Subprime Mortgages*

John Krainer
Federal Reserve Bank of San Francisco

Stephen F. LeRoy
Federal Reserve Bank of San Francisco

Munpyung O
University of California, Santa Barbara

November 18, 2008

Abstract

We study the problem of mortgage loan pricing in the presence of house price changes and borrower default. We use an equilibrium model to show how lenders vary both the loan price and the required loan-to-value ratio in response to changes in assumptions about the sharing of deadweight default costs. The model generates a set of testable predictions for risk and return relationships in mortgage markets, as well as predictions for optimal default by borrowers.

Until the last several years the problem of valuing mortgages and mortgage-based securities was essentially that of pricing prepayment risk. Suboptimal exercise by borrowers of the call option embedded in mortgages (paying off the loan when interest rates have risen or failing to do so when they have dropped) raises the value of the contract to the lender (or to a subsequent purchaser of the loan) relative to the value implied by optimal exercise. The problem for purchasers of mortgages was to estimate the extent of the increase in value.

The other major option granted to borrowers in the mortgage contract, that of defaulting, was largely ignored in such valuation exercises. This was so for two reasons: first, most mortgages and mortgage-backed securities were insured by government agencies, so investors could ignore default risk. Second, lenders

*We are indebted to seminar particpants at UCLA and the China Economics and Management Academy for comments.

rarely financed more than 80% of home purchase prices. Accordingly, borrowers in financial difficulty could almost always sell the home and recover the equity, rendering default unnecessary.

In the current US financial crisis these priorities have been reversed. Fluctuations in prepayment behavior are not a major issue now. However, the increasing use of subprime mortgages, often with initial loan-to-value ratios around 100%, combined with house price drops on the order of 15% or more, have left millions of homeowners with negative equity (here equity is temporarily identified as house value less the book value of the mortgage; see the discussion below), possibly creating a financial incentive to default. To the extent that the resulting wave of defaults involved uninsured mortgages, it caused serious problems for financial institutions around the world. It is true that many of the defaults have been on mortgages insured by government-sponsored enterprises, in which case the investors are not at risk. However, in these cases the government-sponsored enterprises are very much at risk. At this writing high default rates have resulted in the US government taking over Fannie Mae and Freddie Mac. The survival of these agencies in their present form is questionable.

Financial institutions that invest in mortgage-backed securities use formal models to value the impact of the prepayment and default options on mortgages. Most Wall Street mortgage valuation models are "reduced-form" models: future default rates are estimated by extrapolating past default rates, using regression methods to correct for changes in the environment. Such methods are reliable only when changes in the environment are minor, so that the past is a reliable guide to the future. That condition has failed spectacularly over the last several years: the current environment of high loan-to-value mortgages combined with sharply declining collateral values is entirely new. There is no reason to presume that reduced-form models will give accurate estimates of default risk under such circumstances, and it appears that they have not done so. Using such models, bond rating agencies assigned triple-A ratings to the senior tranches of many mortgage-backed securities that subsequently suffered major losses in value.

There exist two possible interpretations. The first is that the models are correct, but events have occurred that lie in the extreme tails of the assumed distributions of exogenous variables: if the shock is two or three standard deviations from the mean, even a correct model will give dramatically incorrect predictions. However, it is unlikely that this is the whole story. It appears as if the reduced-form models used by the rating agencies and many investors greatly underestimated default rates, even conditional on the behavior of real estate prices. At any rate, the opinion that this is the case is frequently expressed.

"Structural" mortgage valuation models specify a setting in which the financial consequences to a borrower of defaulting or not defaulting on a mortgage are explicitly modeled. They incorporate the assumption that the borrower defaults when it is optimal to do so, and only then. Structural valuation models, unlike reduced-form models, can in principle give accurate results even in new environ-

ments, such as that we now face. The most sophisticated structural valuation models—Downing, Stanton and Wallace [2], for example—analyze how changes in home prices and interest rates affect both the prepayment and the default options. The authors just cited found that the value of each option depends importantly on the existence of the other. They concluded (in 2005) that changes in house prices were a major determinant of the value of both options; no doubt they would put that conclusion more strongly today.

For the purpose of analyzing the current crisis our primary interest is in the default option rather than the prepayment option. When equity is negative, as is frequently the case now, prepayment is often not possible. Further, we will be assuming constant interest rates, removing a major motivation for refinancing. However, we cannot entirely neglect the prepayment option in analyzing subprime mortgages because, as discussed below, many borrowers have been willing to borrow using expensive subprime mortgages in the expectation of refinancing on more favorable terms in the near future when, due to capital gains, equity will have risen. It follows that the valuation of subprime mortgages may be strongly affected by existence of the prepayment option, particularly if prepayment costs are low.

The model that we will be using incorporates this motive for prepayment: if house prices rise following initiation of a mortgage, default becomes more unlikely. It follows that the actuarial cost of the default option is less than what borrowers are actually paying, since borrowers' periodic payments are based on the (higher) default probability as of the mortgage initiation date. By refinancing borrowers can reset this cost to the appropriate level.

Finally, we simplify by assuming that mortgages are perpetuities that have no scheduled payment of principal. Since almost all the early payments on 30-year mortgages consist of interest, this specification does not involve major distortion.

We also will take house prices to be exogenous, thereby shutting down any link between mortgage underwriting practices and house prices. This is probably a major problem: it appears likely that lax underwriting standards were a major contributor to the price runup that occurred in the decade prior to 2006. Determining the impact of changes in mortgage underwriting standards on house prices is a major research topic, but one best left for further work.

The focus in this paper, then, is on how different possible mortgage terms alter the effect of house price changes on the incentives to default, and thereby on the valuation of mortgages and mortgage-based securities. A major exogenous specification will be the mortgage payment per dollar of house value, the major determinant of loan-to-value ratios (again, the assumption of exogeneity is questionable). For obvious reasons the size of the mortgage payment affects the likelihood and timing of default. Initially attention is restricted to "voluntary default"—default that occurs as an optimal choice on the part of the borrower rather than as a response to cash shortage. This specification is less restrictive than it might otherwise appear inasmuch as both types of default are strongly

influenced by house price changes, which are modeled here (as noted above, a borrower in financial difficulty will refinance or sell in preference to defaulting if house prices are increasing, but cannot do so otherwise).

Finally, it is assumed that borrowers are homogeneous, implying that we do not incorporate measures of borrowers' ability to repay, such as FICO scores, in the analysis. This specification too is not as restrictive as it might otherwise appear—despite the widespread discussion in the financial press of "Ninja loans" (no income, no job, no assets), it turns out that average credit scores of subprime borrowers have remained approximately constant over the past five years (Jaffee [3]). The reason is clear: rising real estate prices implied that even the weakest borrowers could avoid default by refinancing or selling, implying that their FICO failed to provide an accurate measure of risk. The implication is that taking initial loan-to-value ratios as a proxy for underwriting standards generally is not too bad an approximation. Our model, then, relates default to loan terms and house prices. This is no doubt a misspecification, but other studies (Doms, Furlong and Krainer [6], Gerardi, Shapiro and Willen [5], for example) have found that, taking loan terms as given, house price changes are the most important determinant of default frequencies.

1 Default and Mortgage Valuation

Existing structural mortgage valuation models are based on Merton's classic paper [7]. Several modifications of Merton's framework are necessary if it is to be applied to mortgage valuation. First, Merton assumed that the defaultable asset is a discount bond, and that there exists a fixed time horizon. Thus the problem for the borrower was to decide whether or not to default, and if the decision is to default, whether or not to default before the horizon date. As Merton noted, the absence of coupon payments implies that in this setting the default decision is formally identical to the problem of whether to exercise an American call option on a non-dividend-paying stock. Early exercise is never optimal.

This analysis is correct, but not directly applicable to real estate finance. Mortgages involve periodic payments. The homeowner (or investor) weighs the mortgage payment on one hand against the housing services (or rent) and the prospects for capital gain on the other. He defaults if the former outweighs the latter. Modifying Merton's framework to allow for periodic mortgage payments renders the default decision problem formally identical to that of whether and when to exercise an American call option on a dividend-paying stock. In that case early exercise is sometimes optimal. Contemporary applications of Merton's model to mortgage valuation, including ours, implement this respecification.

As far as we can determine, existing structural default models take no account of the housing services or rent that may, or may not, offset the mortgage payments. This is obviously a major omission, and it is unlikely that quantitative

applications can arrive at accurate representations of the default decision if they take no account of housing services. We include a variable representing housing services, and assume that it is proportional to house value. The specification of proportionality is surely an oversimplication, but at least housing services enter the model, so the model makes economic sense.

Existing analyses characterize optimal mortgage default in a setting that takes the mortgage interest rate and other mortgage terms as given. In contrast, we recognize that borrowers must pay for the default option in the form of a mortgage interest rate that exceeds the risk-free interest rate, and that the magnitude of the excess of the mortgage interest rate over the risk-free rate can be determined as an implication of the optimal default decision.

Analyses of optimal exercise of finance options assume that exercise is free. In the real estate setting that assumption may not be acceptable: both borrowers and lenders may incur costs upon default that they can avoid if borrowers do not default. These costs are difficult to measure directly, but they affect mortgage pricing, and therefore can be estimated indirectly. We extend the analysis to allow for such default costs.

Models of optimal default, such as that of Merton and the model of this paper, contain an implication that, we believe, has not been sufficiently appreciated: that the economic value of a mortgage cannot be identified with its book value. Using book value as a determinant of default appears superficially to be an innocuous specification, at least when interest rates are constant and default is costless: a natural intuition would be that when default is costless existence of the default option is irrelevant for valuation of the mortgage since the lender suffers no loss at the time of default. If so, the value of the mortgage should be unaffected by changes in house prices. Therefore, it would seem, if mortgages are modeled as perpetuities, default optimally occurs when the house price equals the initial value of the mortgage.

In contrast to this intuitive argument, when applied to the real estate setting Merton's analysis implies that the economic value of a mortgage follows a differential equation. As we show here, the constants of integration are such that the value of the mortgage depends on the value of the house. This is so even when costs of default are zero and interest rates are deterministic. If the mortgage is modeled as a perpetuity, its economic value differs from its book value after the origination date, due to appreciation or depreciation of the underlying asset. An important implication is that optimal default does not occur as soon as the book value of equity—the market value of the collateral minus the book value of the mortgage—becomes negative. Rather, there exists a range of house values in which the book value of equity is negative but homeowners optimally do not default. Only when the book value of equity passes a threshhold does default occur. This will become clear in the discussion below.

1.1 The Model Setting

We assume that housing services x follow a geometric Brownian motion:

$$dx = \alpha x dt + \sigma x dz, \tag{1}$$

where α is the average proportional increase over time in the level of housing services, σ is the associated volatility and z is a standard Brownian motion. As applied to investment property, x can be interpreted as the difference between rental income and costs to the owner of utilities, repairs, and so on.

It is assumed that the price of housing equals the expectation of future housing services discounted at rate ρ :

$$P_0 = \int_{t=0}^{\infty} e^{-\rho t} \mathcal{E}(x(t)) dt = \frac{x_0}{\rho - \alpha}, \tag{2}$$

where $x(0) = x_0$. Here $\mathcal{E}$ is the expectations operator. As a scale choice, we will set x_0, the value of housing services at the date the mortgage is initiated, equal to 1 henceforth. Thus mortgages are initiated when the price of housing equals $1/(\rho - \alpha)$. The valuation model (2) can be justified either by interpreting the probability measure in eq. (1) as the risk-neutral measure or more simply by directly assuming that agents are risk neutral. The latter assumption is acceptable here because we are not interested in the effects of risk aversion on mortgage valuation. As noted, mortgages are assumed to be perpetuities (conditional on nonoccurrence of default), and to be indexed by the level of the periodic payment c, which we take as exogenous. For the most part we assume that mortgages are not prepayable.

The cash supplied by the lender toward the purchase equals the present value of the mortgage payments, making appropriate allowance for default as described below. Thus low (high) loan-to-value mortgages can be generated by setting c at a low (high) level. The borrower is assumed to supply the remainder of the purchase price as a down payment.

1.2 Costless Default

It is assumed in this section that default is costless to both the borrower and the lender. Thus the borrower is assumed to be able to terminate the mortgage at any time by turning over the house to the lender. He incurs no costs due, for example, to reduced access to credit or moving costs. In turn, the lender is assumed to suffer no loss of value when the house is turned over to him.

We begin by demonstrating intuitively the error pointed out above arising from assuming that the value of the mortgage is a constant independent of the house value. The incorrect analysis is represented in Figure 1. The mortgage is initiated at $x_0 = 1$, and the cash supplied by the lender equals $M(1)$. The borrower's down payment $E(1)$ equals $P(1) - M(1)$, where $P(x)$ equals house value when housing

services equal x. The line $M(x)$ is taken to be flat in the region $x > x_*$, reflecting the assumption that the economic value of the mortgage after the origination date is independent of house value, and therefore equals its book value. Default is held to occur when the borrower's equity reaches zero, which occurs at x_*, where x_* satisfies $P(x_*) = M(x_*)$.

To see the error in this analysis, observe that when x equals x_* the borrower is represented as being indifferent between defaulting and not defaulting. However, this cannot be the case—at $x = x_*$ the borrower is in a heads-I-win, tails-you-lose situation: if the house price drops below x_* he can default immediately and avoid a loss, whereas if it increases his equity becomes strictly positive. This asymmetry implies that the borrower will strictly prefer not defaulting to defaulting, contradicting the assumption implied by the diagram that the borrower is indifferent between defaulting and not defaulting.

This effect can be avoided if we recognize that the value of the mortgage depends on the value of the house even though there are no costs of default. Thus we must distinguish the economic value of the mortgage, which depends on x, from its book value, which does not. Further, the value of the mortgage as a function of x cannot have a kink at $x = x_*$, since it is the kink that causes the anomaly at $x = x_*$. To avoid the kink the value of the mortgage as a function of x must be tangent to the value of the collateral at the default point. It follows that existence of the default option implies that the value of the mortgage is an increasing function of x even when default is costless.

We now provide a correct determination of the optimal default point, from which we can compute the value of the mortgage and the homeowner's equity as a function of house value. Define $E(x)$ as the owner's equity in the house (house value minus economic mortgage value). Absence of arbitrage implies that the discounted gain on the homeowner's equity follows a martingale. This martingale property implies that $E(x)$ satisfies the stochastic differential equation

$$\rho E(x)dt = \mathcal{E}[dE(x)] + (x - c)dt, \tag{3}$$

where dt is the increment in time and c is the periodic payment on the mortgage. Here $x - c$, which can be either positive or negative, is the analogue of dividends in a stock valuation expression, and $dE(x)$ is the analogue of capital gain. Using Ito's lemma, (3) leads to the (deterministic) differential equation

$$\frac{1}{2}x^2\sigma^2 E''(x) + \alpha x E'(x) - \rho E(x) = -x + c. \tag{4}$$

This differential equation is similar to those arising in finance and also in the real options literature (see Dixit and Pindyck [1], Ch. 7 for detailed and very clear exposition of a model that contains the model here as a special case). The boundary conditions are the value-matching condition

$$E(x_*) = 0 \tag{5}$$

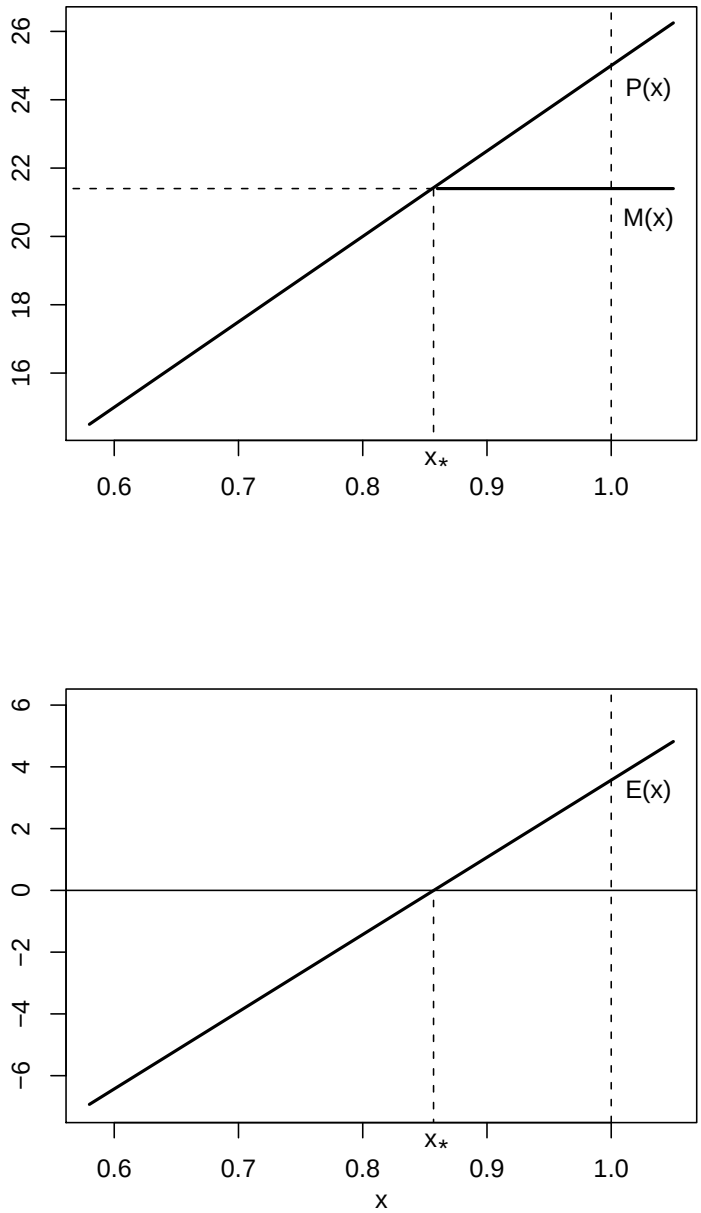

Figure 1: Incorrect analysis of optimal default

and the smooth-pasting condition

$$E'(x_*) = 0. \tag{6}$$

The smooth pasting condition is a necessary condition for optimal exercise of the default option since default is a free boundary. It requires that $M(x)$ be tangent to $x/(\rho - \alpha)$ at x_*. The role of the smooth pasting condition is exactly to avoid the anomaly discussed above that would occur if the greater of $x/(\rho - \alpha)$ and the value of the mortgage had a kink at x_*.

The value of the mortgage equals the value of the house less the borrower's equity. Thus the value-matching condition implies that the value of the mortgage at x_* equals the value of the house, $x_*/(\rho - \alpha)$, and the smooth pasting condition implies that the slope of the mortgage value as a function of x is $1/(\rho - \alpha)$ at $x = x_*$. These boundary conditions allow solution for the constant of integration e.

For $x > x_*$, the solution to the differential equation is

$$E(x) = ex^m + \frac{x}{\rho - \alpha} - \frac{c}{\rho} \tag{7}$$

where e is the constant of integration:

$$e = \frac{1}{m(\alpha - \rho)} \left[\frac{c}{\rho} \left(\frac{m(\rho - \alpha)}{m - 1} \right) \right]^{1-m} \tag{8}$$

and m is the negative root associated with the differential equation (4):

$$m = \frac{-(\alpha - \frac{1}{2}\sigma^2) - \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\rho}}{\sigma^2} \tag{9}$$

(the constant of integration associated with the positive root equals zero).[1]

The value of the mortgage is given by

$$M(x) = P(x) - E(x) = \frac{c}{\rho} - ex^m, \tag{10}$$

from

$$P(x) = \frac{x}{\rho - \alpha} \tag{11}$$

and (7). Because m is negative, when x is high the mortgage value approaches c/ρ asymptotically, agreeing with the value it would have in the absence of default.

[1]For $x < x_*$, $E(x)$ does not follow the solution (7) to the differential equation (4). The simplest way to see this is to observe that for $x < x_*$ immediate default is optimal. That being so, the value of the mortgage equals the value of the house, implying that $E(x) = P(x) - M(x) = 0$. It is easily verified that the (expected) gain on the discounted equity is generally nonzero, and therefore the discounted gain is not a martingale. But the differential equation (4) that characterizes $E(x)$ is based on the martingale assumption.

This makes sense: when x is extremely high default in the foreseeable future is improbable, so it does not materially affect valuation of the mortgage. However, for values of x that are not high, $M(x)$ depends positively on x. It follows that the intuition supplied above—to the effect that when default is costless existence of default should not affect the value of the mortgage—is incorrect.

Having calculated $M(x)$, one can compute the borrower's down payment $E(1)$ as $P(1) - M(1)$, the initial loan-to-value ratio $M(1)/P(1)$, the initial mortgage yield $c/M(1)$ and the recovery rate $P(x_*)/M(1)$.

We now provide a numerical example of the equity values and mortgage values as a function of house services that are implied by this model. Our calibration assumes that $\rho = .07$, $\alpha = .03$, $\sigma = 0.2$, so that the average (real) proportional gain on both the equity and the mortgage is seven per cent, and the initial price-rent ratio is $1/(\rho - \alpha) = 25$ (real-world price-rent ratios are closer to 10 or 15 than 25, but recall that here we are abstracting away from operating costs such as repairs and utilities expense). The standard deviation of the change in housing services, σ, is taken to be 20%. This setting may be too high, but it will do for the purpose of the example.

Under these parameter values a security similar to a conventional mortgage can be generated by specifying that the mortgage payment c is 1.5. In that case the initial loan-to-value ratio is 77.1%. Default occurs when the house value drops to 58.4% of its original value. At that price the recovery rate (ratio of collateral value at the default point to the origination value of the mortgage) is 75.8%. The yield on the mortgage at its origination date, equal to $c/M(1)$, is 7.8%. The difference between this figure and the expected return on the mortgage, 7%, represents compensation for the possibility that the collateral value will drop, possibly leading ultimately to default. The computed functions $M(x)$ and $E(x)$ are displayed in Figure 2.

As the figure indicates, equity as we have characterized it is a nonnegative and increasing function of x for $x > x_*$, and it equals zero at $x = x_*$. Again, it is important to emphasize here that by equity we mean the difference between the value of the house and the economic value of the mortgage, not the difference between the value of the house and the book value of the mortgage. The book value of the mortgage, as a perpetuity, equals its initial value as long as default has not yet occurred. Accordingly, if equity is defined as the value of the house minus the book value of the mortgage, equity becomes negative for values of x strictly above x_*, but as long as $x > x_*$ optimal behavior is not to default. This is so even though default is assumed to be costless.

References to negative equity in the press presume the book value definition of equity; this is unavoidable given the difficulty in estimating the market value of a mortgage in a real-world setting. In this paper, however, as noted the term "equity" denotes the value of the house minus the economic value of the mortgage. Thus in our usage default occurs when equity equals zero; in the absence of default costs, negative equity does not occur.

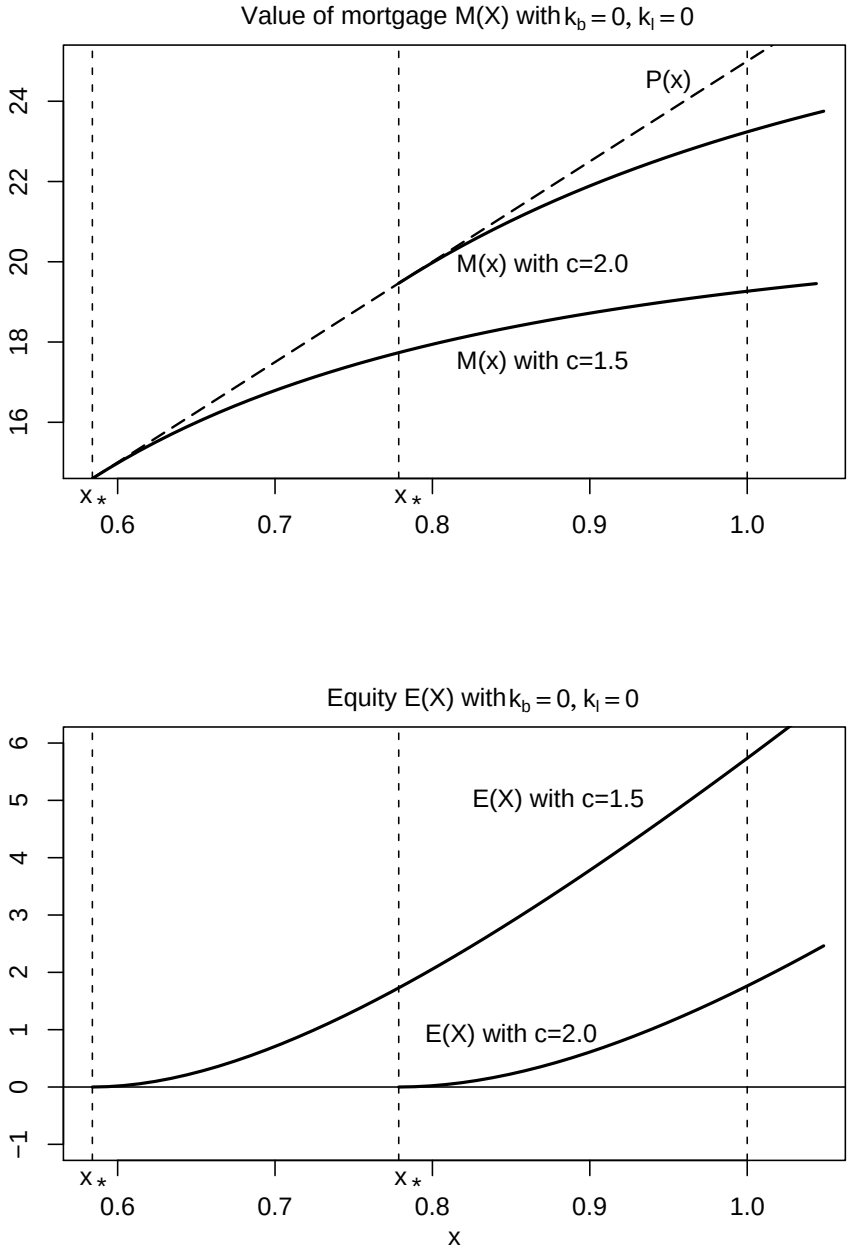

Figure 2: Correct analysis of optimal default

None of this is to say that the book value of a mortgage is irrelevant to its valuation: the prepayment decision is governed by the mortgage's book value in relation to its market value. Prepayment is discussed below.

A security similar to a subprime mortgage results if c is increased from 1.5 to 2.0. This increase in the periodic payment results in an increase in the initiation value of the mortgage; the initial loan-to-value ratio is 93.0%. Default occurs if the house value drops to 78% of its original value. Thus, as would be expected, the magnitude of the price drop that is just sufficient to induce default is much smaller under a subprime mortgage than under a conventional mortgage. Under a conventional mortgage the down payment protects the lender from all but the most drastic price drops, whereas under a subprime mortgage the lender has much less protection. Correspondingly, the recovery rate, 83.7%, is higher under a subprime mortgage than a conventional mortgage.

The fact that default is likelier under a subprime mortgage than a conventional mortgage implies that a higher yield, 8.6%, is needed to compensate the lender for the higher default risk. It is seen that the higher default probability under the subprime mortgage relative to the conventional mortgage more than outweighs the fact that the recovery rate is higher under the subprime mortgage than under the conventional mortgage.

Figures 3, 4 and 5 display computed values of the same mortgage-related variables as functions of c. For now, restrict attention to the lines labeled $k_b = k_l = 0$, which correspond to the case assumed here in which default costs are zero. In all three figures c is plotted horizontally. In Figure 3 the vertical coordinate is the initial loan-to-value ratio, while in Figure 4 the vertical coordinate is the mortgage yield at initiation and in Figure 5 the vertical axis displays the recovery rate. We do not plot the default point x_* but that can be computed as the product of the loan-to-value rate and the recovery rate.

Finally, Figure 6 plots the default point x_* against the loan-to-value ratio. Note, from the identity $LTV \bullet RR = \frac{M(1)}{P(1)} \bullet \frac{P(x_*)}{M(1)} = \frac{P(x_*)}{P(1)} = x_*$, that this figure is implied by Figures 3 and 5. This figure shows the practical importance of the theoretical observation, discussed above, that the economic value of mortgages depends on the level of housing services. Consider, for example, a 98% mortgage (generated by $c = 2.28$). Contrary to the natural intuition that optimal default would occur after a 2% drop in price, in fact a price drop of 11% is required to trigger default ($x_* = 0.89$).

1.3 Costly Borrower Default

In the model of the preceding section the default option is a device that allows sharing between the owner and the lender of the risk of house price decreases. Without a mortgage the owner obviously bears 100% of the risk, whereas when a lender supplies part of the purchase price, the fact that the value of the mort-

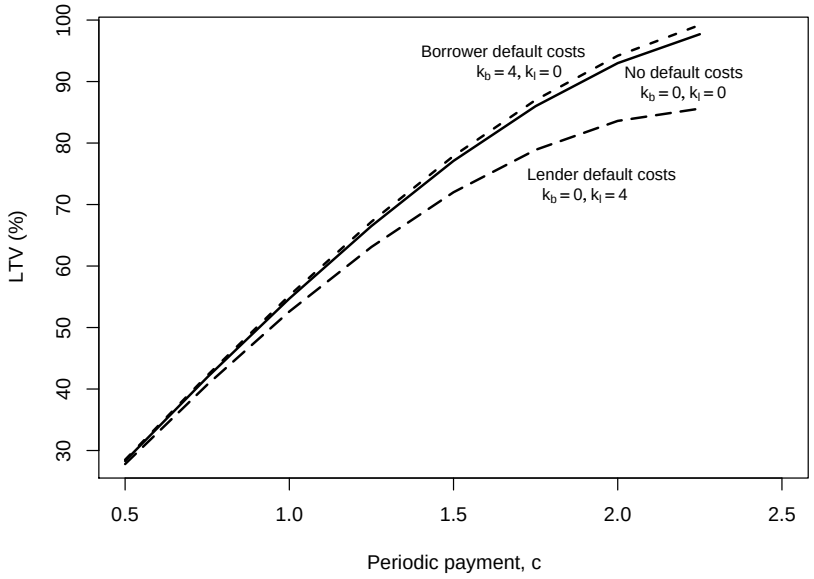

Figure 3: Loan to value ratio as a function of periodic payment under different default scenarios

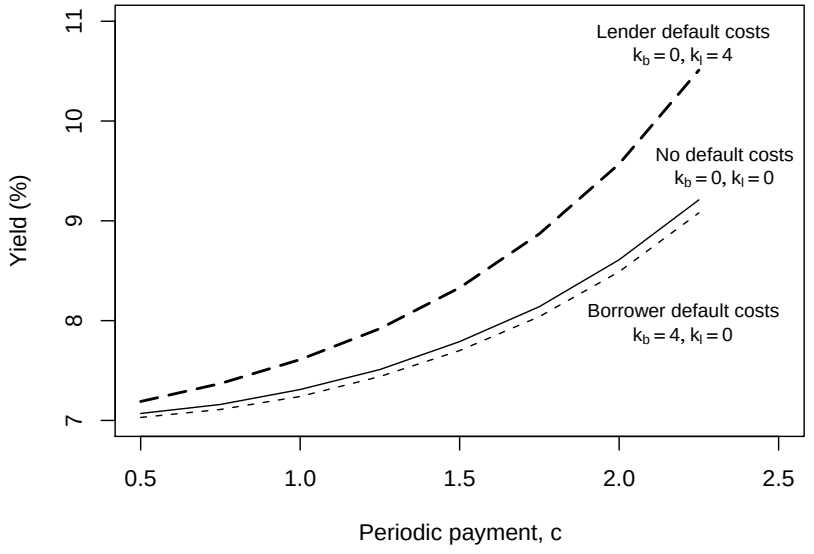

Figure 4: Yield as a function of periodic payment under different default scenarios

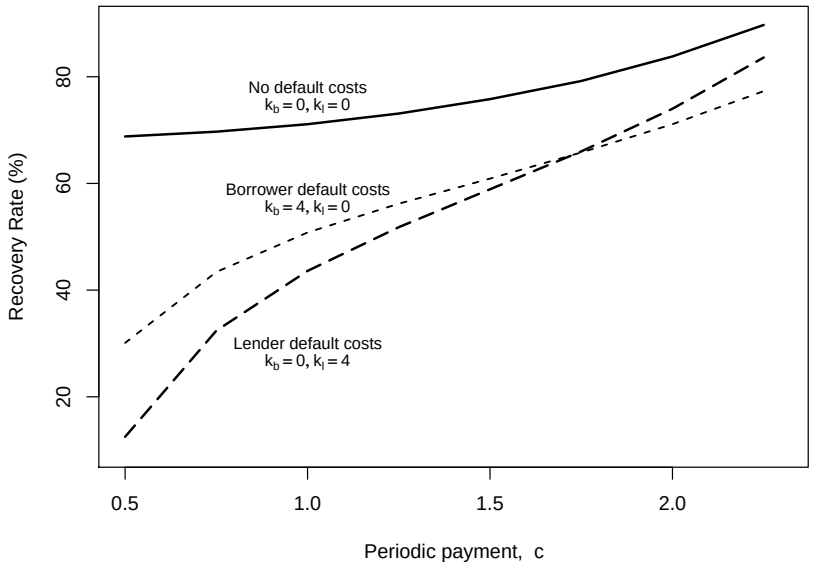

Figure 5: Recovery rates as a function of periodic payment under different default scenarios

gage drops as house value drops implies that he provides partial insurance to the borrower against the risk of loss of value. The proportions in which the loss is shared between the borrower and lender depend on the volatility of house price changes and the terms of the mortgage. Since the associated transfer of risk entails no deadweight loss, subprime loans imply no inefficiency: for any possible house price, that price equals the sum of the mortgage value and the equity value.

To the extent that the preceding analysis, including the assumption of zero default costs, applies to the US crisis of the last several years, the interpretation is that mortgage markets exhibited nothing that can be called an institutional failure. Along these lines, the story incorporated in the model just presented is that house values dropped (on the order of $1 or $2 trillion), with the costs of this drop being shared by homeowners and lenders. The model there incorporated the assumption that subprime mortgages did not contribute to the magnitude of the loss, and also that the loss was a pure transfer involving no deadweight loss.

There is a consensus—although not one that has been documented in a serious way in the financial press or elsewhere—that the assumption of costless default is a major misspecification. It is not entirely clear that this consensus opinion is justified. We noted in the introduction the existence of many subprime mortgages in which the borrowers have not defaulted despite having negative equity. Many observers in the financial press appear to identify this negative equity with

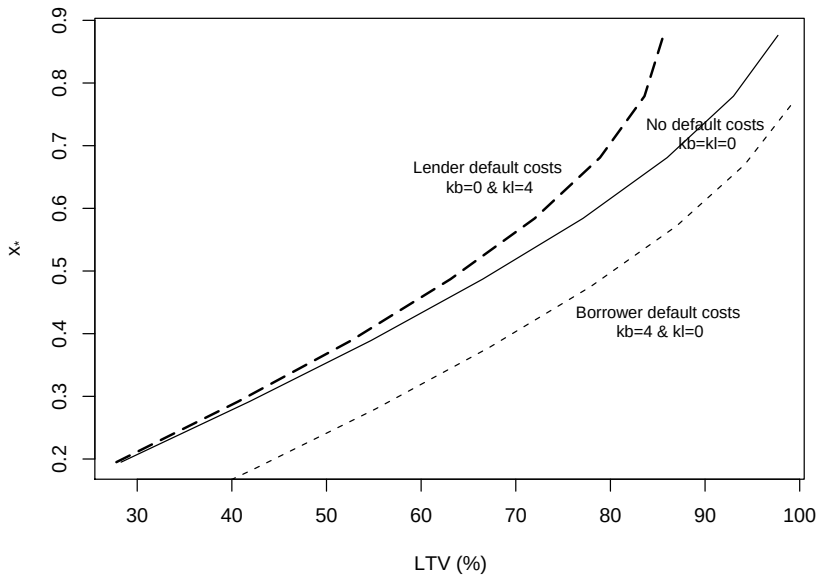

Figure 6: Default point x_* as a function of LTV under different default scenarios

deadweight social costs when it results in default. This observation ignores the distinction noted above between equity based on the economic value of the mortgage and equity based on its book value. When x is low but greater than x_*, equity under the latter definition may be negative, but borrowers will not default because equity under the former definition is still positive. There is no dispute about the fact that many borrowers have negative equity under the book value definition, but that does not establish that they have negative equity under the economic definition, and, as we will see, it is the latter that indicates the existence of positive default costs.

It appears that some subprime loans have been granted that had negative equity at the date of origination. Existence of such mortgages is possible only in the presence of costs of default that induce borrowers to be willing make payments on the loan as long as their equity is only slightly negative. In the absence of such costs borrowers would default immediately upon closing a negative-equity mortgage. Lenders, anticipating this, would never grant negative-equity mortgages. Thus the existence of mortgage loans with negative economic equity is evidence in favor of positive default costs.

Informal evidence suggests the existence of substantial default costs. Default costs to the borrower would surely include the expense of moving necessitated by default, and also the costs implied by the credit downgrades that result from default. In some states the lender can proceed against the other assets of a borrower

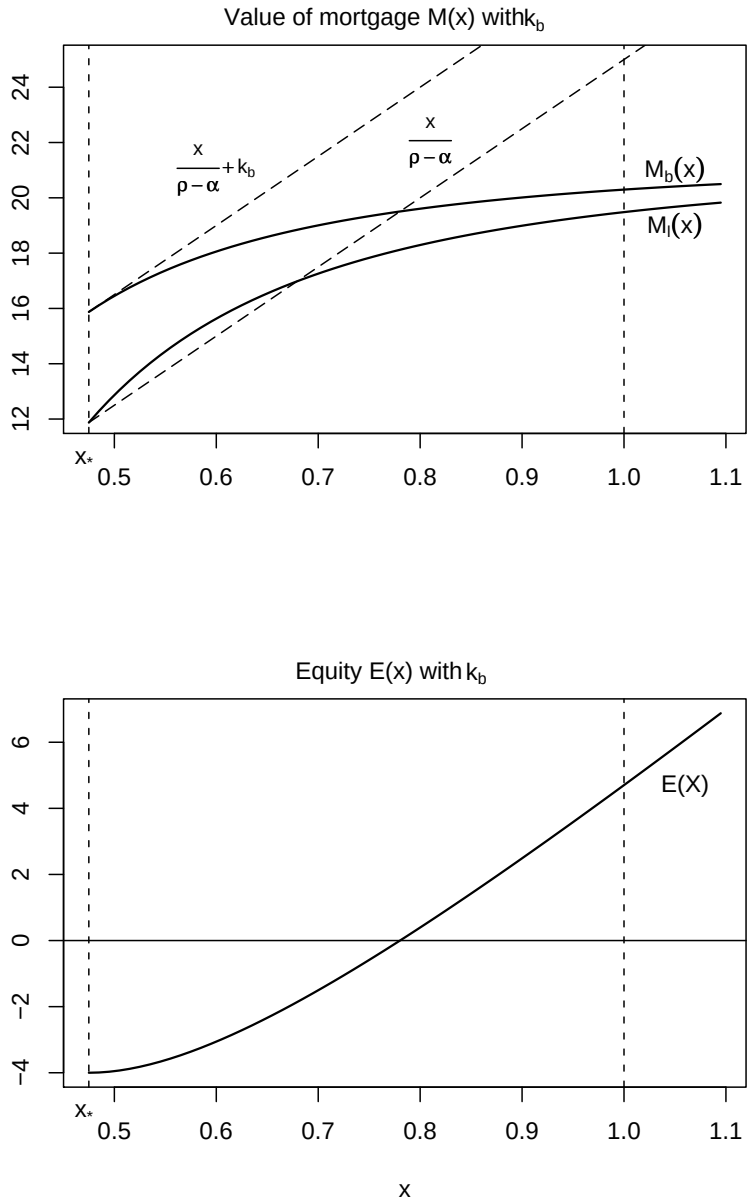

Figure 7: Borrowers' default costs

who defaults, although in other states the lender does not have such recourse. When the lender has recourse the borrower cannot simply avoid any further obligation under the mortgage by turning over the keys to the lender. Costs to the lender include expenses incurred repairing, maintaining and reselling foreclosed properties, and also include foregone income during the period (which may be a year or more) between when the borrower terminates mortgage payments and when the lender can gain possession of the property. The press has reported many instances when defaulting borrowers stole appliances and vandalized the properties upon vacating foreclosed houses. To forestall such behavior, some lenders have offered payments of several thousand dollars to owners of foreclosed properties who turned them over to lenders in satisfactory condition.

To determine the effects of default costs we assume the existence of nonnegative constants k_b and k_l that capture the default costs of the borrower and lender, respectively. It is assumed that these costs are not received as revenue by any party. We also assume that default costs, when strictly positive, are proportional to the initial value of the house at the origination date, so as to be able to carry over the assumption of scale-invariance, which implies that the convention that $x = 1$ at the mortgage origination date is a harmless numeraire choice. Default costs are assumed to be the same for all borrowers and all lenders, implying unrealistically that all mortgages with the same contract terms will default, and at the same time, if any of them do.

Existence of costly default implies that it is necessary to distinguish between the value of a mortgage to the borrower as a liability and its value to the lender as an asset. The easiest way to see this is to observe that when $x = x_*$ the mortgage has liability value of $x_*/(\rho-\alpha)+k_b$ to the borrower and asset value of $x_*/(\rho-\alpha)-k_l$ to the lender. Thus the deadweight loss resulting from default at the time of default equals $k_b + k_l$. We denote the values of the mortgage to the borrower and the lender as $M_b(x)$ and $M_l(x)$. For any x we have $M_b(x) > M_l(x)$; this difference is a decreasing function of x because for high values of x the possibility of default is minor contributor to the value of the loan to either borrower or lender. Both mortgage values converge to c/ρ as x goes to infinity, as occurs when there are no default costs.

It is assumed throughout that there is free entry into, and exit from, the lending industry. Accordingly, lenders adjust loan terms so as to pass the discounted expected value of default costs on to borrowers. As a result, it is necessary to distinguish between the amount of the buyer's down payment, equal to the purchase price less the funds supplied by the lender ($M_l(1)$), and the borrower's equity immediately following purchase, equal to the purchase price less $M_b(1)$. Since $M_b(1) > M_l(1)$, the buyer's equity $E(1)$ at the date of mortgage origination is strictly less than the down payment. Thus purchase of a house using a mortgage entails an immediate deadweight loss to the borrower of $M_b(1) - M_l(1)$, which equals the expected discounted value of default costs. If house prices drop, the

deadweight loss increases, reaching $k_b + k_l$ when prices drop to $x_*/(\rho - \alpha)$.[2]

In this section it is assumed that $k_l = 0$, so that only borrowers have positive default costs.

Incorporating default costs into the model presented above is achieved by replacing the value-matching condition (5) with

$$E(x_*) = -k_b, \tag{12}$$

so that borrowers default when their equity equals $-k_b$. As noted above, the values of the mortgage to the borrower and the lender are then calculated by using the boundary conditions as modified to derive the constants of integration of the differential equations representing $M_b(x)$ and $M_l(x)$.

We now repeat the calculation of default-related mortgage statistics reported in the preceding section assuming $k_b = 4$, so that k_b equals 16% of the assumed purchase price of 25. Comparison of the lines labeled $k_b = k_l = 0$, discussed above, and $k_b = 4, k_l = 0$ in Figures 3, 4 and 5 makes clear the effect of borrowers' default costs on mortgage terms. As one would expect, assuming that borrowers incur costs upon default implies that the default point x_* is lower than was the case with costless default. Therefore default is less probable in the presence of default costs than when they are absent. For any value of c, specifying that borrowers have default costs increases the initial loan-to-value ratio and decreases the mortgage interest rate. Both results are consequences of the fact that default is less probable when borrowers' default costs are present. Figure 4 shows that assuming $k_b > 0$ decreases the recovery rate; this is again a direct result of the fact that x_* decreases in the presence of borrowers' default costs.

[2]The presence of deadweight losses due to costly default implies an inconsistency in the formulation of the model. The model specifies that the discounted value at t of expected housing services equals $x_t/(\rho - \alpha)$ regardless of whether or not a mortgage is used. This specification makes no allowance for the present value of the loss due to default risk when a mortgage is used. Thus when either k_b or k_l is greater than zero, house buyers using mortgages would find that houses are overpriced relative to the expected value of subsequent housing services.

As we will discuss below, this shortcoming of the model has an unfortunate implication for the modeling of prepayable mortgages. As will be noted, borrowers whose homes have appreciated have a motive to refinance because default risk has decreased, implying that they are overpaying for the default option. It would be natural to assume that borrowers will refinance whenever the value of the mortgage liability exceeds the initiation cost plus the prepayment penalty. If the prepayment penalty is less than $M_b(1) - M_l(1)$, however, this specification implies that borrowers would repay their loans immediately upon incurring them. This bizarre implication results from the fact that the model gives no explanation for why individuals use mortgages in the first place. Therefore in the presence of default costs the model cannot give a consistent explanation of when agents refinance as it stands.

One could eliminate the problem just noted by setting the initial price at a level lower than $1/(\rho - \alpha)$. However, the indicated amount of the decrease depends on the loan-to-value ratio. This is different for different values of c, the determination of which we are not modeling, so it is unclear how great a decrease to specify. Thus this respecification does not provide an easy resolution of the problem. We prefer to avoid the major respecifications that would be required to deal with this difficulty.

1.4 Costly Lender Default

In this section we set the borrower's default cost k_b equal to zero and allow the lender's default cost k_l to equal 4. With zero default costs for the borrower, default occurs when $E(x) = 0$. However, the initial condition for $M_l(x)$ at the default point is $M(x_*) = P(x_*) - k_l$ (Figure 7). In Figures 3, 4 and 5 the line marked $k_b = 0, k_l = 4$ shows plots of loan-to-value ratios, mortgage yields and recovery rates that occur when the lender bears a cost upon default. These figures indicate that the effects of lender costs are very different from those of borrower costs. Since it is the borrower who determines the default point, imposing costs on the lender does not directly affect the default point (but doing so does affect the default point indirectly, since lender costs affect the mortgage yield). For given mortgage payment c, imposing default costs on the lender reduces the loan-to-value ratio and increases the mortgage yield. These effects are in the opposite direction of those implies by borrowers' costs. The default point, and therefore the recovery rate, is only slightly affected by imposing default costs on the lender.

It is instructive to plot the mortgage yield and the recovery rate against the loan-to-value ratio, taking all these variables to be implicit functions of c (Figures 8 and 9). Figure 8 shows that the locus of pairs of loan-to-value ratio and mortgage yield when $k_b = 4$ is virtually indistinguishable from the corresponding locus when $k_b = 0$. However, for given loan-to-value ratio, imposing $k_l > 0$ has a strong positive effect on the mortgage yield, particularly for high loan-to-value ratios. In fact, for loan-to-value ratios above about 85% the yield becomes arbitrarily high. This happens because the very high values of c required to generate high loan-to-value ratios imply high default points x_*, which mitigates the positive effect of c on the loan-to-value ratio. It follows that subprime mortgages with initial loan-to-value ratios above 95% implies either that borrower costs are above zero or that lender costs are below 16%, or both, given the assumed values of the other parameters of the model.

1.5 Default Costs and High Loan-to-Value Mortgages

In Section X below we will use mortgage pricing data to make inferences about default costs. The higher the value of c, the higher is the loan-to-value ratio at the mortgage origination date, and the closer is x_* to 1. Thus for values of c that are near the maximum value, default occurs in the next arbitrarily short time interval with arbitrarily high probability. If there are no default costs, the feasible values of c are bounded above by that value of c which results in $M(1) = 1/(\rho - \alpha)$ (in our example this turns out to equal $c =$ xxx). Similarly, if $k_b > 0$ and $k_l = 0$ the upper bound for $M_l(1)$ is $1/(\rho - \alpha) + k_b$. In the case $k_b = 0$ and $k_l > 0$, $1/(\rho - \alpha)$ is an upper bound for $M_b(1)$, which implies that the upper bound for $M_l(1)$ is $1/(\rho - \alpha) - k_l$. In the computed example this means that the upper bound for the loan-to-value ratio is $1 - k_l(\rho - \alpha)$, which equals 84% if $k_l = 4$. Thus the existence

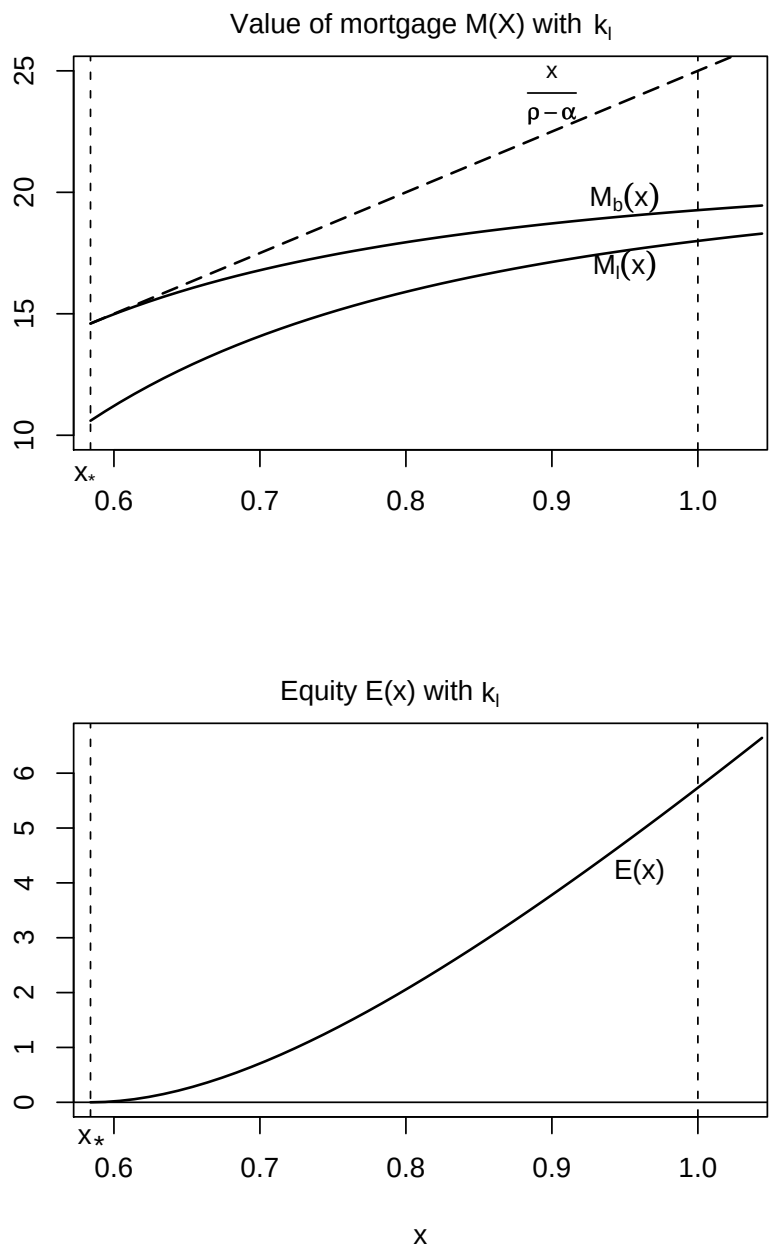

Figure 8: Lender's default cost

of mortgages with initial loan-to-value ratios in excess of 84% is inconsistent with $k_b = 0$ and $k_l = 4\%$; existence of such mortgages implies that either $k_b > 0$ or $k_l < 4\%$.

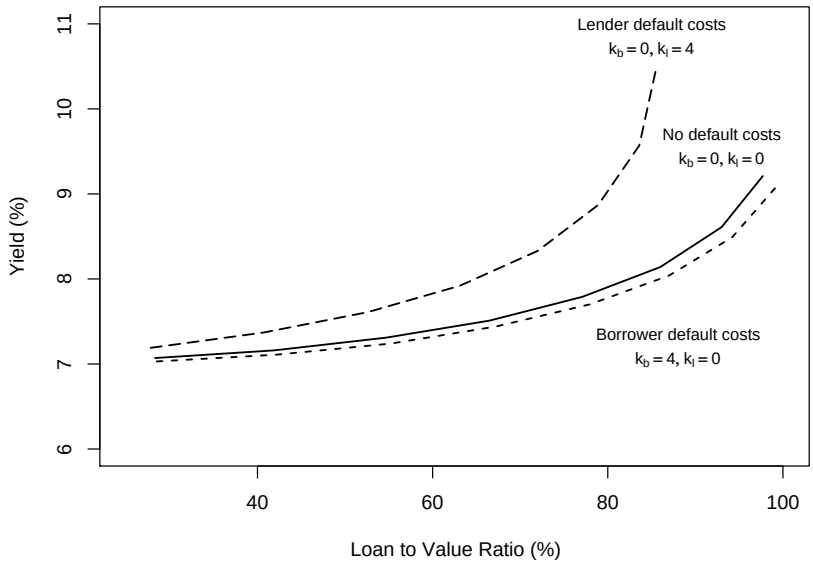

Figure 9: Yields as a function of LTV under different default scenarios

Figure 8 plots mortgage yield as an implicit function of the loan-to-value ratio (these points are generated by varying c). Figure 9 plots recovery rates against loan-to-value ratios. The discussion in the preceding paragraph implies that in both figures the curves that assume $k_l = 0$ have an asymptote at a loan-to-value ratio of 100%, while the curve that assumes $k_b = 0$ and $k_l = 4$ has an asymptote at 84%.

The conclusions are (1) default costs have pronounced effects on the mortgage-related variables, and (2) these effects are different depending on whether they are borne by borrowers or lenders.

1.6 Costly Prepayment

Up to now it has been assumed that mortgages are terminated only by default, so that, conditional on no default, mortgages are perpetuities. This is a major misspecification: borrowers have the option to prepay mortgages at their book values, possibly subject to a prepayment penalty. In the real world they elect to refinance when prevailing interest rates drop, since otherwise the present value of the mortgage payments would exceed the mortgage's book value. Under the present model

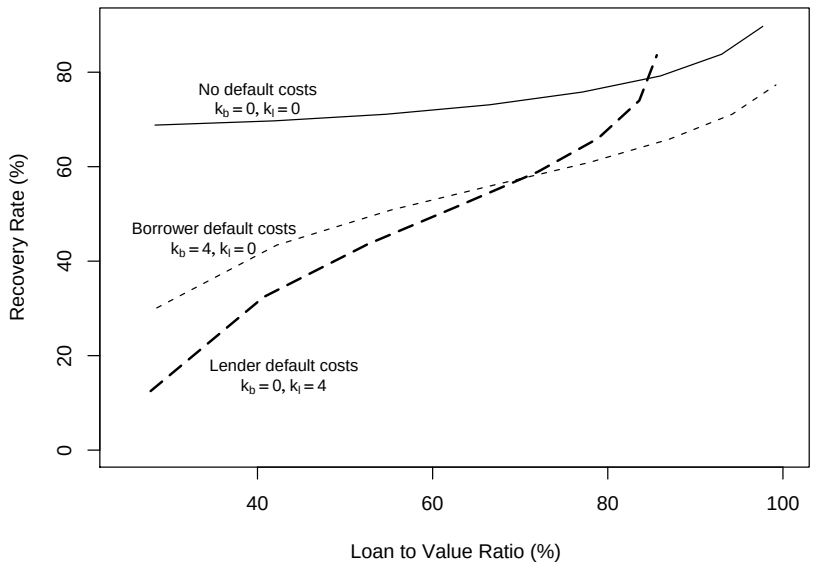

Figure 10: Recovery rates as a function of LTV under different default scenarios

this feature is absent since we are taking interest rates to be deterministic and constant.

As noted in the introduction, in the present model increases in house prices generate a motive for prepayment. This is so because mortgage values increase with x due to the decreased probability of default in the economically relevant future. For $x > 1$ borrowers are paying a higher price for default risk than is justified actuarially, since mortgage payments are based on the default probability appropriate to $x = 1$. Thus borrowers have an incentive to prepay their mortgages when the value of the collateral rises above $1/(\rho - \alpha)$.

These considerations imply that existence of the prepayment option affects the value of subprime mortgages even if interest rates are taken to be nonstochastic, as here. As emphasized in the press, borrowers were willing to make very high periodic payments for high loan-to-value mortgages because they expected to be able to refinance their property on better terms after generating additional equity due to capital gains. Further, they are protected from large losses due to decreases in property values because of the default option and the low level of their equity. Thus lenders are subjected to default losses if house values drop, but, due to prepayment, are unlikely to benefit from gains if they rise. We therefore expect that lenders will require very high periodic payments for high loan-to-value mortgages that are prepayable, particularly if (1) borrowers' default costs are low, so that they will default at high values of x_*, (2) lenders' default costs and high, and

(3) prepayment penalties are low. This is so because they know that default will occur with high probability, resulting in losses, while if the prepayment penalty is low the loan will be terminated with high probability due to refinancing; implying that the maximum gain for the lender equals the prepayment penalty.

In the absence of prepayment costs, borrowers would be motivated to repay continuously whenever $x > 1$. To avoid this unrealistic prediction, we assume that prepayment is subject to a penalty k_p, so that borrowers will prepay only when the value of their mortgages exceed their origination value $M_l(1)$ by at least k_p. It is assumed that prepayment is subject to penalty k_p regardless of when the prepayment occurs. In reality prepayment penalties apply only during the first several years of the mortgage. Our specification has the advantage that it makes possible application of the stationary setting adopted in earlier sections.

Define x^* as the level of x at which borrowers elect to prepay.[3] To derive an upper bound for k_p, note that if $k_p > P(1) - M_l(1)$, the amount the borrower would have to prepay exceeds the value of the collateral. In that case the borrower will never prepay ($x^* = \infty$). Thus in the absence of prepayment costs we have $1 < x^* < \infty$ only when $0 < k_p < P(1) - M_l(1)$. We assume that these inequalities are satisfied.

As shown above, the differential equation defining $E(x)$ has the solution

$$E(x) = e_1 x^{m_1} + e_2 x^{m_2} + \frac{x}{\rho - \alpha} - \frac{c}{\rho}. \tag{13}$$

Here, as opposed to the derivation of $E(x)$ when no prepayment was allowed, we allow both constants of integration e_1 and e_2 to be nonzero. The roots m_1 and m_2 are given by

$$m_1, m_2 = \frac{-(\alpha - \frac{1}{2}\sigma^2) \pm \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\rho}}{\sigma^2}. \tag{14}$$

The boundary conditions are the value-matching conditions

$$E(x_*) = 0 \tag{15}$$

and

$$E(x^*) = \frac{x^* - 1}{\rho - \alpha} + E(1) - k_p. \tag{16}$$

[3]There is a conceptual problem in modeling prepayment in the presence of default costs. If the prepayment penalty is less than the deadweight loss due to default costs at $x = 1$, then we will also have $x^* = 1$. To see this, suppose that $k_p < M_b(1) - M_l(1)$. Then by immediately prepaying their mortgages, borrowers can replace the liability $M_b(1)$ with the cost $k_p + M_l(1)$, which is lower. The bizarre implication that borrowers would prepay a mortgage that they had just incurred reflects the exogenous specification that homeowners borrow despite the deadweight costs of doing so (when k_b or k_l exceeds zero), but are free to terminate the loan at any time upon payment of the penalty k_p (see the discussion in note 2). To ensure that $x^* > 1$ it is sufficient to consider only values of k_p that exceed $M_b(1) - M_l(1)$.

We sidestep this problem by modeling prepayment only when there are no default costs.

The smooth pasting conditions are

$$E'(x_*) = 0 \text{ and } E'(x^*) = \frac{1}{\rho - \alpha}. \tag{17}$$

Intuitively it is clear that higher values of k_p induce higher values of x^*. The default point x_* is lower if prepayment is permitted than if it is not. To see this, suppose that x equals the default point when there is no prepayment option. Then the borrower is indifferent between defaulting and continuing to pay the mortgage. Now suppose that prepayment is permitted. Existence of this option decreases the value of the mortgage as a liability to the borrower, or as an asset to the lender. Therefore the existence of the prepayment option decreases the default point. Lower prepayment penalties increase the value of the option and therefore decrease the default point.

Table 1 shows the values of x that result in default and prepayment for a range of values of c and k_p.

Table 1: x_* and x^* for different values of k_p and c

	No prepayment		$k_p = 1$		$k_p = 2$		$k_p = 3$		$k_p = 4$	
c	x_*	x^*	x_*	x^*	x_*	x^*	x_*	x^*	x_*	x^*
1	0.3893	∞	0.3893	∞	0.3893	∞	0.3893	∞	0.3893	∞
1.5	0.5839	∞	0.5777	1.7708	0.5838	4.8683	0.5839	∞	0.5839	∞
2	0.7786	∞	0.7241	1.3592	0.7487	1.6421	0.7641	2.0272	0.7736	2.7212
2.5	0.9732	∞	0.8182	1.2449	0.8542	1.3893	0.8813	1.5310	0.9033	1.6838
3	1.1679	∞	0.8759	1.1924	0.9161	1.2931	0.9476	1.3825	0.9745	1.4685

As noted above, existence of the prepayment option makes the mortgage more attractive to the borrower and less attractive to the lender. The higher the value of the prepayment option, the higher must be the yield on the mortgage to compensate the lender for giving up the possibility of large capital gains if the house value increases. Figure 11 shows the effect of various prepayment penalties on the tradeoff between the loan-to-value ratio and the mortgage yield. This figure shows that for low loan-to-value ratios the prepayment option has little or no effect on the mortgage yield. This is so because the default option has little value on a low loan-to-value mortgage, implying that the increase in the value of the mortgage if the property appreciates is similarly low. Therefore even a low prepayment penalty will be sufficient to deter prepayment. In contrast, with high loan-to-value mortgages even a small increase in house values leads to a sizeable increase in the mortgage value, motivating prepayment even in the presence of high prepayment costs. (This may explain the very high prepayment speeds of subprime mortgages in the rising real estate markets prior to 2006).

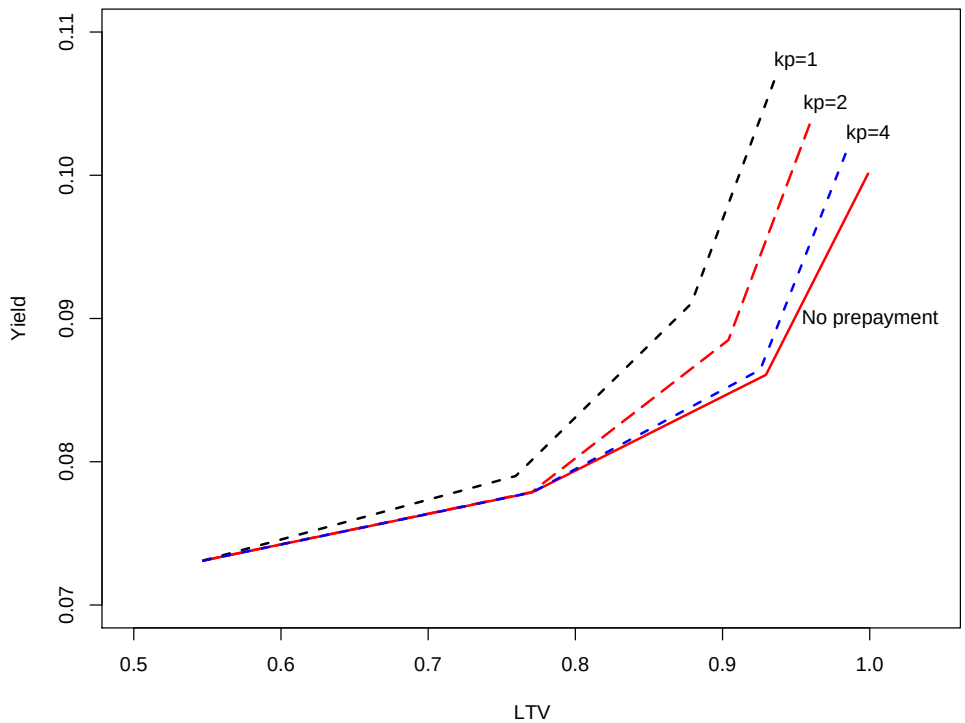

Figure 11: The effect of prepayment penalty on the tradeoff between the LTV and yield

2 Empirical Results

To be supplied.

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