Equilibrium Debt Financing*

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Abstract

Building on Geanakoplos (2009), we study an equilibrium model of an asset market with cash-constrained optimists using their asset positions as collateral to raise debt financing. By using a general binomial setting with time-varying belief dispersion between agents, we highlight an asset's marketability as an important determinant of its collateral value. This is because the availability of secondary market trading allows creditors to sell seized collateral to other optimists with saved cash, which, in turn, not only boosts creditors' initial valuation of the collateral but also motivates optimists to save cash. Our model also establishes the maximum riskless short-term debt as the only debt contract used in the equilibrium and a risk-neutral representation of the equilibrium asset price and prices of debt contracts collateralized by the asset.

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1 Introduction

Economists have increasingly recognized the availability and cost of financing as a key economic factor in determining asset market dynamics. Shleifer and Vishny (1997) highlight redemption risk faced by professional arbitrageurs in limiting their ability to eliminate asset mispricing, while Gromb and Vayanos (2002) and Brunnermeier and Pedersen (2009) emphasize margin risk faced by leveraged institutions in causing them to destabilize asset prices. The difficulty of financial institutions to obtain financing for their investment positions came to the forefront of the recent credit crisis, resulting in the prices of many assets being depressed to levels substantially below their fundamentals. For example, after the failure of Lehman Brothers the prices of corporate bonds became so low that one can earn a positive excess return by buying a corporate bond and a credit-swap (CDS) contract that protects against default on the bond. Garleanu and Pedersen (2010) attribute the underpricing to the inability of financial institutions to use these assets as collateral to raise sufficient financing.[1] This leads to an important economic question: what determines an asset's collateral value? In this paper, we develop a model to highlight the role of an asset's marketability.

Our model builds on the framework proposed by Geanakoplos (2009). This framework consists of a risky asset whose fundamental value fluctuates over time with two possible final states. There is a continuum of agents with heterogeneous beliefs about the fundamental fluctuation probability. Agents with optimistic beliefs but limited cash can use their asset holdings as collateral to raise debt financing from less optimistic creditors. The asset's collateral value depends on the marginal creditor's asset valuation, and determines the buyers' financing capacity to bid up the equilibrium price. A convenient feature of this framework is that optimists always use the maximum one-period riskless debt to finance their investments. As a result, a negative fundamental shock wipes out the initial asset buyers and thus causes both the marginal asset buyer and marginal creditor to shift downward along the belief line, leading to a leverage cycle.

In a follow-up paper, Simsek (2010) considers a static setting in which two groups of agents trade a risky asset with a continuum of fundamental states. He shows that if the agents' belief dispersion is monotonic—i.e., higher for upper states than for lower states, then it is optimal for optimists to use risky debt to borrow from the less optimistic creditors,

[1] See Coffey, Hrung, and Sarkar (2009) for another example of the failure of covered interest rate parity in the dollar interest rate and exchange rates with other currencies during the recent credit crisis.

despite their debt being undervalued. The contrast between the debt financing derived in these models suggests that the optimal form of debt financing may crucially depend on the state specification and agents' belief structure. In this paper, we formally examine this issue by using a general specification to nest the settings of these models.

Specifically, we extend the dynamic framework of Geanakoplos along two important dimensions. First, we allow the asset fundamental to follow a binomial tree, which takes multiple values after multiple periods. This feature makes it possible for risky debt to be optimal. Second, and more importantly, we allow two groups of agents to have time-varying beliefs about the asset fundamental, and, in particular, to have more dispersed beliefs after a negative fundamental shock. To the extent that the more dispersed beliefs make debt refinancing more expensive to optimists (i.e., greater rollover risk), this possibility motivates optimists to use long-term debt to lock in their financing cost, and thus makes their debt maturity choice non-trivial. The greater belief dispersion after a negative shock also implies that agents disagree more about lower states on the final date than about upper states. This belief structure thus goes beyond the monotonic belief structure considered by Simsek.

To uncover the role played by the asset's marketability, we contrast two settings. In our main setting, agents can freely trade in and out of the asset on any date. In the other benchmark setting, agents are allowed to trade the asset only on the initial date. One can interpret the asset in the benchmark setting as non-marketable, which, as highlighted by Longstaff (2009), is a realistic problem for many risky assets during the recent credit crisis. The non-marketability of the asset makes the benchmark setting essentially static and thus analogous to that of Simsek.

Our model delivers several results. First, it highlights the role played by the asset's marketability in determining the joint equilibrium of the asset and credit markets. In the event of default by some optimists on the interim date, the possibility of buying the liquidated collateral on the interim date also motivates some optimists to save cash. The ability to sell the collateral to other optimists with saved cash also boosts the creditors' valuation of the collateral on the initial date. Taken together, these two effects reduce the exposure of the equilibrium asset price to the greater belief dispersion between the borrowers and creditors regarding the probability of the lower states of the final date. In contrast, in the static benchmark setting, agents cannot trade the asset on the interim date. As the creditors suffer losses from the realization of states with asset fundamental lower than the promised

debt payment, their pessimism about these states can severely limit the asset's collateral value and thus constrain optimists' borrowing and the equilibrium asset price. The contrast between the equilibrium outcomes in these two settings demonstrate an asset's marketability as an important determinant of its collateral value.

Second, our model justifies the optimality of using the maximum riskless short-term leverage to finance an optimist's asset position in a general binomial setting. It is true that once an optimist chooses to acquire an asset position, long-term debt financing can dominate short-term debt financing when belief divergence in the down state of the interim date is sufficiently high (i.e., rollover risk is sufficiently high). However, in this situation, saving cash dominates acquiring the position because it allows the optimist to take advantage of the even better investment opportunity when the belief dispersion widens. Thus, it is never optimal to use long-term debt financing in the equilibrium, despite the presence of rollover risk. This result explains the pervasive use of short-term debt in practice. It also verifies the result initially put forward by Geanakoplos. It is important to note that this result builds on optimists' cash saving in the market equilibrium and thus is more subtle than his simple argument that short-term debt allows an optimist to maximize riskless leverage.

Finally, despite the market incompleteness caused by agents' borrowing constraints, we are able to derive a risk-neutral representation of the equilibrium asset price and prices of different debt contracts collateralized by the asset. This representation holds in our binomial setting with any number of periods. It builds on the basic idea that the payoff of a collateralized debt contract is monotonic with respect to the asset value and, due to the binomial uncertainty across each period, needs to provide the same expected return and share the same marginal investor as the asset. This representation facilitates our analysis of the equilibrium and could prove useful in analyzing issues related to collateralized debt financing in more complex settings.[2]

By establishing an asset's marketability as a determinant of its collateral value, our model helps explain the dry-up of financial institutions' funding liquidity during the recent credit crisis through the freeze of market liquidity. This mechanism complements Brunnermeier and Pedersen (2009), who highlight the feedback effect of funding liquidity on market liquidity. Our model also adds to the understanding of borrowers' rollover risk, e.g., Acharya, Gale, and Yorulmaker (2009), and He and Xiong (2009a, b), by demonstrating the important role

[2] See Cao (2010) for a recent study of a dynamic asset pricing model with optimistic agents using short-term collateralized debt contracts to finance their investment positions.

of some agents' cash saving in mitigating rollover risk in equilibrium.

Our model is related to the literature that studies the pervasive use of short-term debt by banks and financial firms. The existing literature has emphasized several advantages of short-term debt. First, short-term debt is a natural solution to a variety of agency problems inside a firm, e.g., Calomiris and Kahn (1991) and Diamond and Rajan (2001). By choosing short-term financing, creditors keep the option to pull out if they discover that firm managers are pursuing value-destroying projects. Second, the short commitment period also makes short-term debt less information sensitive and thus less exposed to adverse-selection problems, e.g., Gorton and Pennacchi (1990). Different from these theories, our model emphasizes short-term debt as a contingent claim for borrowers to trade payoffs with creditors across different future states based on the dispersion of their beliefs.

Diamond (1991) analyzes debt maturity choice for firms with private information about their future credit rating. In his model, firms face a tradeoff between a preference for short maturity due to expecting their credit rating to improve, against liquidity risk due to the loss of their private rents that cannot be assigned to the creditors. Our model also features a similar tradeoff faced by an optimist in using short-term debt—it fulfills his speculative incentive to have a large position at the expense of future rollover risk. Different from Diamond's model, we also incorporate another important feature of asset market equilibrium—each optimist's additional choice to save cash, which dominates establishing an asset position when long-term debt is the preferred financing.

The paper is organized as follows. Section 2 describes the model. We derive the benchmark static setting in Section 3, and then analyze the dynamic setting in Section 4. Finally, Section 5 concludes the paper. We provide the technical proofs in Appendix A and extend the two-period model described in the main text to N periods in Appendix B.

2 The Model

Consider a model with three dates and two periods. The date is indexed by $t = 0, 1, 2$. There is a long-term risky asset, which we interpret either as a stock or a mortgage backed security. The asset pays a final payoff on date 2. The final payoff is determined by the final realization of a publicly observable binomial tree. Figure 1 illustrates the tree. The tree can go either up or down in each period. The tree has four possible paths, which we denote by uu, ud, du, and dd (here, u stands for "up" and d stands for "down"), and three possible

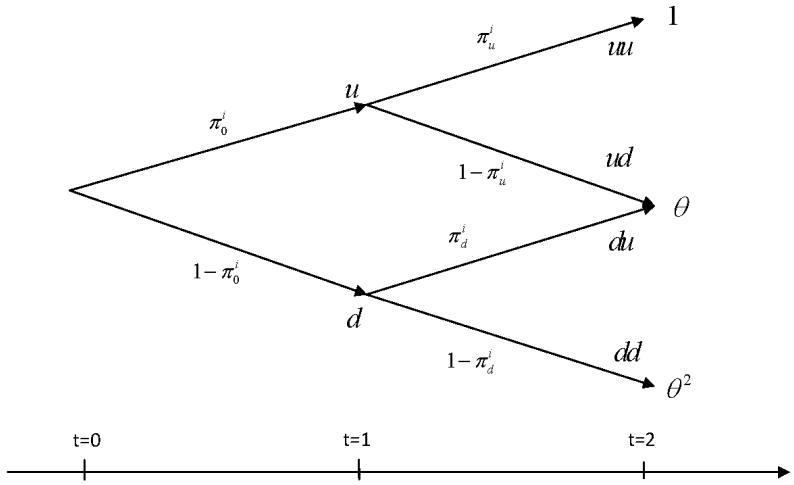

Figure 1: Timeline.

final nodes (paths ud and du lead to the same final node). We normalize the final payoff of the risky asset at the end of path uu as 1, at the end of paths ud and du as θ, and at the end of paths dd as θ^2, where $\theta \in (0,1)$. We denote the asset payoff by $\widetilde{\theta} \in \{1, \theta, \theta^2\}$.

The probability of the tree going up in each period is unobservable. Suppose that there are two groups of risk-neutral agents, who differ in their beliefs about these probabilities. There are three intermediate states on the tree, one on date 0 and two on date 1 (u and d). We collect these intermediate states in the following set: $\{0, u, d\}$. In each of the states, each agent has a belief about the probability of the tree going up in the following period. We collect each agent's beliefs in the following set: $\{\pi_0^i, \pi_u^i, \pi_d^i\}$, where $i \in \{h, l\}$ indicates the agent's type. Throughout this section, we assume that the h-type agents are always more optimistic than the l-type agents across all the intermediate states (here, the superscript "h" and "l" stand for high and low.) That is, $\pi_n^h \geq \pi_n^l$ for any $n \in \{0, u, d\}$. Based on the relative order, we call the h-type agents optimists and the l-type pessimists.

We emphasize that each agent's belief changes over time. The change could be driven either by his learning process or by his sentiment fluctuation.[3] As a result, the belief dispersion between the optimists and pessimists is not constant. Standing at $t = 0$, the difference between π_0^h and π_0^l represents the initial belief dispersion between the two groups about the fundamental fluctuation from date 0 to 1, while the difference between π_d^h and π_d^l represents

[3] See Morris (1996) for a model that studies heterogeneous beliefs with learning in a binomial-tree setting.

Table 1: Asset Payoff and Agent's Belief across Different Paths

	Tree Paths			
	uu	ud	du	dd
Asset payoff	1	θ	θ	θ^2
Optimists' belief	$\pi_0^h \pi_u^h$	$\pi_0^h \left(1-\pi_u^h\right)$	$\left(1-\pi_0^h\right)\pi_d^h$	$\left(1-\pi_0^h\right)\left(1-\pi_d^h\right)$
Pessimists' belief	$\pi_0^l \pi_u^l$	$\pi_0^l \left(1-\pi_u^l\right)$	$\left(1-\pi_0^l\right)\pi_d^l$	$\left(1-\pi_0^l\right)\left(1-\pi_d^l\right)$

the future belief dispersion about the fundamental fluctuation from the state d of date 1 to date 2. As we will show later, these two types of belief dispersion play distinctive roles in determining the optimal debt maturity choice. Later in our analysis, we will highlight the case where the belief dispersion converges in state u (i.e., $\pi_u^h = \pi_u^l$) but diverge further in state d (i.e., $\pi_d^h - \pi_d^l \geq \pi_0^h - \pi_0^l \geq 0$).

We summarize the final asset payoffs at the end of the four possible tree paths and the optimists' and pessimists' belief about each of the paths in Table 1. Note that the optimists assign a higher probability to path uu and a lower probability to path dd. But his beliefs about the middle paths ud and du can be higher or lower than those of the pessimists.

We assume the optimists and pessimists are all risk neutral. We normalize the total supply of the asset, which is initially endowed by pessimists, to be one unit. Without loss of generality, we normalize the measure of optimists to be one. On date 0, optimists are homogeneous, each endowed with c dollars of cash. For simplicity, we assume that pessimists have sufficient cash (deep pocket) to finance optimists' borrowing. We also make a realistic assumption that short-sales of the asset are not allowed.[4]

The focus of our analysis is on the financing of optimists' asset purchases. Since they may not have sufficient cash, they need to borrow from pessimists who sit on the sideline with cash. As pessimists' beliefs affect optimists' cost of financing, their beliefs can indirectly affect the equilibrium asset price. For simplicity, we assume that both the risk-free interest rate and the agents' discount rate are zero. Because pessimists are competitive and have deep pockets, in equilibrium they always demand zero expected return from the asset or debt contract they hold.

[4] In practice, shorting assets other than stocks is difficult and requires expertise. Even for stocks, shorting can be difficult when the supply of shares is limited.

Our setting is related to several strands of the literature. First, our setting extends the model of Geanakoplos (2009) to allow for time-varying beliefs and general final fundamental states. Time-varying beliefs allow us to highlight an optimist's debt maturity choice in the presence of rollover risk, which becomes pronounced when optimists and pessimists' beliefs diverge in state d. We simplify the continuum of constant beliefs considered in Geanakoplos model to two types. As we will show later, by focusing homogeneous optimists, we can isolate an optimist's incentive to save cash from a pure belief effect. The generally specified binomial tree with multiple final states also allows us to have a setting similar to that of Simsek (2010), who studies a static setting with a continuum of fundamental states.

Second, the presence of both heterogeneous beliefs and short-sales constraints is reminiscent of the models that use the joint effect of these two elements to explain asset overvaluation and bubbles, e.g., Miller (1977), Harrison and Kreps (1978), Morris (1996), Chen, Hong, and Stein (2002), and Scheinkman and Xiong (2003).[5] Third, optimists in our model can also be interpreted as rational arbitrageurs trading against pessimistic noise traders, whose belief may divergence further after a negative fundamental shock in the future, i.e., noise-trader risk as in Shleifer and Vishny (1997). Such noise-trader risk in our setting motivates optimists to secure their debt financing.

3 The Static Setting

We first consider a benchmark setting in which agents cannot trade the asset on date 1 and are restricted from using debt contracts that mature on date 1. One can view the non-marketability of the asset as an extreme form of market illiquidity. As highlighted by Longstaff (2009), non-marketability was a serious problem for many assets during the recent credit crisis. Even during normal periods, some assets, such as corporate bonds and subprime mortgages, are highly illiquid. As a result, investors tend to hold them throughout their lives, and their secondary market prices become uninformative and thus unreliable for settling contracts.

In the absence of trading and contract settlements on date 1, this setting is effectively static with three possible final states on date 2. This static setting resembles the one considered by Simsek (2010), who allows two groups of agents with different beliefs to trade a

[5] For simplicity, we rule out flipping of agents' beliefs, which will lead to a resale option to the initial asset owners. Allowing belief flipping does not change the key result of our model.

risky asset with a continuum of states. His analysis focuses on the situation in which the two groups' belief dispersion monotonically increases with the state (i.e., the dispersion is higher for upper states than for lower states). He shows that in this situation, optimists will take on risky debt from pessimists to finance their investments in the risky asset. We will consider both monotonic belief dispersion as well as non-monotonic belief dispersion. In fact, our analysis will show that when the belief dispersion is non-monotonic, optimists will refrain themselves from borrowing risky debt from pessimists in equilibrium.

3.1 Credit and Asset Market Equilibrium

Like Geanakoplos (2009), we assume that the optimists use their asset holdings as collateral to obtain debt financing. We focus on non-contingent debt contracts. A non-contingent debt contract specifies a constant debt payment (face value) at maturity unless the borrower defaults. Non-contingent debt contracts are widely used in practice. Townsend (1979) explains its popularity based on the cost of verifying the state of the world. That is, non-contingent debt contracts circumvent the cost of verifying the value of the collateral as long as the borrower makes the promised payment. Diamond (1984) and Bolton and Scharfstein (1990) also derive the optimality of non-contingent debt based on unobservability of cash flows. In this model, we will restrict borrowers to use only non-contingent debt.

Suppose that the date-0 price of the asset $p_0 \in \left[\mathbb{E}_0^l\left[\widetilde{\theta}\right], \mathbb{E}_0^h\left[\widetilde{\theta}\right]\right)$, which has to hold in equilibrium. Suppose that an optimist uses a long-term debt contract, collateralized by one unit of the asset and with a promised payment of $F \in \left[\theta^2, \theta\right]$ due on date 2. Depending on the asset's final payoff, the debt payment is $\widetilde{\theta} \wedge F \equiv \min\left(F, \widetilde{\theta}\right)$. A pessimistic creditor will grant the following credit for the contract:

$$D_0 = \mathbb{E}_0^l\left[\widetilde{\theta} \wedge F\right].$$

Thus, to establish the position, the optimist has to use $p_0 - \mathbb{E}_0^l\left[\widetilde{\theta} \wedge F\right]$ of his own cash. This amount is exactly the so called haircut. With a cash endowment of c, the optimist can purchase

$$\frac{c}{p_0 - \mathbb{E}_0^l\left[\widetilde{\theta} \wedge F\right]}$$

units of asset. Since each unit gives the optimist an expected payoff of $\mathbb{E}_0^h\left[\widetilde{\theta} - \widetilde{\theta} \wedge F\right]$, he

can maximize his expected value by choosing the debt promise F:

$$\max_F c \frac{\mathbb{E}_0^h\left[\widetilde{\theta}-\widetilde{\theta}\wedge F\right]}{p_0-\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F\right]}. \tag{1}$$

In formulating this objective, we implicitly assume that the credit offered by the pessimistic creditor is the same as the market value of the debt contract. As other optimists may find the debt contract attractive and offer a higher price than pessimists, we need to verify that this is not the case in equilibrium. In fact, as we will show later, this condition does bind in certain situations. Formally, suppose that in equilibrium the optimal debt promise is F^*, which maximizes the optimist's objective in (1). This implies that the marginal value of one dollar, v_0, to an optimist from acquiring the levered asset position is $\frac{\mathbb{E}_0^h\left[\widetilde{\theta}-\widetilde{\theta}\wedge F^*\right]}{p_0-\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F^*\right]}$.We need to verify that this is not lower than his expected value from using an optimal leverage to acquire the debt contract (which has a payoff of $\widetilde{\theta}\wedge F^*$):

$$\frac{\mathbb{E}_0^h\left[\widetilde{\theta}-\widetilde{\theta}\wedge F^*\right]}{p_0-\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F^*\right]} \geq \max_F \frac{\mathbb{E}_0^h\left[\widetilde{\theta}\wedge F^*-\left(\widetilde{\theta}\wedge F^*\right)\wedge F\right]}{\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F^*\right]-\mathbb{E}_0^l\left[\left(\widetilde{\theta}\wedge F^*\right)\wedge F\right]}. \tag{2}$$

Thus, we call a pair of (p_0, F^*) the joint equilibrium of the asset and credit markets if the following two conditions hold:

1. Given the asset price p_0, F^* maximizes an individual optimist's investment objective in (1) and satisfies the constraint in (2).

2. The asset market clears:

$$p_0 = \begin{cases} \mathbb{E}_0^l\left[\widetilde{\theta}\right] & \text{if } c+\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F^*\right] < \mathbb{E}_0^l\left[\widetilde{\theta}\right] \\ c+\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F^*\right] & \text{if } c+\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F^*\right] \in \left[\mathbb{E}_0^l\left[\widetilde{\theta}\right], \mathbb{E}_0^h\left[\widetilde{\theta}\right]\right] \\ \mathbb{E}_0^h\left[\widetilde{\theta}\right] & \text{if } c+\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F^*\right] > \mathbb{E}_0^h\left[\widetilde{\theta}\right] \end{cases}. \tag{3}$$

The market clearing condition requires that if optimists' aggregate purchasing power, which is determined by the sum of their cash endowment c and the borrowed credit $\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F^*\right]$, is lower than pessimists' asset valuation $\mathbb{E}_0^l\left[\widetilde{\theta}\right]$, pessimists will bid up the price to their valuation; if optimists' purchasing power is between pessimists' and optimists' asset valuations, the asset price p_0 is exactly equal to optimists' purchasing power; and if optimists' purchasing power is higher than optimists' asset valuation $\mathbb{E}_0^h\left[\widetilde{\theta}\right]$, they will bid up the price to their valuation.

The following lemma shows that the optimal debt promise must be between θ^2 and θ.

Lemma 1 *The optimist's optimization problem in (1) implies that* $F^* \in \left[\theta^2, \theta\right]$.

3.2 Monotonic Belief Dispersion

We first consider the situation that the belief dispersion between optimists and pessimists about the highest state uu is higher than their belief dispersion about the three upper states $\{uu, ud, du\}$:

$$\frac{\pi_0^h \pi_u^h}{\pi_0^l \pi_u^l} \geq \frac{\pi_0^h + \pi_d^h - \pi_0^h \pi_d^h}{\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l}. \tag{4}$$

This condition implies that the belief dispersion is concentrated in the highest state and is consistent with the monotonic belief ordering imposed by Simsek (2010). As we will discuss later in the dynamic setting, this condition also implies that the rollover risk faced by optimists with short-term debt financing in state d of date 1 is modest relative to their initial speculative incentives on date 0.

We can prove that in this case, the constraint in (2) always holds in equilibrium. Therefore, the equilibrium can be derived from solving an individual optimist's problem in (1) joint with the market clearing condition (3).

Specifically, define

$$P_M \equiv \left(\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l\right) \frac{\pi_0^h \pi_u^h + \left(\pi_0^h \left(1 - \pi_u^h\right) + \left(1 - \pi_0^h\right) \pi_d^h\right) \theta}{\pi_0^h + \pi_d^h - \pi_0^h \pi_d^h} + \left(1 - \pi_0^l\right) \left(1 - \pi_d^l\right) \theta^2$$

as a critical price level, which under the belief condition (4), satisfies

$$P_M \geq \mathbb{E}_0^l \left[\widetilde{\theta}\right].$$

Then, an optimist's optimal debt promise depends on the asset price:

$$F^*\left(p_0\right) = \begin{cases} \theta^2, & \text{if } p_0 \in \left(P_M, \mathbb{E}_0^h \left[\widetilde{\theta}\right]\right); \\ \text{any value in } \left[\theta^2, \theta\right], & \text{if } p_0 = P_M; \\ \theta, & \text{if } p_0 \in \left(\mathbb{E}_0^l \left[\widetilde{\theta}\right], P_M\right]. \end{cases}$$

By using this debt contract, the optimist obtains a credit of

$$C\left(F^*\right) \equiv \left(1 - \pi_0^l\right) \left(1 - \pi_d^l\right) \theta^2 + \left(\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l\right) F^*.$$

Finally, by imposing the market clearing condition (3), we obtain the equilibrium described in the following proposition.

Proposition 2 *Under the belief condition (4), the asset and credit market equilibrium is characterized by the following four cases:*

1. *if* $c \in \left(P_M - \theta^2, \mathbb{E}_0^h\left[\widetilde{\theta}\right] - \theta^2\right]$, *then* $F^* = \theta^2$ *and* $p_0 = \theta^2 + c$;
2. *if* $c \in \left(P_M - C(\theta), P_M - \theta^2\right]$, *then* $F^* = \frac{p_M - c - \left(1-\pi_0^l\right)\left(1-\pi_d^l\right)\theta^2}{\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l}$ *and* $p_0 = P_M$;
3. *if* $c \in \left(\mathbb{E}_0^l\left[\widetilde{\theta}\right] - C(\theta), P_M - C(\theta)\right]$, *then* $F^* = \theta$ *and* $p_0 = C(\theta) + c$;
4. *if* $c \leq \mathbb{E}_0^l\left[\widetilde{\theta}\right] - C(\theta)$, *then* $F^* = \theta$ *and* $p_0 = \mathbb{E}_0^l\left[\widetilde{\theta}\right]$.

This proposition shows that when optimists' initial cash endowment is low (e.g., cases 2, 3, and 4), they will use risky debt to borrow from pessimists to finance their asset positions. This basic result is consistent with that of Simsek (2010).

3.3 Non-monotonic Belief Dispersion

We now consider the situation that the belief dispersion between optimists and pessimists about the highest state uu is lower than their belief dispersion about the three upper states $\{uu, ud, du\}$:

$$\frac{\pi_0^h \pi_u^h}{\pi_0^l \pi_u^l} < \frac{\pi_0^h + \pi_d^h - \pi_0^h \pi_d^h}{\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l}. \tag{5}$$

Interestingly, under this condition, we can show that there does not exist any equilibrium in which optimists use risky debt to borrow from pessimists. The reason is that the belief condition in (5) implies that the belief dispersion between optimists and pessimists about the two middle paths (ud and du), which lead to $\widetilde{\theta} = \theta$, is higher than the belief dispersion about the highest state uu, which gives $\widetilde{\theta} = 1$. As a result, if any risky debt, say a debt contract with face value θ is offered in the equilibrium at the pessimists' valuation, then this risky debt offers a strictly better investment opportunity to any optimist than the risky asset itself. Then, some optimists will choose to withdraw from the risky asset to acquire the risky debt, which, in turn, causes optimists' aggregate leverage and the equilibrium asset price to fall. Due to this logic, the constraint in (2) is violated, and the only equilibrium that can satisfy this constraint is the one with only riskless borrowing by optimists from pessimists. We describe the equilibrium in the following proposition

Proposition 3 *Under the belief condition (5), optimists only borrow riskless debt from pessimists and the equilibrium is characterized by three cases:*

1. *if* $c \in \left(\mathbb{E}_0^h\left[\widetilde{\theta}\right] - \theta^2, \mathbb{E}_0^h\left[\widetilde{\theta}\right]\right]$, *then* $F^* = \mathbb{E}_0^h\left[\widetilde{\theta}\right] - c$ *and* $p_0 = \mathbb{E}_0^h\left[\widetilde{\theta}\right]$;

2. *if* $c \in \left(\mathbb{E}_0^l\left[\widetilde{\theta}\right] - \theta^2, \mathbb{E}_0^h\left[\widetilde{\theta}\right] - \theta^2\right]$, *then* $F^* = \theta^2$ *and* $p_0 = \theta^2 + c$;

3. *if* $c \leq \mathbb{E}_0^l\left[\widetilde{\theta}\right] - \theta^2$, *then* $F^* = \theta^2$ *and* $p_0 = \mathbb{E}_0^l\left[\widetilde{\theta}\right]$.

Taken together, propositions 2 and 3 demonstrate that in the static setting the equilibrium debt financing used by optimists crucially depends on the distribution of the optimists' and pessimists' beliefs across different states. We will use the equilibrium derived in the static setting as the benchmark for understanding the equilibrium in the dynamic setting, which we derive in the next section.

4 Dynamic Setting

We now allow agents to trade the risky asset on the interim date—date 1—after the two possible states, u and d, are revealed to the public. As a result, each optimist can also make endogenous investment and leverage decisions in these two interim states. Moreover, each optimist has the choice to use either short-term debt maturing on date 1 or long-term debt maturing on date 2 to finance his asset position on date 0.

We denote the asset price on date 0 and in states u and d of date 1 by $\{p_0, p_u, p_d\}$. We allow all debt contracts (either long-term or short-term) to be tradable on dates 0 and 1. For a long-term debt contract collateralized by one unit of asset with a promise to pay F on date 2, we denote its market valuation on date 0 and in states u and d of date 0 by $\left\{D_0^L(F), D_u^L(F), D_s^L(F)\right\}$, where the superscript L refers to long-term debt and the subscript refers to the relevant state. For a short-term debt contract with a promise to pay F on the following date, we denote its market value on date 0 and in states u and d of date 0 by $\left\{D_0^S(F), D_u^S(F), D_s^S(F)\right\}$. Note that a long-term debt contract that matures on date 2 becomes identical to a short-term debt contract on date 1. Thus, $D_s^L(F) = D_s^S(F)$ for $s \in \{u, d\}$. Because of the presence of heterogeneous agents in the market, it is important to bear in mind that the marginal investor of these debt contracts in each of the states is determined by the equilibrium—his identity can be either optimists or pessimists and can be different across the states.

4.1 Risk-Neutral Representation of Prices

Before deriving each agent's optimization problem, we first establish a risk-neutral representation of the prices of the risky asset and debt contracts collateralized by the asset. This representation is reminiscent of the standard no-arbitrage risk-neutral price representation in complete markets. Because of the borrowing constraints faced by optimists, the markets in our setting are incomplete even though the asset fundamental follows a binomial tree. Instead, this representation builds on the idea that the payoff of a debt contract collateralized by the asset is monotonic with respect to the asset value and, as a result, it shares the same marginal investor as the asset.

4.1.1 Optimists' Marginal Value of Cash

Given the scarcity of cash to optimists, cash allows them to earn extra rents. To illustrate the idea, consider state d of date 1, one period before the risky asset's liquidation value is revealed. Suppose that the asset price is p_d, which lies between the optimists' and pessimists' asset valuations: $p_d \in \left[\mathbb{E}_d^l\left(\widetilde{\theta}\right), \mathbb{E}_d^h\left(\widetilde{\theta}\right)\right]$, where $\mathbb{E}_d^i$ denotes agent-i's conditional expectation in the state. One dollar of cash allows an optimist to acquire the asset at a discount to his valuation. Thus, its marginal value is higher than 1. More specifically, he can use the cash, levered up by using a debt contract collateralized by one unit of asset and with a promise of θ^2 (the maximum riskless promise), to acquire a position of $\frac{1}{p_d-\theta^2}$ units of the asset. This position gives an expected gross profit of $\frac{\pi_d^h\left(\theta-\theta^2\right)}{p_d-\theta^2}$. This expected profit represents his marginal value of cash in state d, as formally derived in the following proposition.[6]

Proposition 4 *Suppose that in state d the market price of the asset* $p_d \in \left[\mathbb{E}_d^l\left(\widetilde{\theta}\right), \mathbb{E}_d^h\left(\widetilde{\theta}\right)\right]$. *Then, the optimal strategy of an optimist with one dollar of cash is to acquire* $\frac{1}{p_d-\theta^2}$ *units of the asset by using a debt contract with a promise of* θ^2. *As a result, his marginal value of cash is*

$$v_d = \frac{\pi_d^h\left(\theta-\theta^2\right)}{p_d-\theta^2}. \tag{6}$$

[6] This proposition confirms the basic intuition of Geanakoplos (2009) that in a static binomial tree model, an optimist always prefers using the maximum riskless leverage to finance his asset position. It is also useful to note that if the prices of the asset and the debt contract collateralized by the asset are determined by the same marginal investor, the optimist is in fact indifferent between any promise between θ^2 and θ. This is because in this case the cost of the risky part of the debt exactly offsets the expected profit from the asset.

As the asset price p_d is bounded from below by pessimists' asset valuation $\mathbb{E}_d^l\left(\widetilde{\theta}\right) = \pi_d^l\theta + \left(1-\pi_d^l\right)\theta^2$, it is direct to see that the optimist's marginal value of cash is bounded from above: $v_d \leq \pi_d^h/\pi_d^l$.

We can also establish a similar result for state u. Suppose that in state u the market price of the asset is $p_u \in \left[\mathbb{E}_u^l\left(\widetilde{\theta}\right), \mathbb{E}_u^h\left(\widetilde{\theta}\right)\right]$ and that the marginal investor of the debt contracts in the markets is pessimists. Then, his marginal value of cash is

$$v_u = \frac{\pi_u^h\left(1-\theta\right)}{p_u - \theta}. \tag{7}$$

If the beliefs of optimists and pessimists converge in this state, then the asset price has to be equal to their asset valuations and thus the optimist's marginal value of cash is 1, i.e., $v_u = 1$.

Determining the optimist's marginal value of cash on date 0 is more elaborate. On one hand, he can use the cash, levered up by collateralized debt, to acquire the asset. The marginal value of cash depends on the optimal debt financing. As we will show later, the optimal financing is to use the maximum riskless short-term debt, which gives the optimist a marginal value of

$$u_0 = \frac{\pi_0^h v_u\left(p_u - p_d\right)}{p_0 - p_d}. \tag{8}$$

One the other hand, he can save the cash for the next date, which gives him an expected value of $\pi_0^h v_u + \left(1-\pi_0^h\right)v_d$. Optimizing over these two possible choices gives the optimist's marginal value of cash on date 0:

$$v_0 = \max\left(u_0, \pi_0^h v_u + \left(1-\pi_0^h\right)v_d\right). \tag{9}$$

4.1.2 Asset Price

Equipped with optimists' marginal value of cash, the following proposition uses it to establish a risk-neutral representation of the price of the risky asset.

Proposition 5 *Define*

$$\phi_s \equiv \frac{\pi_s^h}{v_s} \in (0,1), \text{ and } \phi_0 \equiv \max\left(\frac{\pi_0^h v_u}{v_0}, \pi_0^l\right) \in (0,1). \tag{10}$$

Then, we have

$$p_u = \phi_u + \left(1-\phi_u\right)\theta, \tag{11}$$

$$p_d = \phi_d\theta + \left(1-\phi_d\right)\theta^2, \tag{12}$$

$$p_0 = \phi_0 p_u + \left(1-\phi_0\right)p_d. \tag{13}$$

Intuitively, ϕ_0, ϕ_u, and ϕ_d reflect the equivalent belief the marginal investor of the asset uses to value the asset. Note that the marginal investor can be either optimists or pessimists. In particular, equation (13) show that on date 0 whether the marginal investor is an optimist or pessimist depends on optimists' cash-value-adjusted belief, $\frac{\pi_0^h v_u}{\max\left(u_0, \pi_0^h v_u + \left(1-\pi_0^h\right) v_d\right)}$, relative to pessimists' belief, π_0^l. Interestingly, a higher expected cash value on date 1, i.e., higher $\pi_0^h v_u + \left(1 - \pi_0^h\right) v_d$, can induce optimists to save cash and, as a result, cause them to have a lower valuation for the asset on date 0 than pessimists despite that $\pi_0^h \geq \pi_0^l$.

4.1.3 Debt Values

Note that collateralized debt contracts are contingent claims whose value monotonically increase with the value of the asset. Given the binomial uncertainty across each period, the return of a debt contract reflects the same risk as the asset. Thus, its valuation has to be in line with the asset, i.e., to share the same marginal investor.

Proposition 6 *In state $s \in \{u, d\}$ of date 1, the market price of a collateralized debt contract with a promise to pay F on date 2 is*

$$D_s^S(F) = \mathbb{E}_s^{\phi_s}\left[F \wedge \widetilde{\theta}\right], \tag{14}$$

where $\mathbb{E}_s^{\phi_s}$ denotes the expectation of an agent who believes that the tree will go up with a probability of ϕ_s in the following period. On date 0, the price of a short-term debt contract with a promise to pay F on date 1 is given by

$$D_0^S(F) = \phi_0 (F \wedge p_u) + (1 - \phi_0)(F \wedge p_d),$$

and the price of a long-term debt with a promise to pay F on date 2 is given by

$$D_0^L(F) = \phi_0 D_u^S(F) + (1 - \phi_0) D_d^S(F),$$

where $D_u^S(F)$ and $D_d^S(F)$ are given in (14).

To understand this proposition, consider a risky debt contract in state d with a promise to pay F on date 2. Its payoff is either $F \wedge \theta$ or $F \wedge \theta^2$ on date 2. If $D_d^S(F) < \phi_d (F \wedge \theta) + (1 - \phi_d)\left(F \wedge \theta^2\right)$, then this debt contract offers a higher expected return than the asset to its marginal investor and thus would motivate him to withdraw from the asset to invest in the contract. This cannot occur in equilibrium. Neither can $D_d^S(F) > \phi_d (F \wedge \theta) + (1 - \phi_d)\left(F \wedge \theta^2\right)$, as this valuation is too high for any one to pay.

4.2 An Optimist's Problem

An individual optimist faces three alternatives on date 0. He can establish an asset position by using his initial cash endowment and short-term debt collateralized by the asset position (we call this strategy S); or by using his initial cash endowment and collateralized long-term debt (we call this strategy L); or he can simply save cash for establishing a position later on date 1 (we call this strategy C). Because the optimist is risk neutral, it is without loss of generality to focus on cornered policies for the optimist's optimization problem.

4.2.1 Optimal Debt Financing

We first study the optimal financing decision for an optimistic buyer to establish an initial asset position on date 0. For each dollar of cash he has, suppose that he establish a position of x units of the asset by using a debt contract $\widetilde{F}$ collateralized by each unit of asset (the debt can be either long-term or short-term). Proposition 6 gives the amount credit the optimist can obtain by selling the debt to the marginal investor in the market:

$$D_0\left(\widetilde{F}\right) = \phi_0 D_u\left(\widetilde{F}\right) + (1-\phi_0) D_d\left(\widetilde{F}\right),$$

where we omit the maturity indicator S or L on debt function D. His budget constraint implies that

$$x = \frac{1}{p_0 - D_0\left(\widetilde{F}\right)}. \tag{15}$$

With this position, his expected value from per unit of cash is

$$\begin{aligned} V_0\left(\widetilde{F}\right) &= x\left[\pi_0^h v_u\left(p_u - D_u\left(\widetilde{F}\right)\right) + \left(1-\pi_0^h\right) v_d\left(p_d - D_d\left(\widetilde{F}\right)\right)\right] \\ &= \frac{\left(\pi_0^h v_u p_u + \left(1-\pi_0^h\right) v_d p_d\right) - \left(\pi_0^h v_u D_u\left(\widetilde{F}\right) + \left(1-\pi_0^h\right) v_d D_d\left(\widetilde{F}\right)\right)}{p_0 - D_0\left(\widetilde{F}\right)}. \end{aligned} \tag{16}$$

This equation reflects two effects. One is a leverage effect: by using a more aggressive debt contract, the optimist can obtain more credit and thus establish a greater position, as shown by the denominator of the formula. The other is a debt-cost effect: a more aggressive debt contract also implies a higher expected debt payment in the future, as shown by the term $\left(\pi_0^h v_u D_u\left(\widetilde{F}\right) + \left(1-\pi_0^h\right) v_d D_d\left(\widetilde{F}\right)\right)$ in the numerator.[7]

[7] Irrespective of the debt maturity, we can conveniently write the optimist's per-asset profit in state s of

4.2.2 Debt Maturity Choice

Given the optimist's expected value from establishing an asset position in (16), we now compare a pair of long-term and short-term debt contracts, which have different promises (F_L and F_S) but give the same initial credit. From the optimist's perspective, he prefers the one with a lower expected debt cost. For these two contracts to give the same initial credit, we must have

$$\phi_0 D_u^S + (1-\phi_0) D_d^S = \phi_0 D_u^L (F_L) + (1-\phi_0) D_d^L (F_L), \tag{17}$$

where with a bit abuse of notation D_s^S (with $s = u$ or d) denotes the value of the short-term debt contract in state s of date 1.

To simplify notation, we define the optimist's cash-value-adjusted probability as

$$\widehat{\pi}_0^h \equiv \frac{\pi_0^h v_u}{\pi_0^h v_u + \left(1-\pi_0^h\right) v_d}. \tag{18}$$

Then, $1 - \widehat{\pi}_0^h = \frac{\left(1-\pi_0^h\right)v_d}{\pi_0^h v_u + \left(1-\pi_0^h\right)v_d}$. With these cash-value-adjusted probabilities, it is direct to show that the short-term debt is preferable if and only if

$$\widehat{\pi}_0^h D_u^S + \left(1-\widehat{\pi}_0^h\right) D_d^S \leq \widehat{\pi}_0^h D_u^L + \left(1-\widehat{\pi}_0^h\right) D_d^L, \tag{19}$$

which, by taking the difference with (17), is equivalent to

$$\left(\widehat{\pi}_0^h - \phi_0\right) \left[\left(D_u^L - D_d^L\right) - \left(D_u^S - D_d^S\right)\right] \geq 0. \tag{20}$$

We can directly show that the term in the second bracket is positive (i.e., the market value of the long-term debt is more sensitive to the date-1 state than the short-term debt), and thus establish the following proposition.

Proposition 7 *Consider two debt contracts, one short-term and the other long-term, which generate the same date-0 credit. Then, the short-term contract gives the lower expected debt cost to an optimistic borrower if and only if*

$$\widehat{\pi}_0^h \geq \phi_0. \tag{21}$$

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date 1 by $p_s - D_s\left(\widetilde{F}\right)$. Then, his date-0 expected profit can be computed based on his belief about the probability of the next-period states adjusted by his marginal values of cash in these states. This argument holds not only for short-term debt, but also for long-term debt. For illustration, consider state u with $F_0^L \in \left(\theta^2, \theta\right)$. The optimist can lever up further by raising $\theta - F_L$ from his one unit of existing asset to purchase $\frac{\theta - F_L}{p_u - \theta}$ additional units of assets, and his total position becomes $1 + \frac{\theta - F_L}{p_u - \theta} = \frac{p_u - F_L}{p_u - \theta}$. As a result, the total value at state u is

$$\frac{p_u - F_L}{p_u - \theta} \pi_u^h (1-\theta) = v_u \left(p_u - D_u\left(\widetilde{F}\right)\right),$$

where we use the definition of v_u in (7) and $D_u\left(\widetilde{F}\right) = F_L$.

This proposition shows that the optimist's debt maturity choice depends on his cash-value-adjusted belief relative to the creditor's belief. The basic intuition works as the following: refinancing of short-term debt allows the borrower to trade a higher payment in the future down state for a lower payment in the up state. This trade explains the insensitiveness of short-term debt to realization of the future state, as paying more in the down state makes the value of the short-term debt less sensitive to the state. Therefore, whether the tradeoff between a higher payment in the down state and a lower payment in the up state is preferable depends on whether the borrower's belief about the probability of the up state is higher than the creditor's belief. As the borrower also needs to consider his marginal value of cash in the future states, his cash-value-adjusted probability is the relevant belief for comparison with that of the creditor.

Proposition 7 shows that long-term debt could dominate short-term debt in financing optimists' asset positions. This result thus contrasts the standard intuition that optimists always prefer short-term debt financing if they have to borrow from pessimistic creditors.

4.2.3 Cash Saving

The optimist can also choose to save cash for the next date. Interestingly, the following proposition shows that saving capital dominates using long-term debt to finance an asset position if (21) is violated.

Proposition 8 *Suppose that $\widehat{\pi}_0^h < \phi_0$. Then, on date 0 the optimist prefers saving cash to establishing an asset position by using long-term debt.*

When $\widehat{\pi}_0^h < \phi_0$, we have

$$p_0 > \widehat{\pi}_0^h p_u + \left(1 - \widehat{\pi}_0^h\right) p_d = \frac{\pi_0^h v_u p_u}{\pi_0^h v_u + \left(1 - \pi_0^h\right) v_d} + \frac{\left(1 - \pi_0^h\right) v_d p_d}{\pi_0^h v_u + \left(1 - \pi_0^h\right) v_d}.$$

We can interpret $\widehat{\pi}_0^h p_u + \left(1 - \widehat{\pi}_0^h\right) p_d$ as the optimist's asset holding value, after taking into account his marginal value of cash in different states of date 1. Then, it is clear that he should not acquire the asset given that the asset price is above his valuation. This proposition provides an interesting result that when long-term debt dominates short-term debt for financing the optimist's asset position, saving cash dominates establishing the position. This result together with Proposition 7 implies that the optimist will never strictly prefer long-term debt in the equilibrium. He could be indifferent between using long-term and short-term debt in financing his position only if $\widehat{\pi}_0^h = \phi_0$.

4.2.4 Leverage Choice

If $\widehat{\pi}_0^h > \phi_0$, we have $p_0 < \widehat{\pi}_0^h p_u + \left(1 - \widehat{\pi}_0^h\right) p_d$, i.e., the asset price is below the optimist's asset valuation. Then, the optimist will find it optimal to establish an asset position by using short-term debt financing. The following proposition shows that the optimist will always use a riskless debt contract with the maximum promise to pay p_d on date 1. The proposition also shows that he is indifferent between saving cash and establishing the position if $\widehat{\pi}_0^h = \phi_0$.

Proposition 9 *When $\widehat{\pi}_0^h > \phi_0$, it is optimal for the optimist to establish an asset position by using a short-term debt contract with a promise to pay p_d on date 1. When $\widehat{\pi}_0^h = \phi_0$, the optimist is indifferent between saving cash and establishing the position.*

Taken together, Propositions 7, 8, and 9 demonstrate that an individual optimist chooses either to save cash for the next period or to acquire an asset position financed by using the maximum riskless short-term debt. The only exception to this result is that when $\widehat{\pi}_0^h = \phi_0$, he can be indifferent between using long-term or short-term debt and between risky or riskless debt. This is because in this situation, the asset and any debt contract are fairly priced from his perspectives.[8]

This result that short-term debt is the only form of debt financing in the equilibrium is useful for several different reasons. First, it explains the dominance of short-term debt using by financial institutions to finance their asset positions. See Brunnermeier (2009) for a description of the pervasive use of short-term debt before the recent credit crisis. Second, it justifies a common practice for dynamic asset pricing models to ignore agents' debt maturity choice in financing their investment positions. Our model demonstrates that in a general binomial setting it is without loss of generality to focus on short-term debt rather than term debt, which can substantially complicate the equilibrium analysis.

It is important to note that the driving force for optimists' preference for using short-term debt in our model is different from the argument put forth by Geanakoplos (2009). He argues short-term debt allows an optimist to maximize riskless leverage. In contrast, our analysis shows that the key difference between short-term and long-term debt demonstrated by our model is refinancing, which allows the borrower to swap debt payment from the interim down state to the interim up state. As a result, the borrower prefers long-term debt exactly

[8] As he is indifferent between different debt contracts in this situation, we will assume that he chooses riskless short-term debt in our later analysis of the equilibrium. We can verify that allowing the optimist to use other contracts in this situation does not affect the asset price in the equilibrium.

when he highly values payoff in the interim down state. But, in this situation he should just save cash to take advantage of the improved investment opportunity in this state. Our result thus builds on optimists' ability to trade the risky asset on the interim date and the resulting incentive to save cash. In this regard, our model highlights the importance of the asset's marketability on the interim date in affecting optimists' financing decision.

4.3 Generalization

We can generalize the key results established in Sections 4.1 and 4.2 under the two-period binomial-tree setting to many periods. We describe the generalization in Appendix B. The key results on the risk-neutral representation of the asset and debt prices and an individual optimist's optimal investment and financing choices naturally extend to a binomial setting with any number of periods.

4.4 A Pessimist's Problem

The problem of an individual pessimist is relatively simple. Because of his risk neutrality and deep pocket, his asset valuation puts a lower bound on the asset's market price and the price of any debt contract collateralized by the asset. As we discussed earlier, optimists always use riskless short-term debt to finance their positions. The market price of the debt contract is always the same as the pessimist's valuation. However, the market price of the risky asset may not always reflect the pessimist's valuation. On date 0, he is indifferent between selling or holding his asset endowment if and only if $\phi_0 = \pi_0^l$. Denote $\lambda \in [0,1]$ as the fraction of pessimists who sell their asset endowments on date 0. Then

$$\lambda = \begin{cases} 1, & \text{if } \phi_0 > \pi_0^l; \\ \text{any value in } [0,1], & \text{if } \phi_0 = \pi_0^l. \end{cases} \tag{22}$$

4.5 Equilibrium

4.5.1 General Characterization

The joint equilibrium of the asset and credit markets can be characterized by five key elements: the prices of the asset on date 0 and in states u and d of date 1: $\{p_0, p_u, p_d\}$, which also determine the price of any collateralized debt contract based on the risk-neutral representation derived in Proposition 6; the fraction of optimists who establish asset positions on date 0: $\alpha \in [0,1]$; and the fraction of pessimists who sell their asset endowments on date 0: $\lambda \in [0,1]$.

State u of date 1 For a given asset price p_u, Proposition 5 implies that the implied belief of the marginal investor is

$$\phi_u = \frac{p_u - \theta}{1 - \theta}.$$

In equilibrium, $\phi_u \in \left[\pi_u^l, \pi_u^h\right]$. Among optimists, $(1-\alpha)$ fraction of them have chosen to save cash on date 0 and thus have an aggregate cash of $(1-\alpha)\,c$; the other α fraction have chosen to acquire in aggregate λ units of the asset by using a one-period debt contract with a promise of p_d and thus have a net worth of $\lambda\,(p_u - p_d)$. Based on our earlier analysis, in this state any optimist will choose to establish an asset position financed by the maximum debt with a promise of θ. In order for these optimists to buy out the total 1 unit of the asset, their total purchasing power, $\lambda\,(p_u - p_d) + (1-\alpha)\,c + \theta$, needs to be above the required purchasing price, p_u. If it falls short of this price, pessimists will fill in only if $\phi_u = \pi_u^l$. Thus, the market clearing condition in state u is

$$\begin{cases} p_u \geq \lambda\,(p_u - p_d) + \theta + (1-\alpha)\,c & \text{if } \phi_u = \pi_u^l \\ p_u = \lambda\,(p_u - p_d) + \theta + (1-\alpha)\,c & \text{if } \phi_u \in \left(\pi_u^l, \pi_u^h\right) \\ p_u \leq \lambda\,(p_u - p_d) + \theta + (1-\alpha)\,c & \text{if } \phi_u = \pi_u^h \end{cases}. \tag{23}$$

State d of date 1 For a given asset price p_d, Proposition 5 implies that the marginal investor's implied belief is

$$\phi_d = \frac{p_d - \theta^2}{\theta - \theta^2}.$$

In equilibrium, $\phi_d \in \left[\pi_d^l, \pi_d^h\right]$. Those optimists who have taken asset positions financed by the maximum riskless debt are now wiped out and those who choose to save cash on date 0 have an aggregate cash of $(1-\alpha)\,c$. Then, any optimist with cash will acquire the asset financed by the maximum debt with a promise of θ^2. In order for these optimists to buy out the total 1 unit of the asset, their total purchasing power, $(1-\alpha)\,c + \theta^2$, needs to be above the required purchasing price, p_d. If it falls short of this price, pessimists will fill in only if $\phi_d = \pi_d^l$. Thus, the market clearing condition in state d is

$$\begin{cases} p_d \geq (1-\alpha)\,c + \theta^2 & \text{if } \phi_d = \pi_d^l \\ p_d = (1-\alpha)\,c + \theta^2 & \text{if } \phi_d \in \left(\pi_d^l, \pi_d^h\right) \\ p_d \leq (1-\alpha)\,c + \theta^2 & \text{if } \phi_d = \pi_d^h \end{cases}. \tag{24}$$

Date 0 For a given asset price p_0, the marginal investor's implied belief is

$$\phi_0 = \frac{p_0 - p_d}{p_u - p_d}.$$

In equilibrium, $\phi_0 \in \left[\pi_0^l, \widehat{\pi}_0^h\right]$, where $\widehat{\pi}_0^h$ is an optimist's cash-value-adjusted belief on date 0 given by equation (18). Proposition 9 implies that each optimist prefers to establish an asset position financed by a one-period debt contract with a promise of p_d if $\phi_0 < \widehat{\pi}_0^h$; and is indifferent between establishing a position or saving cash if $\phi_0 = \widehat{\pi}_0^h$. Thus, each optimist's optimization leads to the following condition:

$$\begin{cases} \alpha = 1 & \text{if } \phi_0 < \widehat{\pi}_0^h \\ \alpha \in (0,1) & \text{if } \phi_0 = \widehat{\pi}_0^h \end{cases}. \tag{25}$$

As we discussed before, each pessimist will choose to hold his asset endowment only if $\phi_0 = \pi_0^l$, and thus leading to the following condition:

$$\begin{cases} \lambda = 1 & \text{if } \phi_0 > \pi_0^l \\ \lambda \in [0,1] & \text{if } \phi_0 = \pi_0^l \end{cases}. \tag{26}$$

Finally, the market clearing condition requires that optimists who choose to acquire asset positions are able to finance their asset purchases:

$$\lambda p_0 = \alpha c + \lambda p_d, \tag{27}$$

where λp_0 is the total market value of the λ units of asset sold by pessimists and $\alpha c + \lambda p_d$ is the total purchasing power of the α fraction of optimists who have an aggregate cash of αc and are able to obtain an aggregate credit of λp_d by using one-period debt contract with a promise of p_d.

Taken together, equations (23), (24), (25), (26), and (27) allow us to determine the five unknowns: p_0, p_d, p_u, α, and λ.

4.5.2 The Case with Non-monotonic Belief Dispersion

To illustrate the role of the secondary market trading on the equilibrium, we focus on analyzing the case in which optimists' and pessimists' beliefs diverge from each other in state d of date 1. More specifically, we suppose that

$$\pi_u^h = \pi_u^l, \ \frac{\pi_0^h}{\pi_0^l} < \frac{\pi_0^h + \left(1 - \pi_0^h\right)\pi_d^h}{\pi_0^l + \left(1 - \pi_0^l\right)\pi_d^l}. \tag{28}$$

These conditions implies the non-monotonic belief structure we analyzed in the static setting. The inequality condition also implies that

$$\frac{\pi_0^h}{\pi_0^l} < \frac{\left(1 - \pi_0^h\right)\pi_d^h}{\left(1 - \pi_0^l\right)\pi_d^l}, \tag{29}$$

i.e., optimists and pessimists disagree more about the second-period asset fundamental conditional on a negative shock in the first period than about the first-period fundamental. The greater belief dispersion in state d implies that it is more costly for an optimist to refinance his risky debt from pessimists in this state, as well as a better investment opportunity if the asset price is determined by pessimists.

In the benchmark static setting, where the asset is not marketable on the interim date, Proposition 3 shows that optimists will only use debt contracts with a promise of θ^2 to finance their asset positions and the equilibrium asset price p_0, if between the pessimists' and optimists' expected asset payoff, is determined by optimists' aggregate cash endowment c plus θ^2, i.e., $p_0 = c + \theta^2$.

When optimists have the choice to trade the asset on the interim date, this option creates incentives to save cash on date 0 to take advantage of the better investment opportunity in state d. Cash saving by some optimists boosts the asset price in state d, which, in turn, makes it easier for other optimists to finance their initial asset positions by collateralizing their positions. This feedback mechanism further boosts the asset price on date 0. In particular, we will show that this feedback mechanism can make the price and collateral value of the asset in the dynamic setting higher than those in the benchmark static setting.

We now derive the equilibrium. Given the convergence of optimists' and pessimists' beliefs in state u, the equilibrium in this state is simple:

$$p_u = \mathbb{E}_u^h\left[\widetilde{\theta}\right] = \mathbb{E}_u^l\left[\widetilde{\theta}\right], \ v_u = 1, \text{ and } \phi_u = \pi_u^h = \pi_u^l.$$

Furthermore, the following lemma establishes that there is always some optimists saving cash on date 0 in the equilibrium.

Lemma 10 *Under the conditions in (28), there is always a fraction of optimists saving cash on date* 0*, i.e.,* $\alpha < 1$.

In analyzing the equilibrium, we focus on two key variables, ϕ_0 and ϕ_d. They summarize the asset prices on date 0 and in state d of date 0, and determine both optimists' and pessimists' investment decisions. Note that since it is always optimal for optimists to save cash on date 0, $v_0 = \pi_0^h + \left(1 - \pi_0^h\right) v_d$, and thus

$$\phi_0 = \max\left(\frac{\pi_0^h}{\pi_0^h + \left(1 - \pi_0^h\right) v_d}, \pi_0^l\right).$$

We describe the equilibrium in the following proposition.

Proposition 11 *Under the conditions in (28), the equilibrium is characterized by the following five cases classified by the level of optimists' aggregate cash endowment c going from high to low:*

1. $c \geq \mathbb{E}_0^h\left[\widetilde{\theta}\right] - \theta^2$. *In this case, optimists have sufficient cash to bid up the asset price to their expected asset payoff in state d of date 1:*

$$\phi_d = \pi_d^h \text{ and } p_d = \mathbb{E}_d^h\left[\widetilde{\theta}\right]$$

and on date 0:

$$\phi_0 = \pi_0^h \text{ and } p_0 = \mathbb{E}_0^h\left[\widetilde{\theta}\right].$$

Pessimists all sell their asset holdings on date 0, $\lambda = 1$; and only a fraction of optimists need to purchase the asset using one-period debt with a promise of $\mathbb{E}_d^h\left[\widetilde{\theta}\right]$, $\alpha = \left(\mathbb{E}_0^h\left[\widetilde{\theta}\right] - \mathbb{E}_d^h\left[\widetilde{\theta}\right]\right)/c$.

2. $c \in \left[\frac{\pi_0^l}{\pi_0^h}\left[\pi_0^h\left(p_u - \theta^2\right) + \left(1-\pi_0^h\right)\pi_d^h\left(\theta - \theta^2\right)\right], \mathbb{E}_0^h\left[\widetilde{\theta}\right] - \theta^2\right]$. *In this case, optimists cannot maintain the asset price at their expected future asset payoff, but nevertheless remain as the marginal investor of the asset. They determine the asset price based on the financing they can obtain: in state d of date 1:*

$$\begin{aligned}
\phi_d &= \frac{c\left(1-\pi_0^h\right)\pi_d^h}{\pi_0^h\left(p_u - c - \theta^2\right) + \left(1-\pi_0^h\right)\pi_d^h\left(\theta-\theta^2\right)} \geq \phi_d^* \equiv \frac{\pi_0^l\left(1-\pi_0^h\right)\pi_d^h}{\left(1-\pi_0^l\right)\pi_0^h} > \pi_d^l, \\
p_d &= \theta^2 + \frac{c\left(\theta-\theta^2\right)\left(1-\pi_0^h\right)\pi_d^h}{\pi_0^h\left(p_u - c - \theta^2\right) + \left(1-\pi_0^h\right)\pi_d^h\left(\theta-\theta^2\right)};
\end{aligned}$$

and on date 0:

$$\begin{aligned}
\phi_0 &= \frac{\pi_0^h}{\pi_0^h + \left(1-\pi_0^h\right)\pi_d^h/\phi_d} \geq \pi_0^l, \\
p_0 &= c + \theta^2.
\end{aligned}$$

Pessimists all sell their asset holdings on date 0, $\lambda = 1$; and only a fraction of optimists need to purchase the asset by using one-period debt with a promise of p_d, $\alpha = \left(p_0 - p_d\right)/c$.

3. $c \in \left[\frac{\pi_0^l\left(1-\pi_0^h\right)\pi_d^h}{\pi_0^h\left(1-\pi_0^l\right)}\left(\theta-\theta^2\right), \frac{\pi_0^l}{\pi_0^h}\left[\pi_0^h\left(p_u-\theta^2\right) + \left(1-\pi_0^h\right)\pi_d^h\left(\theta-\theta^2\right)\right]\right]$. *On date 0, the marginal investor of the asset shifts to pessimists and optimists become indifferent*

between acquiring the asset and saving cash. In state d, optimists remain as the marginal investor of the asset. As c changes inside the given range, the marginal investor's equivalent belief and the asset price remain at constant levels:

$$\begin{aligned}\phi_d &= \phi_d^* \equiv \frac{\pi_0^l\left(1-\pi_0^h\right)\pi_d^h}{\left(1-\pi_0^l\right)\pi_0^h} > \pi_d^l, \\ p_d &= p_d^* \equiv \theta^2 + \frac{\pi_0^l\left(1-\pi_0^h\right)\pi_d^h}{\pi_0^h\left(1-\pi_0^l\right)}\left(\theta-\theta^2\right),\end{aligned}$$

due to the adjustment of the fraction of optimists acquiring positions on date 0:

$$\alpha = 1 - \frac{p_d^* - \theta^2}{c} \geq 0.$$

As a result, the asset price on date 0 is also a constant:

$$\begin{aligned}\phi_0 &= \pi_0^l, \\ p_0 &= p_0^* = \pi_0^l p_u + \left(1-\pi_0^l\right)p_d^*.\end{aligned}$$

The fraction of pessimists who sell their asset endowments is $\lambda = \frac{\alpha c}{p_0^* - p_d^*}$.

4. $c \in \left[\pi_d^l\left(\theta-\theta^2\right), \frac{\pi_0^l\left(1-\pi_0^h\right)\pi_d^h}{\pi_0^h\left(1-\pi_0^l\right)}\left(\theta-\theta^2\right)\right]$. *In this case, optimists all choose to save cash ($\alpha = 0$ and $\lambda = 0$) and manage to maintain the asset price in state d at*

$$p_d = c + \theta^2$$

with the implied marginal investor's belief of

$$\phi_d = \frac{c}{\theta - \theta^2} \geq \pi_d^l.$$

Pessimists are the marginal investor of the asset on date 0:

$$\begin{aligned}\phi_0 &= \pi_0^l, \\ p_0 &= \pi_0^l p_u + \left(1-\pi_0^l\right)\left(c+\theta^2\right).\end{aligned}$$

5. $c < \pi_d^l\left(\theta-\theta^2\right)$. *In this case, pessimists become the marginal investor of the asset in both state d:*

$$\phi_d = \pi_d^l \text{ and } p_d = \mathbb{E}_d^l\left[\widetilde{\theta}\right],$$

and on date 0:

$$\phi_0 = \pi_0^l \text{ and } p_0 = \mathbb{E}_0^l\left[\widetilde{\theta}\right].$$

Optimists all save cash on date 0: $\alpha = \lambda = 0$.

Cases 3 and 4 in Proposition 11 highlight the key difference between the dynamic and static settings (Proposition 3)—by saving their limited cash from date 0 to state d where the marginal value of cash is highest, optimists can effectively support the asset price in state d, which in turn induces higher asset collateral value on date 0 and motivates pessimists to value the asset higher.

4.5.3 An Illustration

To illustrate this effect, we use a set of baseline parameter values:

$$\theta = 0.4, c = 0.15, \pi_0^h = 0.6, \pi_0^l = 0.55, \pi_u^h = \pi_u^l = 0.5, \pi_d^h = 0.75, \pi_d^l = 0.4. \tag{30}$$

It is easy to verify that the specified belief structure of the two groups satisfies the condition in (28).

Figure 2 illustrates the effects of varying optimists' cash endowment c from 0.42 to 0.07. Panel A depicts the date-0 asset price p_0. The two horizontal lines at the levels of 0.5 and 0.556 represents pessimists' and optimists' expectation of the asset's final liquidation value, i.e., $\mathbb{E}_0^l\left[\widetilde{\theta}\right]$ and $\mathbb{E}_0^h\left[\widetilde{\theta}\right]$. The equilibrium price has to fall between these two benchmark levels. The dotted-and-dashed line is the asset price under the static setting given in Proposition 3. When c is above 0.4, the equilibrium price is equal to optimists' valuation (case 1 of Proposition 3). Once c falls below 0.4, the asset price starts to fall with c and is determined by optimists' purchasing power rather than their valuation (case 2 of Proposition 3). As c falls below 0.336, the marginal investor of the asset shifts to pessimists and the asset price is equal to their valuation (case 3 of Proposition 3).

The solid line in Panel A gives the asset price in the dynamic setting derived in Proposition 11. Like the dotted-and-dashed line, when c is above 0.4, the equilibrium price is equal to optimists' valuation (case 1 of Proposition 11). When c falls below 0.4, the asset price starts to fall with c because optimists cannot afford to support the price at their valuation (case 2 of Proposition 11).

Interestingly, as c drops below 0.357 but above 0.14, the price flattens out at the level 0.523 (case 3 of Proposition 11). In this range, the asset price is substantially higher than that in the static setting. Panel C reveals the key source of this difference—as c falls, a greater fraction of optimists $(1-\alpha)$ choose to save cash on date 0 and more pessimists hold their asset endowments (λ). As further shown by Panels B and D, in this range the optimists' cash saving maintains the asset price p_d and the implied belief of the marginal investor ϕ_d

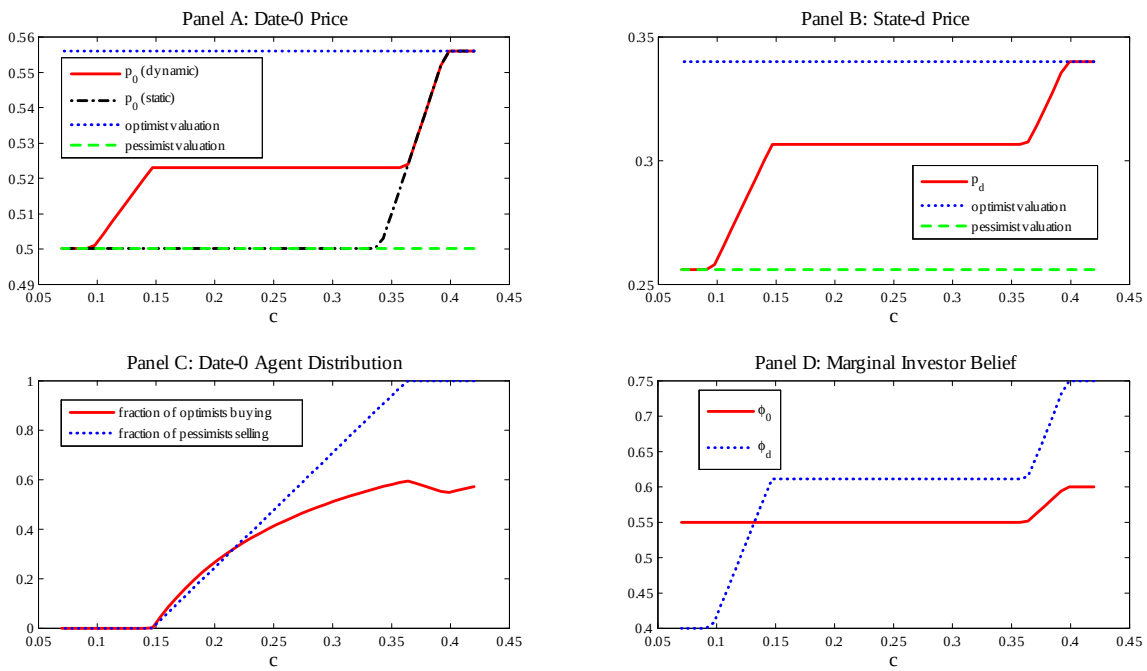

Figure 2: The effects of optimists' cash endowments. This figure is based on the baseline parameters given in (30).

in state d of date 1 at constant levels, and, in particular, ensures optimists as the marginal investor in this state (i.e., $\phi_d > \pi_d^h$). Panel D also shows that the marginal investor on date 0 is pessimists ($\phi_0 = \pi_0^l = 0.55$). Despite this, the date-0 asset price p_0 is higher than that in the static setting because pessimists anticipate that p_d is determined by optimists.

As c gets lower than 0.14 but higher than 0.09, we obtain case 4 of Proposition 11 with all optimists saving cash on date 0 ($\alpha = 0$) and all pessimists holding their asset endowments ($\lambda = 0$). In this case, optimists remain as the marginal investor of the asset in state d. As optimists have saved all of their cash endowments to state d, the asset price in this state now falls with c, which in turn causes the asset price on date 0 to fall with c as well. Nevertheless, the date-0 price p_0 is higher than that in the static setting because pessimists, the marginal investor on date 0, anticipate selling their asset holdings to optimists in state d. Only when c falls below 0.09, pessimists become the marginal investor of the asset both on date 0 and in state d of date 1 (case 5 of Proposition 11). As a result, the prices in the dynamic and static settings coincide.

Taken together, cases 3 and 4 illustrated in Figure 2 demonstrate a significant impact of the asset's marketability on date 1 in inducing optimists to preserve cash from date 0 to

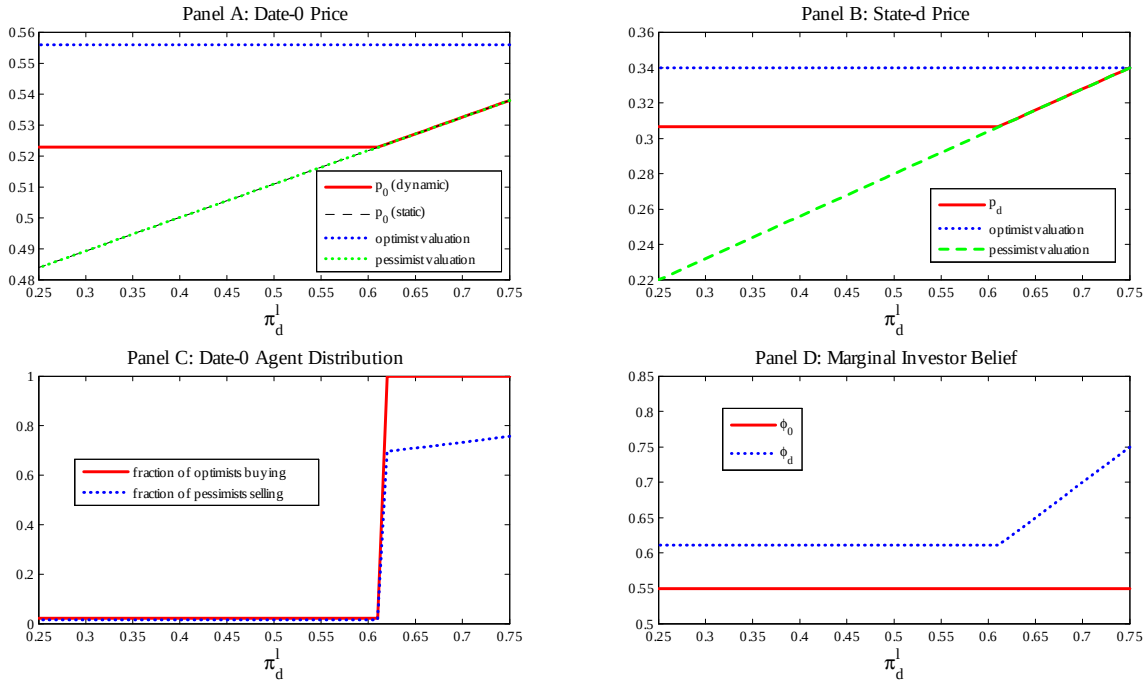

Figure 3: The effects of belief dispersion in state d. This figure is based on the baseline parameters given in (3).

state d, where the marginal value of cash is highest. The optimists' cash saving supports the asset price in state d, which in turn motivates pessimists to assign a higher collateral value to the asset on date 0.

Figure 3 illustrates the effects of the belief dispersion between optimists and pessimists in state d of date 1. In particular, we fix optimists' cash endowment c at its baseline value and vary π_d^l—pessimists' belief in state d, from 0.75 to 0.25. As π_d^l drops, the belief dispersion between optimists and pessimists ($\pi_d^h - \pi_d^l$) widens. The widened belief dispersion can make refinancing more costly to optimists in this state if they choose to use short-term debt on date 0. When π_d^l is above 0.61, the monotonic belief condition in (4) holds, which also means that (28) does not hold. Panel D indicates that in this range the marginal investor is pessimists as $\phi_0 = \pi_0^l$ and $\phi_d = \pi_d^l$. As a result, both date-0 price p_0 and state-d price p_d fall as π_d^l falls (Panels A and B). Interestingly, as π_d^l falls below 0.61, optimists start to save cash on date 0, as shown by Panel C. As a result, the marginal investor in state d shifts to optimists with the marginal investor's equivalent belief ϕ_d fixed at a constant level 0.61 (panel D) and the asset price p_d fixed at 0.307 (panel B). This in turn allows pessimists, the marginal investor of the asset on date 0, to value p_0 at a constant level 0.523. Taken together, Figure 3 shows

that cash saving by some optimists on date 0, induced by the better opportunity in state d, can alleviate the rollover risk faced by other optimists who choose to acquire asset positions on date 0 by using short-term debt.

5 Conclusion

To be added.

Appendix A Proofs for Propositions

A.1 Proof of Lemma 1

We first rule out the case of $F^* < \theta^2$. Suppose that this case holds true. Then, the debt contract is risk free across all of the four possible paths, i.e., $\widetilde{D} = F$. As a result, the optimist can obtain a credit of F^* and his expected debt cost is also F^*. Then, his expected value in (1) becomes

$$\frac{c}{p_0 - F^*}\left[\mathbb{E}_0^h\left(\widetilde{\theta}\right) - F^*\right].$$

Now, consider increasing the debt promise by a tiny amount ϵ. The debt contract is still risk free, and the optimist's expected value becomes

$$\frac{c}{p_0 - F^* - \epsilon}\left[\mathbb{E}_0^h\left(\widetilde{\theta}\right) - F^* - \epsilon\right].$$

Since $p_0 < \mathbb{E}_0^h\left(\widetilde{\theta}\right)$, this expression is increasing with ϵ. In other words, the optimist is better off by borrowing more. This contradicts with F^* being the optimal debt promise. Thus, the optimal debt promise cannot be lower than θ^2.

Next, suppose that $F^* > \theta$. Since the debt promise is higher than θ, the optimist always defaults on the debt contract except at the end of the path uu. Thus, the optimist's profit from his position is $1 - F^*$ at the end of the path uu, and 0 at the end of the other paths. Then, his expected value is

$$\frac{c}{p_0 - \mathbb{E}_0^l\left(\widetilde{\theta} \wedge F^*\right)}\mathbb{E}_0^h\left(\widetilde{\theta} - \widetilde{\theta} \wedge F^*\right).$$

Consider reducing the debt promise by a small amount ϵ, which only affects the payoff at the end of the path uu. The optimists expected value is now

$$\frac{c\mathbb{E}_0^h\left[\widetilde{\theta} - \widetilde{\theta} \wedge \left(F^* - \widetilde{\theta} \wedge F^*\right)\right]}{p_0 - \mathbb{E}_0^l\left[\widetilde{\theta} \wedge \left(F^* - \widetilde{\theta} \wedge F^*\right)\right]} = \frac{c\left[\mathbb{E}_0^h\left(\widetilde{\theta} - \widetilde{\theta} \wedge F^*\right) + \pi_0^h \pi_u^h \epsilon\right]}{p_0 - \mathbb{E}_0^l\left(\widetilde{\theta} \wedge F^*\right) + \pi_0^l \pi_u^l \epsilon}.$$

This expression is increasing with ϵ if

$$\frac{\mathbb{E}_0^h\left(\widetilde{\theta}-\widetilde{\theta}\wedge F^*\right)}{p_0-\mathbb{E}_0^l\left(\widetilde{\theta}\wedge F^*\right)}\leq\frac{\pi_0^h\pi_u^h}{\pi_0^l\pi_u^l}.$$

Note that since $p_0\geq\mathbb{E}_0^l\left(\widetilde{\theta}\right)$, we have

$$\frac{\mathbb{E}_0^h\left(\widetilde{\theta}-\widetilde{\theta}\wedge F^*\right)}{p_0-\mathbb{E}_0^l\left(\widetilde{\theta}\wedge F^*\right)}\leq\frac{\mathbb{E}_0^h\left(\widetilde{\theta}-\widetilde{\theta}\wedge F^*\right)}{\mathbb{E}_0^l\left(\widetilde{\theta}-\widetilde{\theta}\wedge F^*\right)}=\frac{\pi_0^h\pi_u^h}{\pi_0^l\pi_u^l}.$$

Thus, the optimist's expected value increases with ϵ, which contradicts with F^* being the optimal debt promise. This suggests that the optimal debt promise cannot be higher than θ.

A.2 Proof of Proposition 2

The optimist's date-0 expected value by using a debt contract with promise $F\in\left[\theta^2,\theta\right]$ is

$$\max_F\ c\frac{\pi_0^h\pi_u^h+\left(\pi_0^h\left(1-\pi_u^h\right)+\left(1-\pi_0^h\right)\pi_d^h\right)\theta-\left(\pi_0^h+\pi_d^h-\pi_0^h\pi_d^h\right)F}{p_0-\left(1-\pi_0^l\right)\left(1-\pi_d^l\right)\theta^2-\left(\pi_0^l+\pi_d^l-\pi_0^l\pi_d^l\right)F}\tag{31}$$

Direct algebra shows that this objective increases with F if and only if

$$p_0\leq P_M\equiv\left(\pi_0^l+\pi_d^l-\pi_0^l\pi_d^l\right)\frac{\pi_0^h\pi_u^h+\left(\pi_0^h\left(1-\pi_u^h\right)+\left(1-\pi_0^h\right)\pi_d^h\right)\theta}{\pi_0^h+\pi_d^h-\pi_0^h\pi_d^h}+\left(1-\pi_0^l\right)\left(1-\pi_d^l\right)\theta^2.\tag{32}$$

This implies that the optimist should choose $F^*=\theta^2$ if $p_0<P_M$; $F^*=\theta$ if $p_0>P_M$; be indifferent in using any $F\in\left[\theta^2,\theta\right]$ if $p_0=P_M$.

It is direct to verify that under the belief condition (4), $P_M\geq\mathbb{E}_0^l\left[\widetilde{\theta}\right]$. This is because

$$\begin{aligned}
&P_M-\mathbb{E}_0^l\left[\widetilde{\theta}\right]\\
=\ &\left(\pi_0^l+\pi_d^l-\pi_0^l\pi_d^l\right)\frac{\pi_0^h\pi_u^h+\left(\pi_0^h\left(1-\pi_u^h\right)+\left(1-\pi_0^h\right)\pi_d^h\right)\theta}{\pi_0^h+\pi_d^h-\pi_0^h\pi_d^h}-\pi_0^l\pi_u^l-\left(\pi_0^l\left(1-\pi_u^l\right)+\left(1-\pi_0^l\right)\pi_d^l\right)\theta\\
=\ &\left[\frac{\left(\pi_0^l+\pi_d^l-\pi_0^l\pi_d^l\right)\pi_0^h\pi_u^h}{\pi_0^h+\pi_d^h-\pi_0^h\pi_d^h}-\pi_0^l\pi_u^l\right](1-\theta).
\end{aligned}$$

Then, it is direct to use the market clearing condition to derive the equilibrium price listed in the proposition.

To show that the constraint (2) never binds in equilibrium, we only need to verify the case in which the optimists' equilibrium debt promise $F^*>\theta^2$. This occurs only when

$p_0 \leq P_M$. From (31), an optimist's marginal value of using \$1 to purchase the risky asset (by borrowing with a optimal promise of $F^* = \theta$) is

$$\begin{aligned} v_0^* &= \frac{\left[\pi_0^h \pi_u^h (1-F^*) + \left(\pi_0^h \left(1-\pi_u^h\right) + \left(1-\pi_0^h\right)\pi_d^h\right)(\theta - F^*)\right]}{p_0 - \theta^2 - \left(\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l\right)\left(F^* - \theta^2\right)} \\ &\geq \frac{\pi_0^h \pi_u^h (1-\theta)}{P_M - \theta^2 - \left(\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l\right)\left(\theta - \theta^2\right)} \\ &= \frac{\pi_0^h \pi_u^h (1-\theta)}{\left(\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l\right) \frac{\pi_0^h \pi_u^h + \left(\pi_0^h\left(1-\pi_u^h\right) + \left(1-\pi_0^h\right)\pi_d^h\right)\theta}{\pi_0^h + \pi_d^h - \pi_0^h \pi_d^h} - \left(\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l\right)\theta} \\ &= \frac{\pi_0^h + \pi_d^h - \pi_0^h \pi_d^h}{\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l}. \end{aligned}$$

On the other hand, if the optimist wants to acquire the risky debt with payoff $\widetilde{\theta} \wedge F^*$, one can directly verify that it is optimal to finance the position by using riskless debt with a promise of θ^2. Then, his expected value is

$$v_0' = \frac{\left(\pi_0^h + \pi_d^h - \pi_0^h \pi_d^h\right)\left(F^* - \theta^2\right)}{\mathbb{E}_0^l\left[\widetilde{\theta} \wedge F^*\right] - \theta^2} = \frac{\pi_0^h + \pi_d^h - \pi_0^h \pi_d^h}{\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l} \tag{33}$$

which is demonstrated by v_0^*. Therefore, the constraint in (2) always holds.

A.3 Proof of Proposition 3

Suppose that in equilibrium some optimists use risky debt contract with a $F^* \in \left(\theta^2, \theta\right]$ from pessimists. Then, their marginal value from \$1 is

$$v_0^* = \frac{\mathbb{E}_0^h\left[\widetilde{\theta} - \widetilde{\theta} \wedge F^*\right]}{p_0 - \mathbb{E}_0^l\left[\widetilde{\theta} \wedge F^*\right]} = \frac{\left[\pi_0^h \pi_u^h (1-F^*) + \left(\pi_0^h \left(1-\pi_u^h\right) + \left(1-\pi_0^h\right)\pi_d^h\right)(\theta - F^*)\right]}{p_0 - \mathbb{E}_0^l\left[\widetilde{\theta} \wedge F^*\right]}. \tag{34}$$

Because $p_0 \geq \mathbb{E}_0^l\left[\widetilde{\theta}\right]$, v_0^* reaches its maximum when

$$p_0 = \mathbb{E}_0^l\left[\widetilde{\theta}\right] \text{ and } F^* = \theta.$$

Thus, we have

$$\begin{aligned} v_0^* &\leq \frac{\pi_0^h \pi_u^h (1-\theta)}{\mathbb{E}_0^l\left[\widetilde{\theta}\right] - \theta^2 - \left(\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l\right)\left(\theta - \theta^2\right)} \\ &= \frac{\pi_0^h \pi_u^h (1-\theta)}{\pi_0^l \pi_u^l \left(1-\theta^2\right) + \left(\pi_0^l\left(1-\pi_u^l\right) + \left(1-\pi_0^l\right)\pi_d^l\right)\left(\theta - \theta^2\right) - \left(\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l\right)\left(\theta - \theta^2\right)} \\ &= \frac{\pi_0^h \pi_u^h (1-\theta)}{\pi_0^l \pi_u^l \left(1-\theta^2\right) - \pi_0^l \pi_u^l \left(\theta - \theta^2\right)} = \frac{\pi_0^h \pi_u^h}{\pi_0^l \pi_u^l} \end{aligned}$$

Then, it is direct to see that v_0' in (33) dominates v_0^* in (34). Therefore, the constraint in (2) is always violated if there is any risky debt in equilibrium. In the absence of risky debt, it is direct to derive the market equilibrium described in the proposition.

A.4 Proof of Proposition 4

Suppose that the creditor has an equivalent belief of ϕ_d, which will be determined later in equilibrium and can be π_d^l if he is a pessimist. If the optimist promises to pay F on date 2, he can obtain the following credit:

$$D_d^S(F) = \begin{cases} F & \text{if } F \leq \theta^2 \\ \phi_d F + (1-\phi_d)\theta^2 & \text{if } F \in \left[\theta^2, \theta\right] \end{cases}.$$

This debt contract allows him to use \$1 to establish a position of $1/\left(p_d - D_d^S(F)\right)$ units of the risky asset with an expected profit of

$$V(F) = \frac{\pi_d^h(\theta - F \wedge \theta) + \left(1-\pi_d^h\right)\left(\theta^2 - F \wedge \theta^2\right)}{p_d - \phi_d(F \wedge \theta) - (1-\phi_d)\left(F \wedge \theta^2\right)}.$$

If $F \leq \theta^2$,

$$V(F) = \frac{\pi_d^h\theta + \left(1-\pi_d^h\right)\theta^2 - F}{p_d - F}$$

which is clearly increasing in F as $p_d \leq \pi_d^h\theta + \left(1-\pi_d^h\right)\theta^2$. This is because a higher riskless promise allows the optimist to establish a greater position without incurring a higher financing cost.

If $F > \theta^2$,

$$V(F) = \frac{\pi_d^h(\theta - F)}{p_d - \theta^2 - \phi_d\left(F - \theta^2\right)}.$$

It is direct to show that

$$\frac{dV(F)}{dF} \propto -\pi_d^h\left[p_d - \left(\phi_d\theta + (1-\phi_d)\theta^2\right)\right]$$

As the asset price p_d has to be above the asset valuation of the creditor ($p_d \geq \phi_d\theta + (1-\phi_d)\theta^2$), we have $\frac{dV(F)}{dF} \leq 0$. In fact, if $p_d = \phi_d\theta + (1-\phi_d)\theta^2$ (i.e., the creditor is also the marginal investor of the asset,) the optimist is indifferent between any promise F in $\left[\theta^2, \theta\right]$. This is because the risky part of the debt has the same valuation as the asset, which makes the optimist indifferent between borrowing risky debt to take a greater position in the asset.

Overall, the optimist's expected profit is maximized at $F = \theta^2$.

A.5 Proof of Proposition 5

Equations (11) and (12) are simply reformulation of the definitions of v_u and v_d in (7) and (6).

Equations (8) and (9) imply that

$$\begin{aligned} p_0 &= \frac{\pi_0^h v_u}{u_0} p_u + \left(1 - \frac{\pi_0^h v_u}{u_0}\right) p_d \\ &\geq \frac{\pi_0^h v_u}{v_0} p_u + \left(1 - \frac{\pi_0^h v_u}{v_0}\right) p_d \end{aligned} \tag{35}$$

i.e., p_0 has to be above optimists' asset valuation derived from their marginal value of cash in states u and d of date 1. On the other hand, pessimists' deep pocket implies their asset valuation as another lower bound for p_0:

$$p_0 \geq \pi_0^l p_u + \left(1 - \pi_0^l\right) p_d.$$

Combining these two inequalities leads to

$$p_0 \geq \phi_0 p_u + (1 - \phi_0) p_d,$$

with $\phi_0 = \max\left(\frac{\pi_0^h v_u}{v_0}, \pi_0^l\right)$. Furthermore, the equality must bind. Suppose that $p_0 > \phi_0 p_u + (1 - \phi_0) p_d$. Then, it must be that (35) holds strictly, i.e., $v_0 = \pi_0^h v_u + \left(1 - \pi_0^h\right) v_d > u_0$, which in turn implies that optimists prefer saving cash on date 0. Then, the marginal investor of the asset has to be pessimists. However, $p_0 > \phi_0 p_u + (1 - \phi_0) p_d \geq \pi_0^l p_u + \left(1 - \pi_0^l\right) p_d$, which is a contradiction. Thus, (13) has to hold.

A.6 Proof of Proposition 6

Follow up with the example considered in the main text regarding the debt contract in state d with a promise to pay F on date 2. Suppose that $D_d^S(F) < \phi_d (F \wedge \theta) + (1 - \phi_d)\left(F \wedge \theta^2\right)$. If $F < \theta^2$, then the debt is riskless, and the debt price constitutes arbitrage to the marginal investor of the asset. Suppose that $F \in \left(\theta^2, \theta\right)$, i.e., the debt is risky. If the marginal investor of the asset is pessimists (i.e., $\phi_d = \pi_d^l$), then any pessimist will be able to bid up the debt price. Thus, this cannot hold in equilibrium. If the marginal investor is optimists, by borrowing $F \wedge \theta^2$ to purchase the debt contract, an optimist can reach a value that is strictly above the marginal value of cash v_d in state d:

$$\frac{\pi_d^h \left(F \wedge \theta - F \wedge \theta^2\right)}{D_d^S(F) - F \wedge \theta^2} > \frac{\pi_d^h}{\phi_d} = v_d.$$

This is again a contradiction. Thus, $D_d^S(F) < \phi_d (F \wedge \theta) + (1 - \phi_d)\left(F \wedge \theta^2\right)$ cannot occur.

Neither can $D_d^S(F) > \phi_d (F \wedge \theta) + (1 - \phi_d)\left(F \wedge \theta^2\right)$ occur, because this valuation is too high for anyone to pay.

A.7 Proof of Proposition 7

It is clear that for either long-term or short-term debt contract, the credit the optimist can obtain on date 0 is monotonically increasing with the promise. As the promise, increases

from 0 to 1, the credit increases from 0 to $\phi_0 p_u + (1-\phi_0) p_d$, which is the market valuation of the collateral. Thus, the optimist can find the necessary promise to obtain any credit level in this range by using either a long-term contract or a short-term contract.

We now prove that the market value of the long-term contract is more sensitive to the date-1 state than the short-term contract. Suppose the long-term debt gives a promise of F_0^L and short-term debt gives a promise of F_0^S. Then, equation (17) implies that

$$\phi_0 \left[\left(D_u^S - D_d^S \right) - \left(D_u^L - D_d^L \right) \right] = D_d^L - D_d^S.$$

Thus, we only need to show that $D_d^L - D_d^S \leq 0$. This inequality is equivalent to $D_d^S \left(F_0^L \right) - F_0^S \wedge p_d \leq 0$. Suppose that it does not hold. Then, $D_d^S \left(F_0^L \right) > F_0^S \wedge p_d$. Since $D_d^S \left(F_0^L \right) \leq p_d$, we must have $p_d > F_0^S$ and $D_d^S \left(F_0^L \right) > F_0^S$. Since $p_u > p_d > F_0^S$, $\phi_0 \left(F_0^S \wedge p_u \right) + (1-\phi_0) \left(F_0^S \wedge p_d \right) = F_0^S$, but the market value of long-term debt is $D_d^S \left(F_0^L \right) + \phi_0 (D_u^S \left(F_0^L \right) - D_d^S \left(F_0^L \right)) > F_0^S$. This contradicts the initial assumption that the long-term and short-term contracts have the same value.

A.8 Proof of Proposition 8

The optimist's expected value of saving one dollar is

$$\pi_0^h v_u + \left(1 - \pi_0^h \right) v_d.$$

If the optimist uses a long-term debt contract with a promise of F_L to establish an asset position, his expected value from applying (16) is

$$\begin{aligned} & \left[\pi_0^h v_u + \left(1 - \pi_0^h \right) v_d \right] \frac{\widehat{\pi}_0^h \left(p_u - D_u^L \right) + \left(1 - \widehat{\pi}_0^h \right) \left(p_d - D_d^L \right)}{p_0 - \phi_0 D_u^L - (1-\phi_0) D_d^L} \\ = & \left[\pi_0^h v_u + \left(1 - \pi_0^h \right) v_d \right] \frac{\widehat{\pi}_0^h \left(p_u - D_u^L \right) + \left(1 - \widehat{\pi}_0^h \right) \left(p_d - D_d^L \right)}{\phi_0 \left(p_u - D_u^L \right) + (1-\phi_0) \left(p_d - D_d^L \right)}, \end{aligned}$$

One can directly verify that

$$p_u - p_d \geq D_u^L - D_d^L,$$

i.e., the difference of the optimist's equity value across the up and down state is higher than that of the market value of the long-term debt contract. Then, we can show that

$$\begin{aligned} & \widehat{\pi}_0^h \left(p_u - D_u^L \right) + \left(1 - \widehat{\pi}_0^h \right) \left(p_d - D_d^L \right) - \left[\phi_0 \left(p_u - D_u^L \right) + (1-\phi_0) \left(p_d - D_d^L \right) \right] \\ = & \left(\widehat{\pi}_0^h - \phi_0 \right) \left(p_u - p_d - \left(D_u^L - D_d^L \right) \right) \leq 0, \end{aligned}$$

as $\widehat{\pi}_0^h < \phi_0$. Therefore,

$$\left[\pi_0^h v_u + \left(1 - \pi_0^h \right) v_d \right] \frac{\widehat{\pi}_0^h \left(p_u - D_u^L \right) + \left(1 - \widehat{\pi}_0^h \right) \left(p_d - D_d^L \right)}{\phi_0 \left(p_u - D_u^L \right) + (1-\phi_0) \left(p_d - D_d^L \right)} < \pi_0^h v_u + \left(1 - \pi_0^h \right) v_d.$$

Thus, saving capital is dominant.

A.9 Proof of Proposition 9

Suppose that $\widehat{\pi}_0^h > \phi_0$. Since $p_0 = \phi_0 p_u + (1-\phi_0)\, p_d$, we have

$$p_0 < \widehat{\pi}_0^h p_u + \left(1 - \widehat{\pi}_0^h\right) p_d.$$

Then the optimist's expected value from using short-term debt to establish a position, e.g. (16), is

$$\left[\pi_0^h v_u + \left(1-\pi_0^h\right) v_d\right] \left[\frac{\widehat{\pi}_0^h p_u + \left(1-\widehat{\pi}_0^h\right) p_d - \widehat{\pi}_0^h \left(F_0^S \wedge p_u\right) - \left(1-\widehat{\pi}_0^h\right)\left(F_0^S \wedge p_d\right)}{p_0 - \phi_0 \left(F_0^S \wedge p_u\right) - (1-\phi_0)\left(F_0^S \wedge p_d\right)}\right]. \tag{36}$$

As in the proof of Proposition 4, we can easily show that it is optimal to use $F_0^S = p_d$, and the second bracket becomes

$$\frac{\widehat{\pi}_0^h \left(p_u - p_d\right)}{p_0 - p_d} > 1.$$

As a result, by using a short-term debt contract with a promise of p_d to purchase the asset, the optimist obtains a value strictly higher than that from saving cash.

The equivalence of cash saving and establishing position is trivial when $\widehat{\pi}_0^h = \phi_0$, as (36) collapse to $\pi_0^h v_u + \left(1-\pi_0^h\right) v_d$, which is the value from saving cash.

A.10 Proof of Lemma 10

Suppose that every optimist takes a position on date 0 by using one-period debt contract with a promise of p_d. Then, all optimists will be wiped out in state d of date 1, and thus $p_d = \mathbb{E}_d^l\left[\widetilde{\theta}\right]$. This price in turn implies that optimists' marginal value of cash in the state reaches its maximum: $v_d = \pi_d^h/\pi_d^l$. The, an optimist's cash-value-adjusted belief on date 0 is

$$\widehat{\pi}_0^h = \frac{\pi_0^h}{\pi_0^h + \left(1-\pi_0^h\right) v_d} = \frac{\pi_0^h}{\pi_0^h + \left(1-\pi_0^h\right) \pi_d^h/\pi_d^l}. \tag{37}$$

By the inequality in (28), we can show that

$$\frac{\pi_0^h + \left(1-\pi_0^h\right)\pi_d^h}{\pi_0^l + \left(1-\pi_0^l\right)\pi_d^l} < \pi_0^h + \left(1-\pi_0^h\right)\pi_d^h/\pi_d^l,$$

which is equivalent to

$$\pi_0^h \left[\frac{1}{\pi_0^l + \left(1-\pi_0^l\right)\pi_d^l} - 1\right] < \left(1-\pi_0^h\right)\pi_d^h \left[\frac{1}{\pi_d^l} - \frac{1}{\pi_0^l + \left(1-\pi_0^l\right)\pi_d^l}\right],$$

which in turn is equivalent to a direct implication of the inequality in (28):

$$\frac{\pi_0^h}{\pi_0^l} < \frac{\left(1-\pi_0^h\right)\pi_d^h}{\left(1-\pi_0^l\right)\pi_d^l}.$$

Then, equation (37) implies that $\widehat{\pi}_0^h < \pi_0^l \leq \phi_0$, which in turn motivates the optimist to save cash according to Proposition 8. This is a contradiction. Thus, at least some optimists have to save cash on date 0 so that the asset price on in state d won't fall too low.

A.11 Proof of Proposition 11

We focus on cases 2 and 3.

Case 2 In this case, $\phi_d \geq \pi_d^l$ and $\phi_0 \geq \pi_0^l$. Thus, $\lambda = 1$, and equations (25) and (27) imply that

$$\left.\begin{array}{r} p_d = (1-\alpha)\, c + \theta^2 \\ p_0 = \alpha c + p_d \end{array}\right\} \Rightarrow p_0 = c + \theta^2.$$

To determine ϕ_d, note that

$$p_0 = \frac{\pi_0^h}{\pi_0^h + \left(1-\pi_0^h\right) v_d} p_u + \frac{\left(1-\pi_0^h\right) v_d}{\pi_0^h + \left(1-\pi_0^h\right) v_d} p_d. \tag{38}$$

Proposition 5 implies $p_d = \theta^2 + \phi_d \left(\theta - \theta^2\right)$, and Proposition 4 implies that

$$v_d = \frac{\pi_d^h \left(\theta - \theta^2\right)}{p_d - \theta^2} = \frac{\pi_d^h}{\phi_d}.$$

By substituting these equations back to (38), we obtain

$$p_0 \left[\pi_0^h \phi_d + \left(1-\pi_0^h\right) \pi_d^h\right] = \pi_0^h p_u \phi_d + \left(1-\pi_0^h\right) \pi_d^h \left[\theta^2 + \phi_d \left(\theta - \theta^2\right)\right],$$

which implies that

$$\begin{aligned} \phi_d &= \frac{\left(1-\pi_0^h\right) \pi_d^h \left(p_0 - \theta^2\right)}{\pi_0^h \left(p_u - p_0\right) + \left(1-\pi_0^h\right) \pi_d^h \left(\theta - \theta^2\right)} \\ &= \frac{c \left(1-\pi_0^h\right) \pi_d^h}{\pi_0^h \left(p_u - c - \theta^2\right) + \left(1-\pi_0^h\right) \pi_d^h \left(\theta - \theta^2\right)}. \end{aligned} \tag{39}$$

Therefore,

$$\begin{aligned} p_d &= \theta^2 + \phi_d \left(\theta - \theta^2\right) = \theta^2 + \frac{c \left(\theta - \theta^2\right) \left(1-\pi_0^h\right) \pi_d^h}{\pi_0^h \left(p_u - c - \theta^2\right) + \left(1-\pi_0^h\right) \pi_d^h \left(\theta - \theta^2\right)}, \\ \alpha &= \frac{p_0 - p_d}{c}. \end{aligned}$$

Then, in order for $\phi_0 \geq \pi_0^l$, we require

$$\phi_0 = \frac{\pi_0^h}{\pi_0^h + \left(1-\pi_0^h\right) \pi_d^h / \phi_d} \geq \pi_0^l,$$

which is equivalent to

$$\phi_d \geq \phi_d^* \equiv \frac{\pi_0^l \left(1-\pi_0^h\right) \pi_d^h}{\left(1-\pi_0^l\right) \pi_0^h}. \tag{40}$$

Note that $\phi_d^* \geq \pi_d^l$ because of the inequality condition in (29). Thus, as c falls, ϕ_0 hits π_0^l before ϕ_d hits π_d^l. By substituting (39) into (40), we obtain a lower bound for c to ensure that $\phi_0 \geq \pi_0^l$:

$$c \geq \frac{\pi_0^l}{\pi_0^h} \left[\pi_0^h \left(p_u - \theta^2\right) + \left(1-\pi_0^h\right) \pi_d^h \left(\theta - \theta^2\right)\right].$$

Case 3 In this case, c gets further lower. In this case, ϕ_0 already reaches its lower bound π_0^l. This in turn implies that pessimists become indifferent between holding or selling their asset endowments. The lower c also makes ϕ_d lower and thus marginal value of cash $v_d = \pi_d^h/\phi_d$ higher, which motivates optimists to save cash. Thus, in order for optimists to purchase the asset sold by pessimists, we need

$$\widehat{\pi}_0^h = \frac{\pi_0^h}{\pi_0^h + \left(1-\pi_0^h\right)\pi_d^h/\phi_d} = \pi_0^l.$$

This implies that

$$\phi_d = \phi_d^*$$

where ϕ_d^* is defined in (40). Thus, both p_0 and p_d stay at constant levels as c falls down:

$$\begin{aligned} p_d &= p_d^* \equiv \theta^2 + \frac{\pi_0^l\left(1-\pi_0^h\right)\pi_d^h}{\pi_0^h\left(1-\pi_0^l\right)}\left(\theta-\theta^2\right) > \mathbb{E}_d^l\left[\widetilde{\theta}\right], \\ p_0 &= p_0^* \equiv \pi_0^l p_u + \left(1-\pi_0^l\right)p_d^* > \mathbb{E}_0^l\left[\widetilde{\theta}\right]. \end{aligned}$$

As c falls, the adjustment in the equilibrium is through λ, the fraction of pessimists who sell their asset endowments. The market clearing condition in state d (equation (24)) implies that

$$p_d = (1-\alpha)c + \theta^2 \Rightarrow \alpha = 1 - \frac{p_d^* - \theta^2}{c}.$$

Thus, α drops as c falls, i.e., more optimists choose to save cash on date 0. The market clearing condition in state d (equation (27)) implies that

$$\lambda p_0 = \alpha c + \lambda p_d \Rightarrow \lambda = \frac{\alpha c}{p_0^* - p_d^*}.$$

Thus, λ also drops as c falls, i.e., more pessimists choose to hold their asset endowments.

c needs to be above a lower bound in order for $\alpha \geq 0$:

$$c \geq p_d^* - \theta^2 = \frac{\pi_0^l\left(1-\pi_0^h\right)\pi_d^h}{\pi_0^h\left(1-\pi_0^l\right)}\left(\theta-\theta^2\right).$$

B Generalizing the Model to N Periods

Consider a setting with T periods. The asset's fundamental follows a binomial tree, i.e., given the current state it can either go up or down in the following period. The liquidation value of the asset on the final date (date T) can take $T+1$ possible values, denoted by $j = 1, ..., T+1$ from low to high. On a prior date N, there are $N+1$ possible states. Following our model setting in the main text, suppose that there are two groups of agents, with one group holding more optimistic beliefs about the tree while the other group more pessimistic. Denote the group-i's ($i \in \{h, l\}$) belief in state j of date n by $\pi_{n,j}^i$ with $\pi_{n,j}^h \geq \pi_{n,j}^l$

for $\forall n \in \{0, 1, ..., T-1\}$ and $\forall j \in \{1, ..., n+1\}$. In addition, suppose that the optimists have limited capital while the pessimists always have sufficient capital.

We can establish the following proposition.

Proposition 12 *Consider a binomial-tree setting with T periods.*

1. *There exists a set of equivalent beliefs $\{\phi_{n,j} \in [0,1]\}$ of the marginal investor, which permits a risk-neutral representation of the risky asset and any debt contract collateralized by the asset on each node of the tree.*

2. *Consider the j-th node of date $n < T$. Suppose that $\widehat{\pi}^h_{n,j}$ is an optimist's cash-value-adjusted belief on the node. If $\widehat{\pi}^h_{n,j} > \phi_{n,j}$, then he finds it optimal to acquire an asset position by financing the position by using the maximum riskless one-period debt; if $\widehat{\pi}^h_{n,j} < \phi_{n,j}$, then he prefers saving cash for the next period; and if $\widehat{\pi}^h_{n,j} = \phi_{n,j}$, then he is indifferent between saving cash and acquiring an asset position with any debt financing.*

The proof follows. Suppose that the equilibrium price of the asset exists and is denoted by $p_{n,j}$. As suggested by our analysis of the two-period setting, the asset price determines the optimists' investment opportunities and thus marginal value of cash. The asset price also offers a benchmark to set the market value of debt issued by the optimists. First, note that since the pessimists always have sufficient capital, the asset price on any state $p_{n,j}$ must be bounded from below by the pessimists' expected next-period price:

$$p_{n,j} \geq p_{n+1,j} + \pi^l_{n,j} \left(p_{n+1,j+1} - p_{n+1,j}\right), \tag{41}$$

with the equality binding when pessimists take asset positions and thus are the marginal investor of the asset.

The scarcity of optimist capital implies that the optimists can earn rent on their capital. Let's denote their marginal value of cash by $v_{n,j}$ in state j of date n. Suppose that an optimist has one dollar in this state, he can invest it in the asset and lever up the position using the maximum riskless one-period debt (with a promise of $p_{n+1,j}$), which gives him a position of $1/\left(p_{n,j} - p_{n+1,j}\right)$ shares and a per-share profit of $p_{n+1,j+1} - p_{n+1,j}$ in the up state of the next period and 0 in the down state. Given the optimist's belief $\pi^h_{n,j}$ about the up state and the marginal value of capital in the state as $v_{n+1,j+1}$, the expected value from having the position is

$$u_{n,j} = \frac{\pi^h_{n,j} v_{n+1,j+1} \left(p_{n+1,j+1} - p_{n+1,,j}\right)}{p_{n,j} - p_{n+1,j}}. \tag{42}$$

Given the price lower bound in (41),

$$u_{n,j} \leq \frac{\pi^h_{n,j} v_{n+1,j+1}}{\pi^l_{n,j}}.$$

In other words, the expected value from establishing the position is bounded by the pessimists' contemporaneous belief.

Alternatively, the optimist can also choose to save his cash for the next period, which gives him an expected value of $\pi_{n,j}^{h} v_{n+1,j+1} + \left(1 - \pi_{n,j}^{h}\right) v_{n+1,j}$. Thus, his marginal value of cash in the state is

$$v_{n,j} = \max\left(u_{n,j}, \quad \pi_{n,j}^{h} v_{n+1,j+1} + \left(1 - \pi_{n,j}^{h}\right) v_{n+1,j}\right)$$

This equation implies that the optimists' marginal value of cash is a supermartingale: $v_{n,j} \geq \mathbb{E}\left[v_{n+1,\cdot}\right]$, as they have the option to save, and that $v_{n,j} \geq u_{n,j}$. The latter, together with (42), implies that

$$p_{n,j} \geq p_{n+1,j} + \frac{\pi_{n,j}^{h} v_{n+1,j+1}}{v_{n,j}} \left(p_{n+1,j+1} - p_{n+1,j}\right), \tag{43}$$

with the equality binding when the optimists take positions in the asset and are the marginal investor. One of the equalities in (41) and (43) should bind in any given state. Thus, we have

$$p_{n,j} = p_{n+1,j} + \phi_{n,j} \left(p_{n+1,j+1} - p_{n+1,j}\right), \tag{44}$$

where

$$\phi_{n,j} = \max\left\{\frac{\pi_{n,j}^{h} v_{n+1,j+1}}{v_{n,j}}, \pi_{n,j}^{l}\right\}. \tag{45}$$

Note that $\phi_{n,j} \in (0,1)$. This process $\left\{\phi_{n,j}\right\}$, which accounts for the alternation of the marginal investor between the optimists and pessimists, acts as risk-neutral probabilities for representing the equilibrium asset price.

This representation of the equilibrium asset price also applies to any contingent claim whose price is fully determined by the state of the binomial tree, such as any debt contract collateralized by the asset. Denote its market price in state j of date n by $q_{n,j}$. Then, this price has to satisfy

$$q_{n,j} = q_{n+1,j} + \phi_{n,j} \left(q_{n+1,j+1} - q_{n+1,j}\right). \tag{46}$$

To verify this, suppose that $q_{n,j} < q_{n+1,j} + \phi_{n,j} \left(q_{n+1,j+1} - q_{n+1,j}\right)$. Then, whoever is the marginal investor of the asset, either the optimists (if $\frac{\pi_{n,j}^{h} v_{n+1,j+1}}{v_{n,j}} > \pi_{n,j}^{l}$) or the pessimists (if $\frac{\pi_{n,j}^{h} v_{n+1,j+1}}{v_{n,j}} < \pi_{n,j}^{l}$) or both (if $\frac{\pi_{n,j}^{h} v_{n+1,j+1}}{v_{n,j}} = \pi_{n,j}^{l}$), the marginal investor would find buying this debt contract more attractive than buying the asset. This is a contradiction. Now suppose that $q_{n,j} > q_{n+1,j} + \phi_{n,j} \left(q_{n+1,j+1} - q_{n+1,j}\right)$. The expected return across the following period implied by this price is lower than the asset to any market participant. As a result, no one would be willing to buy this debt contract. Again, this is a contradiction. Thus, (46) has to hold.

Based on the representation in (44) and (46), we provide a lemma to show that monotonicity of asset prices is preserved across different dates on the tree.

Lemma 13 *Consider any contingent claim on the asset. If its final payoff is monotonically increasing with respect to the state of the final date, then its price on any prior date is also monotonically increasing with the state.*

Proof. *Without loss of generality, we consider two states j and $j+1$ on date n, which is prior to its maturity date. Then, the price of the claim in these states are*

$$\begin{aligned} q_{n,j} &= q_{n+1,j} + \phi_{n,j}\left(q_{n+1,j+1} - q_{n+1,j}\right), \\ q_{n,j+1} &= q_{n+1,j+1} + \phi_{n,j+1}\left(q_{n+1,j+2} - q_{n+1,j+1}\right). \end{aligned}$$

Then,

$$q_{n,j+1} - q_{n,j} = \left(1 - \phi_{n,j}\right)\left(q_{n+1,j+1} - q_{n+1,j}\right) + \phi_{n,j+1}\left(q_{n+1,j+2} - q_{n+1,j+1}\right).$$

Thus, if $q_{n+1,j+2} \geq q_{n+1,j+1} \geq q_{n+1,j}$, then $q_{n,j+1} \geq q_{n,j}$. Using this property, backward induction on the tree implies that the monotonicity is preserved across any date. ■

This lemma directly implies the price of the asset $p_{n,j}$ and the market value of any long-term debt contract $D_{n,j}$ collateralized by one share of the asset monotonically increase with the state on the tree. Furthermore, note that on the maturity date of the debt contract, the difference before the asset price and the debt payoff is also monotonically increasing with the state. Thus, this lemma implies that across any two adjacent states j and $j+1$ on any date n prior to the debt maturity, $p_{n,j+1} - D_{n,j+1} \geq p_{n,j} - D_{n,j}$, which, in turn, implies that

$$p_{n,j+1} - p_{n,j} \geq D_{n,j+1} - D_{n,j}. \tag{47}$$

This inequality suggests that the asset price is more sensitive to the state than price of any debt contract collateralized by the asset.

We are ready to show that in any state on the tree, an individual optimist finds it optimal to either acquire an asset position by using the maximum riskless one-period debt or to simply save cash for the future period. Consider state j of date n. Define the optimist's cash-value-adjusted belief by

$$\widehat{\pi}^h_{n,j} \equiv \frac{\pi^h_{n,j} v_{n+1,j+1}}{\pi^h_{n,j} v_{n+1,j+1} + \left(1 - \pi^h_{n,j}\right) v_{n+1,j}}.$$

Suppose that $\widehat{\pi}^h_{n,j} \geq \pi^l_{n,j}$, then optimists are the marginal investor of the asset. We can verify that for an optimist financing an asset position by using the maximum riskless short-term debt dominates using any long-term debt contract. Let's compare the financing of a one-period debt contract with a promise of F^S on date $n+1$ and another long-term debt contract with a promise of F^L on date $N > n+1$. Denote the price of the short-term contract by $D^S_{n,j}$ and the price of the long-term contract by $D^L_{n,j}$. As both contracts if issued have to be purchased by the pessimists, to generate the same credit requires

$$\phi_{n,j} D^S_{n+1,j+1} + \left(1 - \phi_{n,j}\right) D^S_{n+1,j} = \phi_{n,j} D^L_{n+1,j+1} + \left(1 - \phi_{n,j}\right) D^L_{n+1,j}. \tag{48}$$

The short-term debt being less costly to the optimist is equivalent to

$$\widehat{\pi}_{n,j}^{h} D_{n+1,j+1}^{S} + \left(1 - \widehat{\pi}_{n,j}^{h}\right) D_{n+1,j}^{S} \leq \widehat{\pi}_{n,j}^{h} D_{n+1,j+1}^{L} + \left(1 - \widehat{\pi}_{n,j}^{h}\right) D_{n+1,j}^{L}. \tag{49}$$

By taking the difference between (49) and (48), (49) is equivalent to

$$\left(\widehat{\pi}_{n,j}^{h} - \phi_{n,j}\right) \left[\left(D_{n+1,j+1}^{L} - D_{n+1,j}^{L}\right) - \left(D_{n+1,j+1}^{S} - D_{n+1,j}^{S}\right)\right] \geq 0.$$

Because $\widehat{\pi}_{n,j}^{h} \geq \pi_{n,j}^{l}$, this holds if and only if

$$\left(D_{n+1,j+1}^{L} - D_{n+1,j}^{L}\right) - \left(D_{n+1,j+1}^{S} - D_{n+1,j}^{S}\right) \geq 0. \tag{50}$$

Equation (48) implies that

$$\left(D_{n+1,j+1}^{L} - D_{n+1,j}^{L}\right) - \left(D_{n+1,j+1}^{S} - D_{n+1,j}^{S}\right) = \frac{1}{\phi_{n,j}} \left(D_{n+1,j}^{S} - D_{n+1,j}^{L}\right)$$

Note that the short-term debt contract matures on $n+1$ and its payoffs are

$$D_{n+1,j+1}^{S} = F^{S}, \text{ and } D_{n+1,j}^{S} = \min\left(F^{S}, p_{n+1,j}\right).$$

Thus, (50) is equivalent to

$$\min\left(F^{S}, p_{n+1,j}\right) \geq D_{n+1,j}^{L}.$$

Suppose that $D_{n+1,j}^{L} > \min\left(F^{S}, p_{n+1,j}\right)$. As $D_{n+1,j}^{L} \leq p_{n+1,j}$, we must have $F^{S} < p_{n+1,j}$ and $F^{S} < D_{n+1,j}^{L}$. Then,

$$\begin{aligned} & \phi_{n,j} D_{n+1,j+1}^{S} + \left(1 - \phi_{n,j}\right) D_{n+1,j}^{S} \\ = \; & \phi_{n,j} F^{S} + \left(1 - \phi_{n,j}\right) \min\left(F^{S}, p_{n+1,j}\right) \\ < \; & F^{S} < D_{n+1,j}^{L} \leq \phi_{n,j} D_{n+1,j+1}^{L} + \left(1 - \phi_{n,j}\right) D_{n+1,j}^{L} \end{aligned}$$

which contradicts (48). Thus, (50) holds and the short-term contract dominates the long-term contract.

Now suppose that $\widehat{\pi}_{n,j}^{h} < \phi_{n,j}$. We can show that saving cash dominates using any long-term debt to finance an asset position. Saving cash gives an expected value of

$$\pi_{n,j}^{h} v_{n+1,j+1} + \left(1 - \pi_{n,j}^{h}\right) v_{n+1,j}.$$

If an optimist uses a long-term contract to finance an asset position, the expected value is

$$\begin{aligned} & \frac{\pi_{n,j}^{h} v_{n+1,j+1} \left(p_{n+1,j+1} - D_{n+1,j+1}^{L}\right) + \left(1 - \pi_{n,j}^{h}\right) v_{n+1,j} \left(p_{n+1,j} - D_{n+1,j}^{L}\right)}{p_{n,j} - \phi_{n,j} D_{n+1,j+1}^{L} - \left(1 - \phi_{n,j}\right) D_{n+1,j}^{L}} \\ = \; & \frac{\pi_{n,j}^{h} v_{n+1,j+1} + \left(1 - \pi_{n,j}^{h}\right) v_{n+1,j}}{p_{n,j} - \phi_{n,j} D_{n+1,j+1}^{L} - \left(1 - \phi_{n,j}\right) D_{n+1,j}^{L}} \left[\widehat{\pi}_{n,j}^{h} \left(p_{n+1,j+1} - D_{n+1,j+1}^{L}\right) + \left(1 - \widehat{\pi}_{n,j}^{h}\right) \left(p_{n+1,j} - D_{n+1,j}^{L}\right)\right] \\ = \; & \left[\pi_{n,j}^{h} v_{n+1,j+1} + \left(1 - \pi_{n,j}^{h}\right) v_{n+1,j}\right] \frac{\widehat{\pi}_{n,j}^{h} \left(p_{n+1,j+1} - D_{n+1,j+1}^{L}\right) + \left(1 - \widehat{\pi}_{n,j}^{h}\right) \left(p_{n+1,j} - D_{n+1,j}^{L}\right)}{\phi_{n,j} \left(p_{n+1,j+1} - D_{n+1,j+1}^{L}\right) + \left(1 - \phi_{n,j}\right) \left(p_{n+1,j} - D_{n+1,j}^{L}\right)} \\ \leq \; & \pi_{n,j}^{h} v_{n+1,j+1} + \left(1 - \pi_{n,j}^{h}\right) v_{n+1,j}. \end{aligned}$$

The last inequality is due to that

$$\left(\phi_{n,j} - \widehat{\pi}^h_{n,j}\right)\left[\left(p_{n+1,j+1} - p_{n+1,j}\right) - \left(D^L_{n+1,j+1} - D^L_{n+1,j}\right)\right] \geq 0$$

because of (47). Thus, saving cash dominates.

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