Stochastic House Appreciation and Optimal Mortgage Lending*

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Abstract

We characterize the optimal mortgage contract in a continuous time setting with a risky borrower who needs to be given incentives to repay his debt, costly foreclosure and stochastic house appreciation. We show that many features of subprime lending observed in practice are consistent with rational behavior of both borrowers and lenders. In particular, when house prices are expected to grow, it is optimal to provide the risky borrowers with a lower initial rate, which is to increase over time, and to increase the borrowers' access to credit. The housing slump results in the tightening of the borrowers' access to credit and default clustering among the least creditworthy ones. We also find that house appreciation makes it profitable to give loans to less creditworthy borrowers who otherwise would be excluded from the housing market, and that these loans could generate substantial ex-ante utility gains for these borrowers. Bailing out the most distressed borrowers in the slump phase is not incentive compatible, as it encourages irresponsible financial behavior in the boom phase. On the other hand, it makes sense to help borrowers who were in good standing before the crisis by reducing their mortgage balance and interest rate.

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1 Introduction

The recent housing market crisis has brought attention to the subprime mortgage market, which experienced exponential growth over the past few years. The share of subprime mortgages to total originations increased from 6% in 2002 to 20% in 2006. As of 2006, the value of U.S. subprime mortgages was estimated at $1.5 trillion, or 15% of the $10 trillion residential mortgage market.[1] Subprime mortgages account for a significant part of the recent increase in household mortgage debt in the United States, from about 60% of GDP in 2003 to above 75% of GDP in 2006.[2] It is widely believed that subprime lending has played a major role in the housing market meltdown of 2007.

Unlike traditional prime mortgages, subprime mortgages have been issued to higher-risk borrowers who bought pricey houses relative to their income level and make little or no downpayment. Many of these mortgages came with incentives including lower initial teaser rates, which were to be reset higher in the future. The most common examples of this are the 2/28 and the 3/27 loans, which carry a lower introductory rate for the first two or three years. After that, the interest rate resets to a higher level for the remaining 28 or 27 years of the loan.

Because of high default rates among subprime borrowers and big losses to subprime investors in the declining housing market, subprime lending has lately caused a storm of controversy. Some critics accuse subprime lenders of predatory lending to naive borrowers who do not fully understand mortgage terms. Others say that subprime underwriters issued mortgages to people who could not afford to pay them back, and then quickly sold the mortgages to outside investors in the form of mortgage-backed securities. Most critics agree that subprime loans do not make economic sense and should have not been originated in the first place. On the other hand, other experts argue that during the housing boom many homes were bought with little or no money down because both buyers and lenders bet on additional home-price appreciation to create equity. These bets, while risky, gave less creditworthy borrowers a chance at homeownership.

A related controversy concerns how mortgage lenders should react to defaults and foreclosures when house prices decline. Federal Reserve Chairman Ben Bernanke called on lenders to aid struggling homeowners by reducing their principal – the sum of money they borrowed – to lessen the likelihood of foreclosure.[3] Similarly, the House of Representatives passed legislation in May 2008 encouraging mortgage companies to reduce the principal on troubled loans.[4] On the other hand, many mortgage lenders and investors are reluctant to help borrowers with troubled loans. They emphasize that the risk of distorting the borrower's incentives adds to the complexity of the issue as once the borrowers anticipate to be bailed out they might stop paying back

[1] See, for example, Agarwal and Ho (August 2007).

[2] The mortgage debt data are from Flow of Funds Accounts of the United States, Federal Reserve Board, and the GDP data are from Bureau of Economic Analysis.

[3] See Bernanke (2008).

[4] See House of Representatives (2008). See also Hubbard and Mayer (2008) for a recent outline of the mortgage modification program featuring mortgage balance and interest rate reduction.

their debt.

Despite the economic significance of subprime lending and the extent of the surrounding controversy, there has been thus far no attempt at formal analysis of efficient lending in a stochastic house appreciation environment. Such an analysis would allow addressing a number of questions. Could a housing boom followed by a housing slump explain certain features of subprime lending and its subsequent meltdown if both lenders and borrowers have rational expectations about the future states of the housing market? Could subprime mortgages improve benefits to borrowers and lenders relative to traditional mortgages when house prices are expected to grow? What is a lender's optimal response when house prices start to decline, taking into account the borrowers' incentives to repay their debt?

In this paper, we formally approach this issue by addressing a more general normative question. Assuming rational behavior of borrowers and lenders, what is the best possible mortgage contract between a home buyer and a financial institution when house prices are expected to grow but also there is risk of housing crisis? Instead of considering a particular class of mortgages, we derive an optimal mortgage contract as a solution to a general dynamic contracting problem without imposing restrictive assumptions on the payments between the borrower and the lender or the circumstances under which the home is repossessed. We then compare features of existing mortgage contracts with the derived best possible contract.

We focus our attention on a simple setting that nevertheless allows us to capture three aspects that we believe are central to subprime lending. These are: (i) a risky borrower who needs to be given incentives to repay his debt, (ii) costly foreclosure and (iii) stochastic house price appreciation. This setting allows us to focus on the fundamental feature of the borrowing-lending relationship with collateral in a stochastic house price environment, which is how to efficiently provide a borrower with incentives to repay his debt using a threat of a costly liquidation taking into account a possibility of house appreciation and a risk of price fall.

We adopt a two-step approach. First, assuming rational behavior, we derive an optimal mortgage contract in the stochastic house appreciation environment, i.e., the best possible incentive-compatible contract that maximizes a total surplus of the relationship between the borrower and the lender, as a solution to a general dynamic contracting problem. Then we examine whether main features of existing mortgage contracts are consistent with the properties of the optimal contract. As the optimal contract contains provisions of what should be done in the housing slump, we can address this question as well.

Specifically, we consider a continuous-time setting in which a borrower with limited liability needs outside financial support from a risk-neutral lender in order to purchase a house. Home ownership generates for the borrower a public and deterministic utility stream. The distribution of the "excess" income, which the borrower can use to pay back his debt, is publicly known, although its realizations are privately observable by the borrower. The moral hazard problem is that the borrower can conceal a part of his income and then divert it for personal consumption rather than repay debt.

We assume that the housing market at time zero is in the boom phase, during which time a home

appreciates at a constant rate. At any time, however, a boom can turn into a slump with a certain probability, in which case a home loses its value and the housing market becomes illiquid. The price process is exogenous, and the borrower and the lender have rational expectations.[5] We assume that the borrower and the lender are sufficiently small so that their actions have no effect on macroeconomic variables such as the market price of the home.

Before the purchase of the house, the borrower and the lender sign a contract that will govern their relationship in the future. The contract specifies transfers between the borrower and the lender, conditional on the history of the borrower's income reports and the state of the housing market, as well as the circumstances under which the lender would foreclose the loan and seize the home. The borrower has limited liability and can default on the mortgage contract at any time. The borrower has also the option to sell the home. We assume that selling the home in the boom phase is more efficient than liquidating it through the repossession process due to associated dead-weight losses.

We characterize the optimal contract, i.e. the arrangement between the borrower and the lender that maximizes their combined total surplus, using three state variables: the state of the housing market (i.e., the boom or slump), the market home price, and the borrower's continuation utility (i.e., the expected payoff to the borrower provided he acts optimally given the terms of his contract with the lender). We then show that the optimal contract can be implemented using a mortgage in the form of a home equity loan (HEL) with embedded access to credit (or negative amortization limit). The features of the optimal contract do not depend on the competitive structure of the lending industry. However, a lender with more bargaining power extracts a bigger surplus by charging a higher underwriting fee. For demonstration purposes, we consider numerical examples in which the lender breaks even.

When house prices are expected to grow, the optimal mortgage provides the borrowers with a lower initial rate, which is to increase over time. The least creditworthy borrowers are given an additional reduction in the rate during the boom, while the more creditworthy borrowers are charged an insurance premium, which allows them to participate in the loan modification program when house prices start to decline. It is also optimal to increase the borrower's access to credit in the boom phase (by increasing the credit line limit or negative amortization limit). The housing slump results in the tightening of the borrowers' access to credit and default clustering among the least creditworthy borrowers. The more creditworthy borrowers (those with lower debt levels) are partially insured against the slump through their participation in a loan modification program featuring a balance reduction.

The features of the optimal mortgage contract can be explained by the incentive-compatibility constraints and the dual-optimization objective of the contracting problem: minimization of liquidation inefficiencies and

[5] Such a process can be derived in a general equilibrium framework. For example, Kahn (2008) introduces a Markov regime-switching specification for productivity growth in the non-housing sector, and shows that such regime switches are a plausible candidate for explaining–both qualitatively and quantitatively–the large low-frequency changes in housing price trends.

maximization of the value of the option to sell the home in the future at a higher price.

The interest rate goes up with time during the housing boom because the lender has to make money on the loan in the good states, as he is likely to lose money in the bad states. Hence, the lender increases the interest rate as the home appreciates. On the other hand, the borrower can afford to pay higher interest rates and is less likely to default in the good states, since he can borrow more as the home appreciates.

The access to credit provides flexibility for the borrower to cover possible low income realizations, which in turn lowers chances of default inefficiencies. The credit limit, which is determined by incentive compatibility constraints, takes into account the value of the option to sell. The option value increases with the home price and so does the credit limit. During the slump, the option to sell the home loses its value, which results in the credit limit reduction.

The additional interest rate reduction of the least creditworthy borrowers during the boom is driven by the maximization of the option value, while the default clustering during the slump happens because of the borrower's limited liability and the incentive-compatibility constraints. Bailing out the most distressed borrowers in the slump phase is not incentive compatible, as it would encourage irresponsible financial behavior in the boom phase. The insurance of more creditworthy borrowers is explained by the fact that these borrowers are not likely to default during the boom, but are more vulnerable during the slump.

The features of the optimal mortgage are parallel to some key aspects of subprime lending. In particular, during the boom it is optimal to increase the interest rates over time. This finding is in line with the fact that many subprime mortgages originated in the US had teaser rates, which were to be reset to higher rates. It is also optimal to provide the borrowers with increasing access to credit over time as long as house prices continue to grow. This result is consistent with the increasing indebtness of households during the housing boom. We also find that house appreciation makes it profitable to give loans to less creditworthy borrowers who otherwise would be shut out of the housing market. This is consistent with an observed extension of mortgage credit to less creditworthy borrowers during the housing boom.[6] According to the parametrized example we consider, the extension of credit to less creditworthy borrowers could generate substantial ex-ante utility gains for these borrowers. The efficient lending also implies default clustering among the least creditworthy borrowers during the slump, which is consistent with an observed rapid increase in defaults among the borrowers with the highest loan-to-value ratios once house prices started to decline.[7]

We conclude that many features of subprime lending are consistent with rational behavior of both borrowers and lenders in a stochastic house appreciation environment. Thus our model may help to explain the emergence of this market in recent years. It is important, however, to stress one limitation of our efficiency results. The features of mortgage lending that are optimal at the individual level may have negative con-

[6] See Mayer and Pence (2008).

[7] Demyank and Van Hemert (2008) find that in 2006 the increases in the adjusted delinquency and foreclosure rates are almost exclusively caused by the worsening performance of loans with a combined LTV of 80 percent or more.

sequences at the aggregate level, since the borrower and the lender do not take into account the potential negative externalities that their optimal contract might impose.

In terms of policy implications, our model suggests that bailing out the most distressed borrowers in the slump phase is not incentive compatible, as it encourages irresponsible financial behavior in the boom phase. On the other hand, it makes sense to help borrowers who were in good standing before the crisis by reducing their debt balance and interest rates.

Related Literature

This paper belongs to the growing literature on dynamic optimal security design, which is a part of the literature on dynamic optimal contracting models using recursive techniques that began with Green (1987), Spear and Srivastava (1987), Abreu, Pearce and Stacchetti (1990) and Phelan and Townsend (1991), among many others.[8] To the best of our knowledge our paper is a first study in the optimal financing literature that considers a dynamic environment with stochastic asset appreciation and an option to sell the asset.

The most closely related paper to ours is Piskorski and Tchistyi (2007) who study the optimal mortgage design in a stationary continuous time setting with volatile and privately observable income of the borrower and a stochastic market interest rate. They show that the optimal mortgage takes the form of an option ARM, and that default rates and interest rates correlate positively with the market interest rate. In this paper, we assume a constant interest rate and focus on a stochastic house price environment where the borrower has the option to sell the home. The findings of this study are complementary to those of Piskorski and Tchistyi (2007).

The two other studies closely related to ours are DeMarzo and Fishman (2007a) and its continuous-time formulation by DeMarzo and Sannikov (2006). These papers study long-term financial contracting in a setting with privately observed cash flows, and show that the implementation of the optimal contract involves a credit line with a constant interest rate and credit limit, long-term debt, and equity.[9]

There is a sizeable real estate finance literature that addresses the design of mortgages.[10]. In terms of this literature, to our knowledge, our paper is the first study of optimal mortgage design in a dynamic moral hazard environment, and the first study that addresses the optimality of alternative mortgage products. Our paper is distinct from this literature, as we do not impose any exogenous restrictions on the type of contracts the lender can offer, and instead derive an optimal mortgage in our environment.

There is also a number of studies that address the optimal strategy of the mortgage borrower in environments with stochastic house prices (see for example Kau, Keenan, Mueller, and Epperson (1992) and Deng, Quigley, and Van Order (2000), Campbell and Cocco (2007)). This literature restricts its attention to

[8] Sannikov (2006) provides a recent treatment of continuous-time techniques for a principal-agent problem.

[9] See also related papers by Biais et al. (2006), Clementi and Hopenhayn (2006), DeMarzo and Fishman (2007b), DeMarzo and Sannikov (2008), Tchistyi (2006), He (2007), DeMarzo, Fishman, He and Wang (2008), Philippon and Sannikov (2007).

[10] See for example Dunn and Spatt (1985), Chari and Jagannathan (1989), LeRoy (1996), Stanton and Wallace (1998).

a specific class of contracts, however, and thus, unlike this paper, does not address the question of optimal lending. Lustig and Van Nieuwerburgh (2005) study consumption insurance and risk premia in a model with housing collateral and limited commitment of the households. Unlike this paper they do not consider a moral hazard problem and their model does not feature default in equilibrium.

The paper is organized as follows. Section 2 presents the continuous-time model with stochastic house appreciation. Section 3 describes the dynamic contracting problem. Section 4 derives the optimal contract using the three state variables: the state of the housing market, the market home price and the borrower's continuation utility. Section 5 presents the implementations of the optimal contract using financial arrangements that resemble the ones used in the residential mortgage market. Section 6 concludes.

2 The Model

Time is continuous and infinite. There is one borrower (a homebuyer) and one lender (a large financial institution).[11] The lender is risk neutral, has unlimited capital, and values a stochastic cumulative cash flow $\{f_t\}$ as

$$E\left[\int_0^\infty e^{-r} df_t\right],$$

where r is the discount rate.

The borrower's consumption consists of two categories. The first is "necessary" consumption, which includes grocery food, medicine, transportation, and other goods and services essential for the survival. The cumulative minimum level of necessary consumption is given by an exogenous stochastic process $\{\eta_t\}$ that incorporates shocks such as medical bills, auto repair costs, fluctuations of food and gasoline prices, and so on. The second is discretionary consumption, which, among many other things, may include such items as restaurant dining, vacation trips, buying a new car, et cetera. We assume that the borrower must use his available funds to first cover the necessary expenses η_t before spending on "discretionary" consumption or potential debt repayment.[12]

The borrower values cumulative discretionary consumption flow $\{C_t\}$ as

$$E\left[\int_0^\infty e^{-rt} dC_t\right],$$

[11] Without loss of generality, we can think about the lender as a group of investors who maximize their combined payoff from the relationship with the borrower. How the investors divide proceeds among themselves is not relevant for the purpose of designing an optimal contract between the borrower and the investors.

[12] This specification is similar in flavor to the one used by Ait-Sahalia, Parker and Yogo (2004), who propose a partial resolution of the equity premium puzzle by distinguishing between the consumption of basic goods and that of luxury goods. In their model, households are much more risk averse with respect to the consumption of basic goods, of which a certain amount is required in every period, which is consistent with the subsistence aspect of basic goods and the discretionary aspect of luxuries.

where $dC_t \geq 0$. The zero consumption ($dC_t = 0$) means that the borrower consumes only necessities.

Let $\bar{Y}_t \geq 0$ denote the borrower's total income up to time t. We will focus on the borrower's "excess" income $Y_t \equiv \bar{Y}_t - \eta_t$, which represents a better measure of the borrower's ability to pay for a house than the total income. A standard Brownian motion $Z = \{Z_t, \mathcal{F}_{2,t}; 0 \leq t < \infty\}$ on $(\Omega_2, \mathcal{F}_2, m_2)$ drives the borrower's income process, where $\{\mathcal{F}_{2,t}; 0 \leq t < \infty\}$ is an augmented filtration generated by the Brownian motion. The borrower's income up to time t, denoted by Y_t, evolves according to

$$dY_t = \mu dt + \sigma dZ_t, \tag{1}$$

where μ is the drift of the borrower's disposable income and σ is the sensitivity of the borrower's income to its Brownian motion component. The excess income is negative when the necessary expense shock η_t is greater than the total income $\bar{Y}_t$. In this case, the borrower has to cover the deficit by drawing on his saving account or by borrowing more, and if he is no able to do so he will declare bankruptcy. From now on, we will refer to Y_t and C_t simply as the borrower's income and borrower's consumption.

We assume that the lender knows μ and σ, but does not know realizations the borrower's excess income shocks Z_t, so the borrower has the ability to misrepresent his income. Thus, realizations of the borrower's income are not contractible. These assumptions are motivated by the observation that lenders use a variety of methods[13] to determine a type of the borrower (represented here by (μ, σ) pair) before the loan is approved, but henceforth do not condition the terms of the contract on the realizations of the borrower's income, likely because the borrower's necessary spending shocks and possibly his total income as well are too costly or impossible to monitor.

The borrower is allowed to maintain a private savings account. The private savings account balance S grows at the interest rate ρ, where $\rho \leq r$. The borrower must maintain a non-negative balance in his account.

The borrower wants to buy a home at date $t = 0$. Home ownership would generate him the public and deterministic utility stream θ. We assume that this utility streams remains constant as long as the borrower stays in the same house.[14] The price P_0 of the home is greater than the borrower's initial wealth Y_0, i.e., $0 \leq Y_0 < P_0$.[15] Thus, the borrower must obtain funds from the lender to finance the house purchase. We assume that the borrower and the lender are sufficiently small so that their actions have no effect on macroeconomic variables such as the market interest rate.[16]

[13] Like credit score, current job status and so on.

[14] For simplicity, we do not consider a possibility that the borrower can make modifications that can either increase or decrease the quality of the house.

[15] The price P_0 is considered as a macroeconomic variable, which is not affected by actions of the borrower and the lender. It is reasonable to expect that the home price P_0 is increasing in its utility θ, and the borrower optimizes over the set of available (θ, P_0) pairs. This optimization is not considered in the paper. This clearly does not lead to a loss of generality, since our analysis applies to any (θ, P_0) pair.

[16] In a general equilibrium framework, actions of mortgage lenders and homebuyers on the aggregate level can affect macroeconomic variables. However, as long as the economic agents on the individual level have no market power, they should regard macroeconomic variables as exogenous in equilibrium.

The housing market is expected to go through two phases. The initial phase, housing boom, is characterized by fast housing appreciation. Housing slump is the absorbing state, characterized by a housing market recession and price stabilization. Let the process $\{N_t\}$ denote the phase of the housing market in the period t:

$$N_0 = 0,$$
$$P\left[N_{t+s} = 0 \text{ for all } s \in [t, t+\Delta) \,|N_t = 0\right] = e^{-\delta\Delta},$$
$$P\left[N_{t+s} = 1 \text{ for all } s \in [t, t+\Delta) \,|N_t = 0\right] = 1 - e^{-\delta\Delta},$$
$$P\left[N_{t+s} = 0 \text{ for all } s \in [t, t+\Delta) \,|N_t = 1\right] = 0.$$

where $N_t = 0$ means the housing boom continues in period t, and $N_t = 1$ means the housing slump phase in period t. Formally, the process $N = \{N_t, \mathcal{F}_{2,t}; 0 \leq t < \infty\}$ is a standard compound Poisson process with an intensity $\delta(N_t)$ on a probability space $(\Omega_2, \mathcal{F}_2, m_2)$, such that $N_0 = 0$ and

$$\delta(N_t) = \begin{cases} \delta & \text{if } N_t \text{ is even} \\ 0 & \text{if } N_t \text{ is odd} \end{cases}.$$

The stopping time $\tau^h = \inf\{t \geq 0 : N_t = 1\}$ denotes the arrival time of the housing slump phase. The market price of the home grows at the rate $g > 0$ per year during the boom, while it remains constant during the slump:

$$P_t = \begin{cases} P_0 e^{gt} & \text{for all } 0 \leq t < \tau^h \\ P_{\tau^h}(1-\alpha) & \text{for all } t \geq \tau^h \end{cases},$$

where $\alpha \in [0, 1]$ measures the extent of house price depreciation.

Before purchase of the house, the borrower and the lender sign a contract that will govern their relationship after the purchase is made. The contract obligates the borrow to report his income realizations to the lender. Conditional on the history of house prices and the borrower's income reports, the contract specifies transfers between the borrower and the lender and the circumstances under which the lender repossesses the home and the circumstances under which the borrower becomes a full homeowner.

Default Option

If the borrower violates the terms of the contract or defaults at time t, he loses the home and receives his reservation value equal to A, which for simplicity we assume to be equal to the expected present value of the borrower's future income, $\frac{\mu}{r}$ plus any private savings he might have.[17] We assume $\mu \geq \theta$ and so $A \geq \frac{\theta}{r}$. The lender sells the repossessed house at a foreclosure auction and receives the payoff $L_t = (1-l)P_t$, where

[17] This assumption is made for expositional clarity and it does not affect any of our results.

$l \in (0,1)$ measures the liquidation costs.

Option to Sell

In each period t, the borrower can put the home on the market. If he does so, the probability of finding a buyer at time t equals $\lambda(N_t)$. For simplicity, we assume that the home can be sold immediately during the boom $\lambda(0) = 1$, while it is impossible to find a buyer during the slump $\lambda(1) = 0$. The house sale generates P_t. The selling is more efficient than liquidation (as $l < 1$).

Definition 1 *Let v_t be the value of full homeownership at time t (e.g., the borrower's continuation utility at time t provided the borrower has no debt). If the housing market is in the boom phase we set $v_t = v_t^0$ and if the housing market is in the slump phase $v_t = v^1 = \frac{\mu+\theta}{r}$.*

We note that as the housing slump is an absorbing state, the continuation utility of the borrower who has no debt in the slump phase does not depend on t.

The sale of the home is not contractible. The borrower puts the home on the market at the time when it maximizes his expected payoff. We assume that if he sells the home the borrower has to pay to the lender $(v_t - a_t)$, i.e., the difference between the full homeownership utility v_t^0 and his continuation utility a_t under the existing mortgage contract. As we will verify in Section 5, the outstanding balance B_t is related to the borrower's continuation utility a_t under the optimal contract as follows:

$$B_t = v_t - a_t.$$

Thus, this assumption means that the borrower pays the outstanding balance B_t to the lender when he sells the home.

In the boom phase of the housing market we have $v_t = v_t^0$. Thus, after the house sale the borrower's continuation payoff is given by

$$A_t^S(a_t) = A + P_t - (v_t^0 - a_t),$$

while the lender receives

$$L_t^S(a_t) = v_t^0 - a_t.$$

Note that the borrower will want to sell whenever

$$A_t^S(a_t) \geq a_t,$$

which is equivalent to

$$A + P_t \geq v_t^0. \tag{2}$$

The optimal selling time determined by equation (2) does not depend on the outstanding balance or the continuation utility of the borrower. This is because the outstanding balance is linear in the borrower's continuation utility, and they cancel each other out. The optimal selling time also does not depend on the liquidation value L_t of the home. This is due to the fact that the borrower does not take into account dead-weight costs associated with liquidation. Equation (2) simply states that the borrower sells the home whenever the value of his outside option A plus the proceeds from the sale exceed his continuation utility under full homeownership.

Proposition 1 *The optimal time for the borrower to sell the home is given by*

$$t_s^* = \frac{1}{g} \log \left(\frac{\theta}{r\left(1 - \frac{g}{r+\delta}\right) P_0} \right), \tag{3}$$

and the value of full homeownership at time $t \leq t_s^*$ in the boom phase of the housing market is equal to

$$v_t^0 = v^1 + \underbrace{e^{-(r+\delta)(t_s^* - t)} \left[P_{t_s^*} - \frac{\theta}{r} \right]}_{\text{value of option to sell at } t}.$$

Proof In the Appendix. ■

Note that v_t^0 is increasing with time. This is because the borrower has the option to sell the home, and this option becomes more valuable as the price of the home increases.

Full Homeownership

If the borrower becomes a full homeowner at time t (the borrower repays his debt) the contract is terminated and the borrower receives the value of full homeownership equal to v_t.

3 Dynamic Moral Hazard Problem

At time 0, the funds needed to purchase the home in the amount of $P_0 - Y_0$ are transferred from the lender to the borrower. A contract, (τ^f, τ^d, I), specifies a time at which the borrower becomes a full homeowner, τ^f, a default time, τ^d, and transfers between the lender and the borrower, all of which are based on the borrower's report of his income and the realized house price process. Let $(\Omega, \mathcal{F}, m) := (\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, m_1 \times m_2)$ be the product space of $(\Omega_1, \mathcal{F}_1, m_1)$ and $(\Omega_2, \mathcal{F}_2, m_2)$. Let $\hat{Y} = \left\{ \hat{Y}_t : t \geq 0 \right\}$ be the borrower's report of his income, where $\hat{Y}$ is (Y, P)-measurable ($\mathcal{F}_t-$measurable). The contract transfers the reported amount, $\hat{Y}_t$, from the borrower to the lender, and $I_t(\hat{Y}, P)$ from the lender to the borrower. Below we formally define a contract.

Definition 2 *A contract $\xi = (\tau^f, \tau^d, I)$ specifies a time at which the borrower becomes a full homeowner τ^f, a default time τ^d, and transfers from the lender to the borrower $I = \{I_t : 0 \leq t \leq \tau\}$, that are based on $\hat{Y}$ and P (N). Formally, τ^f and τ^d are $(\hat{Y}, P)$-measurable stopping times, and I is a $(\hat{Y}, P)$-measurable continuous-time process, such that the process*

$$E\left[\int_0^{\min(\tau^f,\tau^d)} e^{-rs} dI_s \,|\mathcal{F}_t\right]$$

is square-integrable for $0 \leq t \leq \min(\tau^f, \tau^d)$ and $\hat{Y} = Y$.

Definition 3 *Let $\tau = \min(\tau_s^*, \tau^f, \tau^d)$, be the expected time of termination of the relationship due to sale (which happens at t_s^* provided that house boom last until s), full homeownership, or default, respectively, implied by the contract (τ^f, τ^d, I).*

The borrower can misreport his income. Consequently, under the contract ξ, up to time $t \leq \tau$, the borrower receives a total flow of income equal to

$$\underbrace{(dY_t - d\hat{Y}_t)}_{misreporting} + dI_t,$$

and his private savings account balance, S, grows according to

$$dS_t = \rho S_t dt + (dY_t - d\hat{Y}_t) + dI_t - dC_t, \tag{4}$$

where dC_t is the borrower's consumption at time t, which must be non-negative. We recall that $\rho \leq r$.

In response to a contract $\xi = (\tau^f, \tau^d, I)$, the borrower chooses a feasible strategy that consists of his consumption choice, the report of his income, and the selling time in order to maximize his expected utility. Below we formally define the feasible strategy of the borrower.

Definition 4 *Given a contract $\zeta = (\tau^f, \tau^d, I)$, a feasible strategy for the borrower is a pair $(C, \hat{Y})$ such that*

(i) $\hat{Y}$ is a continuous-time process adapted to (Y, P),

(ii) C is a nondecreasing continuous-time process adapted to (Y, P),

(iii) the savings process defined by (4) stays non-negative.

We haven't included explicitly the borrower's selling decision in the definition of his strategy. As we discussed in Section 2 the optimal selling time is determined by equation (2). Equation (2) simply states that it is optimal for the borrower to sell the home whenever the value of his outside option A plus the proceeds

from the sale exceed his continuation utility under full homeownership (which happens at t_s^* provided that house boom last until s).

The borrower's strategy is incentive compatible if it maximizes his lifetime expected utility in the class of all feasible strategies given a contract $\zeta = (\tau^f, \tau^d, I)$. As a result, we have the following definition.

Definition 5 *Given a contract $\zeta = (\tau^f, \tau^d, I)$, the borrower's strategy $(C, \hat{Y})$ is incentive compatible if*

(i) given a contract ζ, the borrower's strategy $(C, \hat{Y})$ is feasible,

(ii) given a contract ζ, the borrower's strategy $(C, \hat{Y})$ provides him with the highest expected utility among all feasible strategies, that is

$$E\left[\int_0^\tau e^{-rt}(dC_t + \theta dt) + e^{-r\tau}(1_{\tau=\tau_s^*} A_\tau^S + 1_{\tau=\tau^f} v_\tau + 1_{\tau=\tau^d} A + S_\tau)\,|\mathcal{F}_0\right] \quad \geq$$

$$E\left[\int_0^\tau e^{-rt}(dC_t' + \theta dt) + e^{-r\tau}(1_{\tau=\tau_s^*} A_\tau^S + 1_{\tau=\tau^f} v_\tau + 1_{\tau=\tau^d} A + S_\tau')\,|\mathcal{F}_0\right]$$

for all the borrower's feasible strategies $(C', \hat{Y}')$, given a contract ζ.

The above definition does not explicitly include the participation constraint imposing the condition that the borrower's utility from the continuation of the contract should be at least as large as the borrower's outside option, A, which he can receive at any time by quitting. As the borrower can always under-report and steal at rate rA until a termination time, any incentive compatible strategy would yield the borrower utility of at least A.

The above definition of an incentive compatible strategy allows us to define the incentive compatible contract as follows.

Definition 6 *An incentive compatible contract is a contract $\zeta = (\tau^f, \tau^d, I)$, together with the recommendation to the borrower, $(C, \hat{Y})$, where $(C, \hat{Y})$ is a borrower's incentive compatible strategy given a contract ζ.*

The contract is optimal if it provides the borrower with his initial expected utility a_0 and maximizes the expected profit of the lender in the class of all contracts that are incentive compatible. Below we provide a formal definition of the optimal contract.

Definition 7 *Given the continuation utility to the borrower, a_0, a contract $\zeta = (\tau^{*f}, \tau^{*d}, I^*)$, together with a recommendation to the borrower $(C^*, \hat{Y}^*)$ is optimal if it maximizes the lender's expected utility:*

$$E\left[\int_0^\tau e^{-rt}(d\hat{Y}_t - dI_t) + e^{-r\tau}(1_{\tau=\tau_s^*} L_\tau^S + 1_{\tau=\tau^d} L_\tau)\,|\mathcal{F}_0\right]$$

in the class of all incentive-compatible contracts that satisfy the following promise keeping constraint:

$$a_0 = E\left[\int_0^{\tau} e^{-rt}(dC_t + \theta dt) + e^{-r\tau}(1_{\tau=\tau_s^*} A_\tau^S + 1_{\tau=\tau^f} v_\tau + 1_{\tau=\tau^d} A + S_\tau)\,|\mathcal{F}_0\right].$$

We note that maximizing the lender's expected utility is equivalent to maximizing the lender's profit, which equals the lender's expected utility less the loan amount to the borrower $[P_0 - Y_0]$, which we take as given.

In the following lemma, we show that searching for optimal contracts, we can restrict our attention to contracts in which truth-telling and zero savings are incentive compatible.

Lemma 1 *There exists an optimal contract in which the borrower chooses to tell the truth and maintains zero savings.*

Proof In the Appendix. ■

The intuition for this result is straightforward. The first part of the result is due to the direct-revelation principle. The second part follows from the fact that it is weakly inefficient for the borrower to save on his private account ($\rho \leq r$) as any such contract can be improved by having the lender save and make direct transfers to the borrower. Therefore, we can look for an optimal contract in which truth-telling and zero savings are incentive compatible.

4 Derivation of the Optimal Contract

In this subsection, we formulate recursively the dynamic moral hazard problem and determine the optimal contract. First, we consider a problem in which the borrower is not allowed to save. We determine the optimal contract[18] in this environment, achieving this in two steps. First, we present and explain the optimal contract after the house price slump occurred. Next, given the post-slump value function, we derive the optimal contract in the boom environment.

We know from Lemma 1 that it is sufficient to look for optimal contracts in which the borrower reports truthfully and maintains zero savings, and so the optimal contract of the problem with no private savings, for a given continuation utility to the borrower, yields to the lender at least as much utility as the optimal contract of the problem when the borrower is allowed to privately save. We will conclude by showing that the optimal contract of the problem with no private savings is fully incentive compatible, even when the borrower can maintain undisclosed savings, justifying our approach.

Methodologically, our approach is based on continuous-time techniques used by DeMarzo and Sannikov (2006) and extended to a setting with a Lévy process by Piskorski and Tchistyi (2007).

[18] This is the allocation satisfying the properties of Definition 7 and the additional constraint that $S = 0$.

4.1 The Optimal Contract without Hidden Savings

Consider for a moment the dynamic moral hazard problem in which the borrower is not allowed to save. First, we will find a convenient state space for the recursive representation of this problem. For this purpose, we define the borrower's total expected utility received under the contract $\xi = (\tau^f, \tau^d, I)$ conditional on his information at time t, from transfers and termination utility, if he tells the truth and follows the optimal selling rule:

$$V_t = E\left[\int_0^\tau e^{-rs}\left[dI_s + \theta ds\right] + e^{-r\tau}(1_{\tau=\tau_s^*}A_\tau^S + 1_{\tau=\tau^f}v_\tau + 1_{\tau=\tau^d}A)\,|\mathcal{F}_t\right].$$

Lemma 2 *The process $V = \{V_t, \mathcal{F}_t; 0 \le t < \tau\}$ is a square-integrable $\mathcal{F}_t$-martingale.*

Proof follows directly from the definition of process V and the fact that this process is square-integrable, which is implied by Definition 2. ■

Below is a convenient representation of the borrower's total expected utility received under the contract $\xi = (\tau^f, \tau^d, I)$ conditional on his information at time t, from transfers and termination utility, if he tells the truth. Let $M = \{M_t = N_t - t\delta(N_t), \mathcal{F}_{1,t}; 0 \le t < \infty\}$ be a compensated compound Poisson process.

Proposition 2 *There exists $\mathcal{F}_t$-predictable processes $(\beta, \psi) = \{(\beta_t, \psi_t); 0 \le t \le \tau\}$ such that*

$$\begin{aligned} V_t &= V_0 + \int_0^t e^{-rs}\beta_s dZ_s + \int_0^t e^{-rs}\psi_s dM_s = \\ & V_0 + \int_0^t e^{-rs}\beta_s \underbrace{\left(\frac{dY_s - \mu ds}{\sigma}\right)}_{dZ_s} + \int_0^t e^{-rs}\psi_s(dN_s - \delta(N_s)ds). \end{aligned} \tag{5}$$

Proof We note that the couple (Z, N) is a Brownian-Poisson process, and it is an independent increment process, which is a Lévy processes, on the space $(\Omega, \mathcal{F}, m)$. Then, Theorem III.4.34 in Jacod and Shiryaev (2003) gives us the above martingale representation for a square-integrable martingale adapted to $\mathcal{F}_t$ taking values in a finite dimensional space (the process V). ■

According to the martingale representation (5), the total expected utility of the borrower under the contract ξ, truth telling, and optimal option execution time conditional on his information at time t equals its unconditional expectation plus two terms that represent the accumulated effect on the total utility of, respectively, the income uncertainty revealed up to time t (Brownian motion part), and the house price uncertainty that has been revealed up to time t (compensated compound Poisson part).

According to Proposition 2, when the borrower reports truthfully, his total expected utility under the contract ξ conditional on the termination time τ equals

$$V_\tau = V_0 + \int_0^\tau e^{-rs}\beta_s\left(\frac{dY_s - \mu ds}{\sigma}\right) + \int_0^\tau e^{-rs}\psi_s dM_s.$$

As I and (τ^f, τ^d) depend exclusively on the borrower's report $\hat{Y}$ and the public house price process P, when the borrower reports $\hat{Y}$, by (5) he gets the expected utility, $a_0(\hat{Y})$, which equals

$$a_0(\hat{Y}) = E\left[V_0 + \int_0^\tau e^{-rt}\beta_t\left(\frac{d\hat{Y}_t - \mu dt}{\sigma}\right) + \int_0^\tau e^{-rt}\psi_t dM_t + \underbrace{\int_0^\tau e^{-rt}(dY_t - d\hat{Y}_t)}_{\text{utility from stealing}} |\mathcal{F}_0\right] =$$

$$E\left[V_0 + \int_0^\tau e^{-rt}\beta_t\left(\frac{dY_t - \mu dt}{\sigma}\right) + \int_0^\tau e^{-rt}\left(1 - \frac{\beta_t}{\sigma}\right)\left(dY_t - d\hat{Y}_t\right) + \int_0^\tau e^{-rt}\psi_t dM_t |\mathcal{F}_0\right]. \tag{6}$$

Note that because the process $(\beta, \psi) = \{(\beta_t, \psi_t); 0 \leq t \leq \tau\}$ is $\mathcal{F}_t-$predictable, as for any $t \geq 0$, $s \geq 0$, $E_0\left[Z_{t+s} - Z_t |\mathcal{F}_0\right] = E_0\left[M_{t+s} - M_t |\mathcal{F}_0\right] = 0$, and given that $E\left[V_0 |\mathcal{F}_0\right] = V_0$, we have that

$$a_0(\hat{Y}) = V_0 + E\left[\int_0^\tau e^{-rt}\left(1 - \frac{\beta_t}{\sigma}\right)\left(dY_t - d\hat{Y}_t\right) |\mathcal{F}_0\right]. \tag{7}$$

Representation (7) leads us to the following formulation of incentive compatibility.

Proposition 3 *If the borrower cannot save, truth-telling is incentive compatible if and only if* $\beta_t \geq \sigma$ $(m - a.s.)$ *for all* $t \leq \tau$.

Proof Immediately follows from (7). ■

It is important to stress that in providing incentives for truth-telling one can neglect an impact of reporting strategies on the magnitude of the adjustment, ψ, in the borrower's continuation utility that occurs when the house price boom ends. It follows from (6) that, though in principle the reporting strategy of the borrower does affect the magnitude of these adjustments, from the perspective of the borrower such adjustments have zero effect on the borrower's expected utility, whatever his reporting strategy. This property considerably simplifies the formulation of incentive compatibility.

To characterize the optimal contract recursively, we define the borrower's continuation utility at time t if he tells the truth as

$$a_t = E\left[\int_t^\tau e^{-r(s-t)}\left[dI_s + \theta ds\right] + e^{-r(\tau-t)}[1_{\tau=\tau_s^*}A_\tau^S + 1_{\tau=\tau^f}v_\tau + 1_{\tau=\tau^d}A] |\mathcal{F}_t\right].$$

Note that for $t \leq \tau$ we have that

$$V_t = \int_0^t e^{-rs}(dI_s + \theta dt) + e^{-rt}a_t.$$

But this, together with (5), implies the following law of motion of the borrower's continuation utility:

$$da_t = ra_t dt - \theta dt - dI_t + \beta_t dZ_t + \psi_t dM_t = (ra_t - \theta - \psi_t\delta(N_t))\, dt - dI_t + \beta_t dZ_t + \psi_t dN_t. \tag{8}$$

Below we discuss informally, using the dynamic programming approach, how to find out the most efficient way to deliver to a borrower any continuation utility $a \geq A$. Proposition 4 and Proposition 5 will formalize our discussion below.

The Optimal Contract in the House Slump Phase

Let $b_1(a, \tau^h)$ be the highest expected utility of the lender that can be obtained from an incentive compatible contract that provides the borrower with utility equal to a given that the house price slump started at τ^h. To simplify our discussion we assume that the function b_1 is concave and C^2 in its first argument. Proposition 4 will establish these arguments formally. Let b_1' and b_1'' denote, respectively, the first and second derivative of b_1 with respect to the borrower's continuation utility a.

We start by observing that transferring lump-sum dI from the lender to the borrower with continuation utility a, moves a contract to that of the borrower's continuation utility of $a - dI$. The efficiency implies that

$$b_1(a, \tau^h) \geq b_1(a - dI, \tau^h) - dI, \tag{9}$$

which shows that for all $a \in [A, \infty)$ the marginal cost of delivering the borrower his continuation utility can never exceed the cost of an immediate transfer in terms of the lender's utility, that is

$$b_1'(a, \tau^h) \geq -1.$$

Define $\bar{a}^1_{\tau^h}$ as the lowest value of a such that $b_1'(a, \tau^h) = -1$.

Lemma 3 *For any $\tau^h \in [0, t_s^*]$, we have that $\bar{a}^1_{\tau^h} = v^1 = \frac{\mu+\theta}{r}$.*

Proof Since the borrower and lender have the same discount factors there is no loss of efficiency in delaying the transfers to the borrower. However paying early to the borrower is costly as long it affects the likelihood of costly liquidation. As long as $a_t < v^1$, the borrower cannot be declared a full homeowner as this would be inconsistent with the borrower's continuation utility. But this implies that as long as $a_t < v^1$, due to incentive compatibility constraint, there is a positive chance of liquidation. Therefore no transfers would be optimal to the borrower in this region as they would lower the borrower's continuation utility and thus increase the likelihood of liquidation. On the other hand, when $a_t \geq v^1$ the borrower can be declared a full homeowner (with accompanied transfer from the lender of $a_t - v^1$). As there are no gains for the lender (and for the borrower) from delaying full homeownership, we conclude that $\bar{a}^1_{\tau^h} = v^1$. ■

Then, conditional on a house price boom ending at τ^h, we have that $\tau^f = \inf\{t \geq \tau^h : a_t = v^1\}$. Full ownership and the option to terminate keep the borrower's continuation utility between A and v^1. But this and (8) imply that when $a \in [A, v^1]$, and when the borrower is telling the truth, his continuation utility

evolves according to

$$da_t = (ra_t - \theta)\, dt + \beta_t dZ_t, \tag{10}$$

where we use the fact that $dN_t = 0$ and $\delta(N_t) = 0$ for all $t \geq \tau^h$.

We next characterize the optimal choice of process (β_t), where $\frac{\beta_t}{\sigma}$ determines the sensitivity of the borrower's continuation utility with respect to his report. Using Ito's lemma, we find that

$$db_1(a_t, \tau^h) = (ra_t - \theta)b_1'(a_t, \tau^h)dt + \frac{1}{2}\beta_t^2 b_1''(a_t, \tau^h)dt + \beta_t b_1'(a_t, \tau^h)dZ_t.$$

Using the above equation, we find that the lender's expected cash flows and the change in the value he assigns to the contract are given as follows:

$$E\left[dY_t + db_1(a_t, \tau^h)\,|\mathcal{F}_t\right] = \left[\mu + (\gamma a_t - \theta)b_1'(a_t, \tau^h) + \frac{1}{2}\beta_t^2 b_1''(a_t, \tau^h)\right] dt.$$

From Proposition 3, we know that if $\beta_t \geq \sigma$ for all $t \leq \tau$ then the borrower's best response strategy is to report the truth, that is, $\hat{Y} = Y$. Because at the optimum, at any time t, the lender should earn an instantaneous total return equal to the interest rate, r, we have the following Bellman equation for the value function of the lender after the house price boom's end at τ^h:

$$rb_1(a_t, \tau^h) = \max_{\beta_t \geq \sigma} \left[\mu + (ra_t - \theta)b_1'(a_t, \tau^h) + \frac{1}{2}\beta_t^2 b_1''(a_t, \tau^h)\right].$$

Given the concavity of the function $b(\cdot, \tau^h)$, setting

$$\beta_t = \sigma$$

for all $\tau^h \leq t \leq \tau$ is optimal.

The lender's value function therefore satisfies the following differential equation

$$rb_1(a_t, \tau^h) = \mu + (ra_t - \theta)b_1'(a_t, \tau^h) + \frac{1}{2}\sigma^2 b_1''(a_t, \tau^h). \tag{11}$$

We need some boundary conditions to pin down a solution to this equation. The first boundary condition arises because the relationship must be terminated to hold the borrower's value to A, so $b_1(A, \tau^h) = L_{\tau^h}$. The second boundary condition comes from the fact that the lender should expect no transfers from the borrower once he becomes a homeowner, that is $b_1(v^1, \tau^h) = 0$. Finally we have that $b_1'(v^1, \tau^h) = -1$.

The proposition below formalizes our findings.

Proposition 4 *Let $b_1(., \tau^h)$ be a function (in a) that solves:*

$$rb_1(a, \tau^h) = \mu + (ra - \theta)b_1'(a_t, \tau^h) + \frac{1}{2}\sigma^2 b_1''(a, \tau^h), \tag{12}$$

when a is in the interval $[A, v^1]$, with boundary conditions $b_1(A, \tau^h) = L_{\tau^h}$, $b_1(v^1, \tau^h) = 0$, and $b_1'(v^1, \tau^h) = -1$.

Then the optimal contract that delivers to the borrower the value a_{τ^h} given that the housing slump occurred at τ^h, takes the following form. If $a_{\tau^h} \in [A, v^1]$, a_t evolves for $t > \tau^h$ as

$$da_t = (ra_t dt - \theta dt) + (d\hat{Y}_t - \mu dt). \tag{13}$$

The default occurs at first time τ^d when a_t hits A. The borrower becomes a homeowner at first time τ^f when a_t hits v^1. The lender's expected utility at any time $t \geq \tau^h$ is given by the function $b_1(a_t, \tau^h)$ defined above, which is strictly concave in a_t over $[A, v^1]$.

Proof Directly follows from DeMarzo and Sannikov (2006) and Bias et al. (2007) as the structure of dynamic moral hazard problem after the house slump corresponds to the one studied in these papers. ■

The evolution of the continuation utility (13) implied by the optimal contract serves three objectives: promise-keeping, incentives, and efficiency. The first component of (13) accounts for promise-keeping. In order for a_t to correctly describe the lender's promise to the borrower, it should grow at the borrower's discount rate, r, less the payment, θdt, which he receives from owning the home, and less the flow of payments, dI_t, from the lender.

The second term of (13) provides the borrower with incentives to report his true income to the lender. Because of inefficiencies resulting from liquidation, reducing the risk in the borrower's continuation utility lowers the probability that the borrower's expected utility reaches A, and thus lowers the probability of costly liquidation. Therefore, it is optimal to make the sensitivity of the borrower's continuation utility with respect to its report as small as possible provided that it does not erode his incentives to tell the truth. The minimum volatility of the borrower's continuation utility with respect to his report of income required for truth-telling equals 1. To understand this, note that under this choice of volatility, underreporting income by one unit would provide the borrower with one additional unit of current utility through increased consumption, but would also reduce the borrower's continuation utility by one unit, so that this volatility provides the borrower with just enough incentives to report a true realization of income. Note that when the borrower reports truthfully, the term $\left(d\hat{Y}_t - \mu dt\right)$ is driftless and equals σdZ_t.

The Optimal Contract in the House Price Boom

Let $b_0(a,t)$ be the highest expected utility of the lender that can be obtained from an incentive compatible contract that provides the borrower with utility equal to a given that we have the house price boom at t ($t < \tau^h$). To simplify our discussion we assume that the function b is concave and C^2 in its first argument. Proposition 4 will establish these arguments formally. Let b_0' and b_0'' denote, respectively, the first and the second derivative of b_0 with respect to the borrower's continuation utility a.

We start by observing that transferring lump-sum dI from the lender to the borrower with continuation utility a_t moves a contract to that of the borrower's continuation utility of $a_t - dI$. The efficiency implies that

$$b_0(a,t) \geq b_0(a - dI, t) - dI, \tag{14}$$

which shows that for all $a \in [A, \infty)$ the marginal cost of delivering the borrower his continuation utility can never exceed the cost of an immediate transfer in terms of the lender's utility, that is

$$b_0'(a,t) \geq -1.$$

Define $\bar{a}_t^0$ as the lowest value of a such that $b_0'(a,t) = -1$.

Lemma 4 *For any $t \in [0, \min(\tau^h, t_s^*))$, we have that $\bar{a}_t^0 = v_t^0$.*

Proof Since the borrower and lender have the same discount factors there is no loss of efficiency in delaying the transfers to the borrower. However paying early to the borrower is costly as long it affects the likelihood of costly liquidation. As long as $a_t < v_t^0$, the borrower cannot be declared a full homeowner as this would be inconsistent with the borrower's continuation utility. But this implies that as long as $a_t < v_t^0$, due to incentive compatibility constraint, there is a positive chance of liquidation. Therefore no transfers would be optimal to the borrower in this region as they would lower the borrower's continuation utility and thus increase the likelihood of liquidation. On the other hand, when $a_t \geq v_t^0$ the borrower can be declared a full homeowner (with accompanied transfer from the lender of $a_t - v_t^0$). As there are no gains for the lender (and for the borrower) from delaying full homeownership, we conclude that $\bar{a}_t^0 = v_t^0$. ■

Then, we have that $\tau^f = \inf\{t \geq 0 : a_t = v_t\}$. The full homeownership and the option to terminate keep the borrower's continuation utility between A and v_t. But this and (8) imply that when $a \in [A, v_t]$, and when the borrower is telling the truth, his continuation utility for $t < \tau^h$ evolves according to

$$da_t = (ra_t - \theta - \psi_t \delta)\, dt + \beta_t dZ_t + \psi_t dN_t, \tag{15}$$

where we use the fact that $\delta(N_t) = \delta$ for $t < \tau^h$.

We need to characterize the optimal choice of process (β_t, ψ_t), where $\frac{\beta_t}{\sigma}$ determines the sensitivity of the borrower's continuation utility with respect to his report, and ψ_t determines the adjustment of the borrower's continuation utility when the house price boom ends. Using Ito's lemma, we find that

$$\begin{aligned} db_0(a_t,t) &= \frac{\partial b_0(a_t,t)}{\partial t}dt + (ra_t - \theta - \psi_t\delta)b_0'(a_t,t)dt \\ &\quad + \frac{1}{2}\beta_t^2 b_0''(a_t,t)dt + \beta_t b_0'(a_t,t)dZ_t + \left[b_1(a_t+\psi_t,t) - b_0(a_t,t)\right]dN_t. \end{aligned}$$

Using the above equation, we find that the lender's expected cash flows and the change in the value he assigns to the contract are given as follows:

$$E\left[dY_t + db_0(a_t,t)\,|\mathcal{F}_t\right] =$$
$$\left[\frac{\partial b_0(a_t,t)}{\partial t} + \mu + (ra_t - \theta - \psi_t\delta)b_0'(a_t,t) + \frac{1}{2}\beta_t^2 b_0''(a_t,t) + \delta\left(b_1(a_t+\psi_t,t) - b_0(a_t,t)\right)\right]dt.$$

From Proposition 3, we know that if $\beta_t \geq \sigma$ for all $t \leq \tau$ then the borrower's best response strategy is to report the truth, that is, $\hat{Y} = Y$. Because at the optimum, at any time t, the lender should earn an instantaneous total return equal to the interest rate, r, we have the following Bellman equation for the value function of the lender

$$rb_0(a_t,t) =$$
$$\max_{\beta_t \geq \sigma,\ \psi_t \geq A - a_t}\left[\frac{\partial b_0(a_t,t)}{\partial t} + \mu + (ra_t - \theta - \psi_t\delta)b_0'(a_t,t) + \frac{1}{2}\beta_t^2 b_0''(a_t,t) + \delta\left(b_1(a_t+\psi_t,t) - b_0(a_t,t)\right)\right], \quad (16)$$

where $t \in [0, t_s^*]$.

Given the concavity of the function setting

$$\beta_t = \sigma$$

for all $t \leq \tau$ is optimal. The concavity of the objective function in ψ_t in the RHS of the Bellman equation (16) also implies that the optimal choice of ψ_t is given as a solution to

$$b_0'(a_t,t) = b_1'(a_t+\psi_t,t), \quad (17)$$

provided that $\psi_t > A - a_t$, and otherwise $\psi_t = A - a_t$. Note that the optimal jump can be expressed as $\psi(a_t,t)$.

The lender's value function in the house price boom therefore satisfies the following differential equation

for $a_t \in [A, v_t^0]$:

$$rb_0(a_t,t) = \frac{\partial b_0(a_t,t)}{\partial t} + \mu + (ra_t - \theta - \psi(a_t,t)\delta)b_0'(a_t,t) + \frac{1}{2}\sigma^2 b_0''(a_t,t) + \delta\left(b_1(a_t + \psi(a_t,t),t) - b_0(a_t,t)\right) \quad (18)$$

with ψ specified above.

We need some boundary conditions to pin down a solution to this equation. The first boundary condition arises because the relationship must be terminated to hold the borrower's value to A, so $b_0(A,t) = L_t$. The second boundary condition comes from the fact that the lender should expect no transfers from the borrower once the borrower becomes a homeowner, that is $b_0(v_t^0,t) = 0$. Moreover $b_0'(v_t^0,t) = -1$. The final boundary reflects the payments to the lender at the selling time, that is $b_0(a_t,t_s^*) = v_{t_s^*}^0 - a_t$. The proposition below formalizes our findings.

Proposition 5 *Let b_0 be a C^2 function (in a) that solves:*

$$rb_0(a_t,t) = \frac{\partial b_0(a_t,t)}{\partial t} + \mu + (ra_t - \theta - \psi(a_t,t)\delta)b_0'(a_t,t) + \frac{1}{2}\sigma^2 b_0''(a_t,t) + \delta\left(b_1(a_t + \psi(a_t,t),t) - b_0(a_t,t)\right) \quad (19)$$

when a is in the interval $[A, v_t^0]$ with boundary conditions

$$b_0(A,t) = L_t,$$
$$b_0'(v_t^0,t) = -1,$$
$$b_0(a_t,t_s^*) = v_{t_s^*}^0 - a_t,$$

where

$$\psi(a,r) = \begin{cases} \text{is a } C^1 \text{ (in a) solution to } b_0'(a,t) = b_1'(a+\psi,t) \text{ for all } (a,t) \\ \text{for which the solution is such that } \psi(a,t) > A - a \\ \\ \text{otherwise it is equal to } A - a \end{cases} . \quad (20)$$

Then the optimal contract that delivers to the borrower the value a_0 takes the following form. If $a_0 \in [A, v_0^0]$, a_t evolves as

$$da_t = (ra_t dt - \theta dt - dI_t) + (d\hat{Y}_t - \mu dt) + \psi(a_t,t)(dN_t - \delta(N_t)dt), \quad (21)$$

The borrower becomes a homeowner at first t time τ^f when a_t hits v_t, The borrower defaults at first t time τ^d when a_t hits A. The borrower sells a house at time t_s^*, provides that $t_s^* < \min(\tau^d, \tau^h)$. The lender's expected utility at any time $t < \tau^h$ is given by the function $b_0(a_t,t)$ defined above, which is strictly concave in a_t over $[A, v_t^0]$. The lender's expected utility at any time $t > \tau^h$ for $a_t \in [A, v^1]$ is given by the function $b_1(a_t, \tau^h)$ defined in Proposition 4.

Proof In the Appendix. ■

The evolution of the continuation utility (21) implied by the optimal contract serves three objectives: promise-keeping, incentives, and efficiency. As in the equation (13), the first component of (21) accounts for promise-keeping while the second term provides the borrower with incentives to report his true income to the lender.

The last term of (21) captures the effect of the stochastic house price appreciation on the borrower's continuation utility. The optimal adjustments, ψ, in the borrower's continuation utility, which are applicable a when house slump occurs, are such that the sensitivity of the lender's expected utility, b, with respect to the borrower's continuation utility, a, is equalized just before and after an adjustment is made.[19] This sensitivity represents an instantaneous marginal cost of delivering to the borrower his continuation utility in terms of the lender's utility, and so the efficiency calls for equalizing this cost across the states. We note that these adjustments imply the compensating trend in the borrower's continuation utility, $-\delta(N_t)\psi(a_t,t)dt$, which exactly offsets the expected effect these adjustments have on the borrower's expected utility.

4.2 The Optimal Contract with Hidden Savings

So far, we have characterized the optimal contract under the assumption that the borrower cannot save. Now we show that, given the optimal contract of the problem with no hidden savings, the borrower has no incentive to save at the solution, and thus the contract of Propositions 4-5 is also optimal in the environment where the borrower can privately save.

Proposition 6 *Suppose that the process $a_t \in [A, v_t]$ solves*

$$da_t = ra_t dt - \theta dt - dI_t + (d\hat{Y}_t - \mu dt) + \psi_t dM_t \tag{22}$$

until stopping time $\tau = \min(\tau^f, \tau^d, \tau_s^*)$, where $\tau^f = \inf\{t > 0 : a_t = v_t\}$ and $\tau^d = \inf\{t > 0 : a_t = A\}$, and where ψ_t is an $\mathcal{F}_t-$predictable process. Then the borrower's expected utility from any feasible strategy in response to a contract (τ^f, τ^d, I) is at most a_0. Moreover, the borrower attains the expected utility a_0 if the borrower reports truthfully and maintains zero savings.

Proof In the Appendix. ■

The above proposition shows that the optimal contract of Proposition 5, remain incentive compatible even if the borrower is allowed to privately save.

[19] Provided that the solution to (17) is interior.

4.3 A Numerical Example

In this section we illustrate the features of the optimal contract in a parametrized example. Table 1 shows the parameters of the model.

Table 1. Parameters of the model

Borrower's type				Home type		Housing market				Interest rate
μ	σ	Y_0	A	θ	P_0	g	δ	l	α	r
1	1	0	20	0.6	8	0.12	0.4	0.25	0.05	0.05

The left hand-side of Figure 1 shows the lender's continuation payoff as a function of the borrower's continuation utility in the boom phase at time zero and the lender's continuation payoff when the boom turns into the slump at time zero. For a given continuation utility to the borrower, the lender's payoff in the boom phase is always greater than it would be in the slump phase. This is because the liquidation value of the home is higher in the boom phase and keeps growing. In addition, the borrower is less likely to default in the boom phase. The full homeownership utility for the borrower is also greater in the boom phase, since it includes the option to sell the home in the future.

The right hand-side of Figure 1 shows the optimal adjustments in the borrower's continuation utility, ψ_t, applicable if the boom turns into the slump at time t for $t = 0$. For $t > 0$, function ψ_t has a similar profile. For any $t \geq 0$, the adjustment is zero at termination, i.e., $a_t = A$, while for the full homeowner, i.e., $a_t = v_t^0$, the adjustment is negative and equal to the lost value of the the option to sell the home $-\left(v_t^0 - v^1\right)$. The jump function ψ_t is convex for smaller a's and concave for bigger a's. Our simulations for different parameters and $t \geq 0$ show that the jump function ψ_t always has a form similar to the one on Figure 1. According to equation (21), the negative jump $\psi(a_t, t)$ at the beginning of the slump phase translates into a positive continuation utility trend "$-\delta(N_t)\psi(a_t, t)dt$" in the boom phase.

The optimal jump ψ_t in the borrower's continuation utility triggered by the slump can be partially allocated to the loss of the option to sell the home. It is therefore natural to decompose the jump as follows:

$$\psi_t\left(a_t\right) = -\frac{v_t^0 - v^1}{v_t^0 - A}\left(a_t - A\right) + \left(\psi_t(a_t) + \frac{v_t^0 - v^1}{v_t^0 - A}\left(a_t - A\right)\right), \tag{23}$$

where the first term represents the reduction in the borrower's continuation utility proportional to his stake in the homeownership due to the loss of the option to sell the home. The solid line connecting points $(A, 0)$ and $\left(v_t^0, -(v_t^0 - v^1)\right)$ on Figure 2 represents the first term of (23).

The actual reduction in the continuation utility is bigger for borrowers with low continuation utility (less creditworthy), and lower for borrowers with high continuation utility (more creditworthy), all relative to their stake in homeownership (their continuation utility). Thus, the less creditworthy borrowers receive

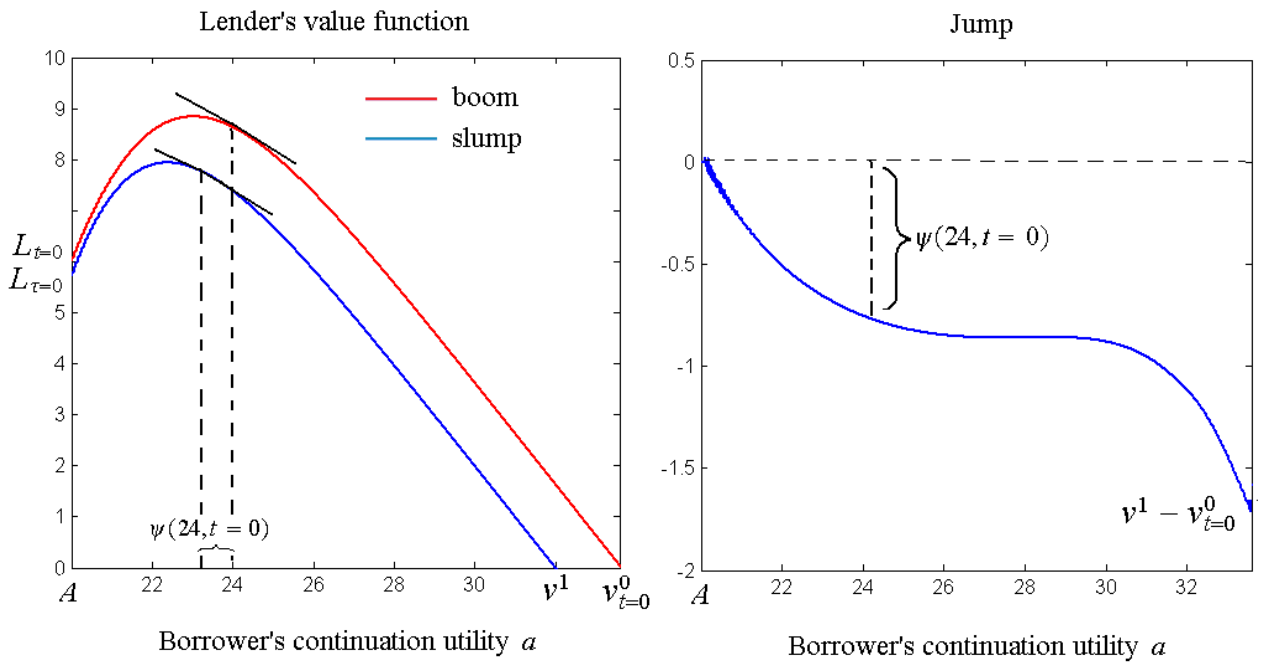


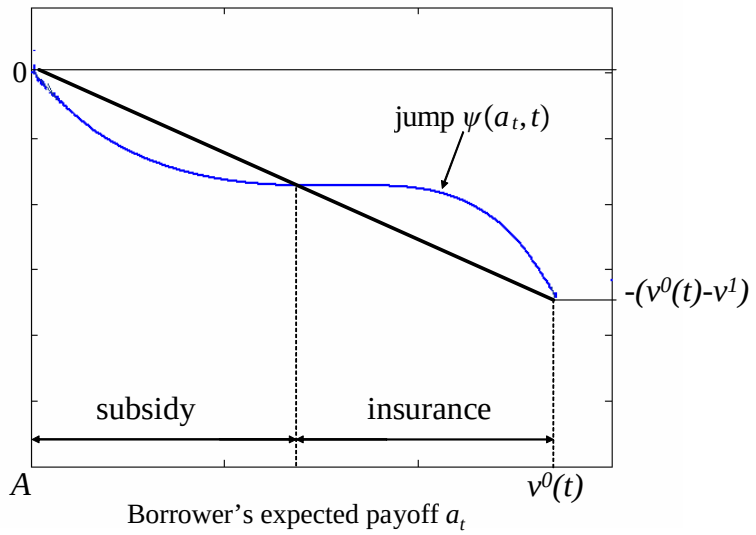
Figure 1: The lender's value function and the optimal adjustments in the borrower's continuation utility.

more preferential treatment, or subsidy, compared to more creditworthy ones during the boom phase, however they receive a larger reduction of their continuation utility in the slump. The more creditworthy borrowers receive a better treatment in the slump as their continuation utility declines less relative to their stake in homownership, which can be interpreted as a form of partial insurance against the slump.

These adjustments can be explained by the incentive-compatibility constraints and the dual optimization objective of the contracting problem: minimization of liquidation inefficiencies and maximization of the value of the option to sell the home in the future at a higher price.

It is efficient to reduce the chances of costly termination when house prices grow, as the relationship between the borrower and the lender is more valuable due to possibility of house appreciation (which increases the future liquidation values and increases the value of option to sell). The preferential treatment of the borrower during the housing boom, which manifests itself as a positive trend in the borrower's continuation utility, accomplishes this task by helping to push away the borrower's continuation utility further away from the liquidation boundary. However, the threat of repossession must be real enough in order for the borrower to share his income with the lender. As a result, it is optimal to increase the chances of repossession during the slump, by instantaneously decreasing the borrower's continuation utility, in order to compensate for the weakened threat of repossession in the boom

Figure 2: Decomposition of the optimal utility jump ψ_t



The additional subsidy to the less creditworthy borrowers, relative to their stake in homewonership, is optimal because it reduces the chances that they default during the boom, which, in turn, maximizes the value of the option to sell the home in the future at a higher price and avoid inefficient bankruptcy procedure. On the other hand, a negative jump in the continuation utility at the beginning of the slump phase leads to the default clustering among the least creditworthy borrowers. This happens because of the borrower's limited liability and the incentive-compatibility constraints. The lower subsidy, and thus effectively a form of insurance of the more creditworthy borrowers, is explained by the fact that more creditworthy borrowers are not likely to default during the boom, but are more vulnerable during the slump.

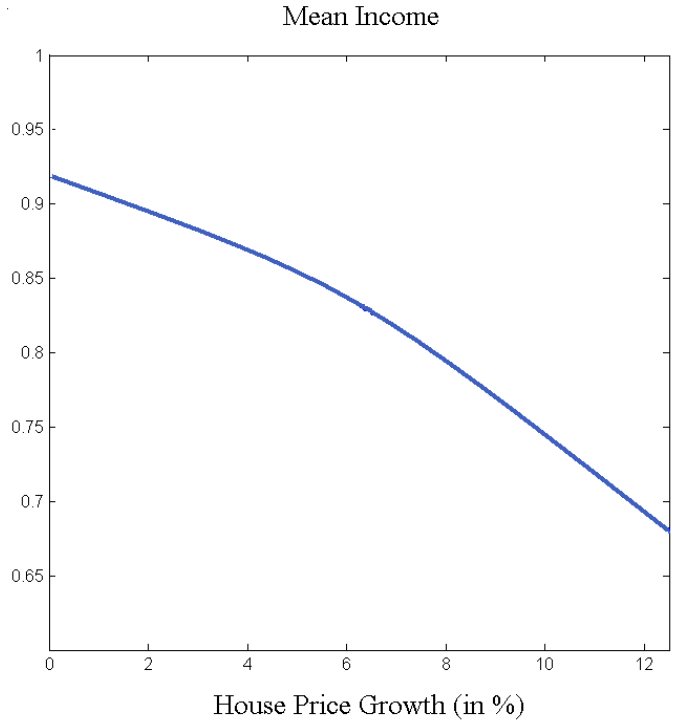
Since less creditworthy receive more preferential treatment in the boom phase, bailing out distressed subprime borrowers in the slump phase is not incentive compatible, as it would encourage irresponsible financial behavior. Indeed, borrowers in good standing would prefer not to pay the mortgage in order to qualify for the subsidy in the boom phase and then get help in the slump phase.

4.3.1 Utility Gains for the Risky Borrowers

First, we show that home appreciation leads to inflow of less creditworthy home buyers into the housing market. Then, we demonstrate that the home ownership generates substantial ex-ante utility gains for these borrowers who otherwise would be shut out of the housing market. Our findings provide theoretical evidence that the housing boom can cause a growth of the subprime market.

For the other parameters as in Table 1, Figure 3 shows the lowest expected income μ, for which the lender

Figure 3: The lowest expected income μ, for which the lender breaks even, as a function of the home price growth rate g.



breaks even, as a function of the home price growth rate g. Faster house appreciation makes it profitable to give loans to borrowers who otherwise would be shut out of the housing market. For example, with the zero home appreciation ($g = 0$), only the borrower with $\mu \geq 0.92$ can take a mortgage and purchase the home. On the other hand, fast home appreciation with $g = 12\%$ makes homeownership possible for the borrower with the mean income $\mu \geq 0.7$.

For the parameters listed in Table 1 (with $g = 12\%$), Figure 4 shows the net utility gain $(a_0 - A)$ from homeownership as a function of the borrower's mean income μ. Borrowers with $\mu < 0.7$ cannot qualify to take the mortgage and purchase the home. However, for borrowers with $\mu > 0.7$ buying the home results in substantial expected utility gains.

Note that the gains are significantly bigger than zero even for the borrower with the lowest qualifying income due to incentive reasons. Only when the borrower's expected utility is substantially above his reservation payoff A, the borrower has a credible incentive not to default on the mortgage even when his income realization is low. Borrowers who are shut out of the housing market cannot get a loan not only because of their low income but also because of the moral hazard problem.

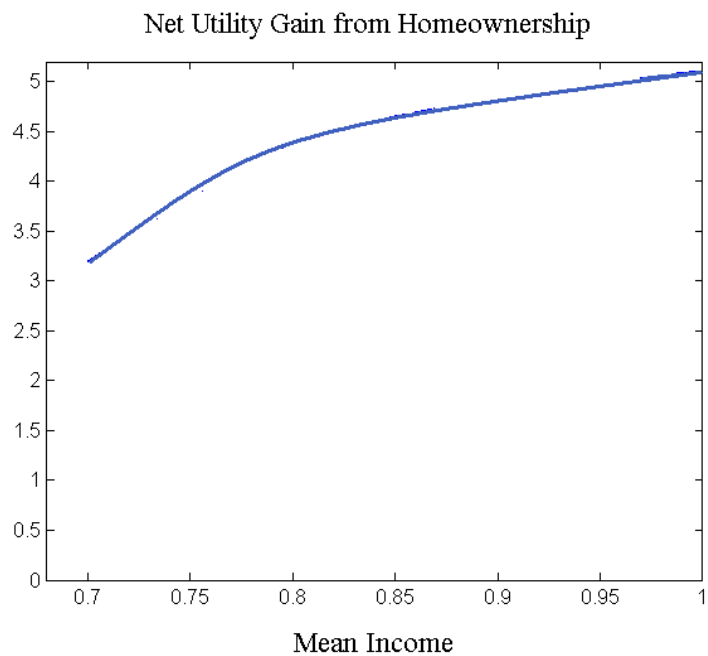


Figure 4: Gains from homeownership

5 Implementation

So far, we have characterized the optimal contract in terms of the transfers between the borrower and the lender and the timing of liquidation, full ownership and selling. In this section, we show that the optimal contract can be implemented using financial arrangements that resemble the ones used in the residential mortgage market. In particular, we consider an option mortgage with a credit line commitment (or a negative amortization limit), which has no minimum payment, and comes with a mortgage modification plan. The definition below provides a formal description of this class of mortgage contracts.

Definition 8 *An optional payment mortgage with adjustable credit limit (or negative amortization limit) consists of:*

- *Mortgage loan with a time-t credit line commitment* C_t^L*,*
- *Interest rate* r_t^M *charged on the balance,* B_t*, of the loan,*
- *Initiation fee* F_0 *charged by the lender at time zero,*
- *A mortgage modification plan, which includes the percentage balance adjustment* BA_t *and the adjustment of interest rate, applicable at the beginning of the housing slump phase.*

- *If the balance exceeds the credit line limit, default occurs, in which case the lender repossesses the home.*

Given the definition of the mortgage, the balance in the boom phase evolves according to

$$dB_t = r_t^M B_t dt - d\hat{Y}_t + dI_t + BA_t B_t dN_t,$$

while the credit line limit is increased by CA_t^L when the housing market switches from the boom to the slump.

Since $b_0'(a,0) \geq -1$ for $a \in \left[A, v_0^0\right]$, it is optimal for the borrower to use all his initial wealth Y_0 as the downpayment, he has to borrow $P_0 - Y_0$ to buy the home at time zero. However, because of the initiation fee F_0, the initial balance will be $B_0 = P_0 - Y_0 + F_0$. We assume that the initiation fee is chosen so that the lender breaks even.

The proposition below shows that the optimal contract can be implemented with an optional payment mortgage with adjustable credit limit (or negative amortization limit).

Proposition 7 *The optimal contract can be implemented using an optional payment mortgage with adjustable credit limit (or negative amortization limit) with the following parameters:*

$$\begin{aligned} C_t^L &= v_t - A, && (24)\\ CA_t^L &= -\left(v_t^0 - v^1\right), && (25)\\ r_t^M &= r + (1 - N_t)\,\delta \frac{v_t^0 - v^1 + \psi_t(v_t^0 - B_t)}{B_t} && (26)\\ BA_t &= \frac{-\left(v_t^0 - v^1\right) - \psi_t(v_t^0 - B_t)}{B_t}. && (27) \end{aligned}$$

The initiation fee F_0 solves

$$b_0\left(v_0^0 - P_0 + Y_0 - F_0, 0\right) = P_0 - Y_0$$

Under the terms of this mortgage, for time $t < \bar{\tau}$, it is optimal for the borrower to use all available income to pay down balance B_t, as long as $B_t > 0$, and the borrower's continuation utility a_t is equal to

$$a_t = A + \left(C_t^L - B_t\right) = v_t - B_t \tag{28}$$

It is optimal to sell the home at time t_s^* given by equation (3).

Proof In the Appendix. ■

We will refer to the implementation of the optimal contract given in Proposition 7 as the optimal mortgage. The following remarks highlight the properties of the optimal mortgage.

Remark 1 *Credit limit C_t^L, interest rate $\bar{r}_t$, and the mortgage modification plan (BA_t and CA_t^L) do not depend on the amount borrowed at time zero B_0.*

Remark 2 *When default happens, the lender receives the liquidation value L_t of the home, and the borrower obtains the value A of his outside option.*

Remark 3 *The interest rate r_t^M in the boom phase can be decomposed into two components:*

$$r_t^M = \underbrace{r + \delta \frac{v_t^0 - v^1}{v_t^0 - A}}_{\bar{r}_t} + \underbrace{\frac{\delta(\psi_t(v_t^0 - B_t) + \frac{v_t^0 - v^1}{v_t^0 - A}\left(v_t^0 - A - B_t\right))}{B_t}}_{q_t(B_t)},$$

where $\bar{r}_t$ is the same for all borrowers, but depends on the price of the home, while $q_t(B_t)$ represents interest rate adjustment based on the credit quality of the borrower.

Remark 4 *The interest rate $\bar{r}_t$ captures the uncertainty of the housing market. It is increasing with time in the boom phase. Figure 5 shows interest rate $\bar{r}_t$ for the parameters listed in Table 1.*

Remark 5 *The second component of the interest rate $q_t(B_t)$ is negative for the borrowers with high debt balance B_t, i.e., the borrowers with continuation utility a_t in the subsidy region on Figure 2, and positive for the borrowers with low debt balance B_t, i.e., the borrowers with continuation utility a_t in the insurance region on Figure 2. Thus, more risky borrowers pay lower interest rates.*

Remark 6 *Proposition 7 says that during the boom it is optimal to increase the interest rates over time and give more preferential interest rates to the borrowers with a lot of debt (less creditworthy). This finding is in line with the fact that many subprime mortgages originated in the US during the period of rapid housing appreciation at the beginning of the 21st century had teaser rates, which were to be reset to higher rates.*

Remark 7 *The credit line limit is equal to the full ownership value of the home. It can be decomposed into two components:*

$$C_t^L = v_t - A = (v^1 - A) + (v_t - v^1),$$

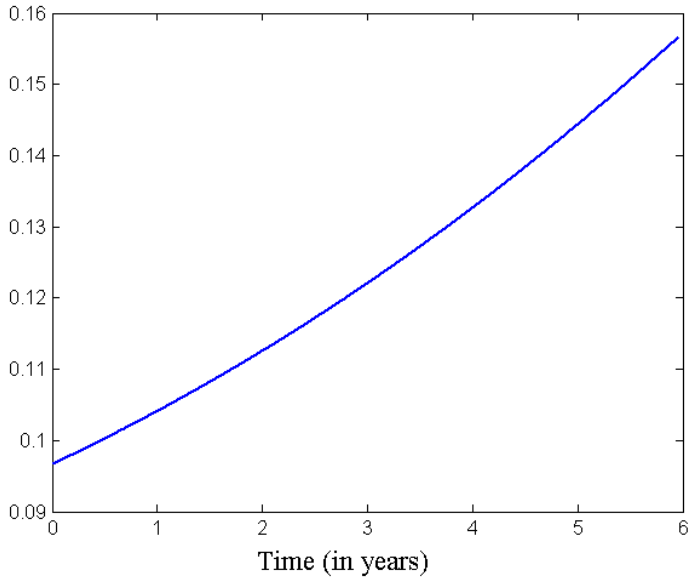
where $(v_t - v^1)$ represents the option value to sell the home in the boom phase, and $\left(v^1 - A\right)$ is the value of the home to the borrower in the slump phase.

Remark 8 *The credit limit is increasing with the price of the home in the boom phase. However, at the time when the boom turns to the slump, the credit limit is reduced to $v^1 - A$ and stays constant. The reduction of the credit limit is equal to the lost value of the option to sell the home $(v_t^0 - v^1)$.*

Remark 9 *At the time when the borrower sells the home, the credit limit is equal to the price of the home:*

$$C_{t_s^*}^L = v_{t_s^*}^0 - A = P_{t_s^*}$$

Figure 5: Interest rate $\bar{r}_t$ as a function of t.



Remark 10 *The credit limit increases should not necessarily be automatic. Instead, the borrower may be required to apply for extra credit whenever he experiences financial needs. In this case, C_t^L can be interpreted as the upper bound on the credit limit at time t.*

Remark 11 *The default clustering happens because of the credit limit reduction. The borrowers whose balance substantially exceeds the new credit limit will default immediately. However, the borrowers whose balance moderately exceeds the new credit limit will be rescued through the balance reduction and the interest rate reduction ($\bar{r}_t$).*

Remark 12 *It is optimal for the lender and the borrower to commit to the mortgage modification ex ante. However, we can also think about it as an anticipated loan modification imposed by the regulators at the time of the crisis.*

Remark 13 *The optimal mortgage contract takes a very simple form in the slump phase: the home equity line of credit (HELOC) with the fixed interest rate and credit limit. This contract is analogous to the optimal contract in DeMarzo and Sannikov (2006).*

Remark 14 *The market value of debt B_t equals $b_0\left(v_t^0 - B_t, t\right)$ in the boom phase and $b_1\left(v^1 - B_t, \tau^h\right)$ in the slump phase.*

How does the optimal mortgage implement the optimal contract? The debt balance works as a memory device that summarizes all the relevant information regarding the borrower's past income realizations revealed

through his payments to the lender. The interest rates charged on the balance, along with the credit line limit, are chosen so that equation (28) always holds. This ensures incentive compatibility of the mortgage because, according to equation (28), for every dollar paid to the lender (borrowed from the lender) the borrower's continuation payoff a_t goes up (down) by the same amount. For example, at any time t the borrower can consume all his available credit $C_t^L - B_t$ and default immediately. However, equation (28) implies that the payoff from this deviation is exactly equal to his equilibrium payoff a_t under the optimal mortgage.

The optimal mortgage provides financial flexibility by allowing the borrower to draw on the credit line, whenever his income is not sufficient to make the interest payment. There is no minimum payment requirement – a low payment from the borrower translates into a higher balance. As a result, the borrower does not need to maintain precautionary savings, because the credit commitments by the lender provide a safety net.

The interest rate r_t^M charged on the balance is higher in the good states of the housing market, i.e., the states with high home prices during the housing boom. Although this result may seem counterintuitive, it can be easily explained by the following argument. The lender is likely to lose money in the bad states, i.e., the states with the housing boom ending early. Therefore, in order to break even at time zero, the lender has to make money in the good states. Hence, the lender increases the interest rate as the home appreciates during the boom. On the other hand, the borrower can afford to pay higher interest rates and is less likely to default in the good states, since he can borrow more and more as the home appreciates.

The optimality of the interest rate reduction for the borrowers with a lot of debt is driven by the maximization of the value of the option to sell the home in the future at a higher price. A lower interest rate increases the chances that the distressed borrower survives long enough to sell the home and pay back the loan.

The borrowers with low debt balance (more creditworthy) pay higher interest rate during the boom and get higher balance reduction in the slump. This result is driven by the minimization of liquidation inefficiencies: the more creditworthy borrowers are not likely to default during the boom, but are more vulnerable during the slump.

A peculiar feature of the optimal mortgage is that, in the boom phase, the credit line limit C_t^L is greater than the market value of the home P_t, and is increasing with the price of the home. The borrower with the balance greater than P_t nonetheless has utility gains from continuing to pay down the mortgage and keeping the home rather than defaulting or selling the home early. Indeed, if the borrower defaults or sells the home, his continuation utility is equal to his outside option A. On the other hand, if the borrower sticks with the mortgage, he will keep the house and enjoy utility flow θ. In addition, if he avoids default in the boom phase, the borrower will sell the home for profit at time t_s^*.

It is also optimal for the lender to allow the borrower to borrow above the value of the home in the boom phase. Of course, if the borrower defaults immediately due to either negative income shocks or the arrival of the slump phase, a higher balance translates into higher losses for the lender. However, by increasing the borrowing limit in the boom phase, the lender minimizes the chances of default due to negative income shocks. Avoiding default in the boom phase, in turn, has three positive effects on the value of the mortgage. First, because the home price grows faster than the borrowing limit, default in the future will be less costly for the lender. Second, avoiding default in boom phase increases the chance that the borrower will survive long enough to sell the home at time t_s^* (at which time $C_{t_s^*}^L = P_{t_s^*}$) and fully repay the balance. A higher balance today translates into a higher payoff for the lender when the borrower sells the home. Third, the borrower can receive positive income shocks, fully repay the balance, and become full homeowner.

6 Concluding Remarks

Recent years have seen a rapid growth of the subprime mortgage market followed by the subprime crisis. Because of high default rates among subprime borrowers and big losses among subprime investors accompanied by the crisis in the housing market, subprime mortgages have caused a storm of controversy. In this paper, we show that key features of subprime lending observed in practice are consistent with rational behavior of both borrowers and lenders. In particular, when house prices are expected to grow, it is optimal to provide the risky borrowers with a lower initial rate, which is to increase over time, and to increase the borrowers access to credit. The housing slump results in the tightening of the borrowers' access to credit and default clustering among the least creditworthy ones. We also find that house appreciation makes it profitable to give loans to less creditworthy borrowers who otherwise would be excluded from the housing market, and that these loans could generate substantial ex-ante utility gains for these borrowers.

In terms of policy implications, our model suggests that bailing out the most distressed borrowers in the slump phase is not incentive compatible, as it encourages irresponsible financial behavior. On the other hand, it makes sense to help borrowers who were in good standing before the crisis by reducing their mortgage balance and interest rate.

Our analysis is based on the assumption of full rationality of borrowers and lenders. Thus, it cannot be applied to a situation with irrational borrowers or lenders.

The optimal contracts do not allow borrowers to refinance their mortgages with another lender. Offering this option would limit the ability to provide the borrower incentives to repay debt, resulting in a decrease of efficiency of the contract. Therefore, our results lend support to prepayment penalties on refinancing.[20]

For tractability we considered an environment with a constant interest rate. In a complementary study Piskorski and Tchistyi (2007) study the optimal mortgage design in a constant house price setting with

[20] According to Standard & Poor's, survey more than 80% of subprime mortgages carry prepayment penalties.

volatile and privately observable income of the borrower and a stochastic market interest rate. They show that the optimal mortgage takes form of either an option ARM, and that default rates and interest rates correlate positively with the market interest rate. We believe that the results of both papers would be preserved in an environment featuring, together, stochastic house prices and stochastic market interest rate. This is because the forces and trade-offs that shape them would also be present in a richer environment.[21]

We also ignored inflation, which is an important consideration for home buyers.[22] However, as long as inflation affects the borrower's income and the liquidation value of the home equally, it would not change the properties of the optimal mortgage in real terms. We also did not allow for contract renegotiations, because a possibility of renegotiation would lead to a suboptimal contract. In practice, lenders should be able to commit to the terms of a mortgage contract, as the competition among them would drive those who are unable to do so out of the market.

There are a number of research directions one might pursue from here. One of them is to develop a general equilibrium model of the "housing bubble." The findings of this study let us suggest that optimal features of mortgage lending can contribute to the "housing bubble." During the housing boom, the inflow of subprime borrowers into the housing market may help sustain house appreciation, possibly driving home prices above the equilibrium level. However, default clustering among risky borrowers during the housing slump may exacerbate the crisis in the housing market. In such a model, features of lending that are optimal at the individual level may have negative consequences at the aggregate level, since the borrower and the lender might not take into account the potential negative externalities that their individual optimal contract might impose. Such a model would allow addressing welfare consequences of risky mortgage lending and social costs and benefits of its potential regulation.

[21] Studying formally an optimal mortgage design in such an environment would be challenging due to a number of state variables involved.

[22] See, for example, Campbell and Cocco (2003).

Appendix

Proof of Proposition 1

Let Δ be the length of the period. The optimal selling time t_s under full homeownership conditional on prices growing is the smallest t such that:

$$P_t(1-s) + A \geq e^{-\Delta r}\left[\Delta(\mu+\theta) + e^{-\Delta\delta}(e^{\Delta g}P_t + A) + (1-e^{-\Delta\delta})\left(\frac{\Delta(\mu+\theta)}{1-e^{-\Delta r}}\right)\right],$$

which yields:

$$e^{\Delta r}\left[P_t + A\right] \geq \Delta(\mu+\theta) + e^{-\Delta\delta}e^{\Delta g}P_t + e^{-\Delta\delta}A + (1-e^{-\Delta\delta})\left(\frac{\Delta(\mu+\theta)}{1-e^{-\Delta r}}\right).$$

The above implies that

$$t_s^* = \inf\left\{t < \tau : (e^{\Delta r} - e^{\Delta(g-\delta)})P_t + (e^{\Delta r} - e^{-\Delta\delta})A \geq \frac{(2-e^{-\Delta r} - e^{-\Delta\delta})\Delta(\mu+\theta)}{1-e^{-\Delta r}}\right\},$$

that is

$$t_s^* = \inf\left\{t < \tau : (e^{\Delta r} - e^{\Delta(g-\delta)} - 1 + e^{\Delta(g-\delta-r)})P_t + (e^{\Delta r} - e^{-\Delta\delta})\mu \geq (2-e^{-\Delta r} - e^{-\Delta\delta})\Delta(\mu+\theta)\right\}.$$

Taking limit as $\Delta \to 0$, we obtain:

$$(r+\delta-g)P_{t_s^*} + (r+\delta)A = \frac{(r+\delta)(\mu+\theta)}{r}.$$

As $A = \frac{\mu}{r}$, we have that

$$P_{t_s^*} = \frac{\theta}{r\left(1 - \frac{g}{r+\delta}\right)}.$$

This implies that the optimal selling time satisfies

$$t_s^* = \frac{1}{g}\log\left(\frac{\theta}{r\left(1-\frac{g}{r+\delta}\right)P_0}\right),$$

provided that the solution exists. Therefore, a sale will occur at $0 < t_s^* < \infty$, conditional on prices growing, iff:

$$1 < \frac{\theta}{r\left(1-\frac{g}{r+\delta}\right)P_0} < \infty.$$

We also note that the value of full homeownership for $0 \leq t < t_s$ is given by:

$$v_t^0 = A + \int_t^{t_s^*} e^{-r(s-t)}\theta ds + e^{-r(t_s^*-t)} E_t \left[(1 - N_{t_s^*}) e^{g t_s^*} P_0 + N_{t_s^*} \frac{\theta}{r} \right],$$

which yields:

$$v_t^0 = A + \left[1 - e^{-(r+\delta)(t_s^*-t)} \right] \frac{\theta}{r} + e^{-(r+\delta)(t_s^*-t)} e^{g t_s^*} P_0.$$

We remember that the value of homeownership for any $t \geq \tau^h$ is equal to

$$v^1 = A + \frac{\theta}{r} = \frac{\mu + \theta}{r}.$$

Using this we note that for $0 \leq t \leq t_s^*$:

$$v_t^0 = v^1 + \underbrace{e^{-(r+\delta)(t_s^*-t)} \left[P_{t_s^*} - \frac{\theta}{r} \right]}_{value\ of\ option\ to\ sell\ at\ t}. \quad \square$$

Proof of Lemma 1

Consider any incentive compatible contract $(\tau^f, \tau^d, I, C, \hat{Y})$. We prove the lemma by showing the existence of the new incentive-compatible contract that that has the following properties:

(i) the borrower gets the same expected utility as under the old contract (τ^f, τ^d, I),

(ii) the borrower chooses to reveal the cash flows truthfully,

(iii) the borrower maintains zero savings,

(iv) the lender gets the same or greater expected profit as under the old contract (τ^f, τ^d, I).

We haven't included explicitly the borrower's selling decision in the definition of his strategy. As we discussed in Section 2 the optimal selling time is determined by equation (2). Equation (2) simply states that it is optimal for the borrower to sell the home whenever the value of his outside option A plus the proceeds from the sale exceed his continuation utility under full homeownership (which happens at t_s^*).

Consider the candidate incentive compatible contract $(\tau'^f, \tau'^d, I', C, Y)$ where

$$\begin{aligned} \tau'^f(Y,P) &= \tau^f(\hat{Y}(Y,P), P), \\ \tau'^d(Y,P) &= \tau^d(\hat{Y}(Y,P), P), \\ I'(Y,P) &= C(Y,P). \end{aligned}$$

We observe that the borrower's consumption and the termination time under the new contract and the proposed borrower's response strategy, (C, Y), are the same as under the old contract, so he earns the same expected utility, which establishes property (i). Also, by construction, the proposed response of the borrower to the contract (τ'^f, τ'^d, I') involves truth-telling and zero savings, which establishes properties (ii) and (iii).

Now we will show that (C, Y) is the borrower's incentive compatible strategy under the contract (τ'^f, τ'^d, I'). We note that the strategy (C, Y) yields the same utility to the borrower under the contract (τ'^f, τ'^d, I') as the incentive compatible strategy associated with the contract (τ^f, τ^d, I). Therefore, to show that (C, Y) is the borrower's incentive compatible strategy under the contract (τ'^f, τ'^d, I'), it is enough to show that if any alternative strategy (C', Y') is feasible under the contract (τ'^f, τ'^d, I'), then C' is also feasible under the old contract (τ^f, τ^d, I).

It follows that if C' is feasible under the new contract, then the borrower has nonnegative savings if he reports $\hat{Y}(Y'(Y,P),P)$ and consumes C' under the old contract, and thus C' is also feasible under the old contract (τ^f, τ^d, I). To see this we note that that the borrower's savings at any time $t \leq \min(\tau'^f(Y'(Y,P),P), \tau'^d(Y'(Y,P),P), t_s^*) = \min(\tau^f(\hat{Y}(Y'(Y,P),P), \tau^d(\hat{Y}(Y'(Y,P),P), t_s^*)$ under the old contract (τ^f, τ^d, I) and the borrower's strategy $(C', \hat{Y}(Y'(Y,P),P))$ are equal to

$$\underbrace{\int_0^t e^{\rho(t-s)} \left[dY_s - d\hat{Y}_s(Y'(Y,P),P) + dI_s(\hat{Y}(Y'(Y,P),P) - dC'_s(Y,P) \right]}_{\text{Savings under the old contract, the borrower's strategy } (C',\hat{Y}(Y'(Y,P),P)), \text{ and the realized } (Y,\ P)} =$$

$$\underbrace{\int_0^t e^{\rho(t-s)} \left[dY'_s(Y,P) - d\hat{Y}_s(Y'(Y,P),P) + dI_s(\hat{Y}(Y'(Y,P),P) - dC_s(Y'(Y,P),P) \right]}_{(\geq 0) \text{ Savings under the old contract given the borrower's strategy } (C,\hat{Y}(Y'(Y,P),P)), \text{ and the realized } (Y'(Y,P),P)}$$

$$\underbrace{+ \int_0^t e^{\rho(t-s)} \left[dY_s - dY'_s(Y,P) + \underbrace{dC_s(Y'(Y,P),P)}_{=I'(Y'(Y,P),P)} - dC'_s(Y,P) \right]}_{(\geq 0) \text{ Savings under the new contract, the borrower's strategy } (C,Y'(Y,P)), \text{ and the realized } (Y,\ P)} \geq 0.$$

Finally, to complete the proof, we need to show that under the new contract (τ'^f, τ'^d, I') the lender gets the same or greater expected profit as under the contract (τ^f, τ^d, I). Note that under the new contract the lender does savings for the borrower. As by assumption the lender's interest rate process is always greater or equal from the saving's interest rate available to the borrower (i.e. $r \geq \rho$), the lender's expected profit improves by

$$E_0 \left[\int_0^\tau e^{-rt} (r - \rho) S_t dt \right] \geq 0,$$

where, $\tau = \min(\tau'^f(Y,P), \tau'^d(Y,P), t_s^*) = \min(\tau^f(\hat{Y}(Y),P), \tau^d(\hat{Y}(\hat{Y}(Y),P), t_s^*)$ which shows (iv). $\square$

Proof of Proposition 5

Let b_0 be a C^2 function (in a) that solves:

$$rb_0(a,t) = \frac{\partial b_0(a,t)}{\partial t} + \mu + (ra - \theta - \psi(a,t)\delta)b_0'(a,t) + \frac{1}{2}\sigma^2 b_0''(a,t) + \delta\left(b_1(a+\psi(a,t),t) - b_0(a,t)\right) \tag{29}$$

when a is in the interval $[A, v^0(t)]$, and $b_0'(a,r) = -1$ when $a > v^0(t)$, with boundary conditions $b_0(A,t) = L_t$ and $b_0(v^0(t),t) = 0$, where

$$\psi(a,r) = \begin{cases} \text{is a } C^1 \text{ (in } a\text{) solution to } b_0'(a,t) = b_1'(a+\psi,t) \text{ for all } (a,t) \\ \text{for which the solution is such that } \psi(a,t) > A - a \\ \\ \text{otherwise it is equal to } A - a \end{cases}.$$

First, the concavity of function $b_0(\cdot,t)$ can be established as in Piskorski and Tchistyi (2007) using the fact that the function b_1 is concave (Proposition 4).

Now for any incentive compatible contract $(\tau^f, \tau^d, I, C, Y)$ and $t \leq \min(\tau^h, \tau)$ we define:

$$G_t = \int_0^t e^{-rt}(dY_s - dI_s) + e^{-rt}b(a_t,t), \tag{30}$$

where a_t evolves according to (8). We note that the process G is such that G_t is $\mathcal{F}_t-$measurable.

We remember that under an arbitrary incentive compatible contract, $(\tau^f, \tau^d, I, C, Y)$, a_t evolves for $t \leq \min(\tau^h, \tau^f, \tau^d)$:

$$da_t = (\gamma a_t - \theta - \psi_t\delta)\,dt - dI_t + \beta_t dZ_t + \psi_t dN_t.$$

where $\beta_t \geq \sigma$ m-a.s. for any $0 \leq t \leq \tau$. From Ito's lemma we get that

$$\begin{aligned} db_0(a_t,t) = \frac{\partial b_0(a,t)}{\partial t} &+ \left[(\gamma a_t - \theta - \psi_t\delta)b_0'(a_t,t) + \frac{1}{2}\beta_t^2 b_0''(a_t,t)\right]dt - b_0'(a_t,t)dI_t \\ &+\beta_t b_0'(a_t,t)dZ_t + \left[b_1(a_t+\psi_t,t) - b_0(a_t,t)\right]dN_t. \end{aligned} \tag{31}$$

We note that (30) implies

$$e^{rt}dG_t = dY_t - dI_t + db_0(a_t,t) - rb_0(a_t,t),$$

Substituting from (31) for $db_0(a_t,t)$ in the above and using $dY_t = \mu dt + \sigma dZ_t$ yields

$$\begin{aligned} e^{rt}dG_t \quad = \quad & \frac{\partial b_0(a,t)}{\partial t} + \left[\mu + (\gamma a_t - \theta - \psi_t\delta)b_0'(a_t,t) + \frac{1}{2}\beta_t^2 b_0''(a_t,r_t) - rb_0'(a_t,t)\right]dt - rb_0(a_t,t) \\ & -(1 + b_0'(a_t,t))dI_t + (\sigma + \beta_t b_0'(a_t,t))\,dZ_t + \left[b_1(a_t+\psi_t,t) - b_0(a_t,t)\right]dN_t \end{aligned}$$

Substituting from (29) for $rb_0(a_t,t)$ in the above yields

$$\begin{aligned} e^{rt}dG_t &= \left[\frac{1}{2}\left(\beta_t^2-\sigma^2\right)b_0''(a_t,t)+\delta b_0'(a_t,t)\left[\psi(a_t,t)-\psi_t\right]\right]dt-(1+b_0'(a_t,t))dI_t \\ &\quad +\left(\sigma+\beta_t b_0'(a_t,t)\right)dZ_t+\left[b_1(a_t+\psi_t,t)-b_1(a_t+\psi(a_t,t),t)\right]dN_t, \end{aligned}$$

whenever $a\in[A,v_t^0]$. From the above we have that for any $0\le t<\min(\tau^h,\tau)$:

$$\begin{aligned} e^{rt}dG_t &\le \underbrace{\left[\frac{1}{2}\left(\beta_t^2-\sigma^2\right)b_0''(a_t,t)\right]}_{\le 0}dt\underbrace{-(1+b_0'(a_t,t))dI_t}_{\le 0} \\ &\quad +\underbrace{\delta\left(\left[b_1(a_t+\psi_t,t)-\psi_t b_0'(a_t,t)\right]-\left[b_1(a_t+\psi(a_t,t),t)-\psi(a_t,t)b_0'(a_t,t)\right]\right)}_{\le 0}dt \\ &\quad +\left(\sigma+\beta_t b_0'(a_t,t)\right)dZ_t+\left[b_1(a_t+\psi_t,t)-b_1(a_t+\psi(a_t,t),t)\right]dM_t, \end{aligned} \tag{32}$$

where $dM_t=dN_t-\delta dt$. The first component of the RHS of the above inequality is less or equal to zero because the function b_0 is concave and $\beta_t\ge\sigma$ for any $t\le\min(\tau^h,\tau)$. The second component is less or equal to zero because $b_0'\ge -1$ and $dI_t\ge 0$. The third component is less or equal to zero because, by definition, the function ψ is a solution to

$$\max_{\psi\ge A-a}\left[b_1(a+\psi,t)-\psi b_0'(a,t))\right].$$

The condition (32) implies that the process G is an $\mathcal{F}_t-$supermartingale up to time $t=\min(\tau^h,\tau)$, where we recall that Z and M are martingales. It will be an $\mathcal{F}_t-$martingale if and only if, for $t>0$, $\beta_t=\sigma$ m-a.s., $\psi_t=\psi(a_t,t)$, and $dI_t=0$.

Let

$$b(a,t)=\begin{cases} b_0(a,t) & \text{for } a\in[A,v_t^0] \text{ and } t<\tau^h \\ b_1(a,\tau^h) & \text{for } a\in[A,v^1] \text{ and } t\ge\tau^h \end{cases}.$$

We now evaluate the lender's expected utility for an arbitrary incentive compatible contract (τ^f,τ^d,I,C,Y), which equals

$$E\left[\int_0^\tau e^{-rs}(dY_s-dI_s)+e^{-r\tau}b(a_\tau,\tau)\right].$$

We note that for $\tau=\tau_s^*$ we have that $b(a_\tau,\tau)=v_{t_s^*}^0-a_{t_s^*}$, for $\tau=\tau^d$ we have that $b(a_{\tau^d},\tau^d)=L_{\tau^d}$ as $a_{\tau^d}=A$, and for $\tau=\tau^f$ we have $b(a_{\tau^f},\tau^f)=0$ as $a_{\tau^f}=v_{\tau^f}$. Using this, and the definition of process G,

we have that under any arbitrary incentive compatible contract $(\tau^f, \tau^d, I, C, Y)$ and any $t \in [0, \infty)$:

$$E\left[\int_0^\tau e^{-rs}(dY_s - dI_s) + e^{-r\tau}[1_{\tau=\tau_s^*} L_\tau^S + 1_{\tau=\tau^d} L_\tau]\right] =$$
$$E\left[G_{t\wedge\tau}\right] + E\left[1_{t\leq\tau}\left(\int_t^\tau e^{-rs}(dY_s - dI_s) + e^{-r\tau}[1_{\tau=\tau_s^*} L_\tau^S + 1_{\tau=\tau^d} L_\tau] - e^{-rt} b(a_t, t)\right)\right] \leq$$
$$b_0(a_0, 0) + E\left[1_{t\leq\tau}\left(\int_t^\tau e^{-rs}(dY_s - dI_s) + e^{-r\tau}[1_{\tau=\tau_s^*} L_\tau^S + 1_{\tau=\tau^d} L_\tau] - e^{-rt} b(a_t, t)\right)\right] =$$
$$b_0(a_0, 0) + e^{-rt} E\left[1_{t\leq\tau}\left(E\left[\int_t^\tau e^{r(t-s)}(dY_s - dI_s) + e^{r(t-\tau)}[1_{\tau=\tau_s^*} L_\tau^S + 1_{\tau=\tau^d} L_\tau]\,|\mathcal{F}_t\right] - b(a_t, t)\right)\right], \quad (33)$$

where, the inequality follows from the fact that $G_{t\wedge\tau}$ is supermartingale and $G_0 = b_0(a_0, 0)$. We note that in the above

$$E\left[\int_t^\tau e^{r(t-s)}(dY_s - dI_s) + e^{r(t-\tau)}[1_{\tau=\tau_s^*} L_\tau^S + 1_{\tau=\tau^d} L_\tau]\,|\mathcal{F}_t\right] < \frac{\mu}{r} + P_{t_s^*} - \left(a_t - \frac{\theta}{r}\right),$$

as the RHS of the above inequality is the upper bound on the lender's expected profit under the first-best (public information) contract. Using the above inequality in (33) we have that

$$E\left[\int_0^\tau e^{-rs}(dY_s - dI_s) + e^{-r\tau}[1_{\tau=\tau_s^*} L_\tau^S + 1_{\tau=\tau^d} L_\tau]\right] \leq b_0(a_0, 0) + e^{-rt} E\left[1_{t\leq\tau}\left(\frac{\mu}{r} + \frac{\theta}{r} + P_{t_s^*} - a_t - b(a_t, t)\right)\right].$$

Using $b'(a, t) \geq -1$, we have that, for any $a \geq A$, $-a - b(a, t) \leq -A - L_t$. Applying this to the above inequality yields

$$E\left[\int_0^\tau e^{-ts}(dY_s - dI_s) + e^{-r\tau}[1_{\tau=\tau_s^*} L_\tau^S + 1_{\tau=\tau^d} L_\tau]\right] \leq b_0(a_0, 0) + e^{-rt} E\left[1_{t\leq\tau}\left(\frac{\mu}{r} + \frac{\theta}{r} + P_{t_s^*} - A - L\right)\right].$$

Taking $t \to \infty$ yields

$$E\left[\int_0^\tau e^{-ts}(dY_s - dI_s) + e^{-r\tau}[1_{\tau=\tau_s^*} L_\tau^S + 1_{\tau=\tau^d} L_\tau]\right] \leq b_0(a_0, 0).$$

Let $(\tau^{f*}, \tau^{d*}, I^*, C^*, Y)$ be a contract satisfying the conditions of the proposition. We remember that this contract is incentive compatible as it is feasible and $\beta_t = \sigma \geq \sigma$ for any $t \leq \tau$. Also under this contract the process G_t is a martingale until time τ (note that $b'(a, t)$ is bounded). So we have that

$$E\left[\int_0^{\tau^*} e^{-R_s}(dY_s - dI_s^*) + e^{-r\tau^*}[1_{\tau=\tau_s^*} L_\tau^S + 1_{\tau=\tau^{*d}} L_\tau]\right] =$$
$$b_0(a_0, 0) + e^{-rt} E\left[1_{t\leq\tau^*}\left(E\left[\int_t^{\tau^*} e^{r(t-s)}(dY_s - dI_s^*) + e^{r(t-\tau^*)}[1_{\tau^*=\tau_s^*} L_\tau^S + 1_{\tau^*=\tau^{*d}} L_\tau]\,|\mathcal{F}_t\right] - b(a_t, t)\right)\right].$$

Taking $t \to \infty$ and using

$$\lim_{t\to\infty} e^{-rt} E\left[1_{t\leq\tau^*}\left(E\left[\int_t^{\tau^*} e^{r(t-s)}(dY_s - dI_s^*) + e^{r(t-\tau^*)}[1_{\tau^*=\tau_s^*}L_\tau^S + 1_{\tau^*=\tau^{*d}}L_\tau]\,|\mathcal{F}_t\right] - b(a_t,t)\right)\right] = 0,$$

yields

$$E\left[\int_0^{\tau^*} e^{-rs}(dY_s - dI_s^*) + e^{-r\tau^*}[1_{\tau^*=\tau_s^*}L_\tau^S + 1_{\tau^*=\tau^{*d}}L_\tau]\right] = b(a_0,0). \quad \square$$

Proof of Proposition 6

Let $(C,\hat{Y})$ be any borrower's feasible strategy given the contract $\zeta = (\tau^f, \tau^d, I)$. The borrower's private saving's account balance, S, under the strategy $(C,\hat{Y})$ and the contract ζ grows, for $t \in [0,\tau]$, according to

$$dS_t = \rho S_t dt + (dY_t - d\hat{Y}_t) + dI_t - dC_t, \tag{34}$$

where we remember that $\rho \leq r$. Define the process $\hat{V}$ as

$$\hat{V}_t = \int_0^t e^{-rs}dC_s + \int_0^t e^{-rs}\theta ds + e^{-rt}(S_t + a_t),$$

From the above it follows that

$$e^{rt}d\hat{V}_t = dC_t + \theta dt + dS_t - rS_t dt + da_t - ra_t dt$$

Using (22) and (34) yields

$$\begin{aligned} e^{rt}d\hat{V}_t &= (\rho - r)S_t dt + (dY_t - \mu dt)dt + \psi_t dM_t = \\ &\quad (\rho - r)S_t dt + \sigma dZ_t + \psi_t dM_t. \end{aligned} \tag{35}$$

Noting that $e^{rt} \geq 1$ for any $t \geq 0$, so we have that

$$d\hat{V}_t \leq (\rho - r)S_t dt + \sigma dZ_t + \psi_t dM_t.$$

Recall that Z and M are martingales, $\rho \leq r$, and that the process S is nonnegative. So it follows from the above that the process $\hat{V}$ is supermartingale up to time τ (note that a is bounded from below). Using this and the fact that by definition $a_{\tau^d} = A$, $a_{\tau^f} = v_{\tau^f}$, $a_{\tau_s^*} = A_{t_s^*}^S(a_{t_s^*})$ we have that for any feasible strategy of the borrower,

$$a_0 = \hat{V}_0 \geq E\left[\hat{V}_\tau\right] = E\left[\int_0^\tau e^{-rs}dC_s + \int_0^\tau e^{-rs}\theta ds + e^{-r\tau}(S_\tau + 1_{\tau=\tau_s^*}A_\tau^S + 1_{\tau=\tau^f}v_\tau + 1_{\tau=\tau^d}A)\right], \tag{36}$$

The right-hand-side of (36) represents the expected utility for the borrower under any feasible $\left(C, \hat{Y}, S\right)$. This utility is bounded by a_0. If the borrower maintains zero savings, $S_t = 0$, reports cash flows truthfully, $d\hat{Y}_t = dY_t$, then $\hat{V}$ is a martingale up to time τ, which means that (36) holds with equality and the borrower's expected utility is a_0. Thus, this is the optimal strategy for the borrower. $\square$

Proof of Proposition 7

Consider the candidate mortgage contract. Under this contract the borrower's balance on the credit line evolves according to

$$dB_t = r_t^M B_t dt - d\hat{Y}_t + dI_t + BA_t B_t dN_t, \tag{37}$$

when $B_t \leq C_t^L$, while the borrower's savings evolve according to

$$dS_t = \rho S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) - dC_t, \tag{38}$$

where I_t represents cumulative withdrawal of money from the credit line by the borrower.

Let $(C, \hat{Y}, S)$ be any borrower's feasible strategy under the proposed mortgage contract. For any feasible borrower's strategy $(C, \hat{Y}, S)$ define a process $\hat{V}$ as

$$\hat{V}_t = \int_0^t e^{-rs}(dC_s + \theta ds) + e^{-rt}\left(\tilde{a}_t + S_t\right), \tag{39}$$

where

$$\tilde{a}_t = v_t - B_t \tag{40}$$

It follows from (37), (40), and (24)-(27) that $\tilde{a}$ evolves as

$$d\tilde{a}_t = \left(r\tilde{a}_t - \theta - \delta(N_t)\psi(\tilde{a}_t, t)\right) dt + \left(d\hat{Y}_t - \mu dt\right) - dI_t + \psi(\tilde{a}_t, t) dN_t \tag{41}$$

Using (1), (38), (39), (41) yields

$$\begin{aligned} e^{rt} d\hat{V}_t &= dC_t + \theta dt + d\tilde{a}_t + dS_t - r\tilde{a}_t dt - rS_t dt \\ &= \sigma dZ_t + \psi(\tilde{a}_t, t) dM_t + (\rho - r) S_t dt \end{aligned}$$

Recall that Z and M are martingales, $\rho < r$, and that the process S is nonnegative. So it follows from the above that the process $\hat{V}$ is a supermartingale up to time $\tau\left(C, \hat{Y}, S\right) = \min(\tau_s^*, \tau^f(C, \hat{Y}, S), \tau^d(C, \hat{Y}, S))$ (note that $\tilde{a}$ is bounded). Using this and the fact that by definition $\tilde{a}_{\tau^d} = A$ and $\tilde{a}_{\tau^f} = v_t$ we have that for

any feasible strategy of the borrower, $(C, \hat{Y}, S)$,

$$A + C_0^L - B_0 = a_0 = \tilde{a}_0 = \hat{V}_0 \geq E\left[\hat{V}_{\tau(C,\hat{Y},S)}\right]$$
$$= E\left[\int_0^{\tau(C,\hat{Y},S)} e^{-rs}(dC_s + \theta ds) + e^{-r\tau}(S_{\tau(C,\hat{Y},S)} + 1_{\tau(C,\hat{Y},S)=\tau_s^*} A_\tau^S + 1_{\tau(C,\hat{Y},S)=\tau^f} v_\tau + 1_{\tau(C,\hat{Y},S)=\tau^d} A)\right] \tag{42}$$

The right-hand-side of (42) represents the expected utility for the borrower under any feasible strategy $\left(C, \hat{Y}, S\right)$, given the terms of the mortgage. This utility is bounded by $A + C_0^L - B_0$, where B_0 is the initial draw on the credit line. If the borrower maintains zero savings, $S_t = 0$, uses all its cash flow for repayment till his mortgage balance reaches zero, $d\hat{Y}_t = dY_t$, so that $dC_t = dI_t = dI_t^* = 0$, then $\hat{V}$ is a martingale, which means that (42) holds with equality and the borrower's expected utility is $A + C_0^L - B_0$. Thus, this is the optimal strategy for the borrower.

Reproducing the above argument for the borrower's optimal strategy, $(C, \hat{Y}, S) = (I^*, Y, 0)$, and the process $\hat{V}_{t'}$, $t' \leq \tau(I^*, Y, 0)$, defined as

$$\hat{V}_{t',t} = \int_{t'}^{t} e^{-r(s-t')}(dC_s + \theta ds) + e^{-r(t-t')}\tilde{a}_t, \quad t \geq t', \tag{43}$$

yields that, for any $0 \leq t \leq \tau(I^*, Y, 0)$, $\tilde{a}_t$ is equal to the borrower's continuation utility under the proposed mortgage contract with the initial expected utility for the borrower given by $a_0 = A + C_0^L - B_0$, which establishes (28).

Under the proposed mortgage contract and the borrower's optimal strategy, the lender's expected utility equals

$$E\left[\int_0^{\tau(I^*,Y,0)} e^{-rt}(dY_t - dI_t^*) + e^{-r\tau(I^*,Y,0)}[1_{\tau(I^*,Y,0)=\tau_s^*} L_\tau^S + 1_{\tau(I^*,Y,0)=\tau^d} L_\tau]\,|\mathcal{F}_0\right],$$

where

$$\tau(I^*, Y, 0) = \tau^*,$$

as the borrower's continuation utility, $\tilde{a}$, evolve according to the equations (21) in the boom and (13) in the slump, e.g. as in the optimal contract. Therefore, we conclude that the proposed mortgage contract implements the optimal contract. □

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