)\tilde{\boldsymbol{X}}_t$ and can thus write

$$\tilde{\boldsymbol{X}}_t = \boldsymbol{A}_{xx}\boldsymbol{X}_{t-1}^* + \boldsymbol{B}_{x\varepsilon} = \boldsymbol{A}_{xx}(\boldsymbol{I} - \boldsymbol{K}_x\boldsymbol{H}_x)\tilde{\boldsymbol{X}}_{t-1} + \boldsymbol{B}_{x\varepsilon}\boldsymbol{\varepsilon}_t \tag{92}$$

Moreover, when $\boldsymbol{X}_t = \boldsymbol{X}_t^1$, application of the projection condition translates the $\boldsymbol{K}_x$ into a given Kalman gain for the forward-looking variables: $\boldsymbol{K}_y = \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{K}_x$, where $\boldsymbol{\mathcal{G}}_{yx}$ is known from the full-information solution of the model. The innovation dynamics of the forward-looking variables are then restricted by the following transition equation:

$$\tilde{\boldsymbol{Y}}_{t+1} = \tilde{\boldsymbol{A}}_{yx}\tilde{\boldsymbol{X}}_{t-1} + \boldsymbol{A}_{yy}\tilde{\boldsymbol{Y}}_{t-1} + \boldsymbol{\eta}_{t+1} \qquad \text{with} \quad \tilde{\boldsymbol{A}}_{yx} = \boldsymbol{A}_{yx} - (\boldsymbol{A}_{yx} + \boldsymbol{A}_{yy}\boldsymbol{\mathcal{G}}_{yx})\,\boldsymbol{K}_x\boldsymbol{H}_x \tag{93}$$

where the endogenous forecast errors, $\boldsymbol{\eta}_t$, remain to be derived. As before, we seek $\boldsymbol{\eta}_t = \boldsymbol{\Gamma}_{\eta\varepsilon}\,\boldsymbol{\varepsilon}_t + \boldsymbol{\Gamma}_{\eta b}\,\boldsymbol{b}_t$, with loadings $\boldsymbol{\Gamma}_{\eta\varepsilon}$ and $\boldsymbol{\Gamma}_{\eta b}$ that satisfy the projection condition.

In addition, in order to ensure stationarity of $\tilde{\boldsymbol{Y}}_t$, $\boldsymbol{A}_{yy}$ has to be a stable matrix or further restrictions need to be imposed on $\boldsymbol{\eta}_t$. In our discussion of the general case, as part of Assumption 4, we imposed the requirement that $(\boldsymbol{A}, \boldsymbol{H})$ are detectable. As discussed in Appendix A, this requirement is tantamount to letting the signal vector load on any linear combinations of backward- and forward-looking variables associated with potentially unstable dynamics. In the present context of a signal that does not load on $\boldsymbol{Y}_t$, detectability of $(\boldsymbol{A}, \boldsymbol{H})$ boils down to the requirement that $\boldsymbol{A}_{yy}$ is a stable matrix.

If only the weaker requirement of detectability of $(\boldsymbol{A}_{xx}, \boldsymbol{H}_x)$ is to be imposed, note that the innovation system (92) and (93) has the form of a typical linear rational expectations system as analyzed, among others, by Klein (2000) and Sims (2002). As shown there, when $\boldsymbol{A}_{yy}$ is not a stable matrix, linear combinations of $\boldsymbol{\eta}_t$ associated with unstable dynamics of $\tilde{\boldsymbol{Y}}_t$ need to be set to zero. However, as illustrated in the simple example of Section 2, this is typically not sufficient to uniquely determine equilibrium outcomes.[51]

The projection condition requires $\text{Cov}\,(\tilde{\boldsymbol{Y}}_t, \tilde{\boldsymbol{Z}}_t) = \boldsymbol{\mathcal{G}}_{yx}\,\text{Cov}\,(\tilde{\boldsymbol{X}}_t, \tilde{\boldsymbol{Z}}_t)$. Due to the exogeneity of $\tilde{\boldsymbol{Z}}_t$ — and thus $\text{Cov}\,(\boldsymbol{b}_t, \tilde{\boldsymbol{Z}}_t) = 0$ — this covariance restriction does not affect admissible belief shock loadings $\boldsymbol{\Gamma}_{\eta b}$.[52]

PROPOSITION 6 (Unrestricted Belief-Shock Loadings When the Signal is Exogenous) *When the signal is exogenous, as given by* (90) *and* (91)*, there are no restrictions on* $\boldsymbol{\Gamma}_{\eta b}$ *in a stable, linear, time-invariant equilibrium (Definition 1) where the projection condition (Definition 2) holds.*

Proof. *Formally, we can decompose* $\tilde{\boldsymbol{Y}}_t$ *into two pieces: a component,* $\tilde{\boldsymbol{Y}}^{\varepsilon}_t$*, that reflects the history of fundamental shocks* $\boldsymbol{\varepsilon}^t$ *and another component,* $\tilde{\boldsymbol{Y}}^{b}_t$*, solely driven by belief shocks.*

$$\tilde{\boldsymbol{Y}}_{t+1} = \tilde{\boldsymbol{Y}}^{\varepsilon}_{t+1} + \tilde{\boldsymbol{Y}}^{b}_{t+1} \tag{94}$$

$$\tilde{\boldsymbol{Y}}^{\varepsilon}_{t+1} \equiv \tilde{\boldsymbol{A}}_{yx}\tilde{\boldsymbol{X}}_{t-1} + \boldsymbol{A}_{yy}\tilde{\boldsymbol{Y}}^{\varepsilon}_{t-1} + \boldsymbol{\Gamma}_{\eta\varepsilon}\boldsymbol{\varepsilon}_{t+1} \tag{95}$$

$$\tilde{\boldsymbol{Y}}^{b}_{t+1} \equiv \boldsymbol{A}_{yy}\tilde{\boldsymbol{Y}}^{b}_{t-1} + \boldsymbol{\Gamma}_{\eta b}\boldsymbol{b}_{t+1} \tag{96}$$

with the evolution of $\tilde{\boldsymbol{X}}_t$ *given by* (92) *and* $\tilde{\boldsymbol{A}}_{yx}$ *as defined in* (93)*. When the measurement vector is exogenous, it is uncorrelated with belief shocks at all leads and lags. Accordingly, the central bank's information set does not contain any signal about* $\boldsymbol{b}_{t+h}$*, for any* h*,* $E(\boldsymbol{b}_{t+h}|\boldsymbol{Z}^t) = \boldsymbol{0}$*. Likewise,* $\tilde{\boldsymbol{Y}}^{b}_t$ *is orthogonal to* Z^t*,* $E(\tilde{\boldsymbol{Y}}^{\boldsymbol{b}}{}_t|\tilde{\boldsymbol{Z}}_t) = \boldsymbol{0}$ *for any* $\boldsymbol{\Gamma}_{\eta b}$*. As a consequence,* $\boldsymbol{\Gamma}_{\eta b}$ *does not affect the projection condition* (83)*,* $\boldsymbol{Y}_{t|t} = \boldsymbol{\mathcal{G}}_{yx}\,\boldsymbol{X}_{t|t}$*.* ■

As can be seen from the proof of Proposition 6, the underlying argument rests on the orthogonality between the exogenous signal vector $\boldsymbol{X}_t$ and the belief shocks $\boldsymbol{b}_t$. The argument easily extends to the more general case when the vector of backward-looking variables $\boldsymbol{X}_t$ contains both exogenous and endogenous components $\boldsymbol{X}^1_t$ and $\boldsymbol{X}^2_t$ as described in the previous section.

Finally, for the case when $\boldsymbol{X}_t = \boldsymbol{X}^1_t$, we can derive simple expressions to construct fundamental shock loadings $\boldsymbol{\Gamma}_{\eta\varepsilon}$ that satisfy the projection condition. Let $\tilde{\boldsymbol{W}}_t \equiv \tilde{\boldsymbol{Y}}^{\varepsilon}_t - \boldsymbol{\mathcal{G}}_{yx}\tilde{\boldsymbol{X}}_t$ and note that the projection condition requires $\tilde{\boldsymbol{W}}_{t|t} = \boldsymbol{0}$ and thus $\tilde{\boldsymbol{W}}_t = \tilde{\boldsymbol{W}}^{*}_t$.[53] Equivalently, the projection condition requires $\text{Cov}\,(\tilde{\boldsymbol{W}}_t, \tilde{\boldsymbol{Z}}_t) = \boldsymbol{\Sigma}_{wx}\boldsymbol{H}' = \boldsymbol{0}$ with $\boldsymbol{\Sigma}_{wx} \equiv \text{Cov}\,(\tilde{\boldsymbol{W}}_t, \tilde{\boldsymbol{X}}_t)$. Based on (90) and (95),

[51] Note that (93) describes the innovation dynamics of the forward-looking variables, which do not directly depend on the policy rule coefficients $\boldsymbol{\Phi}$. in (60). Since the policy instrument always lies in the space of observations of the policymaker, it drops out of the innovations dynamics of the forward-looking dynamics. However, in monetary policy models with interest rate rules, it is the appropriate choice of policy coefficients that creates many possibly unstable dynamics (which cannot be part of a stationary equilibrium) and just a single stationary outcome.

[52] Notice that the result also goes through, when part of the vector of backward-looking variables was endogenous, as long as the signal remains exogenous.

[53] In light of Proposition 6, we can neglect the effects of belief shocks and $\tilde{\boldsymbol{W}}_t$ has been defined with reference to $\tilde{\boldsymbol{Y}}^{\varepsilon}_t$, as defined in the proof to Proposition 6.

and with $\boldsymbol{A}_{wx} = \boldsymbol{A}_{yx} - \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{A}_{xx} + \boldsymbol{A}_{yy}\boldsymbol{\mathcal{G}}_{yx}$, the dynamics of $\tilde{\boldsymbol{W}}_t$ are given by

$$\tilde{\boldsymbol{W}}_{t+1} = \boldsymbol{A}_{wx}\boldsymbol{X}^*_{t-1} + \boldsymbol{A}_{yy}\tilde{\boldsymbol{W}}_{t-1} + (\boldsymbol{\Gamma}_{\eta\varepsilon} - \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{B}_{x\varepsilon})\boldsymbol{\varepsilon}_{t+1} \tag{97}$$

$$\boldsymbol{\Sigma}_{wx} = \boldsymbol{A}_{wx}\boldsymbol{\Sigma}^*_{xx}\boldsymbol{A}'_{xx} + \boldsymbol{A}_{yy}\boldsymbol{\Sigma}_{wx}\boldsymbol{A}'_{xx}\left(\boldsymbol{I} - \boldsymbol{K}_x\boldsymbol{H}_x\right)' + (\boldsymbol{\Gamma}_{\eta\varepsilon} - \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{B}_{x\varepsilon})\boldsymbol{B}'_{x\varepsilon} \tag{98}$$

where $\boldsymbol{\Sigma}^*_{xx} = \text{Var}\left(\boldsymbol{X}^*_t\right)$ is known from solving the steady-state Kalman filter. The only unknowns in (98) are $\boldsymbol{\Sigma}_{wx}$ and $\boldsymbol{\Gamma}_{\eta\varepsilon}$ and we seek to find $\boldsymbol{\Gamma}_{\eta\varepsilon}$ such that $\boldsymbol{\Sigma}_{wx}\boldsymbol{H}'_x = \boldsymbol{0}$.

Valid values of $\boldsymbol{\Sigma}_{wx}$ must lie in the nullspace of $\boldsymbol{H}'_x$. Specifically, given a $(N_x - N_z) \times N_x$ matrix $\boldsymbol{N}$ such that $\boldsymbol{N}\boldsymbol{H}' = \boldsymbol{0}$.[54] we can construct valid candidates for $\boldsymbol{\Sigma}_{wx}$ by choosing an arbitrary $N_y \times (N_x - N_z)$ matrix $\boldsymbol{G}$ and let $\boldsymbol{\Sigma}_{wx} = \boldsymbol{G}\boldsymbol{N}$.

For a given candidate $\boldsymbol{\Sigma}_{wx} = \boldsymbol{G}\boldsymbol{N}$, $\boldsymbol{\Gamma}_{\eta\varepsilon}$ must thus satsfy the following condition:

$$\boldsymbol{\Gamma}_{\eta\varepsilon}\boldsymbol{B}'_{x\varepsilon} = \boldsymbol{f}(\boldsymbol{G}) \tag{99}$$

$$\text{where} \quad \boldsymbol{f}(\boldsymbol{G}) \equiv \boldsymbol{G}\boldsymbol{N} + \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{B}_{x\varepsilon}\boldsymbol{B}'_{x\varepsilon} - \boldsymbol{A}_{wx}\boldsymbol{\Sigma}^*_{xx}\boldsymbol{A}'_{xx} - \boldsymbol{A}_{yy}\boldsymbol{G}\boldsymbol{N}\boldsymbol{A}'_{xx}\left(\boldsymbol{I} - \boldsymbol{K}_x\boldsymbol{H}_x\right)' \tag{100}$$

The ability to solve (99) for $\boldsymbol{\Gamma}_{\eta\varepsilon}$ depends on the dimension of the problem. When $N_x = N_\varepsilon$, the number of exogenous variables is identical to the number of exogenous shocks, and $\boldsymbol{B}_{x\varepsilon}$ is invertible. When $|\boldsymbol{B}_{x\varepsilon}| \neq 0$ it is straightforward to solve (99) for $\boldsymbol{\Gamma}_{\eta\varepsilon}$ given an arbitrary $\boldsymbol{G}$:

$$\boldsymbol{\Gamma}_{\eta\varepsilon} = \boldsymbol{f}(\boldsymbol{G})(\boldsymbol{B}'_{x\varepsilon})^{-1} . \tag{101}$$

In case of $N_x > N_\varepsilon$, there is not necessarily a $\boldsymbol{\Gamma}_{\eta\varepsilon}$ that solves (99) for any $\boldsymbol{G}$. Instead, $\boldsymbol{G}$ needs to be chosen such that $\boldsymbol{f}(\boldsymbol{G})\left(\boldsymbol{I} - \boldsymbol{B}_{x\varepsilon}(\boldsymbol{B}'_{x\varepsilon}\boldsymbol{B}_{x\varepsilon})^{-1}\boldsymbol{B}'_{x\varepsilon}\right) = \boldsymbol{0}$, which can be obtained numerically.[55] For such a choice of $\boldsymbol{G}$, a valid $\boldsymbol{\Gamma}_{\eta\varepsilon}$ is given by $\boldsymbol{\Gamma}_{\eta\varepsilon} = \boldsymbol{f}(\boldsymbol{G})\boldsymbol{B}_{x\varepsilon}(\boldsymbol{B}'_{x\varepsilon}\boldsymbol{B}_{x\varepsilon})^{-1}$.

4 Quantitative Analysis

We now solve and analyze our modelling framework quantitatively. We first present results for the simple example economy of Section 2 before turning to a New Keynesian economy that is widely used for the analysis of monetary policy. Both applications consider central bank information sets that are spanned by endogenous signals as discussed in section 3.5. When the central bank's measurement vector reflects endogenous variables, analytical solutions are difficult to obtain and we therefore rely on a numerical procedure.

4.1 Quantitative Results for the Simple Example Model

An analytical characterization of equilibria in the simple example economy under exogenous and endogenous information sets are provided in Propositions 1 and 2 in section 2. Nevertheless,

[54] A matrix $\boldsymbol{N}$ such that $\boldsymbol{N}\boldsymbol{H}' = \boldsymbol{0}$ can readily be obtained from the SVD decomposition of $\boldsymbol{H} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}'$ where $\boldsymbol{U}$ and $\boldsymbol{V}$ are orthonormal, $\boldsymbol{S} = \begin{bmatrix}\boldsymbol{S}_1 & \boldsymbol{0}\end{bmatrix}$ and $\boldsymbol{S}_1$ is a $N_z \times N_z$ diagonal matrix. Partition $\boldsymbol{V}$ conformably into $\boldsymbol{V} = \begin{bmatrix}\boldsymbol{V}_1 & \boldsymbol{V}_2\end{bmatrix}$ such that $\boldsymbol{H} = \boldsymbol{U}\boldsymbol{S}_1\boldsymbol{V}'_1$. Since $\boldsymbol{V}$ is orthonormal we have $\boldsymbol{V}'_2\boldsymbol{V}_1 = \boldsymbol{0}$. Choosing $\boldsymbol{N} = \boldsymbol{V}'_2$ then ensures $\boldsymbol{N}\boldsymbol{H}' = \boldsymbol{0}$.

[55] Note that, as introduced in (62), $\boldsymbol{B}_{x\varepsilon}$ has full rank which ensures that $|\boldsymbol{B}'_{x\varepsilon}\boldsymbol{B}_{x\varepsilon}| \neq 0$.

analytical bounds on parameters for existence of a solution remain difficult to derive, especially in the case of an endogenous information set. In this section, we provide further insight into the mechanics and implications of our framework by solving the model numerically. Our numerical solution algorithm combines elements of standard techniques for solving linear RE models with a fast algorithm to solve the non-linear fixed-point problem for the Riccati equation embedded in the Kalman filter and the projection condition. Specifically, the numerical algorithm searches for shock loadings γ_ε, γ_ν, and γ_b that satisfy the projection condition in the endogenous signal case. Further details can be found in Appendix B.

We consider the baseline case of the policy rule $i_t = \phi \pi_{t|t}$ where the endogenous information set $Z_t = \pi_t + \nu_t$. For purposes of illustration, we set the policy parameter $\phi = 1.5$ and assume that the real rate follows an AR(1) process with persistence $\rho = 0.9$ and a unit innovation variance $\sigma_\varepsilon^2 = 1$. Initial experimentation shows that in this simple example the measurement error on inflation has to be large for an equilibrium to exist. We therefore set the variance of the *i.i.d.* measurement error ν_t to $\sigma_\nu^2 = 2.5^2$.[56] We generate 2000 starting points from which our algorithm is able to find valid equilibria in 99 percent of all cases. Each equilibrium is associated with a triple $(\gamma_\varepsilon, \gamma_b, \gamma_\nu)$ of loadings on the shocks in the forecast error decomposition. The fact that there are multiple such loadings for the parameter space simply reflects that the RE solution is indeterminate. In section I.1 of the Supplementary Appendix, we explore the bounds of the existence region for this parameterization numerically.

We plot impulse responses to each shock for the entire set of equilibria in Figure 1, while impulse responses for a specific equilibrium as an example are displayed in Figure 2. For reference, we also plot the impulse responses for the FIRE specification. In the latter case, a unit innovation to the real rate raises inflation by $1/(\phi - \rho) = 5/3$ which then decays at the constant rate ρ. The interest response follows the same pattern. This simply reflects the Fisher effect in that a higher real rate requires a higher nominal rate and in turn a higher inflation rate. The unique FIRE equilibrium has zero response to the measurement error since the model is not defined as having such error, but also to the belief shock since the solution is unique under the given parameterization.

Under LIRE, however, the set of equilibria is notably different. On impact, a unit innovation in the real rate can either lead to an increase or a decrease in inflation over a range of about $(-1.9, 1.7)$ depending on which indeterminate equilibrium the economy is in. Similarly, the nominal rate response can be positive or negative. In effect, in different LIRE equilibria, inflation and the nominal rate can comove positively or negatively. Figure 2 displays impulse response for one of the possible LIRE equilibria where, in contrast to the FIRE solution, inflation and the nominal interest rate comove negatively. While the nominal rate follows the real rate increase, inflation can fall on account of a negative loading γ_ε on the forecast error (see Proposition 2). The figure also shows that a unit measurement error shock lowers inflation and the nominal rate which indicates a negative loading on ν_t in the solution, $\gamma_\nu < 0$ (see Proposition 2). Therefore, no equilibrium exists

[56] As discussed in section 2, the variance of the belief shock σ_b^2 is normalized to unity without loss of generality.

for positive γ_ν.[57] In contrast, the responses to the belief shock are symmetric and unrestricted, similar to the case of an exogenous information set.[58]

Figure 3 reports the autocorrelation function (ACF) and the standard deviation relative to the full information scenario.[59] As shown in the upper row of panels, the FIRE solution displays the typical autocorrelation pattern of a first-order autoregressive process. Comparing the set of outcomes under LIRE against the unique FIRE equilibrium, the persistence of inflation is generally lower and its serial correlation decays much more rapidly under LIRE, whereas for the nominal rate the ACF closely resembles that under FIRE. Since $i_t = \phi \pi_{t|t} = \frac{\phi}{\phi-\rho} r_{t|t}$ the nominal rate behaves like the real rate projection. Given the solution in Proposition 2, this implies that the Kalman gain κ_r is in a tight neighbourhood around zero. The lower panel of the figure shows ranges of relative standard deviations of outcomes under LIRE relative to the respective FIRE outcomes. The interest rate under LIRE is generally less volatile than its FIRE counterpart despite the presence of two additional shocks. The lower interest-rate volatility echoes our discussion in section 2 that the *effective* response to inflation under LIRE after taking into account the filtering problem constitutes a violation of the Taylor principle. In the LIRE case, imperfect information prevents the central bank from moving the policy rate aggressively in response to actual inflation. Instead, optimal filtering leads to attenuation of the policy response, the flipside of which is heightened inflation volatility.

4.2 Quantitative Results for a New Keynesian Model

We now specify and solve a standard New Keynesian model often used in monetary policy analysis under the assumption that the monetary authority has a limited information set. Akin to the simple example model, we assume that there is a private sector, a household, that has the same information set as households in the full information version of the model. We also assume that the central bank only observes noisy measurements of inflation and the level of real GDP. Finally, the central bank follows a monetary policy rule in which it reacts to its best estimate of inflation and the output gap.

Specifically, we assume that the model includes a New Keynesian Phillips curve with a backward-looking component in inflation π_t:

$$(1-\gamma\beta)\,\pi_t = \beta E_t \pi_{t+1} + \gamma \pi_{t-1} + x_t, \tag{102}$$

where $0 \leq \gamma < 1$ denotes the degree of indexation and governs inflation persistence. x_t is the output gap and the sole driver of inflation in this model. Its evolution is captured by a variant of

[57] This is confirmed by Figure 2 in the Supplementary Appendix

[58] The findings are reminiscent of the observation by Lubik and Schorfheide (2004) that changes in comovement patterns are a hallmark of equilibrium indeterminacy and thereby allow econometricians to identify different sets of equilibria. Moreover, their observation that indeterminate equilibria do impose some restrictions on the behavior of the economy in response to fundamental shocks thus carries over to our framework.

[59] Moments are computed via simulation for 20,000 periods with the first 1,000 periods discarded as burn-in to avoid dependence on initial conditions.

the Euler-equation which relates output to the real rate and policy actions:

$$x_t = E_t x_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1} - r_t\right). \tag{103}$$

$\sigma > 0$ is the intertemporal substitution elasticity and governs the responsiveness of output growth to interest rate movements. The term in parentheses is the gap between the actual real rate of interest $(i_t - E_t \pi_{t+1})$ and its natural rate r_t. Similar to Laubach and Williams (2003), we assume that r_t is related to expected growth in potential real GDP y_t:

$$r_t = \sigma E_t \Delta y_{t+1}. \tag{104}$$

Furthermore, we assume that Δy_t follows an autoregressive process of order one:

$$\Delta y_t = \rho_y \Delta y_{t-1} + \varepsilon_t^y, \tag{105}$$

where the innovation ε_t^y is *i.i.d.* Gaussian with zero mean and finite variance σ_y^2, namely $\varepsilon_t^y \sim N\left(0, \sigma_y^2\right)$.

The central bank follows the feedback rule:

$$i_t = \phi_\pi \pi_{t|t} + \phi_x x_{t|t}, \tag{106}$$

where we assume that the policy coefficients ϕ_π and ϕ_x are such that in the FIRE counterpart of this model the equilibrium is unique. As before, $x_{t|t}$ denotes the output gap projection given information available to the central bank at time t. The signal extraction problem is thus somewhat more involved than in the simple example. The central bank not only has to infer the true level of output from its noisy signal, but it also has to infer the best estimate of the output gap from the available data. We choose this specification as it arguably mirrors more closely the practice of many central banks, including the Federal Reserve.

We introduce two measurement errors in inflation and the level of output, ν_t^π and ν_t^x, respectively. The measurement errors are jointly normally distributed and serially and mutually uncorrelated with variances σ_π^2 and σ_x^2. The level of GDP is by construction equal to the growth rate in potential GDP plus the sum of lagged potential GDP and the current output gap. The measurement error to the level of GDP thus acts like an error to the output gap. This specification implies the following measurement vector $\boldsymbol{Z}_t$:[60]

$$\boldsymbol{Z}_t = \begin{bmatrix} \pi_t + \nu_t^\pi \\ \Delta y_t + y_{t-1} + \nu_t^x + x_t \end{bmatrix}, \text{ with } \begin{bmatrix} \nu_t^\pi \\ \nu_t^x \end{bmatrix} \sim N\left(\mathbf{0}, \begin{bmatrix} \sigma_\pi^2 & 0 \\ 0 & \sigma_x^2 \end{bmatrix}\right). \tag{107}$$

We calibrate the model by choosing standard parameter values in the literature (see Table 1). We set the intertemporal substitution elasticity to $\sigma = 1$ to maintain comparability with the

[60] As written, the measurement vector $\boldsymbol{Z}_t$ contains the *level* of output and thus a unit root. However, this unit root affects only the measurement dynamics and not the linear difference system of the New Keynesian model, given by (102), (103) and (106). The conditions for the existence of a steady state Kalman Filter, described in Appendix A, remain satisfied. Since central bank projections are conditioned on the infinite horizon history of $\boldsymbol{Z}_t$, the measurement vector could equivalently be written in terms of output growth.

simple example, while $\beta = 0.99$. The indexation parameter is chosen as $\gamma = 0.25$ which roughly replicates observed inflation persistence. The policy coefficients ϕ_π and ϕ_x are set to 2.5 and 0.5 which guarantees the existence of equilibria in a wide neighborhood of the parameterization. We calibrate the measurement error processes largely in line with the empirical findings in Lubik and Matthes (2016). For inflation, we choose the standard deviation of our measurement error to match their estimated unconditional standard deviation for the inflation error.[61] For real GDP, we assume that log GDP is measured with *iid* error. This automatically induces autocorrelation in the measurement error for the log-difference of GDP, consistent with their findings.[62]

Figure 4 reports impulse response functions under FIRE and LIRE for the fundamental shock to potential GDP growth in the left column, next to the two measurement errors and the two belief shocks, denoted η^π and η^x. Each row shows the response of the model's endogenous variables. Solid blue lines indicate the responses under FIRE while the lines, or areas, in red capture the responses for different equilibria under LIRE. As the only shock under FIRE that affects outcomes the innovation to potential GDP growth increases the natural real rate via the expectations channel. This prompts a rise in the policy rate and reduces expected output gap growth on impact due to a fall in the current output gap. As actual production ramps up to close the gap, inflation declines from its initial peak, which is driven by the relative reduction in supply on impact.

Under LIRE the impulse responses to the fundamental innovation are qualitatively similar to the FIRE responses although they show a somewhat richer dynamic adjustment pattern. Figure 5 shows impulse responses for a single LIRE equilibrium that are fairly close to their FIRE counterparts. We also note that the FIRE response is in parts an envelope to the area of responses associated with indeterminate LIRE equilibria, a pattern we also observe in the simple model. In the Supplementary Appendix, we report results from a specification where the measurement errors are small so that the signal content of incoming data is high. While this specification still implies indeterminate equilibria under LIRE, the impulse responses to the fundamental shock essentially coincide with the FIRE ones. Moreover, the responses to the other shocks are an order of magnitude smaller and would likely be hard to detect in data. This distinguishes our framework from more standard indeterminacy results where the range of equilibria is considerably wider.[63]

The next two columns in Figure 4 show the responses to the measurement error shocks. A positive shock to ν_t^π prompts the central bank to adjust its inflation projection upwards. This stimulates a contemporaneous rise in the policy rate and generally lowers the output gap due to a fall

[61] In their paper, the measurement error in inflation is estimated to be mildly autocorrelated, with a point estimate of around 0.1 for the autoregressive coefficient. The switch from an autoregressive measurement error process to *iid* seems innocuous.

[62] In that case the standard deviation of the measurement error in the log-difference is twice the standard deviation of the measurement error in levels. We match the standard deviation of the *iid* measurement error to half of the unconditional standard deviation of the estimated measurement error for GDP growth. Standard deviations of all shocks are expressed in annualized percentages.

[63] In addition, in the New Keynesian model the impulse responses do not extend over the zero line and thereby doe not offer varying comovement patterns under indeterminacy as is the case in the simple example or in Lubik and Schorfheide (2004).

in current GDP. The inflation rate falls because of the contractionary central bank policy response to the inflation mismeasurement. This pattern is also evident from Figure 5. However, there are equilibria where this pattern is overturned with a considerably smaller, even negative, interest rate response. What drives these differences are the different values of the endogenous Kalman gain associated with various indeterminate equilibria. That is, equilibria exist where the responsiveness of inflation projections to measurement error is small enough so that the standard adjustment dynamics in response to output gap movements and their projections dominate. Responses to the output measurement error follow a similar but less pronounced pattern. A positive innovation ν_t^x leads to an upward revision of output gap projections and an interest rate hike, followed by a decline in current output and a rise in prices. Adjustment patterns to both measurement errors exhibit slowly adjusting and oscillating dynamics.

Finally, the last two columns in Figure 4 show the responses to the belief shocks, which are identical and symmetric.[64] The graphs confirm the results of Lubik and Schorfheide (2003) and Farmer, Khramov, and Nicolò (2015) that sunspot shocks have a representation that map into belief shocks; that is, they affect expectations directly, but to which expectational variable the belief shocks are appended do not affect outcomes for any given equilibrium. To that point, we show two sample responses to belief shocks for alternative equilibria in Figure 5. Moreover, the set of impulse response functions is symmetric around the zero line since the response to a sunspot shock in each equilibrium is only determined up to its sign. Nevertheless, we can still trace out the effect of, for instance, a positive belief shock to inflation such as in the fourth column of the figure.

Suppose consumers believe inflation to be higher than initially anticipated, the belief being driven by the realization of a sunspot that is interpreted as fundamental. This leads the central bank to raise its inflation projection somewhat, but not fully given its filtering problem. The policy rate rises, but not to the full extent required to invalidate consumers' beliefs. This would occur if the policy response were such that it raised the real rate by enough to reign in increased spending and thus rising prices. Although the central bank obeys the Taylor principle, the wedge between private sector and central bank expectations generated by the filtering process is sufficient for indeterminacy to arise. The less than aggressive interest-rate response thereby leads to output gap movements that validate beliefs to the extent that inflation rises by enough.

As in the simple model, we compute autocorrelation functions and relative standard deviations for the New Keynesian model which are reported in Figure 6. The upper panel shows the ACFs for the three key variables in the model. What is notable and to some extent different from the simple model is that the ACFs now cluster tightly around the corresponding ACF under FIRE. This highlights that the implications of the simple example are somewhat stark in terms of how indeterminacy impacts outcomes. A similar impression is conveyed by the range of relative standard deviations in the lower panel. We find the same pattern as in the simple model, namely higher inflation volatility and a slightly lower interest rate volatility which reflects the less aggressive policy

[64] Small differences between the red areas in the two columns arise solely because of numerical discrepancies.

response under FIRE.

Overall, the conclusion from our quantitative analysis of the two models is that multiple equilibrium scenarios under LIRE are pervasive and introduce deviations from fundamental outcomes driven by measurement error and beliefs. These deviations affect the behavior of model variables in a qualitatively significant manner, but the quantitative importance appears limited. To what extent these two types of environments can be distinguished in aggregate data is an important question which is beyond the scope of this paper.

5 Determinate Outcomes Without Optimal Projection

This section describes an alternative class of policy rules that satisfies the same, if not simpler, informational requirements as (60), but also leads to unique equilibrium outcomes. In this alternative class, policy reactions are characterized as responses to incoming data, Z_t, instead of responses to optimal projections that are endogenously determined. Our general framework considers reaction functions for the policy instrument that, as in (60), respond to optimal projections of backward- and forward-looking variables. Rules of this form could, for example, be motivated by noting that a given rule is deemed desirable under full information and pointing to a certainty equivalence argument. In fact, based on reasoning along those lines, Svensson and Woodford (2004) derive optimal reactions in a form consistent with (60).

A central message of this paper, however, is to note that the interaction of the policymaker's filtering and private-sector agents' forward looking behavior, embodied by the linear difference system in (59), lead to a multiplicity of equilibria that is generally inherent in the class of models studied here. Rules of the form in (60) commit the policymaker only in her responses to *projected* input variables, but not in her responses to *incoming data*. The policymaker's projections are rational expectations, and the sensitivity of those expectations to incoming data depends on the signal-to-noise ratio of the central bank's observables in equilibrium, which results in the potential for multiple equilibria.

We adapt the general environment described in section 3 as follows: Under full information, the policy rule responds only to forward-looking variables and lagged policy instruments:[65]

$$\boldsymbol{i}_t = \boldsymbol{\Phi}_i \boldsymbol{i}_{t-1} + \boldsymbol{\Phi}_y \boldsymbol{Y}_t \tag{108}$$

An example of such a rule is an outcome-based Taylor rule, while policy rules with stochastic intercept are excluded. Forward-looking behavior of the private sector is characterized by an expectational difference system similar to (59) except that, for simplicity, central-bank projections are assumed to enter (at most) only via the policy rule, that is:

$$E_t \boldsymbol{S}_{t+1} = \boldsymbol{A} \boldsymbol{S}_t + \boldsymbol{A}_i \boldsymbol{i}_t \,, \tag{109}$$

[65] We continue to use notation introduced in section 3; policy instruments are denoted $\boldsymbol{i}_t$, forward- and backward-lookign variables, $\boldsymbol{Y}_t$ and $\boldsymbol{X}_t$, and the joint vector of $\boldsymbol{Y}_t$ and $\boldsymbol{X}_t$ is $\boldsymbol{S}_t$.

where $\boldsymbol{S}_t$ continues to denote the stacked vector of backward- and forward-looking variables. As before, we assume the values of the policy-rule coefficients $\boldsymbol{\Phi}_i$ and $\boldsymbol{\Phi}_y$ to be such that, when the reaction function (108) is combined with the difference system in (109), there is a unique full-information rational expectations equilibrium.

The measurement vector $\boldsymbol{Z}_t$ conveys a noisy signal of every forward-looking variable:[66]

$$\boldsymbol{Z}_t = \boldsymbol{Y}_t + \boldsymbol{\nu}_t \qquad\qquad \boldsymbol{\nu}_t \sim N(\boldsymbol{0}, \boldsymbol{\Omega}_{\nu\nu}) \tag{110}$$

A policymaker could also consider to simply replace $\boldsymbol{Y}_t$ by its noisy signal. That is, the policymaker could set the policy instrument according to:

$$\boldsymbol{i}_t = \boldsymbol{\Phi}_i \boldsymbol{i}_{t-1} + \boldsymbol{\Phi}_y \left(\boldsymbol{Y}_t + \boldsymbol{\nu}_t \right) . \tag{111}$$

The economy is then described by the expectational difference system (109) and the policy rule (111). Importantly, equilibrium does not hinge on any signal extraction efforts and its determination can be studied using standard methods as described in section 3 3.2. In particular, the only difference between the full-information system consisting of (108) and (109) and the "signal-rule system", described by (109) and (111), is the presence of additional, exogenous driving variables in the form of $\boldsymbol{\nu}_t$. Both systems share an identical transmission of endogenous variables.

Since we assume that the full-information system given by (108) and (109) satisfies the conditions for a unique equilibrium, it follows directly that the "signal-rule system" of (109) and (111) also has a unique equilibrium. In particular, using notation introduced above in our characterization of full-information outcomes for the general case, equilibrium outcomes have the following form:

$$\boldsymbol{Y}_t = \boldsymbol{\mathcal{G}}_{yx} \boldsymbol{X}_t + \boldsymbol{\mathcal{G}}_{yi} \boldsymbol{i}_{t-1} + \boldsymbol{\mathcal{G}}_{y\nu} \boldsymbol{\nu}_t \tag{112}$$

$$E_t \boldsymbol{X}_{t+1} = \boldsymbol{\mathcal{P}}_{xx} \boldsymbol{X}_t + \boldsymbol{\mathcal{P}}_{yi} \boldsymbol{i}_{t-1} + \boldsymbol{\mathcal{P}}_{x\nu} \boldsymbol{\nu}_t \tag{113}$$

where $\boldsymbol{\mathcal{G}}_{yx}$, $\boldsymbol{\mathcal{G}}_{yi}$, $\boldsymbol{\mathcal{P}}_{xx}$, $\boldsymbol{\mathcal{P}}_{yi}$ are conformable partitions of the full-information solution matrices $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{G}}$, with values identical to (69).[67]

The ability to achieve equilibrium uniqueness might initially appear to be an attractive feature. However, the dependence of endogenous outcomes on signal noise in (112) and (113) can lead to potentially highly undesirable fluctuations caused by measurement noise. Effectively, while maintaining the requirement that policy can only respond to observables spanned by $\boldsymbol{Z}^t$, determinacy is achieved under the "signals rule" by committing policy to respond to incoming noise with the same sensitivity as it does to $\boldsymbol{Y}_t$. In particular, in the context of the Fisher-example described

[66]For simplicity, we continue to assume that $\boldsymbol{\nu}_t$ is serially uncorrelated, though equilibrium uniqueness will not depend on this property.

[67]The full-information coefficient matrices known from (69) can be partitioned as follows:

$$\boldsymbol{\mathcal{P}} = \begin{bmatrix} \boldsymbol{\mathcal{P}}_{ii} & \boldsymbol{\mathcal{P}}_{ix} \\ \boldsymbol{\mathcal{P}}_{ix} & \boldsymbol{\mathcal{P}}_{xx} \end{bmatrix} , \qquad\qquad \text{and} \quad \boldsymbol{\mathcal{G}} = \begin{bmatrix} \boldsymbol{\mathcal{G}}_{yi} & \boldsymbol{\mathcal{G}}_{yx} \\ \boldsymbol{\mathcal{G}}_{ii} & \boldsymbol{\mathcal{G}}_{ix} \end{bmatrix} . \tag{114}$$

in section 2, we have $\boldsymbol{\mathcal{G}}_{y\nu} = \mathbf{0}$ and $\boldsymbol{X}_t$ is purely exogenous. With this particular configuration, the variance bounds established in section 2 indicates that *any* admissible equilibrium under the corresponding projections-based policy rule, that is $\boldsymbol{i}_t = \boldsymbol{\Phi}_i \boldsymbol{i}_{t-1} + \boldsymbol{\Phi}_y \boldsymbol{Y}_{t|t}$, generates less-variable outcomes, at least in this particular example.

As described in the Supplementary Appendix, the variance bound derived in the simple example of section 2 also extends to the general case, where we have $\mathrm{Var}\,(\boldsymbol{Y}_t) \leq \boldsymbol{\mathcal{G}}_{yx}\,\mathrm{Var}\,(\boldsymbol{X}_t)\boldsymbol{\mathcal{G}}'_{yx} + \mathrm{Var}\,(\boldsymbol{\nu}_t)$. As a result, the difference between the variance-covariance matrix of outcomes for the forward-looking variables under the projection-based rule and its counterpart generated by the signal-based rule is positive semi-definite; so that any quadratic loss function over $\boldsymbol{Y}_t$ would at least weakly prefer outcomes under the projections-based rule.

6 Conclusion

This paper studies the implications of imperfect information for equilibrium determination in linear dynamic models when differently informed agents interact. We introduce a single deviation from full information rational expectations: one group of agents is strictly less informed than another. By doing so, we differentiate between types of agents that have different information sets, but where each agent forms rational expectations conditional on available information. The implications of this model structure are stark. We show that indeterminacy of equilibrium is generic in this environment, even if the corresponding full information setting implies uniqueness.

More concisely, even a small amount of noise in this environment, despite the best intentions and optimal filtering of a less informed agent, can produce outcomes where there is a sunspot component to economic fluctuations. Our paper thereby contributes to a recent literature on informational frictions in rational expectations models. Specifically, we discuss the implications for the conduct of monetary policy in a simple model of inflation determination and a richer New Keynesian model. In addition to markedly changing equilibrium outcomes qualitatively, our results show that quantitative differences to the full information benchmark can be economically significant.

Throughout our analysis, we have maintained the assumption that the policymaker's limited information set is nested inside the public's information set, which allows us to treat the private sector as a representative agent. The indeterminacy issues identified by our paper should, however, also extend to richer informational environments as long as the policymaker cannot perfectly observe forward-looking choice variables of the private sector. The key condition behind our indeterminacy results is that the policymaker does not respond one-for-one to belief shocks of the private sector when forward-looking choice variables of the private agents are only imperfectly observed by the policymaker.

The findings in this paper suggest various avenues for further investigation. For example, our framework has strong implications for empirical research: The general model under limited information has a state-space representation like any other linear dynamic framework so that a likelihood function can be constructed. The key difference and main complication with respect to standard frameworks is that the solution of the model is not certainty equivalent. Conditional on the Kalman gains the model implies a standard representation, but the gains are equilibrium objects and depend on second moment properties of the solution. This can be taken into account in solution and estimation, albeit at the cost of posing non-trivial computational challenges.[68]

Nevertheless, empirical work can be facilitated by the distinction between actual outcomes and policymaker projections made in our framework. Conditional on the limited information set of the policymaker the model is simply a linear RE model that is unaffected by the computational issues in the full model. Data on projections could therefore be used to estimate the model only over the space of central bank projections, which would help sharpen inference for the model parameters.

While we illustrate our framework with examples of monetary policy with an imperfectly informed central bank, it is not limited to applications in monetary policy. In section 3 we show that indeterminacy is a generic feature for a general class of economies where private-sector behavior is characterized by a set of expectational linear difference equations, exogenous driving processes are Gaussian, policy is described by a linear rule that responds to the policymaker's projections of economic conditions, and the projections are rational.

A related issue is the choice of the information set. In our examples, we endowed the central bank with specific information sets. Alternatively, one could imagine a scenario where the policymaker chooses an optimal information set that minimizes the impact of sunspot shocks and possibly reduces the incidence of multiplicity. This direction has relevance for policy as central banks operate in a real-time environment fraught with measurement error and regularly face judgment calls on the importance of incoming data.

An important extension should be to look beyond a given class of linear policy rules, as considered here, and model the optimal policy choice for a given set of preferences. Such an exercise could also consider how a desirable policy could be implemented with a suitable policy rule, which requires an analysis of equilibrium selection in the presence of indeterminacy.

Appendix

A The Steady State Kalman Filter

This section describes details of the steady-state Kalman filter for the innovations state space (75) and (76) when Assumption 3 holds. Existence of a steady-state Kalman filter relies on finding an ergodic distribution for $\boldsymbol{S}_t^*$ (and thus $\tilde{\boldsymbol{S}}_t$) with constant second moments $\boldsymbol{\Sigma} \equiv \text{Var}\,(\boldsymbol{S}_t^*)$. When a steady-state filter exists, a constant Kalman gain, $\boldsymbol{K}$ relates projected innovations of $\tilde{\boldsymbol{S}}_t$ to

[68] An alternative empirical approach would be to assume exogenous and constant gains which could nevertheless deliver insights for policy implications in a non-optimal environment. In addition, alternative empirical techniques might also be informative such as impulse response function matching that do not necessarily rely on the full solution of the system.

innovations in the signal, $\tilde{\boldsymbol{S}}_{t|t} = \boldsymbol{K}\tilde{\boldsymbol{Z}}_t$ with:[69]

$$\boldsymbol{K} = \operatorname{Cov}\left(\tilde{\boldsymbol{S}}_t, \tilde{\boldsymbol{Z}}_t\right)\left(\operatorname{Var}\left(\tilde{\boldsymbol{Z}}_t\right)\right)^{-1} = \left(\boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{B}\boldsymbol{D}'\right)\left(\boldsymbol{C}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{D}\boldsymbol{D}'\right)^{-1}. \tag{115}$$

The dynamics of $\boldsymbol{S}_t^*$ are then characterized by

$$\boldsymbol{S}_{t+1}^* = (\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C})\ \boldsymbol{S}_t^* + (\boldsymbol{B} - \boldsymbol{K}\boldsymbol{D})\,\boldsymbol{w}_{t+1} \tag{116}$$

Existence of a steady-state filter depends on finding a symmetric, positive (semi) definite solution $\boldsymbol{\Sigma}$ to the following Riccati equation:

$$\begin{aligned}\boldsymbol{\Sigma} &= (\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C})\boldsymbol{\Sigma}(\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C})' + \boldsymbol{B}\boldsymbol{B}' \\ &= \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}' + \boldsymbol{B}\boldsymbol{B}' - \boldsymbol{K}\left(\boldsymbol{C}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{D}\boldsymbol{D}'\right)\boldsymbol{K}' \\ &= \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}' + \boldsymbol{B}\boldsymbol{B}' - \left(\boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{B}\boldsymbol{D}'\right)\left(\boldsymbol{C}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{D}\boldsymbol{D}'\right)^{-1}\left(\boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{B}\boldsymbol{D}'\right)'\end{aligned} \tag{117}$$

Intuitively, the Kalman filter seeks to construct mean-squared error optimal projections $\boldsymbol{S}_{t|t}$ that minimize $\boldsymbol{\Sigma}$. A necessary condition for the existence of a solution to this minimization problem is the ability to find at least some gain $\hat{\boldsymbol{K}}$ for which $\boldsymbol{A} - \hat{\boldsymbol{K}}\boldsymbol{C}$ is stable; otherwise, $\boldsymbol{S}^*$ would have unstable dynamics as can be seen from (116). Thus, existence of the second moment for the residuals, $\operatorname{Var}(\boldsymbol{S}_t^*) = \boldsymbol{\Sigma} \geq \boldsymbol{0}$, is synonymous with a stable transition matrix $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$.

Formal conditions for the existence of a time-invariant Kalman filter have been stated, among others, by Anderson and Moore (1979), Anderson, McGrattan, Hansen and Sargent (1996), Kailath, Sayed and Hassibi (2000), and Hansen and Sargent (2007). Necessary and sufficient conditions for the existence of a unique and stabilizing solution that is also positive semi-definite depend on the "detectability" and "unit-circle controllability" of certain matrices in our state space. We restate those concepts next.

DEFINITION 3 (Detectability) *A pair of matrices $(\boldsymbol{A}, \boldsymbol{C})$ is detectable when no right eigenvector of $\boldsymbol{A}$ that is associated with an unstable eigenvalue is orthogonal to the row space of $\boldsymbol{C}$. That is, there is no non-zero column vector $\boldsymbol{v}$ such that $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{v}\lambda$ and $|\lambda| \geq 1$ with $\boldsymbol{C}\boldsymbol{v} = \boldsymbol{0}$.*

Detectability alone is already sufficient for the existence of *some* solution to the Riccati equation such that $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$ is stable; see (Kailath et al. 2000, Table E.1). Evidently, detectability is assured when $\boldsymbol{A}$ is a stable matrix, regardless of $\boldsymbol{C}$. To gain further intuition for the role of detectability, consider transforming $\boldsymbol{S}_t$ into "canonical variables" by premultiplying $\boldsymbol{S}_t$ with the matrix of eigenvectors of $\boldsymbol{A}$ — this transformation into canonical variables is at the heart of procedures for solving rational expectations models known from Blanchard and Kahn (1980), King and Watson (1998), Klein (2000), Sims (2002). Detectability then requires the signal equation (76) to provide some signal (i.e. to have non-zero loadings) for any unstable canonical variables.[70]

[69] See also (78) in the paper.

[70] Specifically, let $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$ with $\boldsymbol{\Lambda}$ diagonal be the eigenvalue-eigenvector factorization of $\boldsymbol{A}$ so that the columns of $\boldsymbol{V}$ correspond to the right eigenvectors of $\boldsymbol{A}$. Define canonical variables $\boldsymbol{S}_t^C \equiv \boldsymbol{V}^{-1}\boldsymbol{S}_t$. The signal equation can then be stated as $\boldsymbol{Z}_t = \boldsymbol{C}\boldsymbol{V}\boldsymbol{S}_t^C$ and detectability requires the signal equation to have non-zero loadings on at least every canonical variable associated with an unstable eigenvalue in $\boldsymbol{\Lambda}$.

To establish existence of a solution to the Riccati equation that is unique and positive semi-definite, we follow Kailath et al. (2000) and require unit-circle controllability, defined as follows.

DEFINITION 4 (Unit-circle controllability) *The pair* $(\boldsymbol{A}, \boldsymbol{B})$ *is unit-circle controllable when no left-eigenvector of* $\boldsymbol{A}$ *associated with an eigenvalue on the unit circle is orthogonal to the column space of* $\boldsymbol{B}$*. That is, there is no non-zero row vector* $\boldsymbol{v}$ *such that* $\boldsymbol{vA} = \boldsymbol{v}\lambda$ *with* $|\lambda| = 1$ *and* $\boldsymbol{vB} = \boldsymbol{0}$*.*

In our state space, with $\boldsymbol{BD'} \neq \boldsymbol{0}$, shocks to state and measurement equation are generally correlated. Unit-circle controllability is thus applied to the following transformations of $\boldsymbol{A}$, $\boldsymbol{B}$:[71]

$$\boldsymbol{A}^C \equiv \boldsymbol{A} - \boldsymbol{BD'}\left(\boldsymbol{DD'}\right)^{-1}\boldsymbol{C} \qquad\qquad \boldsymbol{B}^C \equiv \boldsymbol{B}\left(\boldsymbol{I} - \boldsymbol{D'}\left(\boldsymbol{DD'}\right)^{-1}\boldsymbol{D}\right) \tag{118}$$

Based on these definitions, the following theorem restates results from Kailath et al. (2000) in our notation:

THEOREM 3 (Stabilizing Solution to Riccati Equation) *Provided Assumption 3 holds, a stabilizing and positive semi-definite solution to the Riccati equation* (117) *exists when* $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ *is unit-circle controllable and* $(\boldsymbol{A}, \boldsymbol{C})$ *is detectable. The steady-state Kalman gain is such that* $\boldsymbol{A} - \boldsymbol{KC}$ *is a stable matrix; moreover, the stabilizing solution is unique.*[72]

Proof. See Theorem E.5.1 of Kailath et al. (2000); related results are also presented in Anderson et al. (1996), or Chapter 4 of Anderson and Moore (1979). ■

In our context, with $\boldsymbol{C} = \boldsymbol{HA}$ and $\boldsymbol{D} = \boldsymbol{HB}$, the conditions for detectability and unit-circle controllability can also be restated as follows.

PROPOSITION 7 (Detectability of $(\boldsymbol{A}, \boldsymbol{H})$) *With* $\boldsymbol{C} = \boldsymbol{HA}$*, detectability of* $(\boldsymbol{A}, \boldsymbol{C})$ *is equivalent to detectability of* $(\boldsymbol{A}, \boldsymbol{H})$

Proof. When $(\boldsymbol{A}, \boldsymbol{C})$ are detectable, we have $\boldsymbol{Cv} \neq \boldsymbol{0}$ for any right-eigenvector of $\boldsymbol{A}$ associated with an eigenvalue λ on or outside the unit circle, $|\lambda| \geq 1$. With $\boldsymbol{C} = \boldsymbol{HA}$ we then also have $\boldsymbol{Cv} = \boldsymbol{HAv} = \boldsymbol{Hv}\lambda \neq \boldsymbol{0} \Leftrightarrow \boldsymbol{Hv} \neq \boldsymbol{0}$ ■

Furthermore, with $\boldsymbol{C} = \boldsymbol{HA}$ and $\boldsymbol{D} = \boldsymbol{HB}$, the above expressions for $\boldsymbol{A}^C$ and $\boldsymbol{B}^C$ can be transformed as follows:

$$\boldsymbol{A}^C = (\boldsymbol{I} - \boldsymbol{P}^C)\boldsymbol{A} \quad \text{and} \quad \boldsymbol{B}^C = (\boldsymbol{I} - \boldsymbol{P}^C)\boldsymbol{B} \quad \text{with} \quad \boldsymbol{P}^C \equiv \boldsymbol{BH'}\left(\boldsymbol{HBB'H'}\right)^{-1}\boldsymbol{H}. \tag{119}$$

$\boldsymbol{P}^C$ is a non-symmetric, idempotent projection matrix with $\boldsymbol{HP}^C = \boldsymbol{H}$.[73]

[71] Notice that $\boldsymbol{B}^C = \boldsymbol{B\mathcal{M}}^D$ where $\boldsymbol{\mathcal{M}}^D = \boldsymbol{I} - \boldsymbol{D'}\left(\boldsymbol{DD'}\right)^{-1}\boldsymbol{D}$ is a projection matrix, which is symmetric and idempotent, $\boldsymbol{\mathcal{M}}^D = \boldsymbol{\mathcal{M}}^D\boldsymbol{\mathcal{M}}^D$, and orthogonal to the row space of $\boldsymbol{D}$.To appreciate the role of $\mathcal{M}^D$, consider the following thought experiment: $\boldsymbol{\mathcal{M}}^D$ construct the residual in projecting the shocks of the system *off* the shocks in the signal equation, $\boldsymbol{w}_t - E(\boldsymbol{w}_t|\boldsymbol{Dw}_t) = \boldsymbol{\mathcal{M}}^D\boldsymbol{w}_t$.

[72] There may be other, non-stabilizing positive semi-definite solutions.

[73] An idempotent matrix is equal to its own square, that is $\boldsymbol{P}^C = \boldsymbol{P}^C\boldsymbol{P}^C$, and the eigenvalues of an idempotent matrix are either zero or one and we have $|\boldsymbol{P}^C| = \boldsymbol{0}$.

PROPOSITION 8 (Unit-circle controllability of $(\boldsymbol{A}(\boldsymbol{I}-\boldsymbol{P}^C), \boldsymbol{B})$) *With $\boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}$ and $\boldsymbol{D} = \boldsymbol{H}\boldsymbol{B}$, unit-circle controllability of $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ is equivalent to unit-circle controllability of $(\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}^C), \boldsymbol{B})$ with $\boldsymbol{P}^C$ defined in* (119).

Proof. Suppose $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ are unit-circle controllable. Let $\tilde{\boldsymbol{v}} \equiv \boldsymbol{v}(\boldsymbol{I} - \boldsymbol{P}^C)$ and note that left-eigenvectors of $\boldsymbol{A}^C$ associated with eigenvalues on the unit circle cannot be orthogonal to $\boldsymbol{P}^C$ (otherwise we would have $\boldsymbol{v}\boldsymbol{A}^C = \boldsymbol{0}$). Accordingly, $\boldsymbol{v}\boldsymbol{A}^C = \boldsymbol{v}\lambda$ with $|\lambda| = 1$, $\boldsymbol{v}\boldsymbol{B}^C \neq \boldsymbol{0}$ and $\boldsymbol{v} \neq \boldsymbol{0}$ is equivalent to $\tilde{\boldsymbol{v}}\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}^C) = \tilde{\boldsymbol{v}}\lambda$ with $|\lambda| = 1$, $\tilde{\boldsymbol{v}} \neq \boldsymbol{0}$ $\tilde{\boldsymbol{v}}\boldsymbol{B} \neq \boldsymbol{0}$. The converse reasoning applies as well. ■

As discussed in the main text, an upshot of Proposition 8 is that a sufficient condition for unit-circle controllability of $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ is for $\boldsymbol{B}$ to have full rank.

Finally, for convenience, we define the the joint concept of detectability and unit-circle controllability for the triplet $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{H})$.

DEFINITION 5 (Joint detectability and unit-circle controllability) *The triplet $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{H}$ is detectable and unit-circle controllable when $(\boldsymbol{A}, \boldsymbol{H})$ is detectable and $(\boldsymbol{A}(\boldsymbol{I}-\boldsymbol{P}^C), \boldsymbol{B})$ is unit-circle controllable, where $\boldsymbol{P}^C$ is defined in* (119).

B Endogenous Forecast Errors when the Signal is Endogenous

This section of the appendix describes an algorithm to solve numerically for the endogenous forecast errors in the endogenous-signal case of our general setup, described in section 3. Our numerical approach combines elements of standard techniques for solving linear RE models with a fast algorithm to solve the non-linear fixed-point problem for the Riccati-equation embedded in the Kalman-filter while ensuring consistency with the projection condition. The algorithm searches for shock loadings $\boldsymbol{\Gamma}_{\eta\varepsilon}$ and $\boldsymbol{\Gamma}_{\eta b}$ that satisfy the projection condition for a Kalman filter that is consistent with equilibrium outcomes of endogenous and exogenous variables.

As described in section 3, the endogenous-signal case considers a measurement equation of the form

$$\boldsymbol{Z}_t = \boldsymbol{H}_x \boldsymbol{X}_t + \boldsymbol{Y}_t \, . \tag{87}$$

By construction, we have $\boldsymbol{Z}_t = \boldsymbol{Z}_{t|t}$ and thus $\boldsymbol{Y}_t^* = -\boldsymbol{H}_x \boldsymbol{X}_t^*$, so that the innovation system given by (75) and (76) can then be simplified as follows:

$$\tilde{\boldsymbol{X}}_{t+1} = \tilde{\boldsymbol{A}} \boldsymbol{X}_t^* + \tilde{\boldsymbol{B}} \boldsymbol{w}_{t+1} \tag{120}$$

$$\tilde{\boldsymbol{Z}}_{t+1} = \tilde{\boldsymbol{C}} \boldsymbol{X}_t^* + \tilde{\boldsymbol{D}} \boldsymbol{w}_{t+1} \tag{121}$$

$$\text{with} \quad \tilde{\boldsymbol{A}} = \boldsymbol{A}_{xx} - \boldsymbol{A}_{xy} \boldsymbol{H}_x \tag{122}$$

$$\tilde{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B}_{x\varepsilon} & \boldsymbol{0} \end{bmatrix} \tag{123}$$

$$\tilde{\boldsymbol{C}} = \boldsymbol{H}_x \; (\boldsymbol{A}_{xx} - \boldsymbol{A}_{xy} \boldsymbol{H}_x) + \boldsymbol{A}_{yx} - \boldsymbol{A}_{yy} \boldsymbol{H}_x \tag{124}$$

$$\tilde{\boldsymbol{D}} = \begin{bmatrix} (\boldsymbol{H}_x \boldsymbol{B}_{x\varepsilon} + \boldsymbol{\Gamma}_{\eta\varepsilon}) & \boldsymbol{\Gamma}_{\eta b} \end{bmatrix} \tag{125}$$

where $\boldsymbol{A}_{xx}$, $\boldsymbol{A}_{xy}$, etc. denote suitable sub-matrices of $\boldsymbol{A}$; and $\tilde{\boldsymbol{D}}$ embodies a given guess of $\boldsymbol{\Gamma}_{\eta\varepsilon}$ and $\boldsymbol{\Gamma}_{\eta b}$.

For a given $\tilde{\boldsymbol{D}}$, the Kalman-filtering solution to this system generates a Kalman gain $\boldsymbol{K}_x$ which can be used to form projections $\tilde{\boldsymbol{X}}_{t|t} = \boldsymbol{K}_x \tilde{\boldsymbol{Z}}_t$. What remains to be seen is whether this guess for $\tilde{\boldsymbol{D}}$ also satisfies the projection condition. The projection condition requires $\boldsymbol{K}_y = \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{K}_x$. Together with the projection condition, the measurement equation (87) implies $\boldsymbol{I} = \boldsymbol{H}_x\boldsymbol{K}_x + \boldsymbol{K}_y = (\boldsymbol{H}_x + \boldsymbol{\mathcal{G}}_{yx})\,\boldsymbol{K}_x$. All told, we need to find shock loadings that support a gain $\boldsymbol{K}_x$ such that $\boldsymbol{L}\boldsymbol{K}_x = \boldsymbol{I}$ where $\boldsymbol{L} = \boldsymbol{H}_x + \boldsymbol{\mathcal{G}}_{yx}$.

We employ a numerical solver that searches for a $\tilde{\boldsymbol{D}}$ that generates a Kalman gain $\boldsymbol{K}_x$ such that $\boldsymbol{L}\boldsymbol{K}_x = \boldsymbol{I}$. Given a solution for $\tilde{\boldsymbol{D}}$ that satisfies the projection condition $\boldsymbol{L}\boldsymbol{K}_x = \boldsymbol{I}$, we can then back out $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ based on (125).

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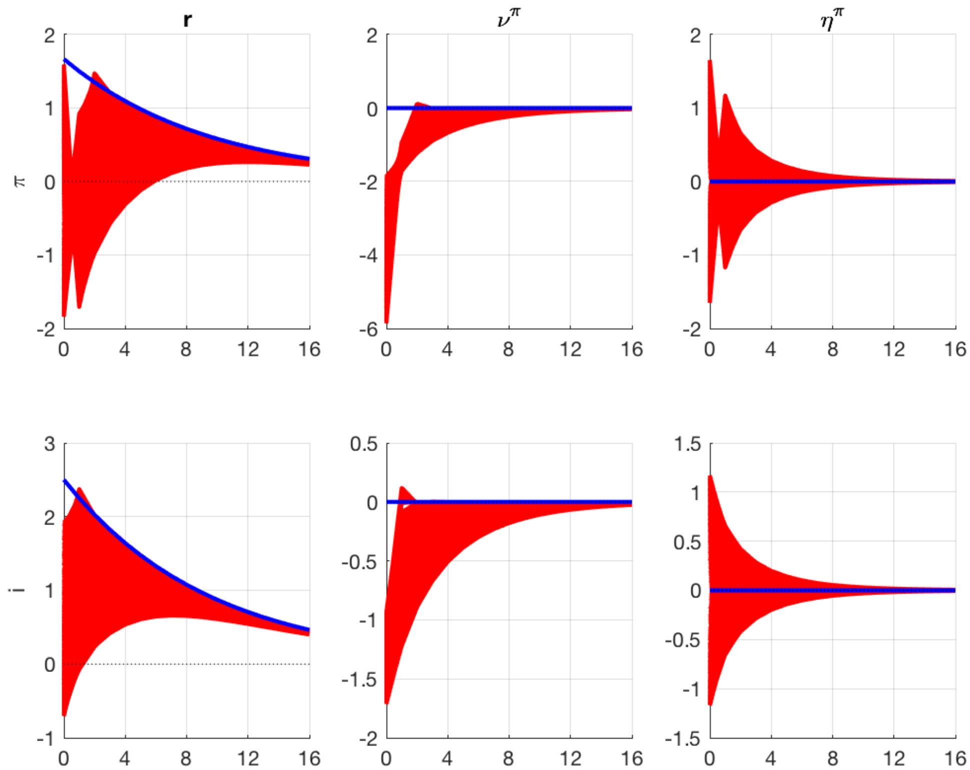
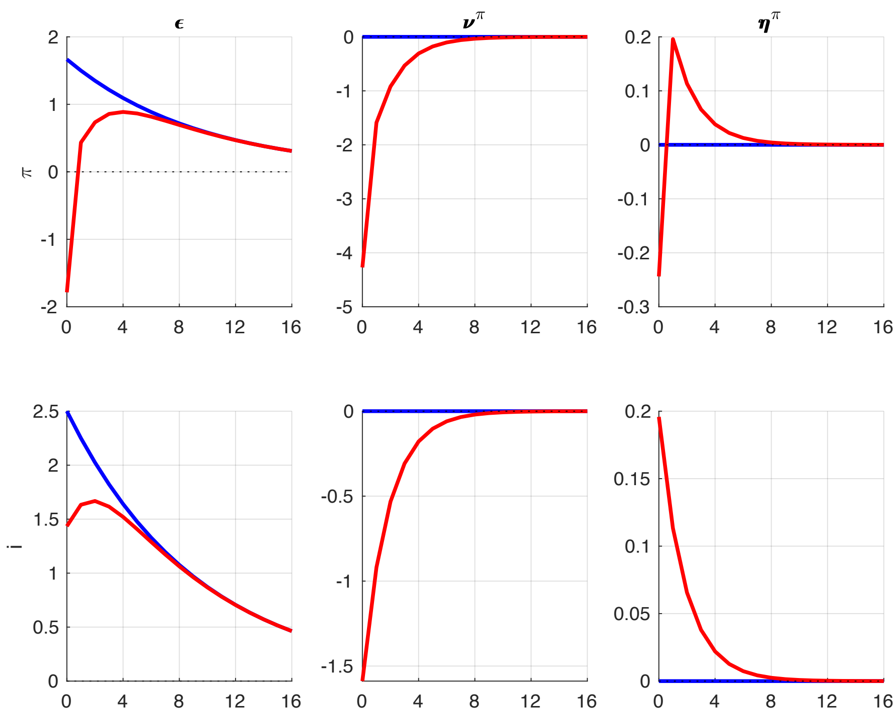
Figure 1: IRFs of Various Equilibria in Fisher economy



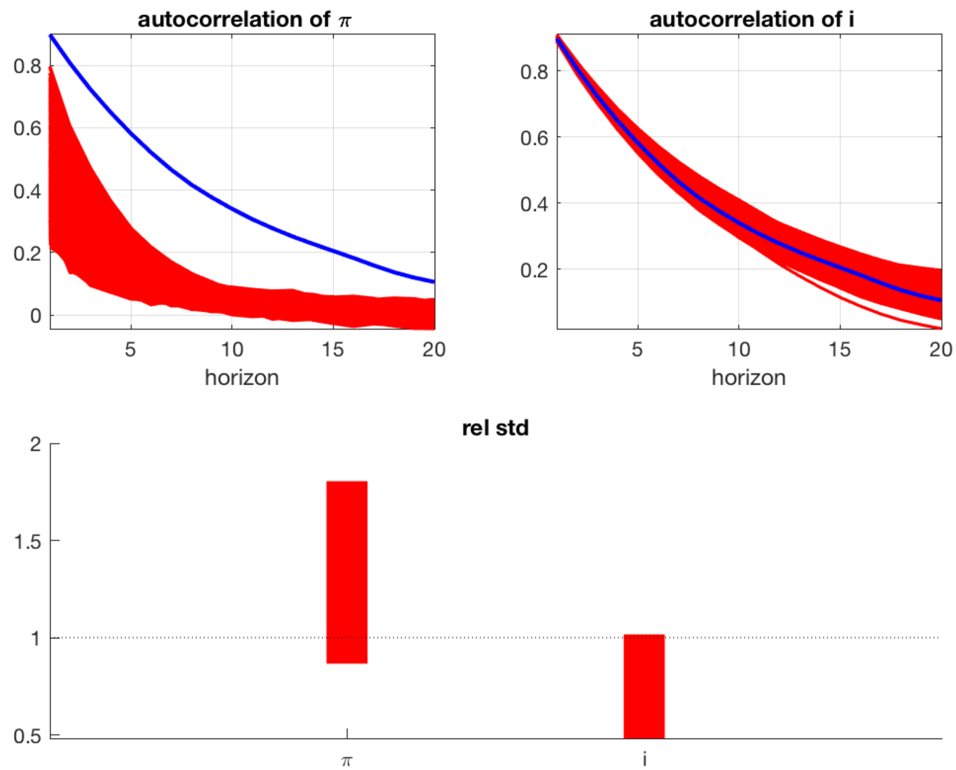
Note: Impulse response functions (IRF) for the Fisher economy model under full information (blue) as well as various limited-information equilibria (red). Each row represents the response of a specific variable to the shocks in the model whereas each column represent the responses of the endogenous variables to a specific shock.

Figure 2: Example IRF from an Equilibrium in the Fisher economy



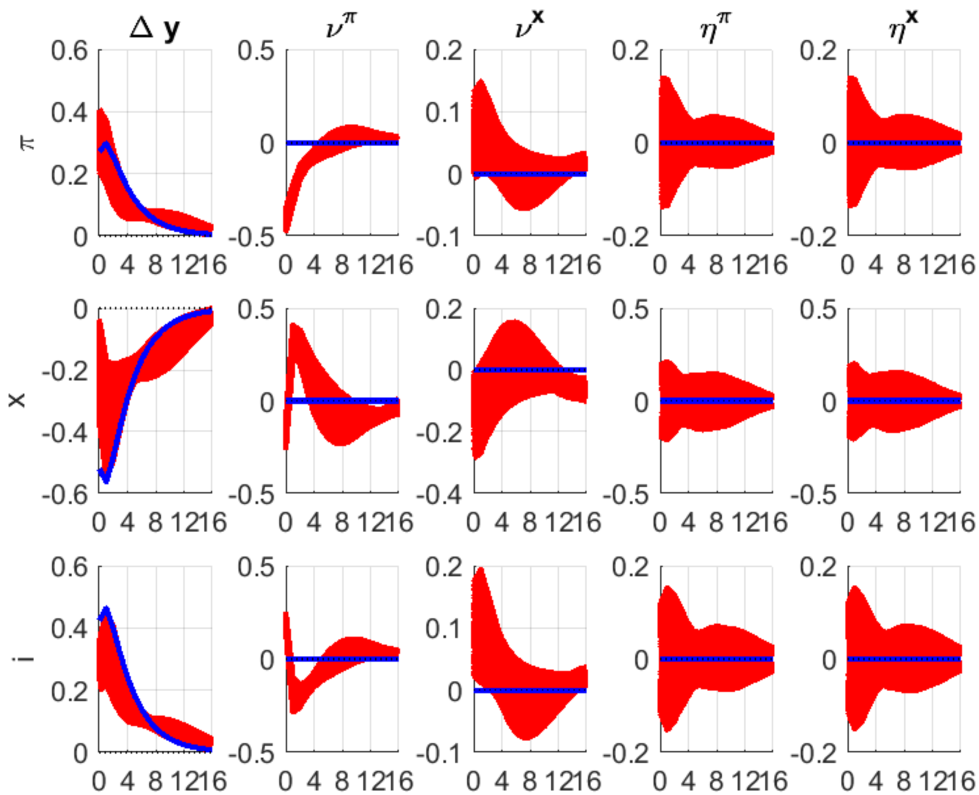
Note: Impulse response functions (IRF) for the Fisher economy model under full information (blue) as well as an example from the limited-information equilibria (red).

Figure 3: Second moments of limited-information equilibria in Fisher economy



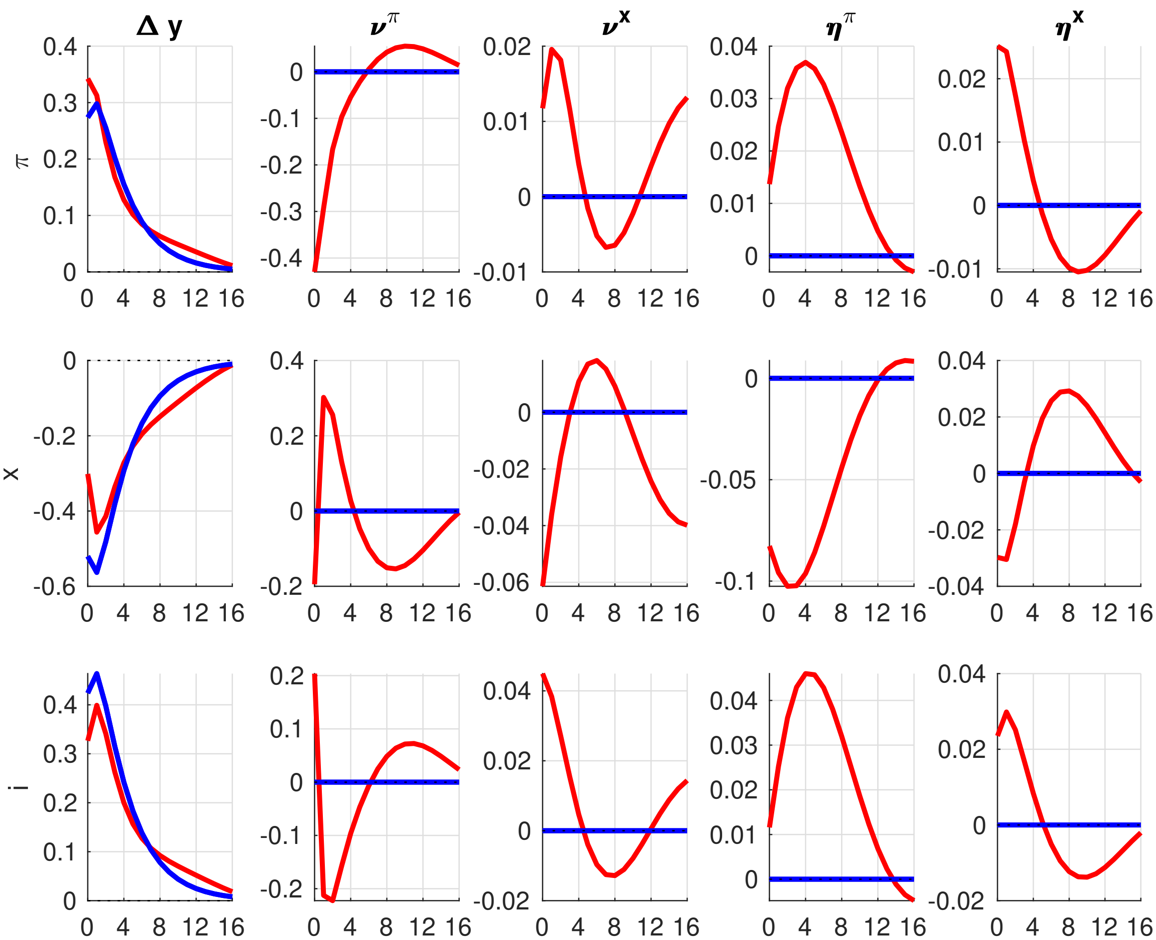
Note: Top panels show moments of endogenous variables for the Fisher economy model under full information (blue) as well as various limited information equilibria (red). Bottom panel reports ranges of relative standard deviations of outcomes under limited information relative to the full-information outcomes.

Figure 4: IRFs of Various Equilibria in New Keynesian model



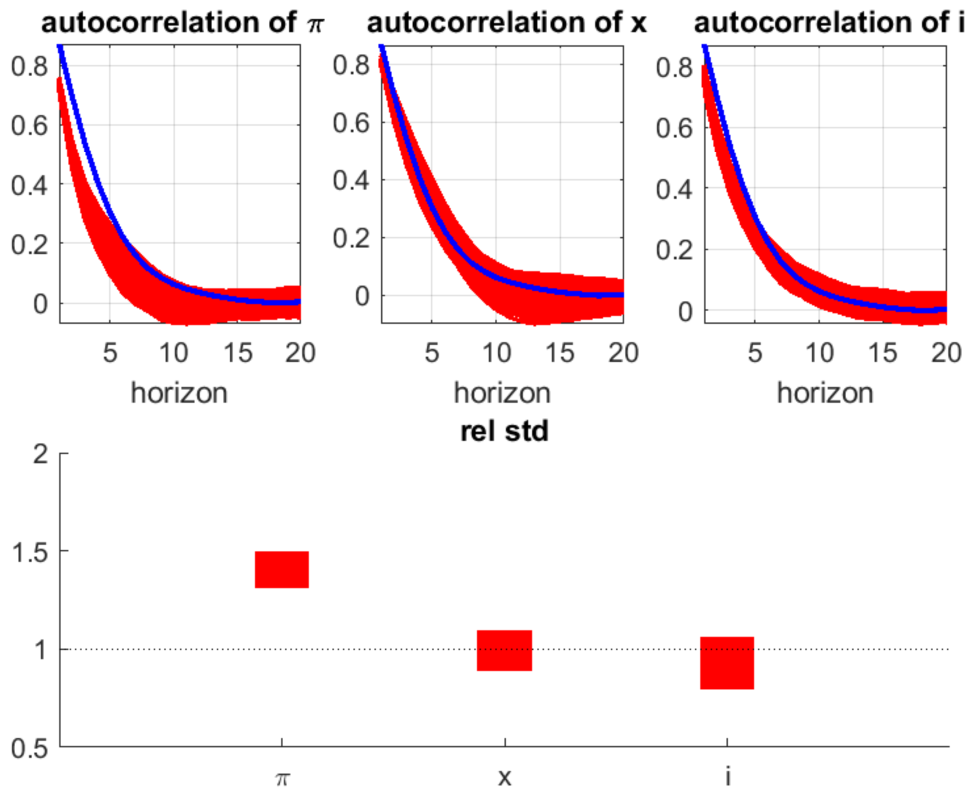
Note: Impulse response functions (IRF) for the New Keynesian model under full information (blue) as well as various limited-information equilibria (red). Each row represents the response of a specific variable to the shocks in the model whereas each column represent the responses of the endogenous variables to a specific shock. An example of the IRF of one of the limited-information equilibira is show in Figure (5).

Figure 5: IRF in New Keynesian model: Example of a limited information equilibrium



Note: Impulse responses for one example (red) of the limited-information equilibria of the New Keynesian model shown in Figure 4. (Full-information IRF in blue.)

Figure 6: Second moments of limited-information equilibria in New Keynesian model



Note: Top panels show moments of endogenous variables for the New Keynesian model under full information (blue) as well as various limited information equilibria (red). Bottom panel reports ranges of relative standard deviations of outcomes under limited information relative to the full-information outcomes.

Table 1: Parameters for NK model

Symbol	Description	Value
β	Discount Factor	0.99
σ	Substitution Elasticity	1.00
ϕ	Labor Elasticity	1.00
γ	Inflation Indexation	0.25
ϕ_π	Policy Coefficient	2.50
ϕ_x	Policy Coefficient	0.50
ρ_y	AR(1) - Coefficient	0.75
σ_y	StD. Output Growth	0.30
σ_π	StD. Measurement Error	0.80
σ_x	StD. Measurement Error	1.39
κ	Composite Parameter	0.17

Note: Parameter values for the numerical analysis of the NK model. Values are standard in the literature.