

# A Leverage-based Model of Speculative Bubbles\*

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## Abstract

This paper explores whether various credit market interventions can give rise to or rule out the possibility of speculative bubbles. As in previous work by Allen and Gorton (1993) and Allen and Gale (2000), a bubble can occur in my model because traders purchase assets with funds borrowed from creditors who cannot perfectly monitor those they lend to. This paper adds to this literature by allowing for more general debt contracts than in previous work, and by allowing dynamic considerations to affect both contracting and trading strategies. These extensions reveal that restricting exotic loan contracts need not rule out bubbles and may actually exacerbate the extent to which assets are overvalued, and that the existence of a bubble hinges not on how low short-term rates fall during a monetary expansion but the level short-term rates are ultimately expected to settle to.

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## Introduction

The spectacular rise and fall of stock prices in the late 1990s and housing prices in the mid 2000s have been cited by many pundits as examples of asset bubbles. Economists typically use the term “bubble” to mean that the price of an asset differs from its “fundamental” value, i.e. the present discounted value of dividends generated by the asset. Whether these episodes truly meet this definition is difficult to ascertain. However, the mere notion that asset prices may have become unhinged from fundamentals during this period has shaped the subsequent debate over macroeconomic policy. For example, some have criticized the aggressive easing pursued by the Federal Reserve in response to the 2001 recession on the grounds that it allowed asset bubbles to arise. Others have faulted the Fed in its regulatory capacity for permitting the proliferation of exotic lending contracts that supposedly encouraged speculation, specifically various types of contracts all premised on low initial payments that rise over the duration of the loan and often referred to as “teaser rate” contracts. Even setting aside the question of whether these episodes were in fact bubbles, it is hard to evaluate the merit of these critiques, since they are often invoked informally rather than derived formally.

The main difficulty with analyzing the role of policy in allowing for bubbles is that in many standard economic models, bubbles cannot occur at all, regardless of credit market policy. This was demonstrated by Tirole (1982), who derived conditions for ruling out the possibility of bubbles in equilibrium. Although several models have been developed that violate these conditions and allow for bubbles, many of these have been criticized as implausible or not conducive for policy analysis. One example are overlapping generation models of money such as Samuelson (1958) and Diamond (1967), which Tirole (1985) interpreted as models of bubbles. Bubbles typically emerge in these models only if the economy grows at least as fast as the riskless rate of return on savings; yet Abel, Mankiw, Summers, and Zeckhauser (1989) show that a generalization of this prediction is rejected empirically. Santos and Woodford (1997) further argue that the bubbles in these models are theoretically fragile, since they would cease to exist as long as even some agents who own a non-vanishing share of the aggregate endowment had infinite horizons. Other models assume agents have different prior beliefs over the fundamental value of the asset, e.g. Harrison and Kreps (1978), Allen, Morris, and Postlewaite (1993), and Scheinkman and Xiong (2003), or that some agents trade in a way that does not depend on fundamentals, e.g. DeLong, Shleifer, Summers, and Waldmann (1990). But without a model for why agents disagree about fundamentals or ignore them when trading, it is hard to predict how policy intervention will affect the possibility of speculative bubbles. Moreover, in none of these models do credit markets play a role in allowing bubbles, despite their central importance inherent in the above critiques.

An alternative theory of bubbles that gives a more prominent role to credit markets was developed by Allen and Gorton (1993) and Allen and Gale (2000). These papers emphasize the role of agency problems as a source of bubbles. In particular, they consider environments in which agents enter into contracts with financiers who cannot monitor what agents do with the funds they receive. Agents who buy assets with

borrowed funds might be willing to buy them even if there were overvalued, so bubbles become possible. This paper adopts such a framework to explore how various credit market interventions can affect whether bubbles are possible. To do so, it extends the Allen and Gale (2000) model to allow for endogenous contracting and dynamics. This is because if we wish to study the role of particular lending contracts, we need to enrich the space of financial contracts to include such contracts in the first place. Likewise, to study the effect of temporary interest rate cuts, we need to introduce a time dimension into the model. Just as importantly, dynamics introduces a “speculative” motive for purchasing overvalued assets in the sense of Harrison and Kreps (1978), i.e. buying an asset in the hope of selling it later for a capital gain. This motive plays an important role in my analysis. Although Allen and Gorton (1993) already considered a dynamic model of speculation, their model is too stylized to explore the questions that motivate this paper. For example, since the asset in their model is intrinsically worthless, one can show that financiers could screen out speculators if they coordinated to extending only debt contracts rather than profit-sharing contracts. This makes their model particularly difficult for exploring the role of leverage in allowing bubbles to arise.

My model offers several new insights regarding the role of policies in either enabling or curtailing the possibility of speculation, thus extending the work of Allen and Gale (2004) on the implications of models of bubbles based on agency problems for policy. First, I argue that outlawing teaser rate contracts need not eliminate bubbles, and such contracts may in fact serve to rein in asset overvaluation rather than contribute to it. This suggests that the focus on contracting arrangements as a cause of bubbles may have been misplaced. As for interest rate policy, the model suggests that the existence of the bubble hinges not on how low rates fall when the Fed lowers them, but how high agents expect rates to be in the future. This is because traders who buy overvalued assets are speculating about their future value, and thus care about future conditions when deciding whether to buy assets for more than their inherent worth.

The paper is organized as follows. Section 1 works through a static version of the model with a restricted set of contracts to illustrate why bubbles can emerge in my framework. Section 2 lays out the full dynamic model with optimal contracting. Section 3 solves the contracting problem between borrowers and financiers. Section 4 discusses the relevance of the model for analyzing a recent episode many suspect involved a bubble. Section 5 explores the effects of various policy prescriptions in the model. Section 6 concludes.

## **1 A Model of Leverage-Based Bubbles**

To understand why the model I develop allows bubbles, it will help to start with a version of the model with only one period. This version is essentially the same as Allen and Gale (2000), a fact that highlights that a bubble emerges in my model for the same reason as in theirs: Those who buy assets can shift risk to their creditors. I then argue that addressing the questions posed in the Introduction requires extending the model to a dynamic setting, and I point out some issues that arise once we move to multiple periods.

In the one period version, agents can purchase assets at the beginning of the period that pay a stochastic dividend  $d$  at the end of the period. Suppose  $d = D > 0$  with probability  $\epsilon$ , where  $0 < \epsilon < 1$ , and  $d = 0$  otherwise. The expected profits from buying an asset at price  $p$  with one's own funds are

$$\epsilon(D - p) + (1 - \epsilon)(-p)$$

This payoff is nonnegative if  $p \leq \epsilon D$ , i.e. a trader will not pay more for the asset than its expected value. But this result need not hold when traders buy the asset with borrowed funds. Following Allen and Gale (2000), suppose a trader with no initial wealth enters a limited-liability debt contract with interest rate  $r$ . The expected payoff from buying an asset at price  $p$  and defaulting if  $d = 0$  is given by

$$\epsilon \cdot \max(0, D - (1 + r)p) + (1 - \epsilon) \cdot 0$$

This payoff is positive if  $p < D/(1 + r)$ , i.e. a trader would be willing to pay up to  $D/(1 + r)$  for the asset. As long as the lender charges a low enough interest rate, specifically if  $1 + r < 1/\epsilon$ , the trader would be willing to buy the asset even if it was a bubble, i.e. if its price  $p$  exceeded its expected value  $\epsilon D$ .

It is easy to confirm that if  $p > \epsilon D$  and  $1 + r < 1/\epsilon$ , lending to an agent so he can buy the asset yields a negative expected return to the lender. Lenders should thus refuse to fund such trades. But in practice, lenders may not be able to tell whether borrowers are buying overvalued assets as opposed to engaging in other activities that are profitable to finance at low  $r$ . For example, while some borrowers buy real estate in case land rents turn out to be high, others wish to buy property they cannot currently afford and must borrow against safe future income to do so. Similarly, while some traders buy equity in case a firm proves to be profitable, others may have an informational advantage about the stocks they trade that makes it profitable to finance them. If lenders cannot distinguish good and bad borrowers but believe enough borrowers are good, they will agree to lend to both at a common low rate. For simplicity, I model good borrowers as entrepreneurs who own no resources but possess a safe technology that converts one unit of resources into  $R > 1/\epsilon$  units. This technology can use at most one unit of input, implying entrepreneurs require a finite amount of resources. Entrepreneurs who choose to produce have nothing to do with the asset. They are only relevant for the asset market because they borrow in the same market as those who buy the asset, but otherwise neither use nor produce the asset. If enough of the borrowers who approach lenders are entrepreneurs wishing to produce, it will be profitable to lend 1 unit to each borrower at rate  $1 + r < 1/\epsilon$  despite expected losses from those who borrow to buy the asset. Imperfectly informed lenders might therefore agree to lend at low  $r$  that make it profitable for leveraged traders to buy overvalued assets.

To verify that assets can in fact be overvalued in equilibrium, I need to introduce additional structure to derive the equilibrium of this economy. Consider the asset market first. I assume assets are available in fixed supply and cannot be sold short. For a bubble to occur, demand for the asset must exceed its fixed supply when  $p = \epsilon D$ . A sufficient condition for this is if the number of traders is at least  $D$  times as large

as the number of assets. Since lenders will not lend to an agent more than an entrepreneur requires, agents will only be able to borrow at most 1 unit of resources. But since the asset will not trade above its maximal payoff of  $D$ , as long as there are at least  $D$  buyers per asset, buyers can collectively borrow more than the stock of assets could ever be worth. Profits from borrowing and buying the asset must then equal zero, or else demand for the asset will exceed its fixed supply. This implies

$$p = D / (1 + r) \tag{1}$$

Note that with zero profits from buying the asset, entrepreneurs will prefer to produce than buy the asset. Thus, we can let entrepreneurs choose whether to buy the asset or borrow to produce.

Next, consider the credit market. I assume free entry by lenders, implying lenders earn zero profits in equilibrium. Let  $\phi$  denote the fraction of borrowers who produce, and  $1 - \phi$  the fraction who each buy  $1/p$  assets. The zero profit condition for lenders is given by

$$\phi r + (1 - \phi) [\epsilon \cdot \min(r, D/p - 1) + (1 - \epsilon)(-1)] = 0 \tag{2}$$

Solving equations (1) and (2) yields

$$\begin{aligned} p &= \phi D + (1 - \phi) \epsilon D \\ r &= \frac{1 - \epsilon}{\epsilon + \phi / (1 - \phi)} \end{aligned}$$

Thus, as long as some agents borrow to produce rather than to buy the asset, i.e.  $\phi > 0$ , and as long as the number of potential buyers for the asset is sufficiently large, the unique equilibrium price  $p$  will exceed  $\epsilon D$ .

The one period example above illustrates how limited information on the part of creditors can give rise to asset bubbles, i.e. situations in which assets trade at prices that exceed their fundamental worth. This insight was already made in Allen and Gale (2000). But to study how various credit-market interventions can affect whether such bubbles could occur, we need to move beyond this insight and extend the model sketched above in at least two ways: allowing for a richer set of contracts and introducing dynamics.

To appreciate the need for richer financing arrangements beyond simple debt contracts, recall that in the wake of the recent housing crisis, some critics argued that exotic financial arrangements with teaser rate features were responsible for luring in buyers and gave rise to a bubble. To determine whether restricting the use of these contracts could eliminate the possibility of bubbles, we need to allow for such contracts in the first place. More generally, allowing a richer contracting environment allows us to explore whether leverage-fueled bubbles are robust to more sophisticated contracts that may make it possible for lenders to screen out those wishing to buy an overvalued asset and impose expected losses on lenders.

To model contracts where payments rise after some time has passed obviously also requires extending the model to include a time dimension. Adding dynamics is also essential for gauging whether interest rate cuts

could give rise to bubbles when these cuts are temporary. But the main advantage of casting the model in a dynamic framework is that it introduces the possibility of *speculative trading* in the sense of Harrison and Kreps (1978), i.e. buying the asset with the hope of selling it later for a higher price. The possibility of resale is absent in a static model, but is nevertheless important for understanding bubbles. Indeed, one implication of my model is that creditors prefer teaser contracts because they encourage those who buy overvalued assets to sell them. We shall also see that the possibility of resale may entice unleveraged agents to buy overvalued assets with their own funds. Finally, modelling speculative trading is especially desirable given that concern about bubbles is often due to evidence of speculation rather than evidence of overvaluation, which is much harder to establish. While Allen and Gorton (1993) already developed a dynamic model in which speculation can arise, they did not use it to explore the role of policy in allowing bubbles. Their model also differs from mine in several key respects, and I point these out below.

Before turning to a version of the model that includes both a richer contracting environment and dynamics, it is worth pausing to discuss some complications that arise simply from introducing dynamics into the model. Towards this end, consider extending the static model above to two periods. That is, suppose assets still pay a single dividend  $d$  at a fixed date, where  $d = D > 0$  with probability  $\epsilon$  and 0 otherwise. But now suppose there are two periods prior to this date in which agents can trade the asset. For simplicity, assume no discounting between periods. Traders who want to buy the asset must secure funds using limited liability debt contracts that are settled after  $d$  is revealed, with rate  $r_1$  on loans made in period 1 and  $r_2$  on loans made in period 2. Let  $p_1$  and  $p_2$  denote the price of the asset in the first and second periods, respectively. Borrowers arrive in the credit market sequentially, some in period 1 and some in period 2. Each period, a fraction  $\phi$  are entrepreneurs who can operate a fixed-scale technology that converts up to one unit of borrowed resources into  $R > 1/\epsilon$  units of output that accrue at the same time  $d$  is revealed.

If  $R$  is sufficiently large, entrepreneurs will prefer to produce than to buy assets with the resources they borrow. Hence, if  $\phi > 0$ , competitive lenders will again charge low  $r_1$  and  $r_2$  that make it profitable for traders to purchase assets at a price above  $\epsilon D$ . The new wrinkle is that we need to determine what traders who buy the asset in period 1 do with it in period 2. Holding on to an asset yields an expected profit of  $\epsilon \cdot \max(0, D - (1 + r_1)p_1)$ , while selling it yields a profit of  $\max(0, p_2 - (1 + r_1)p_1)$ . To rule out uninteresting equilibria in which agents trade the asset even though they never profit from doing so, suppose there is a tiny but positive utility cost from both buying and selling the asset. Since agents who bought the asset in period 1 can guarantee themselves a continuation payoff of zero by holding on to their assets, they will sell only if  $p_2$  exceeds  $(1 + r_1)p_1$  plus the transaction cost. But since  $r_1 \geq 0$ , this implies  $p_2 > p_1$ , i.e. if the asset is resold, its price must increase. Since the absence of discounting implies the fundamental value of the asset is the same in both periods, the asset must become increasingly overvalued with time.<sup>1</sup>

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<sup>1</sup>Note that if  $p_2 > p_1$ , traders who arrive in period 2 borrow more per asset than traders who buy it in period 1. As a result, their willingness to pay for the asset is higher, and there is scope for gains from trade between buyers and sellers.

At the same time, if the price of the asset were expected to rise over time, no agent who owns the asset would agree to sell it early rather than wait to sell it for a higher price. Thus,  $p_2$  cannot exceed  $p_1$  with certainty in equilibrium. This suggests there are two types of equilibria in this economy. In the first type, the asset does not appreciate, i.e.  $p_1 \geq p_2$  with certainty, and assets change hands no more than once, from an original owner to a leveraged buyer. The asset will trade in period 2 in this case only if buyers did not already buy up the entire stock of assets from its original owners in period 1. If the asset trades in both periods, the price must be the same in both, i.e.  $p_1 = p_2$ . Whether this common price exceeds  $\epsilon D$  depends on whether enough traders arrive over the two periods to buy out the entire stock of assets from its original owners at a price of  $\epsilon D$ . If so, the asset will be a bubble, but not a speculative bubble where agents buy the asset in order to resell it. Rather, agents agree to buy an overvalued asset betting  $d$  turns out to be large.

In the other type of equilibria, the price of asset can appreciate, but with probability less than 1. This type of equilibrium arises if demand for the asset in period 1 does not exceed its fixed supply, and traders are unsure whether demand for the asset in period 2 at a price  $\epsilon D$  will exceed the amount of the asset still in the hands of original owners. If a large number of traders show up in the second period,  $p_2$  will exceed  $p_1$ , and some of the traders who arrive in period 2 will have to buy the asset from those who bought it in period 1. Otherwise,  $p_2 \leq p_1$  and all traders who arrive in period 2 buy assets from their original owners, while those who bought assets in period 1 prefer to hold on to them to see if they pay  $d = D$ .

The two-period model illustrates that the equilibrium price of the asset depends on how many potential buyers arrive in the market for the asset and when. If it is certain that not enough traders will arrive to buy out all original asset owners before  $d$  is revealed, the asset must trade at  $\epsilon D$  to keep the original owners indifferent between holding the asset and selling it. If instead it is possible that enough buyers will show up to buy out all original owners, the latter would demand more than  $\epsilon D$  to sell the asset, since the asset will always sell for at least this price and will fetch strictly more if and when a large number of buyers show up. If the number of traders will be enough to buy out the original owners with certainty, the price of the asset will exceed  $\epsilon D$  but not appreciate over time, and the first to arrive will buy assets from their original owners. But if it is uncertain whether enough buyers will arrive in period 2 to buy out the remaining original owners, the price will again start above  $\epsilon D$ , then rise if enough buyers materialize in period 2 and fall otherwise. In the full model with both dynamics and optimal contracts I lay out in the next section, I assume agents are uncertain about future arrivals in a way that implies the price of the asset rises if and when new traders show up. Creditors will take this into account in designing the contracts they offer.

## 2 A Dynamic Model with Optimal Contracting

Building on the model in the previous section, suppose once again that assets pay a single dividend  $d$  at a known terminal date, where  $d = D > 0$  with probability  $\epsilon$  and 0 otherwise. As in Allen and Gorton (1993), I

find it convenient to work in continuous time, and I normalize the date in which  $d$  is revealed to 1. However, it is not essential that  $d$  be revealed at a known date as opposed to a random date. One advantage of using a finite horizon is that I can assume no discounting and economize on notation.

Assets are available in fixed supply and cannot be sold short. I further assume that assets are indivisible and that each agent may purchase no more than one asset. These assumptions remove quantity as a choice variable for the agent, making it easier to solve an agent's trading strategy. But it should be clear from the previous section that the existence of a bubble does not hinge on such restrictions. Since indivisibility implies the price of the asset cannot exceed 1, the most an agent can borrow and bid, I assume  $\epsilon D < 1$  so agents can at least bid the true worth of the asset if they are able to secure funds.

Recall that the equilibrium price of the asset can appreciate only if agents are unsure how many traders arrive before  $d$  is revealed. As such, suppose traders appear in the asset market at random dates in a way that makes it impossible to perfectly forecast how many buyers will arrive before date 1. More precisely, I assume arrivals occur with constant probability  $\lambda$  per unit time, and the number of buyers  $n_t$  who show up at each such arrival is potentially random (and may be zero). The probability of an arrival is thus independent of past arrivals, but the number of traders at each such arrival need not be independent of past arrivals. In other words, traders cannot use past arrivals to predict whether new buyers will show up, but they can potentially use past arrivals to predict demand for the asset in the event that they do show up.

Given that the focus of this paper is on whether various credit market interventions might make speculative bubbles possible, I will proceed as follows. Rather than specify the distribution of the number of agents  $n_t$  at each arrival, I instead ask whether there exist values of  $n_t$  that are consistent with an equilibrium speculative bubble. To do this, I first conjecture that a bubble path exists. Taking this bubble path as given, I derive the optimal contracts lenders would offer and the optimal trading strategies agents would follow. After solving for the strategies of traders, I can check whether there exist values of  $n_t$  that ensure the asset market clears in all dates at the originally conjectured price path. This approach allows me to verify whether a particular speculative bubble is an equilibrium, but it is silent on whether other equilibria exist or how a given equilibrium changes with the underlying environment. Questions regarding uniqueness and comparative statics, while important, are left to a companion paper.

In conjecturing a price path for the asset, note that the price is not uniquely determined when agents fail to arrive; any price that makes it unattractive for those who own the asset to sell it is an equilibrium. Without loss of generality, I set the price to zero in this event. Next, let  $p(t)$  denote the price of the asset at date  $t$  conditional on an arrival at that date. Rather than search through all possible speculative bubble paths, I check whether there exist equilibrium paths that meet the following two conditions:

**A1:**  $p(t)$  is a deterministic and increasing function of  $t$  for  $t \in [0, 1)$ .

**A2:**  $\epsilon D < p(t) \leq 1$  for all  $t \in [0, 1)$ .

Showing that there exists an equilibrium path  $p(t)$  which satisfies A1 and A2 is sufficient to establish that speculative bubbles are possible as equilibria in my model. However, to prove that speculative bubbles are not possible, I would also need to rule out bubbles that do not necessary adhere to A1 and A2. As will become clear below, in those cases where speculative bubbles that adhere to these restrictions fail to exist, the argument for ruling them out can be applied more generally to rule out any bubble.

The reason I look for speculative bubbles that satisfy A1 and A2 is that these conditions greatly simplify the optimal contracting problem. Since the price is deterministic under A1, there is no need to specify contingencies for different price realizations, or to ensure that agents report the price truthfully if lenders cannot observe it. The second condition, A2, must be satisfied by any equilibrium price path respecting A1: The price of the asset is bounded above by 1 given the asset is indivisible, and for a bubble to occur in the first place, the initial price  $p(0)$  must exceed  $\epsilon D$ , which implies  $p(t) > \epsilon D$  for  $t \in (0, 1)$  given  $p(t)$  is increasing in  $t$ . Once I derive the optimal contract and trading strategy taking  $p(t)$  as given, I can check if there exist values of  $n_t$  that ensure this path is an equilibrium. It turns out that for any path  $p(t)$  satisfying A1 and A2, it is quite simple to find values for  $n_t$  that support this path as an equilibrium. In particular, below I show that if  $p(t)$  satisfies A1 and A2, under the optimal contract, agents who buy the asset at date  $t$  will wish to hold it until some cutoff date  $s_t^*$  and then sell it at the next arrival. Suppose we restrict  $n_t$  to two values, 0 and the number of available assets. We then set  $n_t = 0$  if  $t$  is less than the cutoff date of either the original asset owners or the traders who bought the asset most recently, i.e. those who bought at date  $\sup\{\tau < t : n_\tau > 0\}$ , and set  $n_t$  equal to the number of assets otherwise. It is easy to confirm that under these conditions, the asset market will clear at  $p(t)$  in date  $t$  in all states of the world. Searching for bubbles that satisfy A1 and A2 offers a particularly simple way to confirm that a speculative bubble exists.

Demand for the assets is fully characterized by  $\lambda$  and  $n_t$ . But for a bubble to emerge, agents who buy assets must be able to blend in with others who seek credit but do not intend to buy this asset. Thus, as in the previous section, suppose that the  $n_t$  buyers at each arrival are joined by an additional  $\frac{\phi}{1-\phi}n_t$  entrepreneurs. Entrepreneurs can buy the asset as well, but, in contrast to other agents, have the option to operate a technology that converts at most 1 unit of resources into  $R > 1/\epsilon$  units at some future date, and neither uses nor produces the asset. For a bubble to emerge, at least some production must result in output on or after the date that  $d$  is revealed. This is because if all output was produced before  $d$  were revealed, creditors could charge exorbitant rates to those who repay their debt after all production should have ended, making it unprofitable to borrow funds in order to buy the asset.<sup>2</sup> For simplicity, I assume all output accrues exactly at date 1, when  $d$  is revealed, regardless of when production was initiated.

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<sup>2</sup>Clearly, it will be unprofitable to buy and hold the asset if the rate charged for late repayment is high. But buying and selling the asset won't be profitable either. For suppose the first date in which it were unprofitable to buy the asset were strictly positive. One can show that it will be unprofitable to buy the asset just a little before this date, a contradiction.

Sustaining a bubble hinges on entrepreneurs cross-subsidizing those who buy the overvalued asset. Since entrepreneurs can pay up to  $R$  and account for a fraction  $\phi$  of borrowers, and since speculators lose at most what they borrow, the following condition ensures cross-subsidized lending can be profitable:

$$\phi R - 1 > 0. \tag{3}$$

Of course, lending will only be profitable if entrepreneurs produce output rather than buy up the same overvalued assets. To ensure entrepreneurs prefer producing to buying and reselling the asset, the return  $R$  must be large enough to exceed the maximal gains from buying and reselling the asset:

$$R - 1 \geq \sup_{t \in [0,1]} \{p(t)\} - p(0). \tag{4}$$

Since I will focus on the case where  $\epsilon$  tends to 0, producing will also be preferable to buying and holding the asset, which yields at most  $\epsilon D$ . Finally, it must be profitable for agents to buy the asset even when the asset is overvalued. A sufficient condition for this is for  $D$  to be sufficiently large, specifically

$$D > R. \tag{5}$$

This assumption ensures it will be profitable to buy the asset under any contract that induces entrepreneurs to produce. The most entrepreneurs will be asked to pay is  $R$ . If  $D$  is at least as large as  $R$ , non-entrepreneurs can guarantee themselves positive profits by buying the asset and holding it to date 1.<sup>3</sup>

The timing of actions once a cohort of agents arrives is as follows. Agents must initiate all financial transactions immediately upon arrival, i.e. there is no possibility of strategic delay. Agents own no resources, and must borrow funds to undertake any transactions. I assume free entry into the credit market. Agents can approach any potential creditor, but can contract with only one. Since exclusivity imposes fewer constraints on what a contract can achieve, agents would be willing to commit this way. Creditors cannot observe whether an agent who approaches them is an entrepreneur or not. However, they can offer a menu of contracts and let agents select from this menu. The creditor's problem will be laid out more precisely in the next section. Since agents must secure funds immediately, creditors know that if all arriving agents sought to borrow funds, a fraction  $\phi$  of borrowers would be entrepreneurs and the rest would be speculators.

If and when agents secure credit, those who wish to buy an asset or initiate a project must again do so without delay. Thereafter, agents who chose to produce do nothing until date 1 when their output materializes, while agents who purchased an asset must decide whether to sell it if they still own it. If no traders arrive, the equilibrium price will be zero by assumption and agents will prefer to hold on to their

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<sup>3</sup>By contrast, Allen and Gorton (1993) set  $D = 0$ . Such an asset cannot be a bubble in my model: Given equilibrium contracts, asset prices must rise by a non-vanishing increment at each trade, yet the price is bounded by 1. Allen and Gorton can still obtain a bubble because of their timing assumptions. They assume agents borrow before knowing when they will buy the asset. After the agent borrows, lenders are indifferent about offering terms that make it profitable to buy the asset at all dates. The bubble thus arises from the failure of lenders to coordinate and prevent agents from buying the asset at late dates.

asset given transaction costs. If new traders do arrive, the price will equal  $p(t)$  and agents must decide whether to sell or not.<sup>4</sup> Once agents sell the asset, they are assumed to quit the asset market entirely.<sup>5</sup>

Finally, I need to specify what creditors can observe about agents before I can analyze what contracts creditors can offer each cohort of agents when it arrives. Clearly, creditors must not be able to independently learn what an agent did with the funds they borrowed after receiving them. Otherwise, contracts would charge speculators a punitive fee that would deter them from borrowing in the first place. Hence, sustaining a bubble requires that creditors not perfectly observe an agent's wealth. As such, a creditor would be unable to tell if a fellow creditor approached him pretending to be an agent. In what follows, I explicitly allow creditors to pretend to be agents. This constrains the contracts creditors can offer in an important way: It precludes paying non-entrepreneurs not to speculate, since anyone offering such a contract would be flooded by applications from fellow creditors pretending to be non-entrepreneurs.

At the same time, creditors cannot be totally uninformed about borrowers. Otherwise, agents would claim to have run down their wealth and avoid repayment, and creditors would refuse to extend funds *ex ante*. I therefore impose the following assumptions. First, I allow creditors to verify whether an agent has zero or positive wealth at date 1, but not the value of wealth if it is positive. This is meant to capture the fact that creditors can sue agents who claim to have exhausted their wealth, but have no legal standing against those who make no such claims. However, even this coarse information structure allows creditors to deduce the agent's exact wealth, since they can always ask an agent to hand over all of his wealth, verify that the agent has no additional wealth left, and then transfer resources back to the agent. To rule this out, I assume creditors cannot credibly commit to transfer funds. For example, if creditors can make it prohibitively costly for agents to sue them, they cannot be trusted not to shirk their contractual obligations. Agents would then refuse to transfer resources back-and-forth. As I argue in the next section, these assumptions effectively limit contractual arrangements to debt contracts where the creditor transfers resources to the agent when the latter arrives and the agent transfers resources back at subsequent dates.

### 3 Contracting

To recap, the dynamic model assumes agents arrive at random times in the interval  $[0, 1]$ . A fraction of them wish to secure funds to initiate production, and the rest want to buy an overvalued asset. Creditor

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<sup>4</sup>Thus, agents will sell the asset when the expected profits from selling the asset exceed the expected profits from holding on to the asset and trading optimally thereafter. By contrast, Allen and Gorton (1993) assume agents have a bliss point over consumption and sell the asset as soon as they reach this bliss point, regardless of the contract they face.

<sup>5</sup>Since  $p(t)$  is increasing, traders who sell the asset will need to secure some funds to buy it again in the future. If creditors can observe whether agents borrowed in the past, they would turn down agents seeking to borrow a second time, knowing their only use for funds is to speculate. Thus, agents couldn't return to the asset market even if they didn't have to quit.

who are approached by these agents must decide whether to extend funds and under what terms, knowing only that a fraction  $\phi$  of arriving agents are entrepreneurs. I follow the usual route of modelling the contract design problem as a direct revelation mechanism where the creditor designs a mechanism in which those with private information (here, agents) disclose it to those without (here, creditors), and the parties take actions and transfer resources deterministically depending on the information reported.<sup>6</sup> A contract will be defined as *incentive compatible* if it induces those with private information to disclose it truthfully and if it induces those with unverifiable actions to follow the recommendations of the contract. Let  $X$  denote the set of all incentive-compatible contracts. An incentive compatible contract  $x \in X$  is said to be an *equilibrium contract* if there exists no other incentive compatible contract  $x' \in X$  that is strictly preferred to  $x$  by some agents and which yields strictly positive expected profits to the creditor who offers it.

To preview my results, I find that creditors cannot design contracts that deter speculators from borrowing. Their only recourse is to minimize the losses speculators inflict. In particular, they will want to design contracts that encourage speculators who purchased an overvalued asset to sell it rather than hold it. This is done by offering speculators a contract with backloaded payments, i.e. a contract with low initial rates that are eventually reset if agents haven't sold the asset and settled their debt by some specified date.

Formally, a contract requires the agent to reveal his private information, and then recommends actions and transfers to both the agent and the creditor given these announcements. With little scope for verifying the truthfulness of these reports or the actions the parties took, a contract must be designed so that agents agree to report truthfully and both parties agree to follow through with the actions and transfers recommended by the contract. Agents are partly constrained in that they cannot falsely claim to have run down their wealth by date 1. Creditors, by contrast, are unconstrained, and must voluntarily agree to any transfers stipulated by the contract. This restricts what transfers they can credibly commit to. In particular, any funds the creditor transfers *after* the date in which the agent arrives must be transferred back to the creditor in full. This is because by assumption agents have no use for these funds, so transferring them has no effect on what agents can do. Moreover, transferring funds to the agent also does not reveal any information about the agent's wealth. Hence, such transfers do not benefit creditors in any way, and creditors would refuse to make them unless they were recouped by transfers in the opposite direction.

Since transfers from the creditor after the agent arrives must be repaid in full, they cannot be used to provide incentives. I can therefore assume without loss of generality that the equilibrium contract involves no transfers to the agent beyond the initial transfer when he arrives. This initial transfer must be the same for entrepreneurs and non-entrepreneurs. Otherwise, agents would have to disclose their type prior to the transfer, and creditors would refuse to fund non-entrepreneurs. But non-entrepreneurs would then have

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<sup>6</sup>Creditors will find lotteries beneficial in my setup given their different appeal to those who buy the asset and to those who engage in production. But they will not be able to use lotteries to deter speculators from borrowing to buy assets. Since in practice we rarely observe financing arrangements that explicitly rely on randomization, I ignore such contracts in my analysis.

incentive not to be truthful. Any incentive compatible contract must therefore stipulate a transfer from the creditor before the agent makes any announcement. Let  $x_t$  denote the amount the creditor transfers to an agent who arrives at date  $t$ . Since entrepreneurs have no productive use for funds beyond one unit of resources, creditors will not offer more than one unit of resources to any one agent, i.e.  $x_t \leq 1$ .

Once an agent receives the transfer  $x_t$ , he must choose what to do and what to report he did to the creditor. An agent could potentially do nothing, initiate production, or buy the asset, although these choices are limited by  $x_t$  and whether the agent is an entrepreneur. Let  $\omega \in \{\emptyset, e, b\}$  denote his choice, where  $\omega = \emptyset$  implies doing nothing,  $\omega = e$  implies engaging in entrepreneurial activity, and  $\omega = b$  implies buying the asset. Let  $\hat{\omega} \in \{\emptyset, e, b\}$  denote the action the agent reports choosing. A contract would then recommend transfers to the creditor depending on  $\hat{\omega}$ . Since forcing agents to transfer resources as soon as possible can limit their scope to misrepresent themselves as types that can make earlier transfers, there is no reason not to have the contract recommend that agents transfer resources when it is first feasible to do so. Thus, an agent who reports doing nothing, i.e.  $\hat{\omega} = \emptyset$ , will be asked to make a single transfer at date  $t$ . An agent who reports producing, i.e.  $\hat{\omega} = e$ , will be asked to make at most two transfers, one at date  $t$  and one at date 1 when the output from his production is realized. An agent who reports buying the asset, i.e.  $\hat{\omega} = b$ , will also be asked to make at most two transfers, one at date  $t$  and one either when he reports selling the asset or else at date 1 when  $d$  is revealed. In the latter case, the transfer may depend on what he reports as the dividend,  $\hat{d}$ . Finally, since the creditor can verify at date 1 whether the agent has zero or positive wealth, the contract may demand additional transfers from an agent depending on this information. For example, an agent who falsely claims to have no wealth can be forced to pay a fine. Since the agent's exact wealth isn't observable, equity contracts are not enforceable. The only contracts that can be enforced are debt agreements where the repayment amount depends on when the debt is repaid.<sup>7</sup>

After the agent chooses  $\omega$  and reports  $\hat{\omega}$ , he may need to make further choices depending on the action he chose and the action he chose to report. If he does nothing or opts to produce, he faces no additional real choices, other than possibly refusing to make the transfers stipulated under the contract. But since the creditor can always threaten to seize the agent's wealth, a contract can be designed to discourage this. If instead the agent buys the asset, he must choose whether to sell it at each date he still owns it. For any date  $\tau \in [t, 1]$ , let  $h^\tau$  denote the history the agent observes at that date, i.e. any arrivals until date  $\tau$ , the price of the asset until date  $\tau$ , and his own past choices. Let  $\sigma_t(h^\tau) \in [0, 1]$  denote the probability an agent who bought the asset at date  $t$  assigns to selling it after history  $h^\tau$  if he still owns the asset.

In addition, the agent must choose what to report to the creditor. Let  $A_t(\hat{\omega}, \tau)$  denote the set of an-

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<sup>7</sup>By contrast, Allen and Gorton (1993) allow creditors to observe the agent's wealth, but they assume wealth is uninformative about what the agent did. This would be equivalent to making the return to production  $R$  in my setup random in a way that mimics the distribution of positive profits speculators may earn in equilibrium. Since speculators could still pass themselves off as entrepreneurs, equity contracts with limited liability as in their model would still allow speculative bubbles.

nouncements the contract allows an agent to make at date  $\tau$  if he announced  $\widehat{\omega}$  at date  $t$ . Since agents who do nothing or opt to produce face no choices, they should not have anything to report. Thus, we can set  $A_t(\widehat{\omega}, \tau) = \emptyset$  for  $\widehat{\omega} \in \{\emptyset, e\}$  without loss of generality. If instead  $\widehat{\omega} = b$ , i.e. if an agent reports that he bought an asset, then at each date  $\tau$  he should have private information on whether he still owns the asset. At date  $t$ , let us set  $A_t(b, t) = \{0, 1\}$ , where an announcement  $\widehat{a}_t(t) = 1$  means the agent sold the asset at date  $t$  and 0 means he did not. For  $\tau > t$ ,  $A_t(b, \tau)$  is defined recursively. Specifically, let  $A_t(b, \tau) = \{0, 1\}$  if  $\widehat{a}_t(\tau') = 0$  for all  $\tau' \in [t, \tau)$ , and let  $A_t(b, \tau) = \emptyset$  if  $\widehat{a}_t(\tau') = 1$  for some  $\tau' \in [t, \tau)$ . That is, an agent who has yet to report selling the asset will be asked to report if he sold it, while an agent who already reported selling the asset has nothing further to report. If  $\widehat{a}_t(\tau) = 0$  for all  $\tau \in [t, 1]$ , so the agent reported not selling the asset before  $d$  is revealed, he would know its dividend. In that case, let  $A_t(b, 1) = \{0, D\}$ . Otherwise, set  $A_t(b, 1) = \emptyset$ . Let  $a_t(\tau) \in A_t(\omega, \tau)$  denote the true action of the agent given he chose  $\omega$  at date  $t$ .

Next, let  $y$  denote the agent's cumulative income by date 1. That is,  $y = 0$  if the agent did nothing,  $R - 1$  if the agent initiated production,  $p(s) - p(t)$  if the agent bought the asset and sold it at date  $s$ , and  $d - p(t)$  if the agent bought the asset and held it to date 1. I will use the notation  $y = y(\omega, \sigma_t(h^t))$  to reflect the fact that income may depend on the actions of the agent. Let  $x_t^\tau$  denote the transfer the agent would be asked to make at date  $\tau$  under the contract. This transfer would depend on his announcements, i.e.  $x_t^\tau = x_t^\tau(\widehat{\omega}, \widehat{a}_t(\tau))$ . As noted above, we can restrict attention to contracts where agents make transfers as soon as they can, so  $x_t^\tau(\widehat{\omega}, \widehat{a}_t(\tau))$  differs from zero only at a finite number of dates. Agents who announce  $\widehat{\omega} = \emptyset$  will make a single transfer at date  $t$ , so  $x_t^\tau(\emptyset, \widehat{a}_t(\tau)) = 0$  for all  $\tau > t$ . Agents who announce they initiated production will be asked to make at most two positive transfers, at dates  $\tau = t$  and 1. Agents who announce they bought the asset will be asked to make at most two positive transfers, at dates  $\tau = t$  and  $\sup\{\tau' \leq 1 : \widehat{a}_t(\tau') = 0\}$ . The terminal wealth of the agent can thus be expressed as

$$x_t + y(\omega, \sigma_t(h^t)) - \sum_{x_t^\tau(\widehat{\omega}, \widehat{a}_t(\tau)) \neq 0} x_t^\tau(\widehat{\omega}, \widehat{a}_t(\tau))$$

We can now define a contract as incentive compatible if it meets the following conditions:

IC-1: Agents prefer to report  $\omega$  truthfully at date  $t$ , i.e.

$$\omega = \arg \max_{\widehat{\omega}} E \left[ \max_{\omega} \max_{\sigma_t(h^t)} \max_{\widehat{a}_t(\tau) \in A_t(\widehat{\omega}, \tau)} \left\{ x_t + y(\omega, \sigma_t(h^t)) - \sum_{x_t^\tau(\widehat{\omega}, \widehat{a}_t(\tau)) \neq 0} x_t^\tau(\widehat{\omega}, \widehat{a}_t(\tau)) \right\} \right] \quad (6)$$

IC-2: Given they report  $\widehat{\omega} = \omega$  at date  $t$ , agents announce  $a_t(\tau)$  truthfully at all dates  $\tau \in [t, 1]$ , i.e.

$$a_t(\tau) = \arg \max_{\widehat{a}_t(\tau) \in A_t(\omega, \tau)} E \left[ \max_{\sigma_t(h^t)} \left\{ x_t + y(\omega, \sigma_t(h^t)) - \sum_{x_t^\tau(\omega, \widehat{a}_t(\tau)) \neq 0} x_t^\tau(\omega, \widehat{a}_t(\tau)) \right\} \right] \quad (7)$$

IC-3: Creditors will not find it profitable to pretend to be agents and seek credit, i.e.

$$x_t - \sum_{x_t^\tau(\widehat{\omega}, \widehat{a}_t(\tau)) \neq 0} x_t^\tau(\widehat{\omega}, \widehat{a}_t(\tau)) > 0 \text{ only if } y(\omega, \sigma_t(h^t)) = \sum_{x_t^\tau(\widehat{\omega}, \widehat{a}_t(\tau)) \neq 0} x_t^\tau(\widehat{\omega}, \widehat{a}_t(\tau)) - x_t \quad (8)$$

The last IC constraint arises because I assume creditors can disguise themselves as agents and enter into contracts with other creditors. As I alluded earlier, this possibility deters creditors from offering positive net transfers to agents except when they can verify the agent has zero wealth, since any creditor who would offer positive net transfers would be flooded by applications from fellow creditors posing as agents. Only agents who prove their lack of resources will be excused from paying back what they received at date  $t$ .

Not all of the incentive constraints above will be binding. Consider IC-1, the constraint that agents must prefer to accurately report what they did with the funds they received. This incentive constraint will not be binding for agents who contemplate falsely reporting that they did nothing with the funds they receive. This is because IC-3 requires that agents who declare  $\widehat{\omega} = \emptyset$  must transfer  $x_t$  back in its entirety at date  $t$ . Agents who use their funds to purchase assets or engage in production would be unable to do this. Thus, they could not pretend to have done nothing with the funds, even if they preferred to make this report. For the same reason, if  $p(t) < 1$ , agents who engage in production will not be able to pass themselves off as having bought the asset, since the latter would be required to make a positive transfer at date  $t$  that an agent who chose to produce could not make. As long as  $p(t)$  is below 1, the only potentially binding incentive constraint is the one that ensures speculators do not wish to pass themselves off as entrepreneurs.

Next, consider IC-2, which says that agents should continue to report their actions after date  $t$  truthfully. This is only relevant for agents who buy the asset, since only they are asked to make reports beyond date  $t$ . An agent cannot falsely claim to have sold the asset before he did, since he would not be able to transfer any resources at that point. However, he can falsely claim to have sold the asset later than he actually did, or claim not to have sold it at all. To ensure agents report truthfully, the interest rate schedule must be non-decreasing over time, so that those who pay the creditor later must pay back more. Formally,  $x_t^\tau(b, 1)$  must be increasing in  $\tau$  for  $\tau \in [t, 1)$ , and  $x_t^1(b, D) \geq \lim_{\tau \rightarrow 1} x_t^\tau(b, 1)$ .

To summarize, equilibrium contracts are essentially debt contracts with repayment schedules. Agents receive an amount  $x_t$  when they arrive, then choose among possibly multiple repayment schedules that involve (weakly) rising payments over time. I shall now characterize these contracts. I begin with a result concerning the terms of contracts in particular states. Proofs for all claims can be found in an Appendix.

**Claim 1:** In equilibrium, an agent who does nothing or who holds on to an asset which pays no dividends will have zero terminal wealth.

In words, the equilibrium contract confiscates all wealth from agents who fail to earn positive income. This follows directly from (IC-3), since otherwise creditors would have incentive to pass themselves off as agents who earned no income and pocket the resources owed to them under the contract.

The next series of claims characterize what actions agents choose in equilibrium.

**Claim 2:** Let  $\epsilon \rightarrow 0$ . Then in equilibrium  $x_t \geq p(t)$ , so agents will be able to buy the asset under the equilibrium contract if they wanted.

**Claim 3:** Let  $\epsilon \rightarrow 0$ . Then non-entrepreneurs will prefer to buy the asset under the equilibrium contract.

**Claim 4:** Let  $\epsilon \rightarrow 0$ . Then  $x_t = 1$  under the equilibrium contract and entrepreneurs will invest in the project in equilibrium.

**Claim 5:** Under the equilibrium contract, expected profits to the creditor must be zero.

These results can be understood as follows. Assumption (3) ensures that creditors will find it profitable to lend to an agent of unknown type if they could collect all of his output if he were an entrepreneur, even if they collect nothing from non-entrepreneurs. Hence, in equilibrium, creditors will prefer to lend than to stay out of the credit market altogether. Since competition among creditors drives profits to zero, agents who claim to be entrepreneurs will be asked to repay less than  $R$  at date 1. Given  $D \geq R$ , a non-entrepreneur can ensure himself positive expected profits by pretending to be an entrepreneur, buying the asset, then holding it until date 1 to see if it pays out  $D$  and repay the amount demanded from entrepreneurs. Since creditors cannot pay non-entrepreneurs not to speculate, speculation must occur in equilibrium. All creditors can hope to do is minimize the cost of funding speculators by tailoring the terms of the contracts they offer.

I now turn to the terms of the contract offered to the two types. Since  $x_t = 1$  in equilibrium, an agent who announces  $\widehat{w} = e$  at date  $t$  should not have any resources at his disposal until date 1. His contract will thus be a simple debt contract whereby he receives  $x_t = 1$  at date  $t$  and must repay this amount plus an interest charge  $r_t^e = x_t^1(e, \emptyset) - x_t$  at date 1. The next claim shows  $r_t^e > 0$ :

**Claim 6:** Under the equilibrium contract,  $r_t^e = x_t^1(e, \emptyset) - x_t > 0$ .

Next, I turn to the terms offered to those who announce they bought the asset, i.e.  $\widehat{w} = b$ . Let  $V(b, \widehat{w})$  denote the maximal expected utility for an agent who bought the asset but reports  $\widehat{w}$ . (IC-1) requires

$$V(b, b) \geq V(b, e).$$

The next claim establishes that this constraint will hold with equality in equilibrium.

**Claim 7:** In equilibrium, the incentive constraint for type  $b$  will be binding, i.e.  $V(b, b) = V(b, e)$

A non-entrepreneur thus expects to earn the same under the equilibrium contract as he could earn by pretending to be an entrepreneur, buying the asset with the funds he receives, then trading optimally and end up with zero wealth if he cannot afford to transfer  $x_t^1(e, \emptyset)$ . Denote the payoff to the latter strategy

by  $V_0(r_t^e) \equiv V(b, e)$  to reflect that it depends on the terms of the contract  $r_t^e$ . The creditor will choose the terms to offer agents who buy the contract so as to maximize his own expected profits subject to the agent achieving an expected utility of at least  $V_0(r_t^e)$ . In general, these terms will differ from those offered to entrepreneurs. This is because creditors would like to encourage traders to unload the asset as soon as possible. Intuitively, if the trader holds on to the asset, the most the creditor can seize is the dividend  $d$ . But since  $p(t) > \epsilon D$ , this would imply an expected loss for the creditor. Rather than risk this loss, the creditor would rather the agent sell the asset for a price above  $p(t)$  at the next arrival. Hence, creditors will customize the contract to induce agents to sell the asset sooner than they would otherwise.

One way to induce agents to sell the asset earlier is to backload interest payments and charge those who sell the asset early a lower rate than those who sell it late. In fact, creditors would like to charge a negative rate to those who repay their debts early, but this would violate (IC-3). Instead, the best creditors can do is charge a zero interest rate to those who sell the asset early and a high rate to those who sell late. Formally, the optimal contract is characterized by two parameters: a cutoff time  $T_t \in (t, 1]$  and an amount  $R_t^1$  the agent must pay if he fails to sell the asset and it pays  $D$ . If the agent sells the asset before the cutoff date  $T_t$ , he will only have to pay back  $x_t$  in total. If he sells the asset after  $T_t$ , he must hand over all of his wealth. If an agent never sells the asset, he will have to pay  $R_t^1$  at date 1 if  $d = D$ . Formally, we have

$$x_t^t(b, 0) = 1 - p(t), \quad x_t^\tau(b, 1) = \begin{cases} p(t) & \text{if } \tau < T_t \\ p(\tau) & \text{if } \tau \geq T_t \end{cases} \quad (9)$$

and at date 1, the required repayment is given by

$$x_t^1(b, D) = R_t^1 \quad (10)$$

Under the optimal contract, these two parameters are determined jointly, so that  $R_t^1$  equals  $D$  if  $T_t < 1$  and  $T_t$  equals 1 if  $R_t^1 < D$ . In other words, if the contract ever seizes all of the agent's wealth if he sells the asset, it must also do so if he keeps the asset and  $d = D$ . It is easy to show that there exists a unique pair  $(T_t, R_t^1)$  that leaves the agent with utility  $V_0(r_t^e)$ . The next claim proves that this contract is optimal.

**Claim 8:** Given a path  $p(t)$  that satisfies A1 and A2, the contract in (9) and (10) maximizes the expected profits to a creditor among all contracts that deliver utility  $V_0(r_t^e)$  to a non-entrepreneur.

The proof of Claim 8 involves solving for the optimal trading strategy of those who already own the asset. As can be seen in the Appendix, traders who buy the asset at date  $t$  and face the backloaded contract above will hold on to the asset until some cutoff date  $s_t^*$  and then sell it at the next arrival. The supply of the asset at each arrival date is then just the number of traders who are past their cutoff date. Setting the number of buyers  $n_t$  equal to this supply ensures that the asset market will clear at the price  $p(t)$ .

Note that the optimal contract satisfies (IC-2), since speculators cannot gain from claiming they sold their asset later than they did. When  $T_t = 1$  and  $R_t^1 = p(t) + r_t^e$ , the backloaded contract is equivalent

to the one offered to entrepreneurs. Thus, the contracts offered to entrepreneurs and speculators need not be different. In particular, if it ever became apparent that the agents who buy the asset at date  $t$  will not sell it by date 1 under the equilibrium price path even if given a backloaded contract, the terms of the two contracts will necessarily be the same. But if there is some probability that agents sell the asset, the terms will differ. The fact that the equilibrium contract might be separating distinguishes this model from Allen and Gorton (1993) and Allen and Gale (2000), where agents receive identical terms. The difference arises because agents in my model trade strategically, and creditors structure contracts to affect their trading strategies. In a separating equilibrium, creditors know after the fact which of the agents they funded are speculators. But they learn this only after agents accept contracts, and cannot use this information to punish speculators. The existence of a bubble hinges on creditors being unable to distinguish speculators from safe borrowers when they seek credit, not from the contracts they choose.

Introducing dynamics and optimal contracts confirms that a bubble like the one in Allen and Gale (2000) can arise more generally, although the existence of a bubble in a dynamic environment requires stronger restrictions on what creditors can observe and what agents can do. The new insight that emerges from generalizing the model this way is that creditors who anticipate some of their borrowers intend to speculate will design contracts to encourage them to turn around and sell the asset. Contract design is thus important, and restrictions on contractual arrangements can matter for both the credit and asset markets.

## 4 Empirical Relevance of the Model

Given the model produces a dynamic speculative bubble whose existence is due to leverage, it is naturally suited for studying the effect of credit market interventions on the possibility of bubbles. First, though, it is worth pausing to reflect on whether the model is useful for thinking about the very episodes that many have cited as examples of bubbles. In this section, I argue that the model matches some features of the U.S. housing market over the past decade. This does not mean that housing was in fact a bubble, but it suggests the model might be relevant for thinking about this episode if one believes it was.

One assumption that plays a key role in generating a bubble in the model is that agents stake little of their own wealth in the asset, and thus stand to lose little if they fail to sell the asset and its dividends turn out to be low. In that regard, it is noteworthy that the period of rising housing prices in the mid 2000s was accompanied by an increase in real estate purchases involving little or no down payment, or in which the down payment was financed by additional “piggyback” loans rather than with funds put up by the buyer. Since mortgages in many U.S. states are structured as non-recourse loans whereby borrowers can settle their debt obligation by transferring the asset even if its value is less than the outstanding debt, at least some home buyers would have been reasonably protected from a collapse in property values during this period, and thus might have found it profitable to buy property they knew to be overvalued.

Another key element that is necessary for a bubble to emerge in the model is that lenders be willing to extend credit to borrowers even if they know that some of them are intent on speculating. That is, there must be enough borrowers to whom lending offers a positive expected return that they can cover the expected losses from speculators. On this point, it is noteworthy that during this period, financial analysts were arguing that improvements in contract design and securitization created a previously non-existent market that allowed mutually beneficial trade between lenders with a higher tolerance for risk and low-income households willing to pay high rates to obtain access to credit. The notion that speculators could have blended in with borrowers that lenders would have viewed as profitable targets thus seems plausible.

The key conditions the model suggests are needed for a speculative bubble to arise thus seem to have been present in U.S. housing market. At the same time, the model suggests lenders should have sought to protect themselves against speculators, e.g. by offering contracts that encourage unloading bubble assets quickly. It is worth noting that the optimal contracts that emerge in the model indeed resemble some of the mortgage products that became increasingly popular during this period, specifically mortgages that require low initial payments that then rise over time, and that in addition are subject to soft prepayment penalties. Recall that in the model, the contracts offered to speculators involve backloaded payments that encourage them to unload assets more quickly. These contracts are also exclusive, preventing borrowers from escaping the large costs at later dates by refinancing.<sup>8</sup> The mortgage products whose popularity grew rapidly over this period, e.g. interest only loans and option ARMs, all involved the feature that earlier repayments were smaller than later ones. Many of these mortgages, especially for subprime borrowers, imposed prepayment penalties that punish agents who wish to avoid the larger later payments by refinancing. However, since penalties for early prepayment would discourage agents from selling the asset, lenders wishing to replicate the contract in the model should waive the penalty in case the asset was sold. This is precisely what a soft prepayment does. Anecdotal evidence suggests soft prepayment penalties that are waived if the asset is sold were rarely used before the run-up in housing prices, but became quite common during this period, especially for mortgages that require lower initial payments.

The model thus appears to match some of the broad patterns of the U.S. housing market over the recent decade. Again, this need not imply that the housing market over the past decade necessarily involved an asset bubble. That said, there is some evidence of active speculation in the housing market during this period, in the sense that investors bought houses with the express intent of “flipping” them and selling them to others. This suggests there is some merit to exploring the implications of the model in order to gain some insight on the unfolding of this particular episode.

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<sup>8</sup>In principle, lenders in the model would be glad to get rid of speculators by having them refinance with others. However, lenders cannot palm off speculators this way in equilibrium, since new lenders would refuse to refinance borrowers if only speculators applied to be refinanced. Allowing borrowers to recontract with others thus will not allow lenders to selectively shed speculators, but it will constrain the contracts they can offer. As such, lenders would prefer to enter exclusive contracts. For a related discussion on the benefits of exclusivity, see Mayer, Piskorski, and Tchistyi (2008).

## 5 Policy Analysis

I now turn to the question of how various credit market interventions affect the possibility of speculative bubbles in my model. I focus on three types of interventions. First, motivated by the claim that exotic financial contracts encouraged speculation, I consider restrictions on the type of contracts creditors can offer. Next, I consider policies that force agents to use their own funds to buy assets, i.e. down-payment or margin requirements. Lastly, I consider the effects of changes in the opportunity cost of funds for lenders.

### 5.1 Restrictions on the Type of Contracts Creditors can Offer

As noted above, contracts that resemble the ones offered to speculators in the model have come under intense scrutiny lately, with some arguing that teaser-rate features encouraged speculation and contributed to bubbles. Given this, we can ask whether forcing creditors in our model to only offer flat rate contracts would eliminate the possibility of a speculative bubble. A key insight of the model is that these teaser-rate contracts may be a response to a speculative bubble that already exists rather than a force that gives rise to bubbles that wouldn't occur otherwise. Preventing lenders from offering these contracts will not necessarily eliminate bubbles, and, perhaps surprisingly, may only drive asset prices further away from fundamentals.

Formally, as long as  $R \geq 1/\phi$  and  $D \geq R$ , the same arguments used to establish Claims 1-5 apply even if creditors were restricted to simple debt contracts. Thus, a bubble would remain possible. Intuitively, the condition that  $R \geq 1/\phi$  ensures that creditors will prefer to fund an agent drawn at random than to stay out of the credit market. This is because when no credit is supplied, an agent with excess resources could write a contract agents would accept and which extracts enough profits from entrepreneurs to cover the losses on speculators. The condition  $D \geq R$  then ensures non-entrepreneurs can guarantee themselves positive expected profits even if the asset is overvalued, since they can pretend to be entrepreneurs, buy the asset, then hold it to date 1. A restriction on which contracts lenders can offer will thus not prevent agents from buying overvalued assets, but it will affect their incentives to sell the asset once they bought it.

If precluding teaser rate contracts does not eliminate bubbles, will it have other consequences? One natural question is whether these restrictions affect welfare. To address this issue, suppose that all agents in the model – creditors, entrepreneurs, and speculators – act on behalf of a representative household that takes in their profits. Since speculative trading is a zero-sum activity, it has no effect on the aggregate income of the household: Any profits the household earns from speculation correspond to forgone profits from other agents. Entrepreneurial activity, by contrast, creates surplus for the household. But since entrepreneurs can borrow even when lenders are restricted to flat rate contracts, they create the same surplus as before. Restricting contracts would thus have no effect on the welfare of the representative household.

Nevertheless, precluding teaser rate contracts will redistribute resources across the agents acting on behalf of this representative household, even if it does not affect welfare. In particular, restricting contracts will collectively benefit those who trade in the asset at the expense of entrepreneurs. Recall that the feature of the backloaded contract that makes it attractive to creditors is that for a fixed path of asset prices, this contract lowers the price that an agent who already owns the asset would require to sell it. On its own, eliminating teaser contracts would raise the price of the asset going forward beyond when the contract is signed. To study the full effects of changing the contracting environment, we would need to solve for the new equilibrium price path holding fixed the arrival process that we reverse-engineered to sustain the original equilibrium price path. This exercise is somewhat involved, and so I provide only a sketch argument. If agents are more reluctant to sell at a particular point in time, market clearing would drive the price up in that period. This in turn would raise the price of the asset in both earlier and later dates. Those who own the asset earlier will demand a higher price to sell it if they know the price can be higher in the future, and those who buy the asset later would demand a higher price to cover their larger debt obligations. Thus, precluding backloaded contracts would result in higher prices for the asset at all dates, i.e. the asset would become even more overvalued than it was originally. Creditors would thus incur greater losses on speculators, and would need to charge entrepreneurs higher premia. Less of the income the household earns would come from the entrepreneurs who generate it, and more would come from speculators.

If we allowed the return to production  $R$  to vary across entrepreneurs, the fact that creditors need to charge entrepreneurs higher rates when teaser contracts are eliminated could crowd out entrepreneurs with lower values of  $R$ , even though this production is socially efficient when  $R > 1$ . Thus, restricting the set of contracts may lower welfare as well as redistribute resources. But the more important point is that teaser-rate contracts, which have been singled out as a culprit in causing assets to be overvalued, may actually serve to rein in overvaluation. This is because they encourage traders who already purchased overvalued assets to sell them at lower prices than they would under more conventional debt contracts.

## 5.2 Restrictions on Leverage

The next intervention I consider involves restrictions on the degree to which agents can leverage their asset purchases. An important trend during the period of rapidly escalating housing prices was that both households and financial institutions increasingly financed their asset purchases with borrowing rather than their own funds. This is important, since a key element that allows a bubble to emerge in the model is that leveraged agents are sheltered from losses if their purchases fail to be profitable. Restrictions on the extent to which agents can be leveraged may thus prevent bubbles, since if agents must stake some of their own funds they might no longer find it profitable to purchase overvalued assets. The notion that limiting the extent of leverage can safeguard against moral hazard problems has been raised in previous work, e.g. Holmstrom and Tirole (1997). Of course, nothing prevents the lenders in the model from asking agents to

put up their own wealth in the assets they purchase. But the key insight that emerges from the model is that as long as there are enough agents who lack resources but whose activities are still profitable to finance, creditors would be willing to finance even those agents who stake little or none of their own resources. Agents who do stake some of their own resources will likely receive more favorable terms, but those who stake nothing could still secure credit. Policymakers who wish to avoid bubbles might therefore consider forbidding creditors from lending to agents who cannot stake any of their own resources, e.g. by imposing margin or down payment requirements that limit how much agents can leverage their asset purchases.

The model offers two insights on such policies. The first observation is that since bubbles can only occur if there are profitable lenders to cross-subsidize those who borrow to buy overvalued assets, restricting leverage will prevent some beneficial trades from taking place in addition to curtailing speculation. Hence, relying on leverage restrictions to discourage speculative bubbles is likely to incur a social cost that must be balanced against any potential benefits of ruling out bubbles.

The second observation that emerges from the model is that temporary restrictions on leverage will not always prevent speculation, including when these restrictions are put in place. Essentially, if traders expect to be able to sell the asset at a higher price in the future, they might be willing to purchase an overvalued asset even with their own funds. Hence, temporary restrictions on leverage that are lifted too early need not be effective in deterring speculation. This result stands in contrast to Allen and Gorton (1993) and Allen and Gale (2000), where agents would always refuse to buy a bubble if required to do so using their own funds. While leverage restrictions can be used to rule out bubbles in principle, they must be put in place throughout the period in which speculation could occur to be effective.

To demonstrate this claim, I need to modify the model to allow for agents who can shoulder at least part of the cost of the asset on their own. Thus, assume that at each arrival, a negligible fraction of arriving agents – technically, a measure equal to zero in the limit – have enough resources to afford to buy an asset. I assume these agents account for a vanishing measure of all buyers so they have no impact the equilibrium asset price when agents without any resources are able to buy the asset. Once I allow for such agents, I can ask whether they would be willing to buy an overvalued asset with their own funds when margin requirements are in effect. If so, then margin requirements would fail to deter speculation.

Consider a restriction on the amount of leverage agents can assume that is in place between dates 0 and  $t^* \in [0, 1)$  that is lifted thereafter. This restriction precludes agents who lack resources from either buying the asset or investing in the project before date  $t^*$ , since they have no resources to meet any down payment. However, agents who do own some resources could buy the asset before date  $t^*$  if they wanted. Suppose that the number of agents who arrive,  $n_t$ , is as described in Section 2, i.e.  $n_t$  is either zero or equal to the number of available assets. The next claim establishes that they would want to buy the asset.

**Claim 9:** Suppose positive margin requirements are applied only at  $t \in [0, t^*)$ . Then given the assumption on  $n_t$ , there exists an equilibrium in which the price of the asset is the same at  $t \in [t^*, 1)$  as the equilibrium price when  $t^* = 0$ , and at all  $t \in [0, 1]$  the price of the asset exceeds  $\epsilon D$  if an arrival occurs. In addition,

- i. Non-entrepreneurs wish to purchase the asset at all dates, but can only do so from date  $t^*$  on.
- ii. Agents with vast resources would purchase the asset using their own wealth up to some date  $t^{**} \in [0, 1)$ .
- iii. If  $\lambda > 0$ , then  $t^{**} \geq t^*$ , i.e. wealthy agents will buy the asset when margin requirements are in place.

The intuition for Claim 9 is that for the original owner of the asset, opting not to sell the asset is equivalent to paying  $p(t)$  to hold on to it. Thus, if an original owner is willing to keep the asset, an unleveraged agent would be willing to purchase it. Since the number of agents with vast resources is assumed negligible, there must be some original owners who do not sell the asset prior to date  $t^*$ . Hence, prices must be such that the original owners agree to hold on to the asset. But then agents who arrive before date  $t^*$  would be willing to buy it with their own funds. Unlike leveraged buyers who can profit from buying the asset and seeing if its dividend is high, those who buy the asset with their own wealth benefit only by “riding the bubble” in the sense of Abreu and Brunnermeier (2003) and Temin and Voth (2004), i.e. by holding on to an asset whose price is rising and letting go of it before its price “pops”. Riding the bubble will not be profitable unless other buyers can arrive before the bubble pops. Hence, if margin requirements were permanently put in place, i.e. if  $t^* = 1$ , a bubble could not occur: If there were a last date  $t^{**} > 0$  at which agents would be willing to buy the asset using their own wealth, it would be unprofitable to buy the asset just before  $t^{**}$  given the small odds that buyers would arrive before date  $t^{**}$ , which is a contradiction. But if leveraged traders can buy the asset eventually, unleveraged traders would buy it earlier even if it was overvalued.

### 5.3 Changes in the Opportunity Cost of Funds

The last policy intervention I consider involves changes in the opportunity cost of creditors. This policy corresponds to what central banks attempt to do in practice when they intervene in the market for short-term funds. To see how we can incorporate such interventions in the model, suppose creditors in the model are not individuals out to lend their own wealth but banks who can borrow on the overnight market whenever a lending opportunity arises and they do not have funds on hand. The model up to now can be viewed as a special case in which creditors borrow and lend to one another this way at zero interest. The more general interpretation allows us to consider changes in this implicit cost of funds, and explore questions such as whether lowering rates can cause a bubble that would otherwise not have occurred.

Formally, let  $r_t^{FF} \geq 0$  denote the instantaneous rate of return on a loan made at date  $t$ , i.e.  $r_t^{FF}$  represents the limit of the return per unit time on a loan due at date  $t + \Delta$  as  $\Delta \rightarrow 0$ . Let  $R_{t,s}^{FF}$  denote the compound

return between any two dates  $t$  and  $s$ , i.e.

$$R_{t,s}^{FF} = \exp\left(\int_t^s r_x^{FF} dx\right)$$

As the notation suggests,  $r_t^{FF}$  is meant to capture the Federal Funds rate. Note that the overnight rate here represents a real rate, while in practice the Fed sets a nominal rate. I am therefore implicitly assuming that the Fed can affect real rates, at least over the relevant horizon. Agents and creditors are assumed to treat the path of  $r_t^{FF}$  as given. I also assume that any agent in the model who has funds can save them overnight at the same rate  $r_t^{FF}$ . This is equivalent to assuming a competitive banking sector in which depositors earn the same rate of return that banks face in the Federal Funds market.

Before proceeding with the analysis, I first need to examine how allowing  $r_t^{FF}$  to differ from 0 affects the contracting problem between creditors and agents. Since agents can earn an instantaneous return of  $r_t^{FF}$  on their funds, it is immaterial who holds on to resources that are not committed to an asset or a project: either party would earn the return  $r_t^{FF}$ . The timing of payments thus does not matter, other than in constraining agents in what they can credibly report. I can therefore proceed as before in assuming agents must repay lenders as soon as possible, since this limits the ability of agents to misrepresent their types.

Allowing the opportunity cost of funds to be positive has no effect on incentive constraints (IC-1) and (IC-2), but it will slightly alter the formulation of (IC-3) which ensures creditors do not have incentive to falsely enter into contracts with other creditors. If the agent reports he initiated production, i.e.  $\hat{\omega} = e$ , then he would be required to pay back  $x_t - 1$  at date  $t$  and then at least  $R_{t,1}^{FF}$  at date 1, since otherwise a creditor could pretend to be an entrepreneur, earn  $R_{t,1}^{FF}$  rolling one unit over in the overnight market, and then repaying only a part of the profits this strategy nets. Similarly, if the agent reports that he bought the asset, i.e.  $\hat{\omega} = b$ , then he would be required to pay back  $x_t - p(t)$  at date  $t$  and then at least  $R_{t,\tau}^{FF} p(t)$  if he reports selling the asset by date  $\tau$  and at least  $R_{t,1}^{FF} p(t)$  if he reports not having sold it but that it paid a positive dividend. The last condition can be summarized as follows:

$$\begin{aligned} \text{(i)} \quad & x_t^\tau(b, 1) \geq R_{t,\tau}^{FF} p(t) \text{ for } \tau \in (t, 1) \\ \text{(ii)} \quad & x_t^1(b, D) \geq R_{t,1}^{FF} p(t) \end{aligned} \tag{11}$$

Constraint (11) is key for understanding why an increase in the opportunity cost of funds can discourage speculation. Increasing the cost of funds will lead creditors to demand higher repayments from borrowers at all dates. If the cost of funds is sufficiently large, i.e. if it grows faster than  $p(t)$ , agents will be forced to hand over all profits they could earn from speculation, rendering this activity unprofitable. Intuitively, raising the cost of funds creates an alternative that is more profitable for creditors than lending to speculators: lending funds at the Federal Funds rate. By raising the real opportunity cost of funds, the Fed can siphon off the credit that is essential for speculation. Of course, such a policy would also siphon off funds that would have gone to entrepreneurs who need it for socially useful production. Once again, this policy imposes a social cost that must be balanced against any potential benefits of ruling out bubbles.

Although the model implies that setting the Federal Funds rate to a high level should deter speculative trades at least as long as rates remain high, it also shows that the converse is not true: setting a low rate will not necessarily encourage speculative trading while rates are low. This is because the possibility of a bubble depends on the entire future path of the Federal Funds rate rather than its value at a particular point in time. To some extent, this should not be surprising, since speculation is inherently forward looking: Agents are willing to buy an overvalued asset in case it either pays out a large dividend at a future date or in case they can sell it later at a higher price. Whether these bets pay off depends on what agents can keep when they earn these profits and whether future traders would be willing to buy the asset. These in turn depend on the path of the Federal Funds rate in the future. If  $r_t^{FF}$  is expected to be high at future values of  $t$ , a low value today will not be enough to make speculation profitable.

To show this result formally, I now argue that if  $r_t^{FF}$  were set to sufficiently high levels close to the terminal date, a speculative bubble will not emerge even if rates earlier were set arbitrarily close to zero. For technical reasons that will become clear below, relying exclusively on the opportunity cost of funds to discourage agents from buying the asset at dates that are arbitrarily close to the terminal date requires  $r_t^{FF}$  to shoot off to infinity as  $t \rightarrow 1$ . To avoid having a path in which the cost of funds grows without bound, I assume policymakers put in place a margin requirement from some date  $t^* \in [0, 1)$  until date 1. This will preclude non-entrepreneurs from buying the asset beyond date  $t^*$ , regardless of the path of  $r_t^{FF}$  over this period. However, if the opportunity cost of funds  $r_t^{FF} = 0$  for all  $t \in [0, 1]$ , the asset could continue to trade above its fundamental value up until date  $t^*$ . Thus, these margin requirements do not preclude a bubble on their own. But I now argue that if  $r_t^{FF}$  is set to a high but finite level in the interval  $[t^*, 1]$ , a speculative bubble will not be possible in equilibrium even if we set  $r_t^{FF}$  arbitrarily close to 0 before date  $t^*$ .

**Claim 10:** For any  $r^* > 0$ , suppose  $r_t^{FF} = r^*$  for all  $t \in [0, 1 - t^*)$  and  $r_t^{FF}$  for  $t \in [t^*, 1]$  is set to ensure

$$R_{1-t^*, 1}^{FF} \equiv \exp\left(\int_{1-t^*}^1 r_t^{FF} dt\right) \geq \frac{1}{\epsilon} \quad (12)$$

Then a bubble cannot occur in equilibrium.

Note that if we let  $t^* \rightarrow 1$ , we could only satisfy (12) by letting  $r_t^{FF} \rightarrow \infty$  as  $t \rightarrow 1$ . Thus, in the absence of margin requirements, ruling out bubbles requires the opportunity cost of funds to be infinite near the terminal date. The reason for this is that the expected rate of return per unit time from buying the asset and holding it to maturity becomes infinite near the terminal date, since the expected profit from this strategy remains bounded away from zero regardless of how close to the terminal date the asset is purchased. Discouraging agents from buying the asset close to the terminal date would thus require an increasingly larger instantaneous rate of return. Since margin requirements already discourage agents from buying the asset near the terminal date, we do not need to worry about relying on  $r_t^{FF}$  to do so.

Claim 10 demonstrates that a temporary rate cut need not automatically give rise to bubbles: as long

as rates are eventually raised to a high enough level, a bubble could not emerge even if the Federal Funds rate were set to nearly zero at earlier dates. This finding raises an important caveat to the claim that the dramatic cuts in the Federal Funds rate (and in the effective real short rate) following the 2001 recession and the slow pace at which they were reversed were responsible for the bubble that emerged in the housing market. In particular, the existence of a bubble in the model hinges on the entire path of interest rates, not just on interest rates at a point in time. As such, it may well be the case that low interest rates contributed to a bubble that would not have occurred otherwise. However, the case against interest rate policy depends crucially on when agents expected the Fed to reverse their rate cuts and by how much. Ultimately, the existence of a bubble hinges not on how low short-term rates might fall, but on where rates are expected to settle eventually when the policy of easing will eventually be scaled back.

An interesting implication of Claim 10 is that raising the opportunity cost of funds can rule out speculative bubbles without necessitating constant intervention as with margin requirements alone. So long as the Fed is willing to raise rates to high levels in the final stages of the bubble, perhaps in combination with temporarily high margin requirements, it need not take any action beforehand. However, concentrating this intervention over a short period may require setting the cost of funds to very high levels. In particular, discouraging non-entrepreneurs from buying the asset at all dates  $t < t^*$  requires setting  $R_{t^*,1}^{FF}$  high enough to exceed some threshold as in (12). The later is  $t^*$ , the higher the values of  $r_t^{FF}$  must be for  $t \in [t^*, 1]$  to meet a given threshold. Likewise, setting a higher  $r^*$  prior to date  $t^*$  will lower the rates needed beyond date  $t^*$  to curb speculation. In other words, the longer the Fed allows the cost of funds to remain low, the higher rates must be when they eventually clamp down. In addition, such a concentrated intervention only works if traders are forward-looking, as is the case in my model, and understand the unravelling argument for why they would not be able to sell the asset in the future. Continuous intervention does not require traders to reason this way, and might thus prove more robust in preventing speculative bubbles from emerging.

## 6 Conclusion

The dramatic rise and fall in equity and housing prices in the past decade has focused attention on the phenomenon of asset bubbles, i.e. assets that trade at prices above their fundamental value. The prospect of bubbles is often viewed with concern, and recent events have generated considerable debate as to the role of policy in both causing and preventing bubbles. Some have argued that loose credit policy promulgated by the Fed in the wake of the 2001 recession and the slow pace with which it was reversed led to a housing bubble during that period. Others have faulted the Fed in its regulatory capacity for not preventing financing arrangements that supposedly lured in speculators and drove up asset prices. This paper offers a model in which credit plays an essential role in allowing for speculative bubbles that can be used to explore these claims. Yet the model challenges these views: the very contracts many have cited as encouraging bubbles

serve to rein in asset prices in the model, and the possibility of bubbles depends not on how low central banks set rates or how quickly they raise them, but the level rates are expected to settle to.

While the model reveals new insights, it also abstracts from several important issues. This paper focused on the existence of bubbles. But more generally, we also care about uniqueness and comparative statics of equilibria. In particular, we care what drives fluctuations in asset prices, how optimal contracts would take these into account, and how various policy interventions affect not just the existence of bubbles but asset price dynamics if they fail to prevent bubbles. Addressing these questions require further study.

The model also abstracts from uncertainty in terms of when the bubble bursts. In the model, the bubble bursts if  $d$  is revealed to assume its lowest value. This event occurs at a known date. However, information about the asset may arrive at random, as well as at multiple dates. Even if the distribution of the last date in which information arrives has unbounded support, so agents are never sure when the true worth of the asset will be revealed, the model will bear some resemblance to the model analyzed here. This is because the price will naturally be bounded above, either by the amount of resources agents can borrow if we maintain the assumption that assets are indivisible, or by the largest possible realization for dividends. As a result, the asset would cease to trade in finite time, and we can analyze the model by backwards induction as was done here. Despite this similarity, introducing uncertainty as to when information about the asset arrives would add more realism, and may reveal interesting new implications.

Finally, the model as specified is unsuited for exploring whether bursting a bubble is inherently desirable. Recall that a bubble merely redistributes resources across agents, so that if all agents act on behalf of a representative household, bursting the bubble has no effect on welfare. However, with heterogeneity in the return to production, the emergence of a bubble would raise the cost of borrowing, discouraging entrepreneurs with socially valuable but less productive projects from producing and encouraging them to speculate instead. In that case, a bubble may reduce welfare. This result is noteworthy, since bursting bubbles is Pareto worsening in some of the models of bubbles mentioned in the Introduction, and there are few if any existing models in which bursting a bubble is Pareto improving. However, further work is required to determine whether bursting them is indeed desirable. If preventing a bubble requires choking off credit to those with little of their own funds to invest for at least some period of time, the gains from preventing a bubble need to be compared to the costs of cutting off entrepreneurs over that time interval.

## Appendix

**Claim 1:** In equilibrium, an agent who does nothing or who holds on to an asset which pays no dividends will have zero terminal wealth.

**Proof of Claim 1:** Suppose the agent does nothing. Since agents reveal their types truthfully in equilibrium, he would reveal  $\widehat{\omega} = \emptyset$ . Suppose an agent who made this announcement were left with positive terminal wealth. Since his cumulative income  $y = 0$ , this would imply  $x_t^t(\emptyset, \emptyset) < x_t$ . But then it becomes impossible to satisfy (IC-3). Next, suppose an agent holds on to an asset which pays no dividend. In equilibrium, the agent would truthfully reveal that  $\widehat{d} = 0$  at date 1. Since his wealth cannot be negative,  $x_t - x_t^t(b, 0) - p(t) \geq x_t^1(b, 0)$ . But since  $p(t) \geq \epsilon D > 0$ , then  $x_t - x_t^t(b, 0) - x_t^1(b, 0) > 0$ . But (IC-3) then requires that  $y = x_t^t(b, 0) + x_t^1(b, 0) - x_t$ , i.e. his wealth must be zero, as claimed. ■

**Claim 2:** Let  $\epsilon \rightarrow 0$ . Then in equilibrium  $x_t \geq p(t)$ , so agents will be able to buy the asset under the equilibrium contract if they wanted.

**Proof:** Suppose not, i.e.  $x_t < p(t)$ . Agents would then be unable to purchase the asset or invest, and from Claim 1 it follows that agents have zero terminal wealth. Suppose one of the creditors were to offer the following contract:

$$\begin{aligned} \tilde{x}_t &= 1 \\ \tilde{x}_t^1(\widehat{\omega}, \widehat{a}) &= \min\left(\frac{1+\epsilon}{1-\phi}, 1+y\right) \text{ for all } \widehat{\omega} \in \{e, b\} \text{ and } \widehat{a} \neq D \end{aligned} \quad (\text{A.1})$$

for some arbitrarily small and positive  $\epsilon$ . That is, anyone who borrows must pay back a constant interest rate if they can afford it. Since  $\tilde{x}_t = 1$ , an entrepreneur who accepts this contract will be able to invest in the project or purchase the asset. If he invests, his profits will equal

$$\tilde{x}_t + y - \tilde{x}_t^1(e, \emptyset) = R - \frac{1+\epsilon}{1-\phi}$$

Given the restriction that  $(1-\phi)(R-1) - \phi > 0$ , profits will be positive for  $\epsilon$  sufficiently close to zero. Hence, an entrepreneur would strictly prefer this contract to the original equilibrium contract. In addition, for sufficiently small  $\epsilon$ , the entrepreneur will strictly prefer investing in the project to buying the asset. To see this, suppose he bought the asset. If he held it until date 1, his expected profit would equal

$$\epsilon \left( D + 1 - p(t) - \frac{1+\epsilon}{1-\phi} \right)$$

which goes to zero as  $\epsilon \rightarrow 0$ . If he sold the asset, the most he could earn is

$$\sup_s \{p(s)\} + 1 - p(t) - \frac{1+\epsilon}{1-\phi}$$

Given assumption (4),  $R - 1 \geq 1 - p(t) \geq \sup_s \{p(s)\} - p(t)$ , so investing in the project is more profitable than purchasing the asset and selling it could ever be.

Non-entrepreneurs will also accept this contract, and use it to buy assets. In particular, they could always guarantee themselves positive expected profits by buying the asset and holding it to maturity, which would net them

$$\begin{aligned} \epsilon \left( D + 1 - p(t) - \frac{1 + \epsilon}{1 - \phi} \right) &\geq \epsilon \left( R + 1 - p(t) - \frac{1 + \epsilon}{1 - \phi} \right) \\ &\geq \epsilon \left( R - \frac{1 + \epsilon}{1 - \phi} \right) \end{aligned}$$

where recall that the last expression will be positive for small  $\epsilon$ . Since the creditor loses at most 1 unit to non-entrepreneurs who engage in speculation, and since the fraction of agents who are non-entrepreneurs is at most  $\phi$ , expected profits to the creditor are bounded below by

$$(1 - \phi) \frac{\phi + \epsilon}{1 - \phi} - \phi = \epsilon > 0$$

Hence, there exists a contract that agents strictly prefer and which delivers positive expected profits to creditors. The original contract must therefore not have been an equilibrium. ■

**Claim 3:** Let  $\epsilon \rightarrow 0$ . Then non-entrepreneurs will prefer to buy the asset under the equilibrium contract.

**Proof:** Suppose not. Then it follows that non-entrepreneurs do nothing, in which case by Claim 1 we know they will have zero terminal wealth. Since they could buy the asset if they chose according to Claim 2, incentive compatibility requires that buying the asset should yield zero profits under the equilibrium contract regardless of what announcements  $\hat{\omega}$  and  $\hat{a}$  they make. This in turn implies the following restrictions:

$$\begin{aligned} x_t^1(e, \emptyset) + x_t^t(e, \emptyset) &\geq x_t + D - p(t) \\ x_t^1(b, D) + x_t^t(b, 0) &\geq x_t + D - p(t) \\ x_t^1(b, 0) + x_t^t(b, 0) &\geq x_t + D - p(t) \\ x_t^s(b, 1) + x_t^t(b, 0) &\geq x_t + p(s) - p(t) \end{aligned}$$

Each of these conditions implies that the payments required by the contract if the agent purchased the asset are at least as large as the income the agent could earn from buying the asset. Given these restrictions on transfers, an entrepreneur who invests in the project must also earn zero profits, since  $D - p(t) \geq R - 1$ . The entrepreneur would also not earn profits from buying the asset, or doing nothing. But if all agents earn zero profits, both types would strictly prefer the alternative contract (A.1) introduced in the proof of Claim 2, and it would yield positive profits to the creditor who offers it. So non-entrepreneurs not buying the asset cannot be an equilibrium. ■

**Claim 4:** Let  $\epsilon \rightarrow 0$ . Then  $x_t = 1$  under the equilibrium contract and entrepreneurs will invest in the project in equilibrium.

**Proof:** Suppose not, i.e.  $x_t < 1$ . Then an entrepreneur will either do nothing or purchase the asset. From Claim 3, we know that in equilibrium non-entrepreneurs earn positive expected profits from buying the asset. So, if there is an equilibrium in which entrepreneurs do not invest in the project, they will buy the asset when given the opportunity. It then follows from Claim 3 that all agents will purchase the asset.

The proof relies on showing that there exists a contract that only entrepreneurs prefer to the original contract, and which yields positive expected profits to the creditor who offers it. The idea is to allow entrepreneurs to invest in the project but demand a higher transfer payment in return. Since entrepreneurs earn more from investing in a project than from buying the asset, they would prefer this new contract as long as the transfer was not too large. Non-entrepreneurs would avoid this contract, since they would be required to transfer more resources at no benefit to them. Since creditors earn positive profits from lending exclusively to entrepreneurs, the original contract could not have been an equilibrium.

Let  $V_t$  denote the expected utility of an agent who buys the asset at date  $t$  and faces the equilibrium contract  $x$ . Consider simple debt contracts in which the agent receives  $\tilde{x}_t$  at date  $t$  and must repay the minimum of this amount plus interest  $r_t^*$  at date 1 or whatever they can afford, i.e.

$$\begin{aligned}\tilde{x}_t^\tau(\hat{\omega}, \hat{a}_t(\tau)) &= 0 \text{ for } \tau < 1 \\ \tilde{x}_t^1(\hat{\omega}, \hat{a}_t(1)) &= \min\{\tilde{x}_t + r_t^*, \tilde{x}_t + y\} \text{ for all } \hat{\omega}, \hat{a}_t, \text{ and } y\end{aligned}\tag{A.2}$$

I first show that there exists a simple debt contract  $\tilde{x}$  with  $\tilde{x}_t = x_t < 1$  as in the original contract  $x$  that delivers the same expected utility  $V_t$  to agents as the equilibrium contract. I then argue that we can find a sufficiently small  $\varepsilon$  such that a new simple debt contract  $\bar{x}$  of the form

$$\begin{aligned}\bar{x}_t &= 1 > \tilde{x}_t \\ \bar{x}_t^1(\hat{\omega}, \hat{a}_t) &= \min\{1 + r_t^* + \varepsilon, 1 + y\} \text{ for all } \hat{\omega}, \hat{a}_t, \text{ and } y\end{aligned}\tag{A.3}$$

that entrepreneurs will strictly prefer to  $\tilde{x}$ , and thus to the original contract  $x$ , but non-entrepreneurs will not. Lastly, I show the creditor offering contract  $\bar{x}$  will earn positive expected profits.

To prove there always exists an  $r^*$  such that (A.2) delivers  $V_t$  to a trader as the original equilibrium contract  $x$ , I first derive bounds for  $V_t$ , and then show we can choose  $r^*$  to achieve any value within the derived bounds. Consider the terminal wealth of the agent,

$$x_t - \sum_{x_t^\tau(\hat{\omega}, \hat{a}_t(\tau)) \neq 0} x_t^\tau(\hat{\omega}, \hat{a}_t) + y(\omega, \sigma_t(h^\tau))\tag{A.4}$$

If  $y \leq 0$ , we know from Claim 1 that terminal wealth must be zero. If  $y > 0$ , (IC-3) implies that  $\sum_{x_t^\tau(\hat{\omega}, \hat{a}_t(\tau)) \neq 0} x_t^\tau(\hat{\omega}, \hat{a}_t) \geq x_t$ , so terminal wealth is bounded above by  $y$ . At the same time, terminal wealth cannot be negative. This implies  $V_t \geq 0$ , while the upper bound on what a contract can achieve is

if it gives the agent a terminal wealth of  $y$  whenever  $y > 0$ , since this is the most the contract can provide to an agent and satisfy (IC-3). Note that this upper bound is just the expected utility of an agent from a simple debt contract as in (A.2) where  $r_t^* = 0$ . Denote this value by  $\bar{V}_t$ .

I now argue that for any  $V \in [0, \bar{V}_t]$ , there exists an  $r_t^*$  for which the contract  $\tilde{x}$  defined by (A.2) yields an expected utility equal to  $V$ . If we set  $r_t^* = D - p(t)$ , the contract will always leave agents with zero terminal wealth, the lower bound on  $V$ . If we set  $r_t^* = 0$ , the contract will leave agents with a utility of  $\bar{V}_t$ . If we can show that expected utility is continuous in  $r_t$ , it would follow that for any  $V \in [0, \bar{V}_t]$  there exists an  $r_t^*$  that yields a utility of  $V$ . Let  $W(t, s; r_t^*)$  denote the value to the agent at date  $s$  of holding the asset given the contract (A.2). Let  $Q(s)$  denote the probability that there is at least one arrival between date  $s$  and 1, which under my assumptions is given by  $Q(s) = 1 - e^{-\lambda(1-s)}$ . Next, let  $f(x|s)$  denote the probability that the first arrival time after date  $s$  occurs at date  $x$ , which under my assumptions is given  $f(x|s) = \frac{\lambda e^{-\lambda(x-s)}}{1 - e^{-\lambda(1-s)}}$ . Then  $W(t, s; r_t^*)$  must satisfy the integral equation

$$W(t, s; r_t^*) = Q(s) \int_s^1 \max(W(t, \tau; r_t^*), p(\tau) - p(t) - r_t^*, 0) f(\tau|s) d\tau + (1 - Q(s)) \epsilon (D - p(t) - r_t^*).$$

This is because in equilibrium, the agent must prefer not to sell the asset if there are no arrivals, and if there is an arrival the agent will choose between holding the asset or selling, depending on which provides him with a higher utility. This confirms  $W(t, s; r_t^*)$  is continuous (and even differentiable) in  $r_t^*$ . Since the value of the contract to the agent at date  $t$  is  $W(t, t; r_t^*)$ , the claim follows.

Finally, suppose a creditor were to offer the contract  $\bar{x}$  defined by (A.3). In the limit as  $\epsilon \rightarrow 0$ , the amount an entrepreneur could earn under contract  $\tilde{x}$  defined by (A.2) is bounded above by  $\lim_{s \rightarrow 1} p(s) - p(t) - r_t^*$ . By contrast, under contract  $\bar{x}$  defined by (A.3), he could earn  $R - 1 - r_t^* - \epsilon$ . Since we assume  $R - 1 > 1 \geq \lim_{s \rightarrow 1} p(s) - p(t)$ , there exists an  $\epsilon$  small enough such that the entrepreneur will strictly prefer (A.3) to (A.2) and hence to the original equilibrium contract. Non-entrepreneurs, however, will strictly prefer the original contract, since the expected profits under a contract of type (A.2) are decreasing in  $r_t^*$ . Hence, the expected profits to a creditor who offers contract (A.3) are  $r_t^* + \epsilon > 0$ , suggesting the original contract could not have been an equilibrium. ■

**Claim 5:** Under the equilibrium contract, expected profits to the creditor must be zero.

**Proof:** Suppose not, i.e. a creditor expects to earn strictly positive profits in equilibrium. We now argue there exists an alternative contract  $\tilde{x}$  that both creditors and agents strictly prefer, which is a contradiction.

We consider two scenarios. First, suppose the expected transfer by those who announce they bought the asset is strictly positive when they choose  $\hat{a}_t(\tau)$  optimally, which we denote by  $\hat{a}_t^*(\tau)$ . Formally,

$$E \left[ \sum_{x_t^T(\hat{\omega}, \hat{a}_t(\tau)) \neq 0} x_t^T(b, \hat{a}_t(\tau)) \middle| \hat{a}_t(\tau) = \hat{a}_t^*(\tau) \right] > x_t$$

In that case, there exists an  $\varepsilon > 0$  such that we can design a new contract  $\tilde{x}$  where  $\tilde{x}_t = x_t$  and which satisfies

$$E \left[ \sum_{x_t^\tau(\hat{\omega}, \hat{a}_t(\tau)) \neq 0} \tilde{x}_t^\tau(b, \hat{a}_t(\tau)) \middle| \hat{a}_t(\tau) = \hat{a}_t^*(\tau) \right] = E \left[ \sum_{x_t^\tau(\hat{\omega}, \hat{a}_t(\tau)) \neq 0} x_t^\tau(b, \hat{a}_t(\tau)) \middle| \hat{a}_t(\tau) = \hat{a}_t^*(\tau) \right] - \varepsilon$$

where  $\tilde{x}_t^\tau(b, \hat{a}_t(\tau)) \geq \tilde{x}_t = x_t$  for all  $\tau$  and is still monotonic in  $\tau$  to satisfy (IC-2). Since the original contract  $x$  must have been incentive compatible, it follows that for agents who announce they invested in the project must make a positive net transfer to creditors, i.e.

$$x_t^1(e, \emptyset) > x_t$$

For suppose instead that  $x_t(e, \emptyset) = x_t$ . Then non-entrepreneurs would prefer this to their own contract, since as argued in Claim 4, the upper bound on how much an agent can earn is with a zero interest rate contract, which is what agents who announce  $\hat{\omega} = e$  receive, while the contract offered to those who announce  $\hat{\omega} = n$  is equivalent to a simple debt contract as in (A.2) with positive interest. Hence, there exists an  $\varepsilon > 0$  such that

$$\tilde{x}_t^1(e, \emptyset) = x_t(e, \emptyset) - \varepsilon \geq x_t = \tilde{x}_t$$

Since  $\varepsilon > 0$ , both parties will strictly prefer accepting contract  $\tilde{x}$  and telling the truth to accepting contract  $x$  and telling the truth. Moreover, since the original equilibrium contract was incentive compatible, and I subtract the same amount from the expected payoff of both types, non-entrepreneurs would be weakly better off telling the truth under contract  $\tilde{x}$  than passing themselves off as entrepreneurs. Finally, since entrepreneurs strictly prefer to invest under the original contract, they would continue to prefer investing in the project under contract  $\tilde{x}$  for  $\varepsilon$  small enough. Since profits to the creditor under the original contract were strictly positive, they will remain positive under the new contract for  $\varepsilon$  small enough. But then original contract could not have been an equilibrium.

Next, consider the case where the expected transfer by those who announce they bought the asset is zero when they choose  $\hat{a}_t(\tau) = \hat{a}_t^*(\tau)$ , i.e.

$$E \left[ \sum_{x_t^\tau(\hat{\omega}, \hat{a}_t(\tau)) \neq 0} x_t^\tau(b, \hat{a}_t(\tau)) \middle| \hat{a}_t(\tau) = \hat{a}_t^*(\tau) \right] = x_t$$

This is equivalent to giving those who announce  $\hat{\omega} = b$  a contract of type (A.2) with  $r^* = 0$ . Since there will be some non-entrepreneurs who fail to sell an asset which proves to be worthless, the only way for the creditor to earn nonzero expected profits when  $r^* = 0$  is if  $x_t^1(e, \emptyset) > x_t$ , i.e. if entrepreneurs transferred positive resources to the creditor. Suppose a creditor offers a simple debt contract  $\tilde{x}$  of type (A.2) where  $\tilde{x}_t = x_t$  and  $r^* = x_t^1(e, \emptyset) - x_t - \varepsilon > 0$ . Non-entrepreneurs will prefer their original contract, which was essentially a contract with  $r^* = 0$ . Entrepreneurs will prefer the new contract  $\tilde{x}$  to the original one  $x$ ,

and the expected profits to the creditor are  $r^* > 0$ , implying the original contract could not have been an equilibrium. ■

**Claim 6:** Under the equilibrium contract,  $r_t^e = x_t^1(e, \emptyset) - x_t > 0$ .

**Proof:** Suppose not. Recall that an agent who reports an outcome that is not possible is assumed to be left with zero wealth. Given this, if a non-entrepreneur pretended to be an entrepreneur, the fact that  $r_t^e = 0$  implies their expected utility would be the same as under a contract of type (A.2) with  $r^* = 0$ . Incentive compatibility requires that non-entrepreneurs prefer to disclose their information truthfully. But if the equilibrium contract set  $x_t^\tau(b, \hat{a}_t(\tau)) > x_t$  for any  $\tau$ , the agent would be better off pretending to be an entrepreneur. Thus, if  $r_t^e = 0$ , then  $x_t^\tau(b, \hat{a}_t(\tau)) = x_t$  for all  $\tau$ . Since there is positive probability that agents who buy assets will be unable to sell the asset and  $d_t = 0$ , the expected profits to creditors will be negative. But creditors must earn non-negative profits in equilibrium, so the original contract could not have been an equilibrium. ■

**Claim 7:** In equilibrium, the incentive constraint for type  $b$  will be binding, i.e.  $V(b, b) = V(b, e)$

**Proof:** Suppose not, i.e. an agent who buys assets strictly prefers to announce  $b$  than to announce  $e$ . From Claim 6, we know  $x_t^1(e, \emptyset) > x_t$ . Consider a contract  $\tilde{x}$  which offers slightly better terms to those who announce themselves to be entrepreneurs, i.e.  $\tilde{x}_t = x_t$  so entrepreneurs receive the same as under the original contract, but

$$\tilde{x}_t^1(e, \emptyset) - \tilde{x}_t = x_t^1(e, \emptyset) - x_t - \varepsilon.$$

Since  $V(n, n) > V(e, e)$ , we can choose  $\varepsilon$  small enough so that non-entrepreneurs still prefer to reveal themselves truthfully under the original contract than to take the new contract and misrepresent themselves as entrepreneurs. At the same time,  $\tilde{x}$  offers either identical terms to non-entrepreneurs or treats them worse. Then only entrepreneurs will be attracted to the new contract  $\tilde{x}$ , and for  $\varepsilon$  small enough the expected profits to the creditor who offers it will be  $x_t^1(e, \emptyset) - x_t - \varepsilon$  which is strictly positive. Hence, the original contract could not have been an equilibrium. ■

**Claim 8:** Given a path  $p(t)$  that satisfies A1 and A2, the contract in (9) and (10) maximizes the expected profits to a creditor among all contracts that deliver utility  $V_0(r_t^e)$  to a non-entrepreneur.

**Proof:** Consider the problem of what transfers to demand from those who announce they sold at date  $s$ , i.e.  $x_t^s(b, 1)$ , and from those who failed to sell the asset and saw it pay off, i.e.  $x_t^1(b, D)$ , to maximize the expected profits of the creditor while keeping the expected utility of the agent who buys an asset equal to  $V_0(r_t^e)$ . Define  $\Sigma(t)$  as the set of dates under the contract  $x_t$  at which an optimizing agent would opt to sell the asset if there was an arrival. If there is no arrival, we know that the price which clears the asset market would be such that neither the creditor nor the agent would want to sell the asset. Let  $Z(t)$  denote the probability that at least one arrival occurs over this set of dates. Finally, let  $h(s)$  denote the distribution

of the first arrival in the set  $\Sigma(t)$ , conditional on at least one such arrival. Then the creditor would choose  $x_t^s(b, 1)$  and  $x_t^1(b, D)$  to maximize his expected payoff

$$\max_{x_t^s(b, 1), x_t^1(b, D)} (1 - \phi_t) r_t^e + \phi_t Z(t) \int_{\Sigma(t)} [x_t^s(b, 1) - p(t)] h(s) ds + \phi_t (1 - Z(t)) (\epsilon x_t^1(b, D) - p(t)) \quad (\text{A.5})$$

subject to the constraints

1.  $Z(t) \int_{\Sigma(t)} [p(s) - x_t^s(b, 1)] h(s) ds + (1 - Z(t)) \epsilon (D - x_t^1(b, D)) = V_0(r_t^e)$
2.  $p(t) \leq x_t^s(b, 1) \leq x_t^1(b, D)$  and  $x_t^s(b, 1)$  is increasing in  $s$  and

The constraint that  $x_t^s(b, 1) \geq p(t)$  reflects the assumption imposed in the paper that an agent who announces he bought the asset will be required to immediately return  $1 - p(t)$  at date  $t$ , and the total transfers  $\sum_{x_t^\tau(\hat{\omega}, \hat{a}_t(\tau)) \neq 0} x_t^\tau(b, \hat{a}_t(\tau))$  must be at least as large as  $x_t$  to satisfy (IC-3). If we substitute the first constraint into the objective function, we can rewrite this problem as

$$\max_{x_t^s(b, 1), x_t^1(b, D)} (1 - \phi_t) r_t^e + \phi_t \left\{ Z(t) \int_{\Sigma(t)} [p(s) - p(t)] h(s) ds + (1 - Z(t)) (\epsilon D - p(t)) - V_0(r_t^e) \right\}$$

Since the choice of  $x_t^s(b, 1)$  has no effect on  $V_0(r_t^e)$  and  $r_t^e$ , this problem is equivalent to

$$\max_{x_t^s(b, 1), x_t^1(b, D)} Z(t) \int_{\Sigma(t)} [p(s) - p(t)] h(s) ds + (1 - Z(t)) (\epsilon D - p(t)) \quad (\text{A.6})$$

The choices for  $x_t^s(b, 1)$  and  $x_t^1(b, D)$  thus do not enter the objective function in (A.6) directly, but through their effect on  $\Sigma(t)$  and  $Z(t)$ . But the latter is determined by  $\Sigma(t)$ , so the creditor's problem amount to designing the set  $\Sigma(t)$ . But choosing  $\Sigma(t)$  is equivalent to the problem of an agent who buys the asset with his own funds and must decide whether to sell it if an arrival occurs.

Define  $\Pi(t, s)$  as the value for a borrower who bought the asset at date  $t$  with his own funds, opts to wait at date  $s$ , and acts optimally thereafter.  $\Pi(t, s)$  is thus analogous to the value  $W(t, s)$  for an agent who purchases the asset with borrowed funds as opposed to his own. The function  $\Pi(t, s)$  satisfies an analogous integral equation:

$$\Pi(t, s) = Q(s) \int_s^1 \max(\Pi(t, \tau), p(\tau) - p(t)) f(\tau|s) d\tau + (1 - Q(s)) (\epsilon D - p(t))$$

where recall  $Q(s) = 1 - e^{-\lambda(1-s)}$  is the probability of at least one arrival between date  $s$  and the terminal date and  $f(x|s) = \frac{\lambda e^{-\lambda(x-s)}}{1 - e^{-\lambda(1-s)}}$  is the probability that in this event the first arrival after date  $s$  is date  $x$ . Note how this problem compares to that of an agent who borrows funds but faces a zero-interest contract; the equation differs only in the last term, which involves  $\epsilon D - p(t)$  rather than  $\epsilon [D - p(t)]$ . Setting  $p(t) = 0$  yields a value for  $\Pi(t, s)$  that is equal to the expected profits for the original owner who must choose optimally when to sell the asset he was endowed with.

We show that the optimal trading strategy would dictate selling the asset from some cutoff date  $\sigma_t^*$  on. To show this, it will suffice to show that if  $\Pi(t, s) \geq p(s) - p(t)$ , then  $\Pi(t, s') \geq p(s') - p(t)$  for all  $s' < s$ .

Suppose instead that  $\Pi(t, s') < p(s') - p(t)$  for some  $s' < s$ . At date  $s'$ , the agent always has the option of holding on to the asset until date  $s$  and proceeding optimally thereafter. This implies  $\Pi(t, s') \geq \Pi(t, s)$ . But then

$$p(s') - p(t) > \Pi(t, s') \geq \Pi(t, s) \geq p(s) - p(t)$$

which contradicts the fact that  $p(s)$  is assumed to be increasing.

Now, suppose an agent can buy the asset in equilibrium. This means that the original owners of the asset would be willing to sell it: either because they are the ones selling the asset, or if someone else is selling the asset, because the original owners would demand less to sell it than an agent who is leveraged. But this means that the lender would weakly prefer for the agent to sell it, and given  $p(t)$  is increasing, would strictly prefer the agent sell it beyond the current date. This implies that in equilibrium, the contract should be designed to encourage the trader to sell the asset at the first arrival, potentially to the point of paying the agent to unload the asset. But the constraint that  $x_t^s(b, 1) \geq p(t)$  precludes this. The contract that emerges as optimal in equilibrium turns out to be the maximally backloaded contract that yields a utility of  $V_0(r_t^e)$  to the agent. That is, the contract will set  $x_t^s(b, 1)$  to  $p(t)$ , the lowest value it can attain, for as long as possible, and then pile up any transfers above  $p(t)$  to later dates. Formally, define  $T_t \equiv \sup \{s : x_t^s(b, 1) = p(t)\}$  as the last date in which the agent pays back no more than what he borrowed. Then the nature of the contract will be as follows. At any date  $s < T_t$ , by definition the contract will set  $x_t^s(b, 1) = p(t)$ . If  $T_t < 1$ , the contract will further specify

$$x_t^s(b, 1) = p(s) \text{ for } s \geq T, \quad x_t^1(b, D) = D$$

i.e. the agent will have to hand over all of his profits if he fails to sell the asset by date  $T$ . If  $T_t = 1$ , the payment he will be required to make if he fails to sell the asset and it pays out a dividend will satisfy

$$x_t^1(b, D) \in [p(t) + r_t^e, D]$$

and its exact value will be chosen to deliver an expected utility of  $V_0(r_t^e)$  to the agent. It is straightforward to show that there exists a unique pair  $T_t$  and  $x_t^1(b, D)$  that satisfy the above restrictions and deliver the required utility of  $V_0(r_t^e)$ .

Let us refer to the maximally backloaded contract described above which delivers  $V_0(r_t^e)$  as  $x^*$ , and the set of dates during which an agent facing this contract would be willing to sell the asset by  $\Sigma^*(t)$ . By a similar argument as above, the optimal strategy for an agent facing contract  $x^*$  is a cutoff rule, and so  $\Sigma^*(t) = (s_t^*, 1]$  for some  $s_t^* \leq T_t$ . I now argue the contract  $x^*$  satisfies the following property. Pick any contract  $x \neq x^*$  that yields the agent an expected utility of  $V_0(r_t^e)$ , and let  $\Sigma(t; x)$  denote the set of dates at which an agent would choose to sell the asset. Then  $\Sigma(t; x) \subset \Sigma^*(t)$ , i.e. for any date  $s$  that an agent would agree to sell the asset under some contract  $x \neq x^*$ , the agent would also agree to sell it at that date if he faced the contract  $x^*$ .

To see this, pick any  $s \in \Sigma(t; x)$  for some contract  $x \neq x^*$ . If  $s \geq T_t$ , the backloaded contract will guarantee zero terminal wealth no matter what the agent does from time  $s$  on, so an agent facing  $x^*$  will be willing to sell the asset at date  $s$ . Suppose instead that  $s < T_t$ . Since  $s \in \Sigma(t; x)$ , then the agent must prefer to sell the asset to holding on to it, i.e.

$$p(s) - x_t^s(b, 1) \geq W(t, s; x)$$

where  $W(t, s; x)$  denotes the value of waiting given the contract  $x$ . We now argue that  $W(t, s; x) \geq W^*(t, s)$  for  $s < T_t$ . Using the expressions for  $Q(s)$  and  $f(x|s)$ , we have

$$\begin{aligned} W(t, s; x) &= Q(s) \int_s^1 \max[W(t, \tau; x), p(\tau) - x_t^\tau(b, 1)] f(\tau|s) d\tau \\ &\quad + (1 - Q(s)) \epsilon(D - x_t^1(b, D)) \\ &= \int_s^1 \max[W(t, \tau; x), p(\tau) - x_t^\tau(b, 1)] \lambda e^{-\lambda(\tau-s)} d\tau \\ &\quad + e^{-\lambda(1-s)} \epsilon(D - x_t^1(b, D)) \\ &= e^{\lambda s} \left[ \int_s^1 \max[W(t, \tau; x), p(\tau) - x_t^\tau(b, 1)] \lambda e^{-\lambda\tau} d\tau + e^{-\lambda} \epsilon(D - x_t^1(b, D)) \right] \end{aligned}$$

If we differentiate this expression with respect to  $s$ , we get

$$\frac{\partial W(t, s; x)}{\partial s} = \lambda W(t, s; x) - \lambda \max[W(t, s; x), p(s) - x_t^s(b, 1)]$$

Consider any contract  $x \neq x^*$ . Suppose  $W(t, s; x) = W^*(t, s)$  for some  $s < T_t$ . Since any contract must satisfy  $x_t^s(b, 1) \geq p(t)$  by (IC-3), it follows that

$$\begin{aligned} \max[W^*(t, s; x), p(s) - p(t)] &\geq \max[W^*(t, s; x), p(s) - x_t^s(b, 1)] \\ &= \max[W(t, s; x), p(s) - x_t^s(b, 1)] \end{aligned}$$

Note that when  $x = x^*$ , we have  $x_t^s(b, 1) = p(t)$ . This implies that whenever  $W(t, s; x) = W^*(t, s)$ , we have

$$\begin{aligned} \frac{\partial W^*(t, s)}{\partial s} &= \lambda W^*(t, s; x) - \lambda \max[W^*(t, s; x), p(s) - p(t)] \\ &= \lambda W(t, s; x) - \lambda \max[W(t, s; x), p(s) - p(t)] \\ &\leq \lambda W(t, s; x) - \lambda \max[W(t, s; x), p(s) - x_t^s(b, 1)] \\ &= \frac{\partial W(t, s; x)}{\partial s} \end{aligned}$$

i.e.  $\frac{\partial W^*(t, s)}{\partial s} \leq \frac{\partial W(t, s; x)}{\partial s}$  whenever  $W^*(t, s) = W(t, s; x)$ . Since  $W^*(t, t) = W(t, t; x) = V_0(r_t^e)$ , it follows that

$$W^*(t, s) \leq W(t, s; x)$$

as claimed. Hence, we have

$$\begin{aligned} p(s) - p(t) &\geq p(s) - x_t^s(b, 1) \\ &\geq W(t, s; x) \\ &\geq W^*(t, s) \end{aligned}$$

These inequalities imply the agent would be willing to sell the asset and contract  $x^*$ .

Since the creditor would like to have the agent to sell the asset at all dates, it follows that given two contracts  $x$  and  $x'$  that give the agent the same expected utility  $V_0(r_t^e)$  such that  $\Sigma(t; x) \subset \Sigma'(t; x')$ , the creditor will prefer contract  $x'$  over contract  $x$ . The maximally backloaded contract thus solves the planner's problem. ■

**Claim 9:** Suppose positive margin requirements are applied only at  $t \in [0, t^*)$ . Then given the assumption on  $n_t$ , there exists an equilibrium in which the price of the asset is the same at  $t \in [t^*, 1)$  as the equilibrium price when  $t^* = 0$ , and at all  $t \in [0, 1]$  the price of the asset exceeds  $\epsilon D$  if an arrival occurs. In addition,

- i. Non-entrepreneurs wish to purchase the asset at all dates, but can only do so from date  $t^*$  on.
- ii. Agents with vast resources would purchase the asset using their own wealth up to some date  $t^{**} \in [0, 1)$ .
- iii. If  $\lambda > 0$ , then  $t^{**} \geq t^*$ , i.e. wealthy agents will buy the asset when margin requirements are in place.

**Proof:** I first argue that the original price  $p(t)$  remains an equilibrium at dates  $t \in [t^*, 1)$ . Since wealthy agents account for a measure zero of all agents, at date  $t^*$  we can proceed as if no traders arrived before date  $t^*$ . Let  $t'$  denote the first arrival after date  $t^*$  in which a positive number of traders arrive. By assumption, the number of traders who arrive is equal to the number of assets. Since  $p(\cdot)$  is an equilibrium price in the original model without margin requirements, it follows that all buyers would be willing to buy the asset at price  $p(t')$  given the same continuation price path, and that all original owners would be willing to sell at this price given the continuation path of prices is also  $p(\cdot)$ . Hence, the asset market will clear at the prices that clear the market in the original equilibrium, with all assets changing hands. The same argument applies at subsequent arrivals beyond  $t'$ . Since  $p(t) > \epsilon D$  for all  $t \in [t^*, 1)$ , no original owner would sell it for  $\epsilon D$ . When the measure of entrepreneurs is positive, the price must be bounded away from  $\epsilon D$  before  $t^*$ , and will do so in the limit as this measure tends to zero.

Next, I argue that agents with vast resources would purchase the asset up to some date  $t^{**} \geq t^*$ . Before date  $t^*$ , the original owners must be indifferent to selling the asset, since there will not be enough demand to buy out all original owners. However, the problem for the original owners who choose not to sell the asset is equivalent to buying the asset with one's own funds. Thus, buyers would be willing to buy the asset with their own wealth in equilibrium at all dates before  $t^*$ . Beyond date  $t^*$ , a similar argument to Claim 4, their optimal strategy will be a cutoff rule of selling if the price exceeds some cutoff level, i.e. an agent who purchased the asset at date  $t$  will be willing to sell it from some date  $\sigma_t^*$  on, and the expected profits from purchasing the asset at date  $t$  are given by

$$V_0(t) = \max_{\sigma_t^*} Q(\sigma_t^*) \int_{\sigma_t^*}^1 [p(x) - p(t)] f(x|\sigma_t^*) dx + (1 - Q(\sigma_t^*)) [\epsilon D - p(t)]$$

where  $Q(s)$  denotes the probability that at least one more agent who is willing and able to buy the asset will arrive after date  $s$ , and  $f(x|s)$  denotes the density of the first such arrival conditional on there being such an arrival. A creditor will be willing to purchase the asset if maximizing the above expression over all  $\sigma_t^*$  yields a positive expression. Applying the envelope theorem, we have that

$$\frac{dV_0(t)}{dt} = -p'(t).$$

Since  $p(t)$  is increasing, it follows that if there exists a  $t^{**}$  such that  $V_0(t^{**}) = 0$ , creditors will not find it profitable to purchase the asset beyond date  $t^{**}$ . A contradiction argument shows that  $t^{**} < 1$ .

Finally, since the measure of wealthy agents is negligible, once non-entrepreneurs can buy assets, they will be willing to do so given they were willing to do so under the original equilibrium. ■

**Claim 10:** For any  $r^* > 0$ , suppose  $r_t^{FF} = r^*$  for all  $t \in [0, 1 - t^*)$  and  $r_t^{FF}$  for  $t \in [t^*, 1]$  is set to ensure

$$R_{1-t^*,1}^{FF} \equiv \exp\left(\int_{1-t^*}^1 r_t^{FF} dt\right) \geq \frac{1}{\epsilon}$$

Then a bubble cannot occur equilibrium.

**Proof:** Define  $T = \sup\{t \mid \text{Prob}(\text{asset sells at date } t) > 0\}$  as the supremum over all dates at which the asset might trade. Suppose that  $T > 0$ . Consider the expected profit from purchasing the asset at date  $T - \epsilon$ . The probability that at least one trader arrives in this interval is given by  $Q_\epsilon = 1 - e^{-\lambda\epsilon}$ , which tends to 0 as  $\epsilon \rightarrow 0$ . The expected payoff to an agent from buying the asset at date  $t$  is at most

$$Q_\epsilon \max\left\{\lim_{\tau \rightarrow T^-} p(\tau) - p(T - \epsilon) - r^*, 0\right\} + \epsilon(1 - Q_\epsilon) \max\{D - R_{1-t^*,1}^{FF} p(t), 0\}$$

Since  $p(t) > p(0)$ , the second term is equal to zero. Since  $\lim_{\tau \rightarrow T^-} p(\tau) = \lim_{\epsilon \rightarrow 0} p(T - \epsilon)$ , it follows that there exists an  $\epsilon^* > 0$  such that  $p(T) - p(t) < r^*$  for all  $t \in (T - \epsilon^*, T)$ . But then no agent would purchase the asset given the tiny cost of a financial transaction, contradicting the claim that  $T$  is the supremum for the set of dates at which the asset trades. ■

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