S. Supplementary Materials

This appendix contains proofs, theoretical results, and detailed derivations omitted from the text. Equations that are not presented here are in the paper.

S.1. More Proofs

Proof of Lemma 3. For simplicity, both state and agent indices will be kept implicit here. Without loss of generality, let $R_s = 1$ for all s. Consider two different allocations $\left(k, \hat{\theta}, \theta\right)$ and $\left(k', \hat{\theta}', \theta'\right)$ with two different market fundamentals z, z', respectively. The collateral constraints (4) for an agent h in state s with these two allocations be binding:

$$p(z)k + \hat{\theta} + p(z)\theta = 0 \Longrightarrow \hat{\theta} = -p(z)\left(k + \theta\right) \tag{81}$$

$$p(z')k' + \hat{\theta}' + p(z')\theta' = 0 \Longrightarrow \hat{\theta}' = -p(z')\left(k' + \theta'\right) \tag{82}$$

Since we are looking for a counter example, we can pick these two allocations to satisfy

$$\hat{\theta} = \hat{\theta}' < 0 \Longrightarrow p(z)\left(k + \theta\right) = p(z')\left(k' + \theta'\right) > 0 \tag{83}$$

The positivity of the prices implies that $k + \theta > 0$ and $k' + \theta' > 0$.

Now consider a convex combination allocation: $k^\lambda = \lambda k + (1-\lambda)\, k'$, $\hat{\theta}^\lambda = \lambda\hat{\theta} + (1-\lambda)\,\hat{\theta}'$, $\theta^\lambda = \lambda\theta + (1-\lambda)\,\theta'$, $\mathbf{c}^\lambda = \lambda\mathbf{c} + (1-\lambda)\,\mathbf{c}'$, and $z^\lambda = z(\mathbf{c}^\lambda)$, where $0 < \lambda < 1$. Using equations (81)-(82), we can write

$$p(z^\lambda)k^\lambda + \hat{\theta}^\lambda + p(z^\lambda)\theta^\lambda = \left(\frac{k+\theta}{p(z')}\right)\left[\lambda p(z^\lambda)\left(p(z') - p(z)\right) + p(z)\left(p(z^\lambda) - p(z')\right)\right]$$

There is no loss of generality to assume that $p(z) < p(z^\lambda) < p(z')$. Then, pick λ that is smaller than λ^*:

$$\lambda^* = \left(\frac{p(z') - p(z^\lambda)}{p(z') - p(z)}\right)\left(\frac{p(z)}{p(z^\lambda)}\right) \tag{84}$$

Using the condition that $p(z) < p(z^\lambda) < p(z')$, we can show that $0 < \lambda^* < 1$. This condition implies that we can pick $0 < \lambda < \lambda^* < 1$ such that

$$\lambda p(z^\lambda)\left(p(z') - p(z)\right) + p(z)\left(p(z^\lambda) - p(z')\right) < 0 \tag{85}$$

Using $k + \theta > 0$, this clearly violates the collateral constraint (4). Therefore, the attainable set is non-convex. *Q.E.D.*

Proof of Theorem 6.1. Let $(\mathbf{x}, \mathbf{y})$, and $\left(P_{20}, P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)\right)$ be a competitive equilibrium. Suppose the competitive equilibrium allocation is not Pareto optimal, i.e. there is an attainable allocation $\tilde{\mathbf{x}} \in X$ such that $\mathscr{U}\left(\tilde{\mathbf{x}}^h\right) \geq \mathscr{U}\left(\mathbf{x}^h\right)$ for all h and $\mathscr{U}\left(\tilde{\mathbf{x}}^{\tilde{h}}\right) > \mathscr{U}\left(\mathbf{x}^{\tilde{h}}\right)$ for some $\tilde{h}$. With a limited space, we use b as a typical bundle. With local nonsatiation of preferences, we have

$$\sum_b P\left(b\right) x^h\left(b\right) \leq \sum_b P\left(b\right) \tilde{x}^h\left(b\right)$$

for all h, and

$$\sum_b P(b)\, x^{\tilde{h}}(b) < \sum_b P(b)\, \tilde{x}^{\tilde{h}}(b)$$

for some $\tilde{h}$. Summing over all agents with weights $\left(\alpha^h\right)_h$, we have

$$\sum_{(b)} P(b) \sum_h \alpha^h x^h(b) < \sum_{(b)} P(b) \sum_h \alpha^h \tilde{x}^h(b) \tag{86}$$

Using the optimal condition (39) for the broker-dealer's profit maximization problem, we have

$$P(b)y(b) = \left(c_{10} + P_{20}c_{20} + P_{20}k + \widehat{P}_a(\mathbf{z}) \cdot \hat{\theta} + P_a(\mathbf{z}) \cdot \theta + P_\Delta(\mathbf{z}) \cdot \Delta\right) y(b) \tag{87}$$

Using the market-clearing condition for lotteries in period-0, (42), we can substitute $\sum_h \alpha^h x^h(b)$ for $y(b)$ for every bundle b on the left hand side. Then, summing over all bundles b gives

$$\sum_{b \in \mathscr{B}} P(b) \sum_h \alpha^h x^h(b) \quad = \quad \sum_h \alpha^h e_{10}^h + P_{20} \sum_h \alpha^h e_{20}^h \tag{88}$$

This equation is a result of the market-clearing conditions (35)-(37) and (40)-(41).

Similarly, given that $\sum_h \alpha^h \tilde{x}^h(b) \geq 0$, multiplying (39) by $\sum_h \alpha^h \tilde{x}^h(b)$ and then summing over b give

$$\sum_b P(b) \sum_h \alpha^h \tilde{x}^h(b) \quad \leq \quad \sum_h \alpha^h e_{10}^h + P_{20} \sum_h \alpha^h e_{20}^h \tag{89}$$

This equation is a result of the feasibility conditions (23)-(26) and (28).

Using (88) and (89), (86) can be rewritten as

$$\sum_h \alpha^h e_{10}^h + P_{20} \sum_h \alpha^h e_{20}^h < \sum_h \alpha^h e_{10}^h + P_{20} \sum_h \alpha^h e_{20}^h$$

It is clear that the left-hand side is exactly equal to the right-hand side of the inequality. This is a contradiction! *Q.E.D.*

The compensated equilibrium is defined as follows.

Definition 8. A compensated equilibrium is specification of allocation $(\mathbf{x}, \mathbf{y})$, and prices P_{20}, $P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ such that

(i) for each h, $\mathbf{x}^h \in X^h$ solves

$$\min_{\hat{\mathbf{x}}^h} \sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \hat{x}^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \tag{90}$$

subject to

$$\mathscr{U}\left(\hat{\mathbf{x}}^h\right) \geq \mathscr{U}\left(\mathbf{x}^h\right) \tag{91}$$

taking prices as given,

(ii) for the broker-dealer, $\left\{\mathbf{y}, \widehat{P}_a(\mathbf{z}), P_a(\mathbf{z}), P_\Delta(\mathbf{z})\right\}$ solves (38), taking prices as given,

(iii) in period-0, markets for good-1, good-2 and lotteries clear, i.e. (40)-(42) hold,

Note that the sole difference between the compensated equilibrium and the competitive equilibrium is the specification of consumer's problem (90).

Theorem S.1. *Any solution to the Pareto program with Pareto weight $\lambda^h \geq 0, \forall h$ can be supported as a compensated equilibrium.*

Proof. Given that the optimization problems are well-defined concave problems, Kuhn-Tucker conditions are necessary and sufficient. The proof are divided into three steps

(i) Kuhn-Tucker conditions for Pareto Optimal allocations: We will first characterize solutions to the Pareto program using Kuhn-Tucker conditions. Let $\widetilde{P}_{10}, \widetilde{P}_{20}$ be the dual variables on the resource constraints of good-1 and good-2 in period-0 (23)-(24), respectively, and $\alpha^h \widetilde{P}_l^h$ be the dual variable on the probability constraint (19). Let $\widetilde{\widehat{P}}_a(\mathbf{z}) = \left(\tilde{\widehat{P}}_a(z_s, s)\right)_{s=1}^S$, $\widetilde{P}_a(\mathbf{z}) = \left(\widetilde{P}_a(z_s, s)\right)_{s=1}^S$, and $\widetilde{P}_\Delta(\mathbf{z}) = \left(\widetilde{P}_\Delta(z_s, s)\right)_{s=1}^S$ be the dual variables on the resource constraints for contracts paying in good-1 (25), those for contracts paying in good-2 (26), and those for the consistency constraints (28), respectively. All non-negativity constraints are kept implicit for brevity. A solution to the Pareto program satisfies the following condition for $x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$

$$\begin{aligned} \lambda^h\left[U\left(c_{10}, c_{20}\right) + \beta V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right)\right] &\leq \widetilde{P}_{10}c_{10} + \widetilde{P}_{20}c_{20} + \widetilde{P}_{20}k + \widetilde{\widehat{P}}_a(\mathbf{z}) \cdot \hat{\theta} \\ &+ \widetilde{P}_a(\mathbf{z}) \cdot \theta + \widetilde{P}_\Delta(\mathbf{z}) \cdot \Delta + \widetilde{P}_l^h \end{aligned} \tag{92}$$

where the inequality holds with equality if $x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) > 0$. For any bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ with $x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) > 0$ for some h, we can show that

$$\frac{\widetilde{P}_a(z_s, s)}{\widetilde{\widehat{P}}_a(z_s, s)} = \frac{U_{2s}^h}{U_{1s}^h} = p(z_s) \tag{93}$$

This result is derived using a variational principle with respect to $\hat{\theta}_s$ and θ_s, and using the fact that the agent can trade in spot markets at price $p(z_s)$, which implies that $\frac{U_{2s}^h}{U_{1s}^h} = p(z_s)$. This result in fact it is the counterpart of the result in Lemma 7.

(ii) Kuhn-Tucker conditions for equilibrium allocations: We will characterize solutions to the consumers' and intermediary's problems in equilibrium using Kuhn-Tucker conditions. Let $\gamma^h(0)$ and $\gamma^h(l)$ be the Lagrange multiplier for constraint (91), and for the probability constraint (19), respectively. All non-negativity constraints terms are kept implicit for brevity. The optimal condition for $x^h\left(c, k, \hat{\theta}, \theta, \mathbf{z}\right)$ is given by

$$\gamma^h(0)\left[U\left(c_{10}, c_{20}\right) + \beta V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right)\right] \leq P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) + \gamma^h(l) \tag{94}$$

where the inequality holds with equality if $x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) > 0$. Recall that the optimal condition of the broker-dealer's profit maximization problem (38), for each bundle $b = \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$, is

$$P(b) \leq c_{10} + P_{20}c_{20} + P_{20}k + \widehat{P}_a(\mathbf{z}) \cdot \hat{\theta} + P_a(\mathbf{z}) \cdot \theta + +P_\Delta(\mathbf{z}) \cdot \Delta \tag{95}$$

where the condition holds with equality if $y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) > 0$.

(iii) Matching dual variables and prices: we will show that the optimal conditions of the Pareto program are equivalent to the optimal conditions of consumers' and broker-dealer's problems. Recall that good-1 is the numeraire. To match let $\gamma^h(0) = \frac{\lambda^h}{\widetilde{P}_{10}}\ \forall h$, $P_{20} = \frac{\widetilde{P}_{20}}{\widetilde{P}_{10}}$, $\widehat{P}_a(\mathbf{z}) = \frac{\widetilde{\widehat{P}}_a(\mathbf{z})}{\widetilde{P}_{10}}\ \forall \mathbf{z}$, $P_a(\mathbf{z}) = \frac{\widetilde{P}_a(\mathbf{z})}{\widetilde{P}_{10}}\ \forall \mathbf{z}$, and $\gamma^h(l) = \frac{\widetilde{P}_l^h}{\widetilde{P}_{10}}\ \forall h$.
Using the matching conditions specified above, the optimal condition for the constrained optimality (92) becomes

$$\begin{aligned} \gamma^h(0)\left[U\left(c_{10}, c_{20}\right) + \beta V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right)\right] &\leq c_{10} + P_{20}c_{20} + P_{20}k + \widehat{P}_a(\mathbf{z}) \cdot \hat{\theta} \\ &+ P_a(\mathbf{z}) \cdot \theta + P_\Delta(\mathbf{z}) \cdot \Delta + \gamma^h(l) \end{aligned} \tag{96}$$

On the other hand, using (95), the optimal condition for the equilibrium (94) becomes

$$\begin{aligned} \gamma^h(0)\left[U\left(c_{10}, c_{20}\right) + \beta V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right)\right] &\leq c_{10} + P_{20}c_{20} + P_{20}k + \widehat{P}_a(\mathbf{z}) \cdot \hat{\theta} \\ &+ P_a(\mathbf{z}) \cdot \theta + P_\Delta(\mathbf{z}) \cdot \Delta + \gamma^h(l) \end{aligned} \tag{97}$$

which is exactly the same as (96). This shows that a solution to the Pareto program also solves the consumer's and broker-dealer's problems.

Recall that the resource constraints in the Pareto program are identical to the market-clearing conditions in equilibrium. In addition, the consistency constraints in the Pareto program are equivalent to the ones in the broker-dealer's problem. Hence, we have shown that any Pareto optimal allocation is also a compensated equilibrium allocation.

Q.E.D.

The competitive equilibrium with transfers is defined analogously to the competitive equilibrium, but the consumer's problem is modified. Specifically, every agent h will receive feasible redistributed wealth $w^h \in \mathbb{R}$. In addition, a feasible redistributed-wealth allocation $\left[w^h\right]_h$ satisfies

$$\sum_h \alpha^h w^h = \sum_h \alpha^h \left[e_{10}^h + P_{20}e_{20}^h\right] \tag{98}$$

Given redistributed wealth w^h, each agent h solves

$$\max_{\mathbf{x}^h} \sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)\left\{U\left(c_{10}, c_{20}\right) + \beta V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right)\right\} \tag{99}$$

subject to $\mathbf{x}^h \in X^h$, and budget constraint (with transfers)

$$\sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta)} P\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) \leq w^h \tag{100}$$

Definition 9. A competitive equilibrium with transfers is specification of allocation $(\mathbf{x},\mathbf{y})$, and prices $P_{20}, P\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)$ and is a feasible redistributed-wealth allocation in period-0, w^h for every h satisfying (98), such that

(i) for each h, $\mathbf{x}^h \in X^h$ solves (99) subject to (100), taking prices as given,

(ii) for the broker-dealer, $\left\{\mathbf{y}, \widehat{P}_a(\mathbf{z}), P_a(\mathbf{z}), P_\Delta(\mathbf{z})\right\}$ solves (38), taking prices as given,

(iii) in period-0, markets for good-1, good-2 and lotteries clear, i.e., (40)-(42) hold,

Proof of Theorem 6.2. Let $\mathbf{x} = \left(\mathbf{x}^h\right)_h$ be a Pareto optimal allocation. According to Theorem S.1, any Pareto optimal allocation can be supported as a compensated equilibrium. We only need to show that any compensated equilibrium, corresponding to $\lambda > 0$, is a competitive equilibrium with transfers. In particular, we will use a cheaper-point argument to show that the expenditure minimization (90) is equivalent to the utility maximization (99). In order to do so, we shall show that there exists an allocation $\widehat{\mathbf{x}}^h \in X^h$ that costs less than $\mathbf{x}^h$, for every agent h.

Let $(\mathbf{x},\mathbf{y})$ be a compensated equilibrium allocation. Redistributed wealth for an agent h is defined by $w^h = \sum_b P(b)x^h(b)$. Using (87), (35)-(37), (40)-(42), with some algebra, we can show that

$$\sum_h \alpha^h w^h = \sum_{b\in\mathscr{B}} y(b)\left[c_{10} + P_{20}\left(c_{20}+k\right)\right] = \sum_h \alpha^h\left[e^h_{10} + P_{20}e^h_{20}\right]$$

This result shows that the wealth allocation $\left(w^h\right)_h$ is feasible, i.e., satisfying condition (98).

With $\lambda^h > 0$, for every h, an Inada condition ($\lim_{c\to 0} U^h_i(c) = \infty$ for $i = 1,2$) guarantees that a solution to the Pareto program, which is a compensated equilibrium allocation, will not have a strictly positive mass on $c = 0$.

We will then show that there is an alternative allocation, $\hat{\mathbf{x}}$, with $\mathbf{c}_0 = 0$ that is cheaper than the compensated equilibrium allocation, $\mathbf{x}$. In particular, let $0 \in C$; that is, the zero consumption allocation in period-0 is on the grid. Consider an alternative allocation for an agent h, $\widehat{\mathbf{x}}^h$, such that

$$\hat{x}^h\left(0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) = \sum_{\mathbf{c}_0} x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) \tag{101}$$

$$\hat{x}^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) = 0, \quad \text{for any } \mathbf{c}_0 \neq 0 \tag{102}$$

Note that the alternative allocation put strictly positive masses on bundles with $c = 0$. The strictly increasing utility function implies that $P_{20} > 0$. Consequently, the optimal condition of the broker-dealer (39) implies that, for a given bundle $\left(k,\hat{\theta},\theta,\mathbf{z},\Delta\right)$,

$$P\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) > P\left(0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right), \quad \text{for any } c \neq 0 \tag{103}$$

We now would like to compare the equilibrium values of $\mathbf{x}^h$ and $\widehat{\mathbf{x}}^h$.

$$\begin{aligned}
\sum_b P(b)\, x^h(b) &> \sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta)} P\left(0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) \\
&= \sum_{(k,\hat{\theta},\theta,\mathbf{z},\Delta)} P\left(0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) \hat{x}^h\left(0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) \\
&= \sum_{(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta)} P\left(0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) \hat{x}^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)
\end{aligned}$$

where the first inequality follows (103), the first equality results from (101), and the last equality follows (102).

This shows that there exists an allocation $\widehat{\mathbf{x}}^h$ that is cheaper than the compensated equilibrium allocation, $\mathbf{x}^h$, for every agent h. As a result, using the cheaper-point argument, a compensated equilibrium is a competitive equilibrium with transfers. *Q.E.D.*

Proof of Theorem 6.3. For notational convenience, we redefine the grid to include the endowment profiles; the period-0 endowment of an agent h is given by

$$\begin{aligned}
\mathbf{e}^h(b) &= 1, \quad \text{for } b = \left(\mathbf{e}_0^h, k=0, \hat{\theta}=0, \theta=0, \mathbf{z}=0, \Delta=0\right) \\
&= 0, \quad \text{otherwise}
\end{aligned}$$

In addition, the optimal condition of the broker-dealer (39) implies that the price of bundle $\left(\mathbf{e}_0^h, k=0, \hat{\theta}=0, \theta=0, \mathbf{z}=0, \Delta=0\right)$ is $P\left(\mathbf{e}_0^h, 0,0,0,0,0\right) = e_{10}^h + P_{20} e_{20}^h$. Therefore, the total value of period-0 endowment lottery of an agent h, $\mathbf{e}^h$, is given by

$$\sum_b P(b)\, \mathbf{e}^h(b) = P\left(\mathbf{e}_0^h, 0,0,0,0,0\right) = e_{10}^h + P_{20} e_{20}^h \tag{104}$$

which is exactly income in the budget constraint (31).

Let $\mathbf{P} = (P(b))_{b\in\mathscr{B}}$ be the prices of all bundles. In addition, we also add the price of good-2 in period-0, P_{20} into the price space as $P_{20} = P(c=(0,1),0,0,0,0,0)$. In other words, P_{20} is embedded in $\mathbf{P}$. As in Prescott and Townsend (2005), with the possibility of negative prices, we restrict prices $\mathbf{P}$ to the closed unit ball;

$$D = \left\{\mathbf{P} \in \mathbb{R}^n | \sqrt{\mathbf{P}\cdot\mathbf{P}} \leq 1\right\} \tag{105}$$

Note that the set D is compact and convex.

Consider the following mapping $(\lambda, \mathbf{x}, \mathbf{P}) \to (\lambda', \mathbf{x}', \mathbf{P}')$, where $\lambda, \lambda' \in S^{H-1}$, $\mathbf{x}^h \in X^h$. Recall that the consumption possibility set X^h is non-empty, convex, and compact. Let $\overline{X}$ be the cross-product over h of X^h: $\overline{X} = X^1 \times \ldots \times X^H$.

The first part of the mapping is given by $\lambda \longrightarrow (\mathbf{x}', \mathbf{P}')$, where $\mathbf{x}'$ is the solution to the Pareto program given the Pareto weight λ, and $\mathbf{P}'$ is the renormalized prices. With the second welfare theorem, the solution to the Pareto program for a given Pareto weight λ also gives us (compensated) equilibrium prices $\mathbf{P}^*$, where $P^*(c,k,\hat{\theta},\theta,\mathbf{z},\Delta) = \widetilde{P}_{10} c_{10} + \widetilde{P}_{20} c_{20} +$

$\widetilde{P}_{20}k + \widetilde{\widehat{P}}_a(\mathbf{z}) \cdot \hat{\theta} + \widetilde{P}_a(\mathbf{z}) \cdot \theta + \widetilde{P}_\Delta(\mathbf{z}) \cdot \Delta$. The nonlocal satiation of preferences implies that $\mathbf{P}^* \neq 0$. The normalized prices are given by

$$\mathbf{P}' = \frac{\mathbf{P}^*}{\mathbf{P}^* \cdot \mathbf{P}^*}$$

where "$\cdot$" is the inner product operator. Note that $\mathbf{P}' \cdot \mathbf{P}' = 1$. In order to preserve the convexity of the mapping while prices in the unit ball D, we define the convex hull of the normalized prices. Let $\widetilde{D}$ be the sets of all normalized prices, and accordingly $co\widetilde{D}$ be its convex hull. Since $\mathbf{P}' \in \widetilde{D}$, $\mathbf{P}' \in co\widetilde{D}$, which is compact and convex. Note that extending $\widetilde{D}$ to its convex hull does not add any new relative prices. It is not too difficult to show that this mapping, $\lambda \longrightarrow (\mathbf{x}', \mathbf{P}')$, is non-empty, compact-valued, convex-valued. By the Maximum theorem, it is upper hemi-continuous. In addition, the upper hemi-continuity is preserved under the convex-hull operation.

The second part of the mapping is given by $(\lambda, \mathbf{x}, \mathbf{P}) \longrightarrow \lambda'$. The new weight can be formed as follows:

$$\hat{\lambda}^h = \max\left\{0, \lambda^h + \frac{\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right)}{A}\right\} \tag{106}$$

$$\lambda'^h = \frac{\hat{\lambda}^h}{\sum_h \hat{\lambda}^h} \tag{107}$$

where A is a positive number such that $\sum_h \left|\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right)\right| \leq A$. It is clear that this mapping is also non-empty, compact-valued, convex-valued, and upper hemi-continuous. In conclusion, $(\lambda, \mathbf{x}, \mathbf{P}) \to (\lambda', \mathbf{x}', \mathbf{P}')$ is a mapping from $S^{H-1} \times \overline{X} \times S^{n-1} \longrightarrow S^{H-1} \times \overline{X} \times S^{n+1}$. Since each set is non-empty, compact, and convex, so does its cross-product. In addition, the overall mapping is non-empty, compact-valued, convex-valued, and upper hemi-continuous since these properties are preserved under the cross product operation. By Kakutani's fixed point theorem, there exists a fixed point $(\lambda, \mathbf{x}, \mathbf{P})$.

Proved in Theorem S.1, any Pareto optimal allocation can be supported as a compensated equilibrium. In addition, the strictly increasing utility function implies that $P_{20} > 0$. Hence, with positive endowments, an agent h's wealth at the fixed point is strictly positive;

$$w^h = \mathbf{P} \cdot \mathbf{e}^h = e^h_{10} + P_{20} e^h_{20} > 0$$

With strictly positive wealth, a compensated equilibrium is a competitive equilibrium with transfers (using a cheaper-point argument as in the proof of Theorem 6.2).

We now need to show that the budget constraint (without transfers) holds for every agent h; $\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) = 0$. Using (42), (39), (104), (35)-(36), (40)-(42), (37) and some algebra, $\sum_h \alpha^h \mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right)$ is

$$\sum_h \alpha^h \mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) = 0 \tag{108}$$

In addition, at a fixed point $\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right)$ must be the same sign for every h. Hence, $\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) = 0$ for every agent h. This clearly confirms that the budget constraint

(without transfers) of every agent h holds. Hence, a competitive equilibrium (without transfers) exists. Note that supply of the broker-dealer, $\mathbf{y}$, can be recovered from $\mathbf{x}$, and similarly other prices, $\left(P_{20}, \widehat{P}_a(\mathbf{z}), P_a(\mathbf{z}), P_\Delta(\mathbf{z})\right)$, can be inferred from the dual variables from the Pareto program at the fixed point. *Q.E.D.*

Proof of Lemma 5. Let $\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h, \hat{\tau}^h, \tau^h\right)_h$ be an attainable allocation. We will show that we can find an equivalent allocation with no spot trade, i.e., $\hat{\tau}_s'^h = \tau_s'^h = 0$. Consider the following candidate allocation (with $'$)

$$\begin{aligned} \mathbf{c}_0'^h &= \mathbf{c}_0^h, \ \forall h && (109)\\ \hat{\theta}_s'^h &= \hat{\theta}_s^h + \hat{\tau}_s^h, \ \forall s, h && (110)\\ \theta_s'^h &= \theta_s^h + \tau_s^h, \ \forall s, h && (111) \end{aligned}$$

Note that agents here acquire or issue securities on good 1 and good 2 in state s rather than waiting for trade in spot markets. The rest of the proof is similar to the proof of Lemma 4, and hence is omitted (it is available in our Working Paper version). *Q.E.D.*

Proof of Lemma 6. Suppose $\left(\mathbf{c}_0^h, k^h, \hat{\psi}_s^h, \psi_s^h, \hat{\sigma}_s^h, \sigma_s^h, \hat{\nu}_s^h, \nu_s^h, \hat{\tau}^h, \tau^h\right)_h$ is attainable. Consider the following alternative allocation (with $'$) $\left(\mathbf{c}_0^h, k^h, \hat{\psi}_s'^h, \psi_s'^h, \hat{\sigma}_s'^h, \sigma_s'^h, \hat{\nu}_s'^h, \nu_s'^h, \hat{\tau}'^h, \tau'^h\right)_h$ such that

$$\hat{\sigma}_s'^h = \sigma_s'^h = \hat{\nu}_s'^h = \nu_s'^h = 0, \psi_s^h = 0, \ \forall h, s \tag{112}$$

$$\begin{aligned} \hat{\psi}_s'^h &= \left(\hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h\right) + p(z_s)\left(\psi_s^h + \sigma_s^h + \nu_s^h\right) && (113)\\ \hat{\tau}_s'^h &= -p(z_s)\left(\psi_s^h + \sigma_s^h + \nu_s^h\right) + \hat{\tau}_s^h && (114)\\ \tau_s'^h &= \left(\psi_s^h + \sigma_s^h + \nu_s^h\right) + \tau_s^h && (115) \end{aligned}$$

Note that at the alternative allocation, agents will do in spot markets what they might have done in asset-backed security markets. In addition, with active spot markets, there is no need to trade in collateral-backed securities paying in good-2 (trade in the ones paying in numeraire good only). The rest of the proof is similar to the proof of Lemma 5, and hence is omitted (it is available in our Working Paper version). *Q.E.D.*

Proof of Lemma 8. Consider two bundles, $\left(\mathbf{c}_0', k', \hat{\theta}', \theta', \mathbf{z}', \Delta'\right)$ and $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$, that lead to the same consumption allocation for an agent h (conditions (i)-(iv) in Lemma 11 hold). To get rid of an arbitrage opportunity, their prices must be the same:

$$P\left(\mathbf{c}_0', k', \hat{\theta}', \theta', \mathbf{z}', \Delta'\right) - P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) = 0 \tag{116}$$

Using the profit maximization condition of a broker-dealer (39), the above equation becomes

$$P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) = c_{10} + P_{20}c_{20} + P_{20}k + \widehat{P}_a(\mathbf{z}) \cdot \hat{\theta} + P_a(\mathbf{z}) \cdot \theta + P_\Delta(\mathbf{z}) \cdot \Delta \tag{117}$$

Substituting (117) into (116) gives

$$\begin{aligned}
0 &= P\left(\mathbf{c}_0', k', \hat{\theta}', \theta', \mathbf{z}', \Delta'\right) - P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \\
&= c_{10}' + P_{20}c_{20}' + P_{20}k' + \widehat{P}_a(\mathbf{z}') \cdot \hat{\theta}' + P_a(\mathbf{z}') \cdot \theta' + P_\Delta(\mathbf{z}') \cdot \Delta' \\
&\quad - \left[c_{10} + P_{20}c_{20} + P_{20}k + \widehat{P}_a(\mathbf{z}) \cdot \hat{\theta} + P_a(\mathbf{z}) \cdot \theta + P_\Delta(\mathbf{z}) \cdot \Delta\right] \\
&= P_{20}\left(k' - k\right) + P_a(\mathbf{z}) \cdot \left(\theta' - \theta\right) + \sum_s P_\Delta(z_s, s)\left[\left(z_s e_{2s}^h + z_s R_s k' - e_{1s}^h\right) - \left(z_s e_{2s}^h + z_s R_s k - e_{1s}^h\right)\right] \\
&= \left(k' - k\right) P_{20} - \left(k' - k\right) \sum_s P_a(z_s, s) R_s + \left(k' - k\right) \sum_s P_\Delta(z_s, s) R_s z_s \\
&= \left(k' - k\right)\left[P_{20} - \sum_s P_a(z_s, s) R_s + \sum_s P_\Delta(z_s, s) R_s z_s\right]
\end{aligned}$$

where the third equality follows from conditions (i)-(iii) in Lemma 11 and the definition of Δ in (18), and the fourth one follows from $\theta_s' - \theta_s = -R_s\left(k' - k\right)$ (condition (iv) in Lemma 11). This must be true even if $k \neq k'$. As a result, the term in the bracket must be zero. *Q.E.D.*

The following lemma will be used to prove Theorem 3.1.

Lemma 10. *A collateral equilibrium is constrained optimal if and only if all collateral constraints are not binding, i.e.* $\gamma_{cc-s}^h = \mu_{cc-s}^h = 0$ *for all* h *and all* s.

Proof. The proof is based on first-order conditions for Pareto program without lotteries and the ones for a collateral equilibrium. Note that the resource constraints in program 1 and the market-clearing constraints in equilibrium are clearly equivalent. In addition, the collateral constraints are the same in both problems as well. Hence, we only need to match all first-order conditions from both problems. In addition, with limited space, we will focus only on the conditions that lead to an externality. The detailed proof is available in our Working Paper version.

Optimal Conditions for the Pareto Program (12)

Let μ-s be the Lagrange multipliers for the participation, feasibility, and collateral constraints. Combining the first-order conditions with respect to c_{10}^h, k^h, and the complementarity slackness conditions for the collateral constraints gives:

$$\frac{U_{20}^h}{U_{10}^h} = \sum_s \pi_s \beta \frac{U_{2s}^h}{U_{10}^h} R_s \quad + \quad \sum_s \frac{\mu_{cc-s}^h}{\mu_{\tilde{u}}^h U_{10}^h} p(z_s) R_s - \sum_s \frac{\alpha^h}{\mu_{\tilde{u}}^h U_{10}^h} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \hat{\theta}_s^{\tilde{h}} \quad (118)$$

Note that (118) is exactly the same as (14).

Optimal Conditions for a Collateral Equilibrium

Let γ-s be the Lagrange multipliers for the budget constraint, and the collateral constraints. Combining the first-order conditions with respect to c_{10}^h, k^h, and the complementarity slackness conditions for the collateral constraints gives:

$$\frac{U_{20}^h}{U_{10}^h} = \sum_s \pi_s \frac{\beta U_{2s}^h}{U_{10}^h} R_s + \sum_s \frac{\gamma_{cc-s}^h}{U_{10}^h} p(z_s) R_s \quad (119)$$

($\Longleftarrow$) Suppose that $\gamma^h_{cc-s} = \mu^h_{cc-s} = 0$ for all h and all s. We then can show that any collateral equilibrium allocation will also solve the Pareto program by matching all necessary and sufficient conditions. In particular, we can pick $\frac{\mu_{20}}{\mu_{10}} = P_{20}, \frac{\mu_{\hat{\theta}s}}{\mu_{10}} = \widehat{P}_{as}, \frac{\mu_{\theta s}}{\mu_{10}} = P_{as}$, and $\gamma^h_{cc-s} = \frac{\mu^h_{cc-s}}{\mu^h_{\bar{u}}} = 0$. In conclusion, any collateral equilibrium allocation is constrained optimal if $\gamma^h_{cc-s} = \mu^h_{cc-s} = 0$ for all h and all s.

($\Longrightarrow$) Suppose that a collateral equilibrium allocation is constrained optimal, i.e. solves the Pareto program. Hence, it must satisfy (118). Using the same matching conditions, this will be true only if the last terms in (118) is zero. We will prove this by contradiction. Suppose that there are some h with $\mu^h_{cc-s} \neq 0$, and the last terms in (118) is zero:

$$\frac{\alpha^h}{\mu^h_{\bar{u}} U^h_{10}} \sum_s \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \left(\sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \hat{\theta}^{\tilde{h}}_s \right) = 0 \tag{120}$$

This must be true for all h and $\tilde{h}$. We will fist argue that $\sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \hat{\theta}^{\tilde{h}}_s$ has the same sign for every state s. Using the first-order conditions for the Pareto program with respect to $c^h_{10}, \hat{\theta}^h_s$, and the resource constraint for $\hat{\theta}^h_s$, we can show that $\sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \hat{\theta}^{\tilde{h}}_s = -\pi_s \beta \mu_{10} \sum_{\tilde{h}} \frac{U^{\tilde{h}}_{1s}}{U^{\tilde{h}}_{10}} \alpha^{\tilde{h}} \hat{\theta}^{\tilde{h}}_s$. The optimality requires that an agent with a larger IMRS, $\frac{U^{\tilde{h}}_{1s}}{U^{\tilde{h}}_{10}}$, will hold positive $\hat{\theta}^{\tilde{h}}_s$ and vice versa. This implies that the positive term of $\alpha^{\tilde{h}} \hat{\theta}^{\tilde{h}}_s$ will be weighted more than the negative one. Combining with the market-clearing condition $\sum_{\tilde{h}} \alpha^{\tilde{h}} \hat{\theta}^{\tilde{h}}_s = 0$, we can conclude that $-\pi_s \beta \mu_{10} \sum_{\tilde{h}} \frac{U^{\tilde{h}}_{1s}}{U^{\tilde{h}}_{10}} \alpha^{\tilde{h}} \hat{\theta}^{\tilde{h}}_s \leq 0, \ \forall s$. With homothetic preferences, $\frac{1}{\mu^h_{\bar{u}} U^h_{10}} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} < 0$. As a result,

$$-\frac{\pi_s \beta \mu_{10} \alpha^h}{\mu^h_{\bar{u}} U^h_{10}} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \frac{U^{\tilde{h}}_{1s}}{U^{\tilde{h}}_{10}} \alpha^{\tilde{h}} \hat{\theta}^{\tilde{h}}_s \geq 0, \ \forall s \tag{121}$$

As a result, (120) will hold only if

$$\sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \hat{\theta}^{\tilde{h}}_s = -\pi_s \beta \mu_{10} \sum_{\tilde{h}} \frac{U^{\tilde{h}}_{1s}}{U^{\tilde{h}}_{10}} \alpha^{\tilde{h}} \hat{\theta}^{\tilde{h}}_s = 0 \tag{122}$$

This condition implies that $\frac{U^{\tilde{h}}_{1s}}{U^{\tilde{h}}_{10}} = \frac{U^h_{1s}}{U^h_{10}}, \ \forall h, \tilde{h}, s$. Using the fact that $\frac{U^h_{2s}}{U^h_{1s}} = p(z_s)$ for all h, we can also show that $\frac{U^{\tilde{h}}_{1s}}{U^{\tilde{h}}_{1\hat{s}}} = \frac{U^h_{1s}}{U^h_{1\hat{s}}}, \ \forall h, \tilde{h}$. In words, the marginal rate of substitutions across times and states are equalized across agents. Under the assumption 1, these equalities are necessary and sufficient conditions for first-best optimality, which in turn implies that all collateral constraints are not binding, i.e. $\gamma^h_{cc-s} = \mu^h_{cc-s} = 0$ for all h and all s. Hence, we can conclude that a collateral equilibrium is constrained optimal only if all collateral constraints are not binding. *Q.E.D.*

Proof of Theorem 3.1. The proof of the Theorem is by contrapositive. Suppose a collateral equilibrium is constrained optimal. Lemma 10 implies that a necessary condition for a

collateral equilibrium to be constrained optimal is that all collateral constraints are not binding. No binding collateral constraints implies first-best optimality. In short, we have shown that first-best optimality is a necessary condition for constrained optimality. Thus we can conclude that if a collateral equilibrium is not first-best optimal, then it is constrained suboptimal. *Q.E.D.*

S.2. An Indeterminacy of Collateral Allocations

Using a similar argument to proof of Lemma 5, we show that the collateral and securities paying in good-2 allocations are indeterminate; that is, neither k nor θ_s can be pinned down (but the net-claim of good-2, $R_s k + \theta_s$, will be uniquely determined). Roughly speaking, agents are indifferent between buying contracts ($\theta_s > 0$), and holding collateral ($k > 0$) and selling contracts against it ($\theta_s < 0$) as long as they lead to the same consumption allocation in period-1 over state s. Note that storage technology is linear and there is no direct utility per se from holding collateral. The formal result is summarized in the following lemma.

Lemma 11. *For a feasible bundle of an agent type h $\left(c_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$, any bundle $\left(c_0', k', \hat{\theta}', \theta', \mathbf{z}', \Delta'^h\right)$ such that (i) $c_0' = c_0$, (ii) $\mathbf{z}' = \mathbf{z}$, (iii) $\hat{\theta}' = \hat{\theta}$, and (iv) $R_s k' + \theta_s' = R_s k + \theta_s$, $\forall s$, is also be feasible for the agent h, and leads to the same consumption allocation as the original bundle.*

Proof. The proof is similar to the one of Lemma 5. *Q.E.D.*

Condition (iv) implies that there is some indeterminacy between k and θ_s. In particular, if we set $k' = 0$, then we can reach the same consumption allocation by setting the security position to be $\theta_s' = R_s k + \theta_s$. This implies that there is no loss of generality to assume that all collateral is held by an unconstrained type, and the others including constrained agents hold no collateral, $k = 0$, and therefore we will do so, unless stated otherwise.

Thus a constrained agent may hold no collateral and therefore issue no directly-collateralized securities, yet her collateral constraint is binding still. The fact that a constrained agent is effectively not borrowing at all in equilibrium seems counterintuitive at first. In a partial equilibrium setting when the price of collateral good is fixed exogenously, one would imagine that the agent will try to buy more of the collateral good and then borrow against to increase current consumption. In this *general equilibrium* setting where collateral price is determined endogenously, however, the price of the collateral good rises so in effect those transactions will offset each other and lead to a zero net transfer.

S.3. Collateral Constraints and Default

This section shows that contracts that do actually default does not relax the collateral constraints (4); that is, contracts that do default are not necessary. They may exist and get traded, but we can support an equivalent allocation without them. In particular, we now derive a collateral constraint with contracts that do default, and then show that the same net-payoff and same collateral constraint can be reached using no-default contracts. This is, in fact, a result of Lemma 2 but it is nice to be explicit, as the result seems counterintuitive.

Let $\widehat{C}$, C, $\widehat{C}^\sigma$, and C^σ be the collateral levels of defaulting contracts promising to pay a unit of good-1 with good-2 as collateral, promising to pay a unit of good-2 with good-2 as collateral, promising to pay a unit of good-1 with financial assets as collateral, and

promising to pay a unit of good-2 with financial assets as collateral, respectively. Note that, for expositional reasons, we assume that all contracts are contracts that do default. Accordingly, the payoffs of those contracts, which by construction with default, in state s are

$$\widehat{D}_s = \min\left(P_{2s}R_s\widehat{C}, 1\right) = P_{2s}R_s\widehat{C} \tag{123}$$

$$D_s = \min\left(R_sC, 1\right) = R_sC \tag{124}$$

$$\widehat{D}_s^\sigma = \min\left(P_{2s}\widehat{C}^\sigma, 1\right) = P_{2s}\widehat{C}^\sigma \tag{125}$$

$$D_s^\sigma = \min\left(\frac{C^\sigma}{P_{2s}}, 1\right) = \frac{C^\sigma}{P_{2s}} \tag{126}$$

The collateral requirement condition for contracts using physical good-2 as collateral is given by

$$k^h \geq -\widehat{C}\min\left(0, \hat{\psi}_s^h\right) - C\min\left(0, \psi_s^h\right)$$

Multiplying by $P_{2s}R_s$ both sides gives

$$\begin{aligned} P_{2s}R_sk^h &\geq -P_{2s}R_s\widehat{C}\min\left(0, \hat{\psi}_s^h\right) - P_{2s}R_sC\min\left(0, \psi_s^h\right) \\ &= -\widehat{D}_s\min\left(0, \hat{\psi}_s^h\right) - P_{2s}D_s\min\left(0, \psi_s^h\right) \end{aligned} \tag{127}$$

where the last equality follows from (123)-(124).

The collateral requirement condition regarding contracts paying in good-1 using purchased assets as collateral can be written as

$$D_s\max\left(0, \psi_s^h\right) + D_s^\sigma\max\left(0, \sigma_s^h\right) \geq -\widehat{C}^\sigma\min\left(0, \hat{\sigma}_s^h\right) \tag{128}$$

which can be rearranged as

$$\begin{aligned} P_{2s}D_s\max\left(0, \psi_s^h\right) + P_{2s}D_s^\sigma\max\left(0, \sigma_s^h\right) &\geq -P_{2s}\widehat{C}^\sigma\min\left(0, \hat{\sigma}_s^h\right) \\ &= -\widehat{D}_s^\sigma\min\left(0, \hat{\sigma}_s^h\right) \end{aligned} \tag{129}$$

where the last equality follows from (125).

Similarly, the collateral requirement condition regarding contracts paying in good-2 using purchased assets as collateral can be written as

$$\begin{aligned} \widehat{D}_s\max\left(0, \hat{\psi}_s^h\right) + \widehat{D}_s^\sigma\max\left(0, \hat{\sigma}_s^h\right) &\geq -C^\sigma\min\left(0, \sigma_s^h\right) \\ &= -P_{2s}D_s^\sigma\min\left(0, \sigma_s^h\right) \end{aligned} \tag{130}$$

where the last equality follows from (126).

Summing conditions (127)-(130) gives the collateral constraint in state s, for an agent h,

$$\begin{aligned} P_{2s}R_sk^h &\geq -\widehat{D}_s\left[\max\left(0, \hat{\psi}_s^h\right) + \min\left(0, \hat{\psi}_s^h\right)\right] - \left[\max\left(0, \hat{\sigma}_s^h\right) + \widehat{D}_s^\sigma\min\left(0, \hat{\sigma}_s^h\right)\right] \\ &\quad -P_{2s}D_s\left[\max\left(0, \psi_s^h\right) + \min\left(0, \psi_s^h\right)\right] - P_{2s}D_s^\sigma\left[\max\left(0, \sigma_s^h\right) + \min\left(0, \sigma_s^h\right)\right] \\ &= -\left(\widehat{D}_s\hat{\psi}_s^h\right) - \left(\widehat{D}_s^\sigma\hat{\sigma}_s^h\right) - P_{2s}\left(\widehat{D}_s^\sigma\psi_s^h\right) - P_{2s}\left(D_s^\sigma\sigma_s^h\right) \end{aligned} \tag{131}$$

The above collateral constraint shows that what really matter for the collateral constraint is the total payoff of contracts. As a result, we can find equivalent contracts with no-default that satisfy the same collateral constraint, by re-normalizing the original contracts. In particular, consider contracts, with collateral $\widehat{C}' = \frac{1}{P_{2s}R_s}$, $C' = \frac{1}{R_s}$, $\widehat{C}'^{\sigma} = \frac{1}{P_{2s}}$, and $C'^{\sigma} = P_{2s}$. Hence, their payoffs are payoffs of those contracts in state s, respectively, are

$$\widehat{D}'_s = \min\left(P_{2s}R_s\widehat{C}', 1\right) = 1 \tag{132}$$

$$D'_s = \min\left(R_sC', 1\right) = 1 \tag{133}$$

$$\widehat{D}'^{\sigma}_s = \min\left(P_{2s}\widehat{C}'^{\sigma}, 1\right) = 1 \tag{134}$$

$$D'^{\sigma}_s = \min\left(\frac{C'^{\sigma}}{P_{2s}}, 1\right) = 1 \tag{135}$$

Note that these are no-default contracts.

In order to reach the same total payoff as originally, let the agent hold securities $\hat{\psi}'^h_s = \widehat{D}_s\hat{\psi}^h_s$, $\hat{\sigma}'^h_s = \widehat{D}^{\sigma}_s\hat{\sigma}^h_s$, $\psi'^h_s = D_s\psi^h_s$, and $\sigma'^h_s = D^{\sigma}_s\sigma^h_s$. As a result, the collateral constraint (131) becomes

$$P_{2s}R_sk^h + \left(\hat{\psi}'^h_s + \hat{\sigma}'^h_s\right) + P_{2s}\left(\psi'^h_s + \sigma'^h_s\right) \geq 0 \tag{136}$$

which is identical to the collateral constraint (4), derived from no-default contracts only.

S.4. Derivation of a Competitive Equilibrium with the Externality in Environment 1

The endowment profile and the first-best allocation suggest that agent 2 would like to move resources forward from $t = 1$ to $t = 0$, and therefore will be constrained. Hence, we will assume that agents type 2 hold no collateral, i.e. $k^1 = k$ and $k^2 = 0$. We now solve for an equilibrium k. From the market clearing conditions of contracts, we can set $\hat{\theta}^1_1 = \hat{\theta} = -\hat{\theta}^2_1$ and $\theta^1_1 = \theta = -\theta^2_1$. Note that this does not mean agent 1 is demanding both securities. In addition, using the specified collateral allocation, the market fundamental in period-1 is now $z = \frac{4}{4+k}$ (the ratio of endowment of good 1 to the sum of endowment of good 2 and saving), and consequently the spot price of good-2 in period 1 is $p(z) = \left(\frac{4}{4+k}\right)^2$.

With homothetic preferences, the first-order conditions of the problem (15) for both types imply that in spot markets at date $t = 0$

$$P_{20} = \left(\frac{c^1_{10}}{c^1_{20}}\right)^2 = \left(\frac{c^2_{10}}{c^2_{20}}\right)^2 = \left(\frac{4}{4-k}\right)^2 \tag{137}$$

Since agent 1's collateral constraint is not binding, the first-order conditions of her utility-maximization problem (15) with respect to θ^1_1 and c^1_{10} lead to

$$P_1 = \frac{U^1_{21}}{U^1_{10}} = \left(\frac{c^1_{10}}{c^1_{21}}\right)^2 \tag{138}$$

where $U^h_{it} = \frac{\partial U^h}{\partial c_{it}}$ is the marginal utility with respect to c_{it}, and P_1 is the price of a security paying in good 2 in period $t = 1$, θ^h_1. Note that we put superscript h on the utility function

for clarity. Further, the first-order conditions of the consumer's problem (15) with respect to θ_1^1 and k^1 (interior solutions) lead to

$$P_{20} = P_1 \tag{139}$$

Intuitively, this is the case because their payoffs are identical and both are collateralizable. Using (137) and (138), condition (139) implies that

$$\frac{c_{10}^1}{c_{20}^1} = \frac{c_{10}^1}{c_{21}^1} \Longrightarrow c_{20}^1 = c_{21}^1 \tag{140}$$

That is, an unconstrained agent consumes the same amount of good 2 in both periods.

Substituting (137) and (138) into (139) gives

$$\begin{aligned} \left(\frac{4}{4-k}\right)^2 &= \left(\frac{c_{10}^1}{c_{21}^1}\right)^2 \\ \frac{4}{4-k} &= \frac{c_{10}^1}{1+k+\theta} \Longrightarrow (4-k)\, c_{10}^1 = 4+4k+4\theta \end{aligned} \tag{141}$$

where we use $c_{21}^1 = 1+k+\theta$.

On the other hand, an agent type 2's collateral constraint is binding; with $k^2 = 0$,

$$\hat{\theta}^2 + p(z)\theta^2 = 0 \Longrightarrow -\hat{\theta} - p(z)\theta = 0 \Longrightarrow \hat{\theta} = -\left(\frac{4}{4+k}\right)^2 \theta \tag{142}$$

where the second and the last equations use $\hat{\theta}^2 = -\hat{\theta}$ and $\theta^2 = -\theta$, and $p(z) = \left(\frac{4}{4+k}\right)^2$, respectively.

The budget constraint of an agent 1 (16) can be written as

$$c_{10}^1 - 3 + P_{20}\left[c_{20}^1 + k - 3\right] + \hat{P}_1\hat{\theta} + P_1\theta = 0 \tag{143}$$

A standard no-arbitrage argument (similar to the one used in Lemma 7) implies that

$$P_1 = p(z)\widehat{P}_1 \tag{144}$$

It thus true from (144) that

$$\hat{P}_1\hat{\theta} + P_1\theta = \widehat{P}_1\hat{\theta} + \widehat{P}_1 p(z)\theta = \widehat{P}_1\left[\hat{\theta} + p(z)\theta\right] p(z) = 0 \tag{145}$$

where the last equation follows the fact that the term in the bracket is zero, from (142). Now the LHS of the budget constraint (143) can be rewritten as

$$c_{10}^1 + P_{20}\left[c_{20}^1 + k - 3\right] = 3 \tag{146}$$

Using (137), we can replace c_{20}^1 by $\left(\frac{4-k}{4}\right) c_{10}^1$. Then using $P_{20} = \left(\frac{4}{4-k}\right)^2$ gives

$$\begin{aligned} c_{10}^1 + \left(\frac{4}{4-k}\right)^2 \left[\left(\frac{4-k}{4}\right) c_{10}^1 + k - 3\right] &= 3 \\ \Longrightarrow (4-k)\, c_{10}^1 &= \frac{3k^2 - 40k + 96}{8-k} \end{aligned} \tag{147}$$

Substituting (141)into (147) gives

$$\frac{3k^2 - 40k + 96}{8 - k} = 4 + 4\theta + 4k \Longrightarrow 4\theta + 4k = \frac{3k^2 - 36k + 64}{8 - k} \tag{148}$$

With the identical homothetic preferences, the period-1 consumption allocations must satisfy

$$z = \frac{4}{4 + k} = \frac{c_{11}^1}{c_{21}^1} \Longrightarrow \frac{4}{4 + k} = \frac{1 + \hat{\theta}}{1 + k + \theta} \tag{149}$$

Substitute (142) into (163) gives

$$4\theta + 4k = -3k\left(\frac{4 + k}{8 + k}\right) + 4k \tag{150}$$

Using (148) and (150), we have

$$\frac{3k^2 - 36k + 64}{8 - k} = -3k\left(\frac{4 + k}{8 + k}\right) + 4k \Longrightarrow 4k^3 - 384k + 512 = 0 \tag{151}$$

There are three roots for equation (151). Using the condition that $0 \leq k \leq 4$, there is only one feasible solution, i.e. $k \approx 1.3595$. To sum up, the equilibrium collateral allocation is $k^1 = k = 1.3595$ and $k^2 = 0$.

S.5. Derivation of a Competitive Equilibrium with the Externality in Environment 2

We restrict our attention to a symmetric allocation of each type. Using Lemma , we assume that all constrained agents hold no collateral, i.e., $k^h = 0$ for $h = 2, 3$. Let $k^1 = k$.

First, the first-order conditions of the consumer's problem (15) result in

$$\frac{c_{10}^1}{c_{20}^1} = \frac{c_{10}^2}{c_{20}^2} = \frac{c_{10}^3}{c_{20}^3} = \frac{12.5}{12.5 - k} \tag{152}$$

From the endowment profile, it is clear that an agent 1 will not be constrained. The first-order conditions of the consumer's problem (15) with respect to θ^1 and c_{10}^1 lead to

$$\frac{U_{21}^1}{U_{10}^1} = P \tag{153}$$

Further, the first-order conditions of the consumer's problem (15) with respect to θ^1 and k^1 (interior solutions) lead to

$$P_{20} = P \tag{154}$$

Combining (153), (154) and the utility function (57), gives

$$P_{20} = \left(\frac{12.5}{12.5 - k}\right)^2 = P = \frac{U_{21}^1}{U_{10}^1} = \left(\frac{c_{10}^1}{c_{21}^1}\right)^2 \tag{155}$$

This implies that

$$\frac{12.5}{12.5-k} = \frac{c_{10}^1}{c_{21}^1} = \frac{c_{10}^1}{0.5+k+\theta^1} \Longrightarrow (12.5-k)\, c_{10}^1 = 12.5\left(0.5+k+\theta^1\right) \tag{156}$$

where we use $c_{21}^1 = 0.5+k+\theta^1$.

In addition, the market fundamental in period-1 is $z = \frac{12.5}{12.5+k}$, and consequently the spot price of good-2 in period 1 is $\left(\frac{12.5}{12.5+k}\right)^2$. The bindingness of the collateral constraints of agent 2 and agent 3, combining with the market-clearing conditions of securities, imply that

$$\hat{\theta}^1 = -\left(\frac{12.5}{12.5+k}\right)^2 (\theta) \tag{157}$$

A standard no-arbitrage argument (similar to the one used in Lemma 7) implies that

$$P_1 = p(z)\widehat{P}_1 \tag{158}$$

, which can be used to show that

$$\hat{P}_1\hat{\theta}^1 + P_1\theta^1 = \widehat{P}_1\hat{\theta}^1 + \widehat{P}_1 p(z)\theta^1 = \widehat{P}_1\left[\hat{\theta}^1 + p(z)\theta^1\right] p(z) = 0 \tag{159}$$

where the last equation follows the bindingness of the collateral constraints of agent 2 and agent 3, combining with the market-clearing conditions of securities. The budget constraint of an agent 1 (16) can be written as

$$c_{10}^1 - \frac{12.5}{3} + P_{20}\left[c_{20}^1 + k - 11.5\right] = 0 \tag{160}$$

Substituting (152) and (155) into (160), we have

$$(12.5-k)\, c_{10}^1 = \frac{12.5^2\,(11.5-k) + \frac{12.5}{3}\,(12.5-k)^2}{25-k} \tag{161}$$

Substituting (156) into (161), we have

$$12.5\left(0.5+k+\theta^1\right) = \frac{12.5^2\,(11.5-k) + \frac{12.5}{3}\,(12.5-k)^2}{25-k} \tag{162}$$

With the identical homothetic preferences, the period-1 consumption allocations must satisfy

$$z = \frac{12.5}{12.5+k} = \frac{c_{11}^1}{c_{21}^1} \Longrightarrow \frac{12.5}{12.5+k} = \frac{0.5+\hat{\theta}^1}{0.5+k+\theta^1} \tag{163}$$

where the equality follows (157). This can be rewritten as

$$12.5\left(0.5+k+\theta^1\right) = (12.5+k)\left(0.5 - \left(\frac{12.5}{12.5+k}\right)^2 \theta^1\right) \tag{164}$$

Solving (162) and (164) altogether for k and θ^1, we have one feasible root ($0 \le k \le 12.5$) $k = 7.2722, \theta^1 = -4.2772$. To sum up, the equilibrium collateral allocation is $k^1 = k = 7.2722$, and $k^2 = k^3 = 0$.

S.6. Derivation of a Competitive Equilibrium with the Externality in Environment 3

First of all, the price of good-2 in period-0 is given by

$$P_{20} = \left(\frac{2}{2-k}\right)^2 \tag{165}$$

Similarly, the market fundamental in each state s is $z_s = \frac{2}{2+k}$. Hence, the spot price of good-2 in each state s is given by

$$p(z_s) = \left(\frac{2}{2+k}\right)^2, \ \forall s \tag{166}$$

Further, the price of a (collateralized) security paying in good-2 in state s is given by

$$P_s = \max_h \left(\frac{\pi_s U_{2s}^h}{U_{10}^h}\right), \ \forall s \tag{167}$$

The endowment structure implies that agents type 2 will have higher MRS $\frac{\pi_s U_{2s}^h}{U_{10}^h}$ in state 1, and vice versa. Hence, (167) can be rewritten as

$$P_1 = \frac{\pi_s U_{21}^2}{U_{10}^2} = \frac{1}{2}\left(\frac{2}{1+k+\theta}\right)^2 = \frac{\pi_s U_{22}^1}{U_{10}^1} = P_2 \tag{168}$$

Note that the symmetry also implies that $P_1 = P_2$. Using the optimal conditions with respect to k^h and θ_s^h, we can show that

$$P_{20} = P_1 + P_2 \Longrightarrow \left(\frac{2}{2-k}\right)^2 = \left(\frac{2}{1+k+\theta}\right)^2 \tag{169}$$

Next, with the homotheticity of preferences, the ratio of consumption in each state of each agent must be equal to the market fundamental; that is,

$$\frac{1+\hat{\theta}}{1+k+\theta} = z_s = \frac{2}{2+k} \tag{170}$$

Furthermore, the collateral constraint in state $s = 1$ of an agent type $h = 1$ is binding, i.e.

$$p(z_1)k - \hat{\theta} - p(z_1)\theta = 0 \Longrightarrow \hat{\theta} = \left(\frac{2}{2+k}\right)^2 (k-\theta) \tag{171}$$

Note that the same equation can be derived from the binding collateral constraint in state $s = 2$ for an agent type $h = 2$.

We can compute a collateral equilibrium using (169), (170), and (171) to solve for $\left(k, \theta, \hat{\theta}\right)$. We can rewrite (169) as

$$2 - k = 1 + k + \theta \Longrightarrow \theta = 1 - 2k \tag{172}$$

In addition, Substituting (171) into (170) gives

$$1+\left(\frac{2}{2+k}\right)^2(k-\theta)=\left(\frac{2}{2+k}\right)(1+k+\theta) \tag{173}$$

Then, substituting (172) into (173) will give

$$\begin{aligned} 1+\left(\frac{2}{2+k}\right)^2(k-1+2k) &= \left(\frac{2}{2+k}\right)(1+k+1-2k) \\ &\Longrightarrow \quad 3k^2+16k-8=0 \end{aligned} \tag{174}$$

The unique feasible (positive) solution to the above quadratic equation is $k \approx 0.4603$.