Fiscal Policy in Debt Constrained Economies*

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Abstract

We study optimal fiscal policy in a small open economy (SOE) with sovereign and private default risk. The SOE's government uses linear taxation to fund exogenous expenditures and uses public debt to inter-temporally allocate tax distortions. We characterize a class of environments in which the tax on *labor* goes to zero in the long run, while the tax on capital income may be non-zero, reversing the standard prediction of the Ramsey tax literature. The zero labor tax is an optimal long run outcome if the private agents are impatient relative to the international interest rate and the economy is subject to sovereign debt constraints. The front loading of labor taxes allows the economy to build a large (aggregate) debt position in the presence of limited commitment. We show that a similar result holds in a closed economy with imperfect inter-generational altruism.

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1 Introduction

Economies frequently pursue policies that lead to fiscal crises, usually typified by sustained deficits that eventually lead to an inability to increase or roll-over debt (without paying an historically abnormal premium), and an associated sharp increase in tax rates and decline in government expenditure. The recent experience of Greece, Ireland, and Portugal, are only the latest examples of such crises. See Reinhart and Rogoff (2009) for many more examples from the historical record. Policies that run up debt, and eventually encounter borrowing constraints, may be the rational response of citizens (and their politicians) who face a world interest rate that is below their subjective rate of time preference. However, the normative question of whether the observed policies of front loading consumption and back loading taxes is indeed optimal in such an environment has not been thoroughly studied. To this end, this paper studies optimal fiscal policy in economies that are debt constrained, with a specific interest in relatively "impatient" economies for which the debt constraints are particularly relevant.

We consider optimal fiscal policy in a linear-tax framework. The canonical Ramsey formulation of optimal fiscal policy is quite simple: a government funds fiscal expenditures using linear taxes, and chooses the sequence of taxes that maximizes the welfare of the representative private citizen. A well known result in this framework is that capital taxes should be zero in the long run if the economy converges to a steady state (Judd, 1985; Chamley, 1986; Atkeson et al., 1999), and that taxes on labor income should be "smoothed" using government debt (Lucas and Stokey, 1983; Ljungqvist and Sargent, 2004). That is, if a steady state exists, the government will rely solely on labor taxes.[1] This prediction is robust to dropping the Ramsey assumption of full commitment, as shown by Dominguez (2007) and Reis (2008). There are a number of alternative environments in which capital is taxed in the steady state, but our focus is not solely the role of capital taxes in a steady state, but also the role of labor taxes.

This paper explores several variations of the canonical framework. At their core, each variation shares the fact that the inter-temporal marginal rate of substitution (MRS) of private agents may differ from the marginal rate of transformation (MRT) in the long run. In particular, agents are impatient relative to the inter-temporal price of resources. The greater impatience may reflect higher mortality in developing economies, imperfect altruism, or simple preference heterogeneity (with large/rich countries being rich because

[1] If labor is a type of capital, as in environments in which human capital can be accumulated, then labor taxes may also go to zero for the same reason that capital taxes go to zero. See Jones et al. (1997).

they have patient agents). Alternatively, countries with weaker domestic financial markets may export their savings, putting downward pressure on the world interest rate faced by citizens and governments in the rest of the world (Caballero et al., 2008, Mendoza et al., 2009). In this spirit, our primary scenario is a small open economy in which private agents discount at a rate that differs from the world interest rate. If a government faces a borrowing constraint, agents would like to pursue a declining path of consumption given the world interest rate, but are eventually constrained from doing so. Our main result is that in such an environment the tax on labor income converges to zero. That is, the optimal response to impatience and borrowing constraints is to *front load* taxes, driving labor taxes to zero in the limit.

The result is quite general in that the exact mechanism which drives a wedge between the MRS and the MRT is unimportant. An alternative scenario to a debt constrained small open economy is a closed economy in which the government discounts future consumption differently than the private agents. For example, if time indexes different generations and private agents are altruistic, then the inter-generational Pareto problem may feature a welfare function with weights on future period utility that are possibly higher than the weights implied by private altruism (e.g., as considered by Phelan, 2006 and Farhi and Werning, 2007, in models of private information). The optimal fiscal policy in such an economy (eventually) involves a subsidy to capital to sustain the required consumption of future generations. Our results show that, in addition, the labor tax must converge to zero.

The intuition of the result begins with the fact that the economy ultimately faces a borrowing constraint. The class of borrowing constraints we consider are motivated by the need for debt to be "self-enforcing;" that is, utility is at least as great by paying back the debt than it is from defaulting. The appeal of self-enforcing constraints stems from the practical limits of enforcing international debt contracts. We model the constraints in a fairly general way, but at their essence they involve placing a lower bound on equilibrium utility. Impatience absent a borrowing constraint involves long run immiseration, an outcome that will be inconsistent with realistic enforcement mechanisms.

To the extent possible, the optimal response to impatience is to front load consumption and leisure. The flip side of this is to exhaust the economy's borrowing capacity. For a given level of utility, an undistorted allocation of labor maximizes output, and therefore maximizes the amount of debt that can be serviced. In particular, the borrowing constraint places a floor on utility, while the efficient allocation ensures that this utility is delivered in a way that maximizes long run debt payments, freeing up resources for

early consumption. The zero labor tax in the long run is simply the mechanism through which fiscal policy exhausts the front-loading capacity of the economy's debt constraint. Viewed from the international bond market's perspective, the more the government is willing to front load taxation, the more the financial markets will be willing to lend. The closed economy result has a similar intuition. The optimal fiscal policy delivers the desired level of inter-generational altruism in a way that minimizes the need to pass-on physical capital.

In our framework, there is an aggregate borrowing constraint which involves both private and public debt, as both state variables determine the relative benefits of repayment versus default. As noted above, the optimal fiscal policy maximizes the long run aggregate debt of the economy. However, this does not necessarily imply large public debt positions. Indeed, the fact that labor taxes are zero in the long run requires the fiscal authority to fund government expenditures from net claims on private agents (and foreigners) plus any net tax receipts from capital income. We explore an example economy in which the government runs fiscal surpluses on the transition to the steady state. It is the private sector that is indebted in the long run, not the fiscal authority. Private debt is consistent with efficient output, while fiscal debt limits the economy's debt servicing capacity due to the need for distortionary taxes.

The normative implications of our model therefore stand in stark contrast to many observed fiscal trajectories. The optimal policies studied in this paper involve front loading labor taxes, and, correspondingly, reducing the available tax revenue to fund long run fiscal expenditures and to pay interest on public debt. Viewed through our model, observed fiscal crises typified by large pubic debt positions and sharp increases in labor tax rates are not consistent with the optimal response to impatience and debt constraints. We partially bridge this gap by considering an unanticipated and uninsured "bailout," in which a fraction of private debt becomes public debt. The standard incomplete markets intuition (as in Barro (1979)) is that labor taxes adjust up and stay high thereafter to finance the increased stock of government debt. The presence of meaningful long-run debt constraints, however, results in short run labor taxes "overshooting" their long run level, reflecting the need to front load. This allows the economy to continue to borrow despite the shock to public debt.

We stress that we hew fairly closely to the standard Ramsey framework and its close variants. That is, taxation is linear by assumption. We do allow for limited commitment and consider taxes supported by reputational equilibria, as well as incorporate alternative political economy frictions. However, we do not address issues of private information,

heterogeneity, and incompleteness of asset markets. It is well known in these models that optimal capital taxes may not be set to zero in the long run. We leave for future research whether our other normative results, including the zero labor tax, carry over to such environments.

The remainder of the paper is organized as follows: Section 2 presents the model environment; section 3 characterizes the optimal fiscal policy in an open economy setting; section 4 presents a quantitative analysis of the optimal policy and discusses the optimal financing of bailouts; section 5 extends the benchmark model to include private debt constraints (section 5.1) and to a closed economy environment (section 5.2); and section 6 concludes. The appendix contains all proofs.

2 Environment

In this section we describe the environment faced by households, firms, and the government. The key departure from the standard framework concerns debt constraints, which are introduced at the sovereign level. We restrict attention to the a deterministic environment. In this section, we focus on a small open economy, which faces an exogenous world interest rate $r_t^\star$. We characterize the closed economy model in section 5.2.

2.1 Representative Household

The representative private household has time-separable preferences with utility over consumption c (our numeraire) and labor n represented by

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \tag{1}$$

with $\beta \in (0,1)$.

We impose the following restrictions:

Assumption 1. *The utility function u satisfies the following conditions: (i) $u : X \to \mathbb{R}$ where $X \equiv (0,\infty) \times (0,\bar{n})$ with $0 < \bar{n} \leq \infty$; (ii) u is twice differentiable with $u_c > 0$, $u_n < 0$, $u_{cc} < 0$, $u_{nn} < 0$ and $u_{cc}u_{nn} - (u_{cn})^2 > 0$ for all $c, n \in X$; (iii) consumption and leisure are normal goods, $u_{cc}u_n - u_{cn}u_c \geq 0$ and $u_{nn}u_c - u_{cn}u_n \leq 0$ for all $c, n \in X$; and (iv) u satisfies the following*

boundedness assumption on preferences: both $u_c c(u_{cc}/u_c - u_{cn}/u_n)$ *and* $u_c n(u_{cn}/u_c - u_{nn}/u_n)$ *are bounded functions in* $(c,n) \in (\epsilon_c, \infty) \times (0, \epsilon_n)$ *for some* $\epsilon_c > 0$, $\epsilon_n \in (0, \bar{n})$.

The first three assumptions are standard. The last assumption insures that certain key expressions remain well behaved as consumption becomes large or labor approaches zero. This latter assumption holds for several of the preferences commonly used in the macroeconomics literature. For example, if utility takes the usual Cobb-Douglas form $(c^\gamma(1-n)^{1-\gamma})^{1-\sigma}/(1-\sigma)$ with $\gamma \in (0,1)$ and $\sigma > 0$, or the power-separable form $c^{1-\sigma}/(1-\sigma) - \psi n^{1+\gamma}/(1+\gamma)$ with $\sigma, \gamma, \psi > 0$, then the conditions in Assumption 1 are satisfied.

For what follows, we will assume that the consumers are more impatient that the world financial markets in the limit:

Assumption 2 (Impatient Consumers). *There exists* $M > 0$ *and* T *such that* $1 > M > \beta(1 + r_t^\star)$ *for all* $t > T$.

The focus of the paper is to understand the role this assumption plays in tax smoothing, and we will contrast the results with those from the standard assumption $\beta(1 + r_t^\star) = 1$ as we proceed.

The household provides labor in a competitive domestic labor market at a wage w_t, and labor is immobile across borders. Without loss of generality, we assume labor taxes are levied on the firms, so w_t represents wages after taxes.

Let r_t denote the net interest rate (before-taxes) received by consumers on their financial assets from time $t-1$ to t. No arbitrage implies $r_t = r_t^\star$ in an open economy. Let r_t^k denote the rental rate of the domestic capital stock owned by consumers, and δ its depreciation rate. Let ϕ_t^k represents the residence-based tax on capital income received in time t, where "residence-based" refers to the fact that domestic agents pay this tax regardless of the source of the capital income or its location. We introduce "source-based" taxation in the firm's problem below. No arbitrage between bonds and physical capital implies that the after tax return is equalized:

$$1 + (1 - \phi_t^k) r_t = 1 + (1 - \phi_t^k)(r_t^k - \delta).$$

It follows that:

$$r_t^k = r_t + \delta. \tag{2}$$

We define the (after-tax) period-0 price of consumption at time t as:

$$p_t = \prod_{s=1}^{t} \frac{1}{1 + (1 - \phi_s^k) r_s} \tag{3}$$

We normalize $p_0 = 1$. We let a_t denote the wealth of agents, including both financial wealth and capital holdings, net of capital income taxes. To be precise about the timing, if x is invested at the end of period $t-1$, the pre-tax wealth in period t is $(1+r_t)x$, and the after-tax wealth is $a_t = (1+r_t)x - \phi_t r_t x = (1 + (1-\phi_t)r_t)x$. The flow budget constraint of the consumer can be expressed:

$$a_{t+1} = (1 + (1 - \phi_{t+1}^k) r_{t+1})(a_t - c_t + w_t n_t + T_t), \tag{4}$$

where $T_t \geq 0$ denotes a non-negative lump-sum transfer from the government at time t. Beginning from an initial wealth a_0 and imposing a No Ponzi condition, the present-value budget constraint of the consumer is:

$$\sum_{t=0}^{\infty} p_t \left(c_t - w_t n_t - T_t \right) \leq a_0. \tag{BC}$$

Note that our notation implies that a_0 is net of period-0 capital taxes. As is well known, distortionary taxation can be avoided with a large enough initial capital levy. By starting from an a_0 such that distortionary taxes are still required, we are implicitly following the standard practice of assuming a bound on the initial capital levy. To be explicit, assume $\phi_0^k = 0$, which is without loss of generality given that a_0 is treated as an arbitrary initial condition.[2]

Our benchmark environment assumes that private agents do not face constraints on borrowing. We revisit this assumption in section 5.1. To anticipate, we show that it is sufficient to consider a representative private agent who ignores the presence of debt constraints. In particular, we show that any competitive equilibrium allocation that satisfies a private debt constraint can be implemented as an equilibrium in which household's do not directly face a borrowing constraint. This is true because the effect of a private debt constraint on consumer behavior can be replicated by a tax on borrowing plus a lump sum rebate of this tax revenue. However, the presence of private debt constraints alters the government's problem, as discussed below.

[2]This is without loss of generality for the consumer's problem. For the government's problem introduced below, we can adjust period-0 fiscal requirements to reflect any initial capital tax revenue, making the zero tax without loss of generality for that problem as well.

The household's problem is to choose sequences $\{c_t\}$ and $\{n_t\}$ to maximize (1) subject to (BC). This is a standard convex problem that can be solved using Lagrange multiplier techniques. Let θ be the multiplier on the budget constraint. The first order conditions for each t can be written as:

$$\beta^t u_c(c_t, n_t) = \theta p_t \tag{5}$$

$$\beta^t u_n(c_t, n_t) = -\theta p_t w_t. \tag{6}$$

These conditions plus the constraints are necessary and sufficient for the unique solution to the household's problem.

Note that $\theta > 0$, given the strict monotonicity of u, and so the budget constraint holds with equality. We can then write the budget constraint of the consumer as:

$$\sum_{t=0}^{\infty} \beta^t \left(u_c(c_t, n_t)\,(c_t - T_t) + u_n(c_t, n_t) n_t\right) = u_c(c_0, n_0) a_0. \tag{7}$$

As usual, this equation will later form the basis of the "implementability condition" used in Proposition 1.

2.2 Firms

The representative firm operates a constant returns to scale production function $F(k, n)$ and hires labor and rents capital in competitive factor markets to maximize profits. It pays a linear source-based tax τ_t^n on its wage bill and a source-based tax τ_t^k on its rental payments. The firms problem in period t is therefore

$$\max_{k_t, n_t} \left\{ F(k_t, n_t) - (1 + \tau_t^n) w_t n_t - (1 + \tau_t^k) r_t^k k_t \right\}.$$

where w_t is the wage and r_t^k the rental rate.

The first order conditions, necessary and sufficient, are:

$$F_k(k_t, n_t) = (1 + \tau_t^k) r_t^k \tag{8}$$

$$F_n(k_t, n_t) = (1 + \tau_t^n) w_t. \tag{9}$$

2.3 Government Budget Constraints

The government has to fund a sequence of expenditures g_t, which we take to follow a deterministic and exogenous process.[3]

The relevant inter-temporal price for the government is the before tax price, q_t:

$$q_t \equiv \prod_{s=1}^{t} \frac{1}{1+r_s} \tag{10}$$

with $q_0 = 1$. Therefore, the government's budget constraint is:

$$\sum_{t=0}^{\infty} q_t \left(g_t + T_t - \tau_t^n w_t n_t - \tau_t^k r_t^k k_t - \frac{\phi_t^k r_t}{1+(1-\phi_t^k) r_t} a_t \right) \leq -b_0 \tag{11}$$

where b_0 is the initial debt of the government. Recall that a_t is after-tax period-t wealth, so the initial amount invested at the end of period $t-1$ is $a_t/(1+(1-\phi_t^k)r_t)$, which is the tax base for residence-based capital taxation in the expression above. Note as well that we are allowing free disposal of government income.

2.4 Government's Lack of Commitment

The Ramsey approach to optimal fiscal policy assumes the government can commit to a sequence of tax, transfers and debt repayment promises. We are interested, however, in environments in which the government lacks commitment. Towards that goal, we consider the following *sovereign constraints*, that take the form:

$$W_t(\{u_s\}_{s=t}^{\infty}, k_t) \geq 0, \text{ for } t \in \{0, 1, \dots\} \tag{12}$$

where $u_t = u(c_t, n_t)$.

We restrict attention to forward-looking constraints that place a lower bound on utility, rather than an upper bound. Specifically, we assume that

Assumption 3. *W_t is differentiable in all its arguments and $\partial W_t/\partial u_s \geq 0$ for all $t \geq 0$ and $s \geq t$; with $\partial W_t/\partial u_t > 0$ for all $t \geq 0$.*

[3] It is possible to make g_t an endogenous choice with some (potentially time varying) utility value. As our main result holds for arbitrary sequences of government expenditure (subject to boundedness conditions on the equilibrium allocation), we simply treat public expenditures as a primitive.

A straightforward interpretation of these sovereign constraints is a limited commitment environment in which the government cannot commit to its promises on taxes and debt. Consider thus the game played by the government, the representative agent and the international financial markets. Suppose that if the government in power at time t deviates from a prescribe allocation (for example, by defaulting on the debt or expropriating capital), it can guarantee itself a deviation utility $\underline{U}_t(k_t)$. Then, a constraint of the type above ensures that allocation is indeed a subgame perfect equilibrium. In section 5.1 we show that such a constraint also arises when the government has the ability (but not necessarily the desire) to enforce private debt contracts with international financial markets. With this interpretation in mind, we shall refer to these constraints as "sovereign debt constraints" (SDC).

Moreover, the constraints can accommodate political economy distortions that interact with limited commitment. For example, Aguiar and Amador (2011) introduce constraints on allocations of the form:

$$W_t = \sum_{j=0} \beta^j \left(\theta\delta^j + 1 - \delta^j\right) u(c_{t+j}, n_{t+j}) - \underline{U}_t(k_t) \geq 0, \forall t \geq 0, \tag{13}$$

where θ and δ reflect the fact that political incumbents may face political turnover risk and prefer consumption to occur during incumbency. Setting $\theta = 1$ and $\delta = 0$ generates the limited commitment-benevolent government framework.[4] Similarly, Aguiar et al. (2009) model a limited-commitment government with a geometric discount factor $\tilde{\beta} \neq \beta$, with associated constraints:

$$W_t = \sum_{j=0} \tilde{\beta}^j u(c_{t+j}, n_{t+j}) - \underline{U}_t(k_t) \geq 0, \forall t \geq 0. \tag{14}$$

The fact that constraints of the type (12) arise in models of endogenous default motivate the title of the paper. Nevertheless, limited commitment is not the only motivation for such constraints. In a full commitment environment, there may be reason to incorporate constraints of that form. For example, consider a dynastic model in which agents live for one period and bequeath assets to the next generation of the dynasty. Intergenerational altruism is governed by β. Now consider a Pareto problem in which the government maximizes the initial generation's utility as given by equation (1), subject to giving generation t a utility level of at least $\underline{U}_t$. In this environment, the constraint set

[4] See Battaglini and Coate (2008) for a similar representation of the government's utility.

takes the form:

$$W_t = \sum_{s=0} \beta^s u(c_{t+s}, n_{t+s}) - \underline{U}_t \geq 0, \forall t \geq 1.$$

These sovereign constraints motivate environments in which the optimal fiscal policy distorts inter-temporal tradeoffs from the perspective of the private agents, whose decisions underlie the implementability condition. While fairly general, constraints of the type (12) do not encompass environments in which the government has a direct incentive to distort the private agent's static labor-leisure choice. That is, W_t depends on consumption and labor through the utility function $u(c, n)$, but not directly. This rules out situations in which labor supply and labor taxes are not chosen simultaneously (so n_t affects off-equilibrium payoffs, as in Acemoglu et al., 2008 and Acemoglu et al., 2011), or environments in which the government wants to manipulate relative factor prices (see the discussion of this in regard to private debt constraints in section 5.1 below).

With this, we are now ready to define competitive equilibrium in the next sub-section.

2.5 Competitive Equilibrium

We are interested in allocations that can be supported as an equilibrium outcome. For the open-economy benchmark economy, we assume the world interest rate is given by $r_t^\star$. No arbitrage implies that $r_t = r_t^\star$. The representative household's and government's budget constraints imply an aggregate resource constraint for the economy:

$$\sum_{t=0}^{\infty} q_t \left(c_t + g_t + (r_t^\star + \delta)k_t - F(k_t, n_t)\right) \leq A_0. \qquad \text{(RC}^\star\text{)}$$

where A_0 is the initial wealth of the economy: $A_0 = a_0 - b_0$. Note that for the open economy case, the relevant state variable is total wealth. In particular, firms can rent capital from foreigners and domestic agents can trade their physical capital for foreign assets, and so the initial physical capital stock k_0 is an equilibrium outcome and not a predetermined variable.

We define a competitive equilibrium in the standard fashion, augmented by the presence of the sovereign debt constraints:

Definition 1. *An **open economy competitive equilibrium** with an initial asset position (A_0, a_0), consists of sequences of prices $\{r_t, w_t, r_t^k\}$; taxes $\{\phi_t^k, \tau_t^n, \tau_t^k\}$ with $\phi_0^k = 0$; non-negative lump-sum transfers $\{T_t\}$; and quantities $\{c_t, n_t, k_t\}$ such that* **(i)** *inter-temporal prices correspond*

to the small open economy assumption: $r_t = r_t^\star$ *for all* t*, and* r_t^k*,* p_t*,* q_t *satisfy equations (2), (3), and (10);* **(ii)** $\{c_t, n_t\}$ *solve the constrained household problem given initial wealth* a_0*, prices and taxes;* **(iii)** $\{n_t, k_t\}$ *solve the firm's problem given prices and taxes, that is, equations (8) and (9) hold;* **(iv)** *the government budget constraint (11) holds;* **(v)** *the open-economy aggregate resource constraint (RC***) holds; and* **(vi)** *the sovereign debt constraints (12) hold for all* t*.*

Given that the sovereign debt constraints are restrictions on an allocation's quantities (c_t, n_t, k_t), it follows that competitive equilibria can be characterized using a primal approach, as usual:

Proposition 1. *An allocation* $\{c_t, n_t, k_t\}_{t=0}^{\infty}$ *can be implemented as a competitive equilibrium if and only if: (a) the resource constraint (RC***) holds; (b) the sovereign debt constraints (12) hold for all* t*; and (c) the following implementability condition is satisfied:*

$$\sum_{t=0}^{\infty} \beta^t (u_c(c_t, n_t)c_t + u_n(c_t, n_t)n_t) \geq u_c(c_0, n_0)a_0. \tag{IC}$$

There are in principle many competitive equilibria. In the next section, we discuss equilibrium selection.

3 Efficient Allocations

Let us start by defining our notion of efficiency:

Definition 2. *An* **efficient allocation** *is a competitive equilibrium allocation that maximizes household's utility as of time 0.*

Note that the economy is populated by a representative agent as well as a sequence of political incumbents characterized by W_t. This objective function, plus a corresponding sequence of constraints $W_t \geq 0$, implies that our notion of efficiency can be viewed as choosing a competitive equilibrium that lies along the Pareto frontier of the initial representative household and the sequence of political incumbents.[5] It should be clear that efficiency does not imply the lack of political economy distortions in the choice of allocations.

[5] In section 5.2 we reinterpret the Pareto problem as one spanning different generations of private households, rather than political incumbents, with the objective function representing the first generation's welfare.

Efficient allocations are solutions to the following problem:

$$\max_{\{c_t,n_t,k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t,n_t), \tag{P}$$

subject to:

$$\sum_{t=0}^{\infty} q_t\left(F(k_t,n_t)-c_t-g_t-(r_t^\star+\delta)k_t\right)+A_0 \geq 0, \tag{P.1}$$

$$\sum_{t=0}^{\infty} \beta^t\left(u_c(c_t,n_t)c_t+u_n(c_t,n_t)n_t\right)-u_c(c_0,n_0)a_0 \geq 0, \tag{P.2}$$

$$W_t(\{u_s\}_{s\geq t},k_t) \geq 0, \text{ for all } t \geq 0 \tag{P.3}$$

3.1 Necessary Conditions for an Efficient Allocation

Note that implementability and sovereign debt constraints, (P.2) and (P.3), are not necessarily convex. We proceed by characterizing necessary conditions for an interior optimum. Let η denote the multiplier on (P.2), μ denote the multiplier on (P.1), and $\beta^t\lambda_t$, $t=0,1,...$ denote the sequence of multipliers on constraints (P.3).[6]

The first order necessary condition with respect to consumption at time $t \geq 1$ is:

$$\beta^t\left[u_c(c_t,n_t)+\eta\left(u_c(c_t,n_t)+u_{cc}(c_t,n_t)c_t+u_{cn}(c_t,n_t)n_t\right)\right]+ \left[\sum_{s=0}^{t}\beta^s\lambda_s\frac{\partial W_s}{\partial u_t}\right]u_c(c_t,n_t)=q_t\mu. \tag{16}$$

The first term $\beta^t u_c(c_t,n_t)$ represents the value of consumption in period t to the consumer (the objective function). The term following the multiplier η reflects the need to satisfy the implementability condition (P.2). The terms involving W_s represent the fact that increasing consumption in period t relaxes the sovereign constraints in that period, and all preceding periods to the extent that W is forward looking. The final term to the right of the equal sign is the price of consumption at time t, translated into utility terms via the multiplier μ. Note that for period $t=0$, we have the same condition but replace the term

[6]There are technical conditions that must be met to ensure the validity of Lagrangian methods (these relate to the usual regularity conditions plus the existence of summable multipliers when facing an infinite sequence of constraints). In the Appendix, we formally address these issues. For expositional simplicity, in the text we proceed as if the usual Lagrange multiplier approach is valid.

multiplying η with

$$u_c(c_0, n_0) + u_{cc}(c_0, n_0)(c_0 - a_0) + u_{cn}(c_0, n_0)n_0,$$

given the presence of $u_c(c_0, n_0)$ on the right hand side of the implementability condition (P.2).

The first order condition for labor is similar:

$$\beta^t \left[u_n(c_t, n_t) + \eta \left(u_{cn}(c_t, n_t)c_t + u_n(c_t, n_t) + u_{nn}(c_t, n_t)n_t\right)\right] + \left[\sum_{s=0}^{t} \beta^s \lambda_s \frac{\partial W_s}{\partial u_t}\right] u_n(c_t, n_t) = -F_n(k_t, n_t)q_t\mu, \quad (17)$$

for $t \geq 1$. For $t = 0$, replace the term multiplying η with $u_{cn}(c_0, n_0)(c_0 - a_0) + u_n(c_0, n_0) + u_{nn}(c_0, n_0)n_0$.

The first order condition for capital is:

$$F_k(k_t, n_t) = r^\star + \delta + \frac{\beta^t}{q_t\mu}\lambda_t \frac{\partial W_t}{\partial k_t}. \quad (18)$$

Note that capital may be distorted away from the efficient level ($F_k = r^\star + \delta$) as the sovereign's debt constraint may depend on the level of installed capital. In particular, a large capital stock may tempt the government to renege on implicit capital tax promises ($\partial W_t / \partial k_t > 0$), leading to downward distortion of investment when (P.3) binds.[7]

3.2 Optimal Taxation

Using the first order conditions, we can characterize properties of optimal fiscal policy; that is, the nature of taxation in an efficient allocation. Recall that the tax on labor τ^n can be expressed in terms of quantities using the household's and firm's first order conditions:

$$\tau_t^n = \frac{F_n(k_t, n_t)u_c(c_t, n_t)}{-u_n(c_t, n_t)} - 1.$$

[7] See Benhabib and Rustichini (1997), Dominguez (2007), Reis (2008), Aguiar et al. (2009), and Aguiar and Amador (2011) for studies of capital taxation in limited commitment environments.

Dividing (17) by (16) and rearranging, we have:

$$1+\tau_t^n = \frac{1+\eta\left(1+\frac{u_{cc}c_t+u_{cn}n_t}{u_c}\right)+\sum_{s=0}^{t}\beta^{s-t}\lambda_s\frac{\partial W_s}{\partial u_t}}{1+\eta\left(1+\frac{u_{cn}c_t+u_{nn}n_t}{u_n}\right)+\sum_{s=0}^{t}\beta^{s-t}\lambda_s\frac{\partial W_s}{\partial u_t}}. \tag{19}$$

From this expression, we see that to the extent the right hand side differs from one, labor will be distorted by taxation.

Distortionary taxes in (19) stems from $\eta > 0$, as the expression in parentheses multiplying η is different in the denominator and numerator of (19). Recall that η is the multiplier on the implementability condition, which ensures that fiscal policy is implemented in a manner consistent with household and firm optimization. If the government had access to lump sum taxation, or had sufficient initial assets to cover expenditures without distortionary taxes, then η is zero.

However, the fact that $\eta > 0$ is not sufficient to conclude that labor is distorted in the limit. We must first analyze the remaining term in (19). In particular, in both the numerator and denominator there are the cumulative sum of non-negative numbers reflecting the sovereign debt constraints: $\sum_{s=0}^{t}\beta^{s-t}\lambda_s\partial W_s/\partial u_t$. If this sum grows without bound, and the terms involving η are bounded, the expression on the right of (19) converges to one; that is, the labor tax converges to zero. However, for this sum to diverge, it must be that the sovereign debt constraints are binding in the long run, and in principle must be ruling out the Ramsey allocation from the equilibrium set. Note that the Ramsey allocation prescribes a marginal utility of consumption that increases without bound; that is, the country immiserates itself. This obviously requires an extreme form of commitment to the country's foreign liabilities. As an alternative, we consider allocations where this does not happen:

Definition 3 (No Immiseration). *An allocation satisfies* ***no-immiseration*** *if* $\liminf_{t\to\infty} c_t > 0$ *and* $\limsup_{t\to\infty} n_t < \bar{n}$*;*

The lack of commitment provides the natural intuition that a country would rather repudiate its debt obligations than see consumption and leisure go to zero. A sufficient condition on the sovereign debt constraints to ensure no-immiseration is:

Lemma 1. *Suppose that there exist $\epsilon_c > 0$, $\epsilon_n > 0$ and finite T such that for all $t > T$:*

$$\sup_{\{c_s,n_s\}_t^\infty,k_t} \left\{ W_t(\{u(c_s,n_s)\}_{s=t}^\infty, k_t) \middle| c_t = \epsilon_c \right\} < 0, \text{ and}$$
$$\sup_{\{c_s,n_s\}_t^\infty,k_t} \left\{ W_t(\{u(c_s,n_s)\}_{s=t}^\infty, k_t) \middle| n_t = \bar{n} - \epsilon_n \right\} < 0,$$

then an efficient allocation must satisfy no-immiseration.

Many standard frameworks satisfy this condition. For example, suppose $W_t \equiv \sum_{s=t}^\infty \theta^s u_s - \underline{U}(k_t)$ for some discount factor $\theta \in (0,1)$ and outside option $\underline{U}(k_t)$ such that (i) $\underline{U}(k_t)$ is bounded below, and (ii) u is bounded above with $\lim_{c\to 0} u(c,n) = \lim_{n\to\bar{n}} u(c,n) = -\infty$; then an efficient allocation satisfies no-immiseration.[8]

We now show that a non-immiserazing allocation must feature a labor tax that goes to zero:

Proposition 2. *[Zero Labor Tax in the Long Run] Suppose that Assumptions 1, 2 and 3 hold. If an efficient allocation exists and satisfies no-immiseration, then* $\lim_{t\to\infty} \tau_t^n = 0$.

The proof relies on the fact that Assumption 2 implies that β^t/q_t converges to zero, as domestic consumers are impatient relative to the world interest rate. To see why this is important, consider the first order condition for consumption, (16), where we have rearranged terms:

$$\frac{\beta^t}{q_t}\left[1+\eta\left(1+\frac{u_{cc}c_t+u_{cn}n_t}{u_c}\right)+\sum_{s=0}^{t}\beta^{s-t}\lambda_s\frac{\partial W_s}{\partial u_t}\right]=\frac{\mu}{u_c(c_t,n_t)}. \tag{20}$$

The fact that consumption is bounded away from zero and labor bounded away from $\bar{n}$ (no-immiseration), together with $\mu > 0$,[9] implies the right hand side of the above equation is strictly positive. The fact that β^t/q_t is converging to zero implies the limit of the sum must be unbounded (given that Assumption 1 implies that the term multiplying η is bounded). The unbounded sum reflects that the debt constraints are binding in the limit, with the strictly positive multipliers accumulating over time. Looking back at equation (19), if the infinite sum diverges, and the terms in brackets remain bounded, then the labor tax most be going to zero.

[8] For example, if utility takes the usual Cobb-Douglas form $(c^\gamma(1-n)^{1-\gamma})^{1-\sigma}/(1-\sigma)$ with $\gamma \in (0,1)$ and $\sigma > 0$, or the power-separable form $c^{1-\sigma}/(1-\sigma) - \psi n^{1+\gamma}/(1+\gamma)$ with $\sigma, \gamma, \psi > 0$, then conditions (ii) and (iii) hold for $\sigma > 1$, the empirically relevant value.

[9] The fact that the resource constraint binds ($\mu > 0$) is an intuitive result that we prove in the appendix.

The standard Ramsey solution can be recovered by setting $\lambda_s = 0$ for all s. In this case, both sides of (20) converge to zero (while the ratio of q_t to $\beta^t u_c$ may be bounded), and if $\eta \neq 0$ the labor tax wedge may be non-zero. This highlights the fact that the Ramsey full-commitment allocation without debt constraints does not call for zero labor taxes in the long run. If $\beta(1 + r_t^\star) < 1$, the Ramsey allocation calls for increasing marginal utility of consumption, which violates the no-immiseration condition in the proposition. In the presence of limited commitment, such a path of consumption is not self-enforcing and the growing sum of non-zero multipliers reflects the presence of the debt constraints.

Note as well that if $\beta R = 1$, we have $\beta^t / q^t = 1$ and the terms involving η remain relevant in the long run. Therefore, labor taxes do not in general converge to zero when private agents discount at the world interest rate (while capital taxes typically do converge to zero). See Dominguez (2007) and Reis (2008) for the equivalent closed economy analysis in which the marginal rate of inter-temporal substitution equals the return on capital in the steady state. When agents are patient, there is no counter-weight to the incentive to relax the debt constraints by saving, and the constraints become irrelevant in the long run (this is an application of the well known "back loading" result of Ray, 2002).

3.3 Front Loading and Debt Constraints

Proposition 2 states that labor is undistorted in the long run if agents are relatively impatient. That is, that labor tax distortions are *front loaded*. Perhaps one would have thought that distortions should be back loaded due to impatience, or at the least distortions should be smoothed to the extent possible. However, agents wish to front load consumption and leisure. This suggests borrowing to the greatest extent possible subject to the debt constraints. The efficient allocation of labor is the counter-part to servicing a large aggregate debt.

In particular, suppose that the efficient allocation converges to a steady state $(c_\infty, n_\infty, k_\infty)$, with associated steady state interest rate $r^\star$ and fiscal expenditure g. Suppose as well that the constraints W_t are weakly decreasing in capital, so more capital tightens the constraints, although this is not strictly necessary for the following proposition.[10] Let B_∞ denote the steady state aggregate debt position. From the aggregate resource constraint,

$$B_\infty = \frac{1 + r^\star}{r^\star} \left[F(k_\infty, n_\infty) - c_\infty - g - (r^\star + \delta) k_\infty \right].$$

[10] If W_t were increasing in k, the proposition holds with the inequality of the last constraint reversed.

The efficient allocation maximizes steady state debt subject to the debt constraints:

Proposition 3. *[Maximal Debt] Suppose the efficient allocation converges to a steady state. Under the conditions of Proposition 2, the efficient allocation converges to the maximal steady state debt allocation, where the latter allocation is the solution to:*

$$\max_{c,n,k} \left\{ \frac{1+r^\star}{r^\star} \left(F(k,n) - c - g - (r^\star + \delta)k \right) \right\},$$

subject to

$$\begin{aligned} u(c,n) &\geq u(c_\infty, n_\infty), \\ k &\leq k_\infty. \end{aligned}$$

Note that the constraint set in the maximal debt problem is not a singleton, as there are many choices of consumption and leisure that deliver the same level of utility. What the proposition says is that the efficient allocation delivers the steady level of utility in a manner that maximizes net payments to foreign financial markets. The benefit of such an allocation to an impatient economy is that a large steady state debt finances front loading of consumption and leisure. The implication for fiscal policy is to leave labor decisions undistorted to maximize output.

We should emphasize that aggregate debt is maximized in the long run, not government debt. The debt referred to in the proposition is the aggregate debt of the country, which is the sum of private and public foreign liabilities. Even if the decentralization is such that the only foreign liabilities are government liabilities, these can be balanced by government claims against the domestic private sector, making the private sector the ultimate debtor. Indeed, the fact that the government does not tax labor income in the long run requires the government to hold enough claims against private agents or foreigners to fund government expenditures plus any net subsidy to capital income. This generates the somewhat surprising implication that the government of an impatient economy may eventually run fiscal surpluses on the transition path, a point we discuss in detail in our example economy below.

From the perspective of the private agents, the efficient allocation also maximizes private steady state debt. In particular, the net payments a private agent makes to financial markets (including the capital rental market) at any point in time is $w_t n_t - c_t$. For a given level of utility, taxes on labor reduce the available income for net debt payments, and in this sense zero labor taxes allows private agents to maximize long run debt. In this re-

gard, we stress that Proposition 2 exploits the fact that private agents are impatient, and not just (or necessarily) the government. As noted above, the proof of Proposition 2 relies on the fact that the compounded discount factor relevant for the IC constraint approaches zero faster than q_t, and (IC) is derived from the private agents' problem independently of the government's objective and the debt constraints. In the absence of taxation, private agents' would like to pursue a path in which consumption and leisure are front loaded and long run debt is maximized, and the efficient fiscal policy faced with such a constraint ensures that this is accomplished to the extent allowable.

We note that the desire to minimize long run labor distortions, to service the maximal long run debt, holds regardless of the discount factor SDC or if we replaced the objective function with that of a patient agent. It is the private agent's discount factor present in the implementability constraint that generates the result. Thus, at the root of the result is minimizing tax distortions. This begs the question of why a large private sector debt is the avenue for minimizing tax distortions.

The canonical result in minimizing tax distortions is zero tax on capital and an (approximately) stable tax on labor. This holds in our framework if the private agents discount at the world interest rate. When the private agent's are relatively impatient, combined with a lack of commitment, we have that the private agent's Euler Equation – evaluated at the world interest rate – must be distorted in the long run. This is true whether the distortion is implemented by a tax on borrowing or by rationing by the foreign financial markets which lend directly to the private agents (see section 5.1 for a discussion). Therefore, there is necessarily a first order distortion to the private agent's Euler Equation stemming from the debt constraint, and a zero-capital-tax steady state is not self-enforcing. Now suppose that labor taxes were constant, but not zero, in such a steady state. If a constant labor tax minimizes labor market distortions, a small perturbation will have second order costs. However, as we have shown, a decline in the future labor tax relaxes the debt constraint today, generating a *first order* gain by reducing the capital market wedge in the private Euler Equation. Thus, efficiency calls for a declining labor tax wedge that asymptotes to zero.

4 An Example Economy with Sovereign Debt Constraints

We now study a specific open-economy environment to render the constraints (12) concrete and fully characterize the efficient allocation. We also use the example economy to

study the optimal tax policy to finance an unanticipated government "bailout" of private debtors.

4.1 Preferences and Technology

Consider a small open economy without capital and with a linear production technology: $f(n) = n$. Assume also that g_t is constant and that the economy can borrow from abroad at a constant risk-free interest rate of $R \equiv 1 + r^\star$. We impose that $\beta R \leq 1$. The per-period utility function of the domestic representative agent is:

$$u(c, n) = \frac{c^{1-\sigma}}{1-\sigma} - \omega \frac{n^{1+\nu}}{1+\nu}. \tag{21}$$

At any point, the government of the economy can decide to cut all access from the international financial markets by defaulting on their external obligations. Such a move guarantees a utility level of $\underline{U}$ to the representative agent, where this payoff $\underline{U}$ can include the costs of default imposed to the country by the foreigners in a default event. For our analysis, we do not need to specify anything more about how $\underline{U}$ gets determined, except that it places a lower bound on equilibrium consumption and leisure:

Assumption 4. *There exists $\tilde{c} > 0$ and $\tilde{n} < \bar{n}$ such that $\underline{U} > u(\tilde{c}, \tilde{n})/(1-\beta)$.*

The sovereign constraints (P.3) ensure that domestic citizens are never made better off by default:

$$W_t \equiv \sum_{j=t}^{\infty} \beta^{j-t} \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \omega \frac{n_t^{1+\nu}}{1+\nu} \right) \geq \underline{U}; \text{ for all } t. \tag{22}$$

The implementability condition takes the following form:

$$\sum_{t=0}^{\infty} \beta^t \left\{ c_t^{1-\sigma} - \omega n_t^{1+\nu} \right\} \geq c_0^{-\sigma} a_0 \tag{23}$$

4.2 The Efficient Allocation

The efficient allocation for this economy can then be characterized as follows. Suppose that constraint (22) is not binding. Then the first order condition for consumption, equa-

tion (16), can be written in this environment as

$$(\beta R^\star)^t (1+\eta(1-\sigma)) = \mu c_t^\sigma, \text{ while } W_t > \underline{U}, t > 0.$$

If $\beta R^\star = 1$, then c_t is constant, while if $\beta R^\star < 1$, c_t is strictly declining due to impatience. Similarly, the first order condition for labor, equation (17), takes the form:

$$(\beta R^\star)^t (1+\eta(1+\nu)) = \omega^{-1}\mu n_t^{-\nu}, \text{ while } W_t > \underline{U}, t > 0.$$

The right hand side is strictly declining in n_t. If $\beta R^\star = 1$, then n_t is constant, while if $\beta R^\star < 1$, n_t is strictly increasing due to impatience.

The labor tax wedge along this unconstrained path is:

$$\tau_t^n = \frac{\eta(\nu+\sigma)}{1+\eta(1-\sigma)}, \text{ while } W_t > \underline{U}, t > 0.$$

Regardless of relative impatience, while unconstrained by the borrowing constraint, labor tax is a constant. If $\beta R^\star = 1$, we see that consumption, labor, and taxes are all constant, which accords with tax smoothing.

If $\beta R^\star < 1$ taxes are constant while unconstrained, but the fact that c_t is falling and n_t is increasing over time implies that $u_t < u_{t-1}$ and this path is not sustainable in the long run. It must be then that at some point $W_t = \underline{U}$ by Assumption 4. The following lemma states that $W_t = \underline{U}$ is an absorbing state for the utility level:

Lemma 2. *Suppose that $W_t = \underline{U}$ for $t > 0$, then $W_{t+s} = \underline{U}$ for all $s > 0$.*

We can also obtain an explicit solution for the behavior of the labor tax. In our current environment, $\partial W_s / \partial u_t = \beta^{t-s}$ for $s \leq t$, and zero otherwise. Substituting this into (16) and (17), we have

$$(\beta R^\star)^t \left(1+\eta(1-\sigma)+\sum_{j=0}^{t}\lambda_j\right) = \mu c_t^\sigma$$
$$(\beta R^\star)^t \left(1+\eta(1+\nu)+\sum_{j=0}^{t}\lambda_j\right) = \omega^{-1}\mu n_t^{-\nu}.$$

where we have used that $f'(n) = 1$. Taking the ratio of these two expressions to solve for

the labor tax, we have:

$$\tau_t^n = \frac{\eta\,(\nu+\sigma)}{1+\eta(1-\sigma)+\sum_{j=0}^{t}\lambda_j}.$$

The fact that $\lambda_t > 0$ whenever the borrowing constraint binds implies that the summation $\sum_{j=0}^{t}\lambda_j$ is increasing over time along a constrained path, and the labor tax is falling. In the limit, $\tau_t^n = 0$, as implied by Proposition 2.

The efficient allocation is graphically depicted in Figure 1. The figure represents indifference curves in consumption-labor space. The curve $\underline{u} \equiv (1-\beta)\underline{U}$ denotes the flow utility that delivers $\underline{U}$. The tax wedge can be read off the slope of the indifference curve: $\frac{1}{1+\tau^n} = -\frac{u_n}{u_c} = \left.\frac{dc}{dn}\right|_u$. We consider the initial allocation (after time 0) at point A, which is associated with some initial assets in private and public hands (a_0, A_0).

We overlay the dynamics of the efficient allocation on the indifference curves, using the figure as a phase diagram. For $u_t > \underline{u}$, the allocation is unconstrained by the borrowing limit. As noted above, the unconstrained allocation features falling consumption, increasing labor, and a constant tax wedge. The constant tax wedge implies parallel shifts across indifference curves as utility falls. Point B is the first point at which utility reaches $\underline{u}$. Once $u_t = \underline{u}$, utility is constant and we move along the indifference curve. As we move along, consumption and labor both increase and the tax wedge falls, eventually reaching zero in the limit. Point C represents the steady state of the economy.

The intuition while the economy is unconstrained by the borrowing limit is straightforward. Impatience relative to $r^\star$ makes a declining path of consumption and leisure optimal. Standard tax smoothing insights makes a constant tax rate efficient as well. The fact that the economy ultimately hits the constraint is also straightforward. However, the question remains why point B is not a steady state. That is, once the economy hits the borrowing constraint, why are there further dynamics, and why do these involve a declining labor tax and increasing consumption?

The intuition is related to Proposition 3. Output at point B is distorted downward by labor taxes, reducing the ability of the economy to service a large steady state debt. As the tax on labor is reduced, output increases. In particular, output increases more than consumption as long as the tax rate is strictly positive, raising the steady state net exports (i.e., net debt payments) of the economy. To see that consumption increases less than output (i.e., labor), note that along the indifference curve, $dc = -u_n/u_c dn = dn/(1+\tau^n)$. By moving along the indifference curve to the zero-tax steady state C, the government

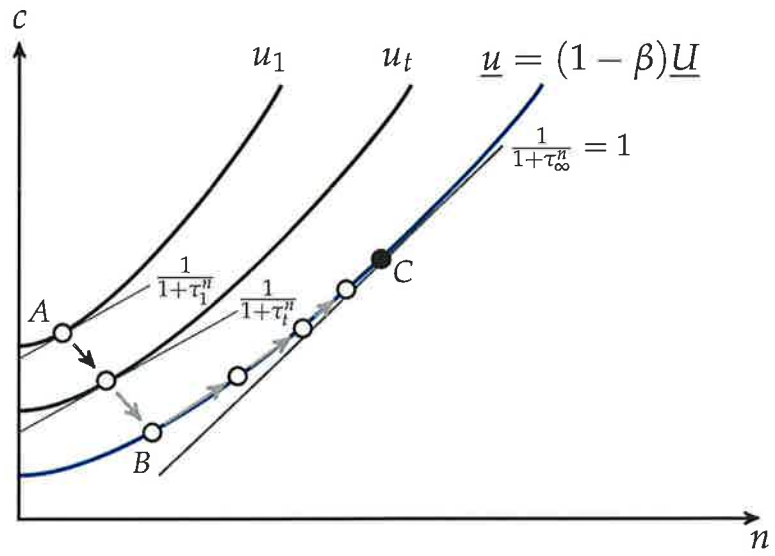


Figure 1: Path of Efficient Allocation when $\beta R < 1$. The curves represent intra-period indifference curves in the (c, n) space, and movements to the southeast represent reductions in utils. The arrows represent time, and the white circles the static allocation at a given point in time. The black circle represents the steady state.

increases its capacity to service steady state debt. The benefit of this policy for impatient agents is that consumption and leisure can be front loaded when the debt is incurred.

4.3 A Decentralization: Managing the portfolio of sovereign and domestic debt

We now explore the time path of aggregates of the economy described above. Starting from a positive tax economy, we have that labor taxes are falling over time and both the private agent and the economy as a whole are running down assets. It is not clear, however, what is happening to government debt itself, which is the difference between private and aggregate wealth.[11] In particular, the government may be borrowing from abroad to retire domestic debt. We can shed light on fiscal deficits through a numerical example.

Figure 2 depicts the path of a particular parameterization of the example economy.[12]

[11] More precisely, because a is after-tax private wealth, the accounting identity is $a_t R/(1+(1-\phi_t^k)r^\star) = A_t + b_t$. For $t = 0$, we have $\phi_0^k = 0$, and so $a_0 = A_0 + b_0$.

[12] Specifically, let $u(c,n) = \log c - \omega \frac{n^{1+\nu}}{1+\nu}$. We set $\nu = 0.5$ and $g = 0.20$, and select ω and $\underline{u}$ so that steady

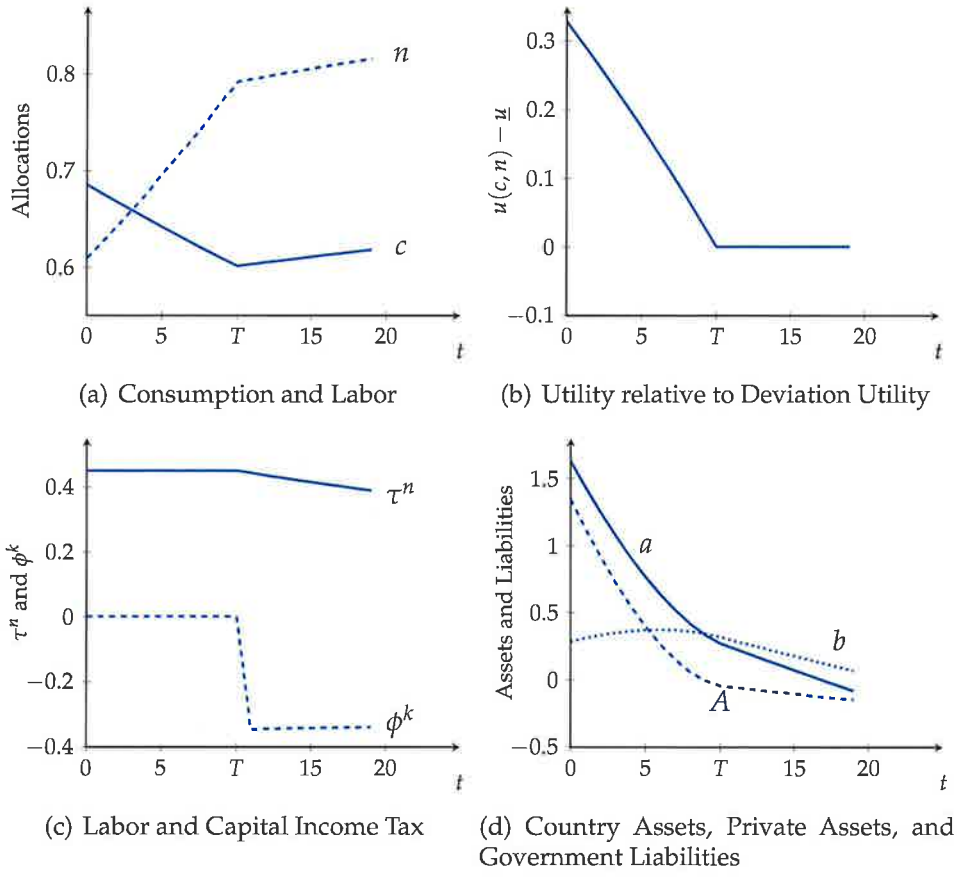
Given that the model is quite simple, we present figure 2 as a guide to intuition rather than a quantitative exercise. Time is the horizontal axis in each panel, and period T denotes the first period in which the borrowing constraint binds. The first panel depicts consumption and labor, and the second panel depicts utility relative to $\underline{u}$. Consumption and leisure decline prior to period T, as the borrowing constraint is slack and agents are relatively impatient. At period T, utility has fallen to $\underline{u}$, as seen in panel (b), and remains there. Nevertheless, as discussed above, dynamics continue, with leisure continuing to fall but consumption now rising. This is supported by a declining tax on labor. In particular, as depicted in panel (c), labor taxes are constant prior to T, but then begin to fall. This is the front loading of taxes that is optimal when the economy is constrained. Eventually, the labor taxes converges to zero, as stated in Proposition 2.

The tax on capital income is zero when the economy is unconstrained, and then becomes negative (a tax on borrowing) after period T. The tax on borrowing is necessary to keep the private agents from violating the aggregate borrowing limit. In the steady state, we have $\phi^k_\infty = -\frac{1-\beta R}{r^\star \beta}$, so that agents choose a constant path of consumption at the after-tax interest rate. The fact that consumption is increasing after period T implies that ϕ^k undershoots its steady state level.

Panel (d) depicts the corresponding path of assets and liabilities. The country's aggregate net foreign asset position A falls rapidly while the borrowing constraint is slack. Once constrained, the economy continues to draw down assets and starts to accumulate foreign liabilities, although the process is slower after period T. Private assets a also fall, both before and after T. The "deceleration" at period T is less pronounced for private assets than it is for aggregate assets. This reflects that private assets need to be reduced in order to make labor efficiency consistent with the implementability constraint in the steady state. That is, the flip side of front-loading labor taxes is that the government pays down domestically held public debt. The government's total debt, b, is also depicted in panel (d), which represents the difference between private wealth and aggregate wealth. As might be expected, government debt is initially increasing while the economy is unconstrained. However, at some point *before* the economy becomes constrained, the government starts paying down its debt, and continues to do so after period T. In the limit, debt is reduced to the point that labor taxes are no longer necessary to fund government expenditures.

state labor/income is 1 and steady state net foreign liabilities are 65 percent of aggregate income. The international interest rate is 0.05, and $\beta = 0.94$, so private agents discount at a higher rate than the world interest rate.

Figure 2: Time Path of Economic Aggregates



(a) Consumption and Labor

(b) Utility relative to Deviation Utility

(c) Labor and Capital Income Tax

(d) Country Assets, Private Assets, and Government Liabilities

The dynamics depicted in figure 2 highlight the important role that government debt and net foreign assets play in supporting the convergence to first best labor. The model does not make a clean prediction for the relative quantities of public debt held domestically and abroad without further restrictions. That is, if private agents can hold foreign assets directly, there is an indeterminacy in the following sense: The government can borrow from a domestic resident directly, or indirectly by having the domestic resident lend to foreign investors and then borrowing the same amount from foreign markets.

For many developing economies, private residents do not hold large net foreign asset positions, and the vast majority of international borrowing and lending is implemented by the government. If we set private agent foreign assets to zero, then we know that domestic assets a are equivalent to domestically held government debt.[13] Moreover, the

[13]This follows from the absence of physical capital, but even in an economy with capital, we continue to

country's net foreign assets equal government foreign reserves minus sovereign debt. Therefore, panel (d) of figure 2 indicates that domestically held government debt is falling (a), while net sovereign liabilities are increasing (or foreign assets A are falling). Therefore, the path to efficient labor is one in which the public debt held abroad is increasing while the public debt held domestically is falling. As a normative prediction, the model states that the government should pay off domestically held debt first, while continuing to borrow from abroad. This is the optimal path to zero labor taxes for an impatient economy.

4.4 Financing Bailouts

We now stretch the model to address the following scenario: suppose at time t', the government nationalizes a portion of private debt. In our deterministic model, we treat this event as unanticipated, which can be a rough proxy for a model in which such an event is uninsurable and occurs with low probability. We do not address why the government nationalizes private debt; indeed, in our framework, it is never optimal to do so. Rather, we address how to finance the increase in public debt ex post.

To be precise, suppose that the government makes a transfer to the private agent at time t'.[14] The unanticipated transfer requires the government to choose a new path of taxes. We assume that the transfer is not so large as to induce a default by the government. Moreover, the fact that the government re-optimizes tax policy raises the familiar incentive to tax capital ex post and manipulate the period-zero value of the public debt. We assume that the government does not try to immediately reclaim it's bailout in such a manner. Specifically, we assume that $u_c(c_{t'}, n_{t'})(a_{t'} + T) \geq \bar{T}$ after the bailout, placing a floor on period-t' private resources evaluated at the new equilibrium prices. The implementability condition is therefore:

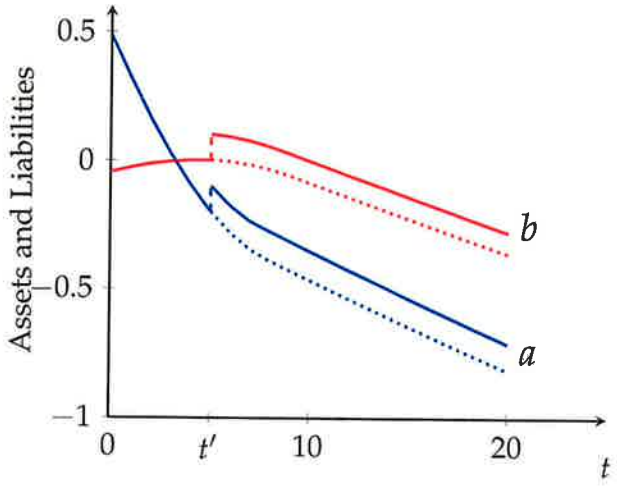
$$\sum_{s=0}^{\infty} \beta^s \left\{ c_{t'+s}^{1-\sigma} - \omega n_{t'+s}^{1+\nu} \right\} \geq \bar{T}. \tag{24}$$

The remaining elements of the problem remain the same, as the period-t' aggregate net foreign asset position $A_{t'}$ has not changed.

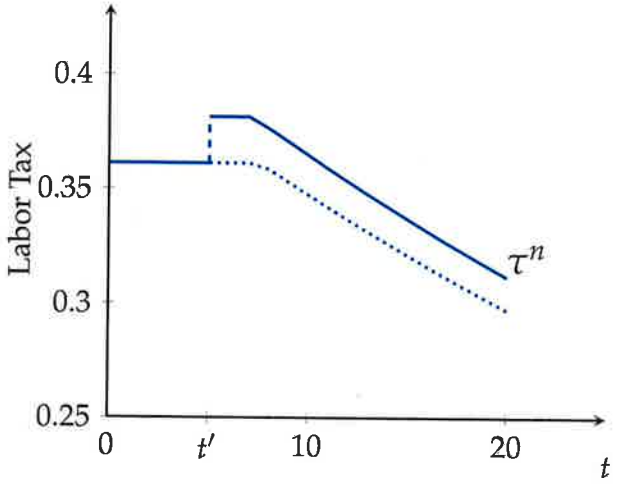
have a sharp prediction for domestically held debt as long as private foreign assets are zero.

[14] Our scenario is particularly reminiscent of Ireland's bailout of its banks in 2008. Ireland had consistently run fiscal surpluses before the financial crisis of 2008. Nevertheless, public debt as a ratio to GDP ratio went from 25 percent in 2007 to over 100 percent by 2011, in large part due to the nationalization of private bank liabilities during the financial crisis.

Figure 3: Bailouts



(a) Country Assets, Private Assets, and Government Liabilities



(b) Labor Tax

Figure 3(a) depicts the experiment:[15] at t', private assets (a) increase by 0.10, which is ten percent of steady state output, and government debt b increases by the same amount. The dotted line shows the original path absent the bailout. Panel 3(b) shows that labor taxes immediately jump. Once the debt constraint binds, which occurs sooner given the bailout, labor taxes start to decline. That is, labor taxes *overshoot* in response to the shock, reflecting the incentive to front load distortions. This contrasts with the standard "ran-

[15]The parameters are the same as in figure 2. We assume $A = a = -.20$ and $b = 0$ at the time of the bailout.

dom walk" intuition of optimal taxation in the presence of uninsurable shocks (Barro, 1979, Aiyagari et al., 2002). Indeed, when $\beta R = 1$ in our framework, taxes would jump up at t' and then remain constant thereafter. However, in the patient environment, the country does not face binding debt constraints in the long run and thus has no need to front load taxes.

The path of b indicates that the government starts to run a surplus soon after the bailout and b begins to decline. We do not depict the change in allocation, but both consumption and labor fall. The fact that labor declines, despite the income effect of the drop in consumption, reflects the increase in labor taxes. In this particular example, the substitution effect wins out. Also not depicted is the tax on capital income, which remains at zero until the debt constraint binds and then becomes negative.

While highly stylized, this scenario suggests the value of front loading the financing of bailouts. In this environment, the sharp increase in tax revenue is necessary for the government to pay down its newly acquired debt and allow the private sector to resume borrowing. While this is highly distortionary, it is necessary to ensure the economy services its large stock of steady state debt. Or, viewed from the perspective of financial markets, absent the front loading of tax revenues, the market will not be willing to allow the continued accumulation of net foreign liabilities.

5 Extensions

In this section we consider how our results translate to two alternative environments. The first is the case when the household faces a debt constraint. This analysis studies the link between private debt constraints and the sovereign debt constraints studied above, and we show how our analysis carries over. The second case is a closed economy. Given the focus on the relatively low world interest rate, it may not be clear at this point that the result regarding front loading labor taxes carries over to a closed economy. It is therefore useful to show that the result rests on the wedge between the inter-temporal marginal rate of substitution and the marginal rate of transformation, whether the MRT is determined by the domestic capital stock or the world interest rate.

5.1 Household Debt Constraints

In this subsection we extend our benchmark model to incorporate limited commitment on the part of the private agent. Specifically, we assume that agents face "household debt constraints" (HHDC) of the form:

$$V_t(\{u_s\}_{s=t}^{\infty}|\{w_s, T_s, \phi_s\}_{s=t}^{\infty}) \equiv \sum_{s=t}^{\infty} \beta^{s-t} u_s - \underline{V}_t\left(\{w_s, T_s, \chi_s\}_{s\geq t}\right) \geq 0 \text{ for all } t \geq 1, \quad \text{(HHDC)}$$

where $\underline{V}_t$ represents the value of repudiating debt at time t. Following Kehoe and Levine (1993), the constraint captures a limited commitment environment in which agents that default are permanently excluded from assets markets, but continue to participate in spot labor markets. In our environment, this implies working in the labor market. We also allow for additional enforcement mechanisms that the government may have available, such as wage garnishment, which we parameterize by $\chi_t \in [0, 1]$. In particular, let

$$\underline{V}_t\left(\{w_s, T_s, \chi_s\}_{s\geq t}\right) \equiv \sum_{s=t}^{\infty} \beta^{s-t} \max_{n_s} u(\chi_s w_s n_s + T_s, n_s). \tag{25}$$

Note that after a default, consumption is labor income ($\chi_s w_s n_s$) minus wage garnishment plus transfers if any (T_s).

The presence of equilibrium prices in the constraint set opens up room for externalities stemming from private borrowing constraints, a feature of limited commitment environments emphasized by Kehoe and Levine (1993) and a large subsequent literature. For a given wage and transfer sequence, note that V_t is differentiable and concave in household consumption and labor.

We do not include an initial period debt constraint given that maximizing (1) will satisfy the initial constraint, if feasible, and we assume that it is always feasible. In particular, we assume that the constraint set has a non-empty interior. That is, a_0 is such that for any bounded sequence of positive prices there exists an interior point in the constraint set. A sufficient condition for this is that it is feasible for the agent to supply $n < \bar{n}$ in the initial period and save part of that period's labor income.

The households problem is now the same as before, with the addition of the sequence of constraints (HHDC). Let $\beta^t \gamma_t \geq 0$ be the multiplier on period t's constraint, and, as before, θ be the multiplier on the household budget constraint (BC).[16]

[16]We solve the household's and government's problem using Lagrange multiplier techniques. As noted

The revised first order conditions are:

$$\beta^t u_c(c_t, n_t)\left(1 + \sum_{s=0}^{t} \gamma_s\right) = \theta p_t \tag{26}$$

$$\beta^t u_n(c_t, n_t)\left(1 + \sum_{s=0}^{t} \gamma_s\right) = -\theta p_t w_t. \tag{27}$$

These conditions, the constraints, and the complementary slackness conditions characterize the unique solution to the household's problem.

Note that as before $\theta > 0$, and so the debt constraints affect inter-temporal decisions, but not the static consumption-leisure choice:

$$\frac{-u_n(c_t, n_t)}{u_c(c_t, n_t)} = w_t.$$

The debt constraints (and their associated multipliers) can be viewed as an inter-temporal wedge. In particular, it will be convenient to define

$$\Gamma_t \equiv 1 + \sum_{s=1}^{t} \gamma_s,$$

with $\Gamma_0 = 1$ as the first period's constraint is slack. The first order conditions can then be written as

$$\beta^t \Gamma_t u_c(c_t, n_t) = \theta p_t$$
$$\beta^t \Gamma_t u_n(c_t, n_t) = -\theta p_t w_t.$$

Using that $T_t \geq 0$, it follows that we can write the budget constraint of the consumer as:

$$\sum_{t=0}^{\infty} \beta^t \Gamma_t (u_c(c_t, n_t)c_t + u_n(c_t, n_t)n_t) \geq u_c(c_0, n_0)a_0, \tag{28}$$

which is the revised implementability condition. This expression differs from (P.2) due to the presence of Γ_t, reflecting the shadow costs of the household's borrowing constraint.

The definition of a competitive equilibrium remains the same as the benchmark, with the stipulation that prices and allocations are consistent with the household's optimiza-

before, there are technical issues regarding the nature of the multipliers in infinite dimensional problems. We proceed by assuming the existence and validity of these multipliers. A sufficient condition in this context is that u is bounded. See the appendix for more details.

tion in the presence of debt constraints, given a wage-garnishment policy χ_t.

Recall that the primal approach to solving for efficient allocations treated the implementability condition as a constraint. The difficulty here is that (28) depends on the multipliers Γ_t as well as the allocation. The presence of debt constraints implies that an equilibrium allocation may be supported by multiple price sequences as the Euler Equation holds only with weak inequality. Fiscal policy determines not only the allocation, but also the prices conditional on an allocation. For each of these candidate price sequences, we have associated multipliers γ_t to satisfy the household's first order conditions. Nevertheless, the following proposition states we can purse the primal approach despite the private borrowing constraints:

Proposition 4. *An allocation $\{c_t, n_t, k_t\}_{t=0}^{\infty}$ can be implemented as a competitive equilibrium if and only if: (a) the resource constraint (RC*) holds; (b) the household debt constraint holds at the equilibrium wage sequence and zero transfers:*

$$V_t\left(\{u_s\}_{s=t}^{\infty} | \{w_s, 0, \chi_s\}\right) \geq 0, \text{for all } t \geq 1, \tag{29}$$

where $u_s = u(c_s, n_s)$, $w_s = -u_n(c_s, n_s)/u_c(c_s, n_s)$; (c) the sovereign debt constraints hold; and (d) the implementability condition of proposition 1 is satisfied:

$$\sum_{t=0}^{\infty} \beta^t (u_c(c_t, n_t)c_t + u_n(c_t, n_t)n_t) \geq u_c(c_0, n_0)a_0. \tag{IC}$$

This proposition states that we can search for efficient allocations without concerning ourselves with Γ_t, as long as we restrict attention to allocations that satisfy the household debt constraints. The reason we can ignore the wedge in the private agent's Euler Equation stems from their equivalence to a tax on borrowing. In particular, consider an agent that wishes to borrow more but is constrained from doing so. The multiplier γ_t is the shadow cost of this constraint. The government can replace γ_t with a tax on borrowing, so the agent is indifferent to the constraint at the original allocation (that is, the Euler Equation holds with equality given the tax). This replaces the shadow cost with an actual tax. This is always feasible for the government as long as the tax raises revenue. That is, as long as asset positions are non-positive when the debt constraint binds. As the deviation utility balances the budget period-by-period by definition (as agents lose access to financial markets), and delivers the same utility when the constant binds, this will always be the case. In this manner, the government can raise revenue without distorting the allocation by taxing borrowing at the debt constraint. The intuition is the same as the well

known equivalence of an import tariff and a import quota – both restrict the quantity of imports, but differ only in how the tariff revenue is shared.

The only difference between the problem with and without private debt constraints is therefore the constraint (29) and the potential availability of additional enforcement tools χ_t. Absent these additional tools, the government will have an incentive to distort spot wages to relax the household's borrowing constraint. In this way, labor taxes will, in general, not be zero, even in the absence of fiscal expenditures. This is a crude mechanism to enforce private debt contracts, as the distortion affects all workers along the equilibrium path.

A more realistic scenario is where additional tools (ranging from wage garnishment to debtor's prisons) may be available. The question then becomes one of whether the government is *willing* to enforce contracts directly. If the household accumulates enough foreign debt, the government may not choose to enforce these contracts, even if it can. However, this is the same question as whether the government repays its own debt, and thus equivalent to the type of sovereign debt constraints explored in our benchmark formulation. We conclude by formally stating this equivalence in regard to long run labor taxes:

Corollary 1 (Corollary to Proposition 2 with Private Debt Constraints). *Suppose the government can enforce private debt contracts directly (i.e., $\chi_t = 0$ is feasible for all t), then under the assumptions of Proposition 2* $\lim_{t\to\infty} \tau_t^n = 0$.

5.2 A Closed Economy

In this subsection, we show how our results carry over to a closed economy. In a closed economy, all capital is owned by domestic agents, and the source tax on capital, τ_t^k, is equivalent to the residence based tax on capital, ϕ_t^k, so we can set the last one to zero without loss of generality. It follows then that $p_t = q_t$ and the aggregate resource constraint is the national income accounting identity:

$$c_t + g_t + k_{t+1} - (1-\delta)k_t \leq F(k_t, n_t). \tag{RC}$$

Market clearing requires that the private agents' initial wealth corresponds to their holdings of government debt plus the domestic capital stock: $a_t = b_t + p_{t-1}k_t/p_t$, where the fact that k_t is adjusted by the inter-temporal price p_{t-1}/p_t reflects the fact that a_t and b_t are in period-t (after tax) units, while k_t is the amount invested at the end of period $t-1$.

We restrict $\tau_0^k = 0$ to eliminate the initial capital levy solution.

For simplicity, we assume that the private agent's can commit to debt contracts, and the only constraints are on the sovereign. This will avoid the need to discuss the additional notation of the previous subsection.

The requirements of a closed economy competitive equilibrium are the same as their open economy counterparts, with the appropriate adjustment to the resource constraint, and recognition that initial capital is a state variable in a closed economy:

Definition 4. *A **closed economy competitive equilibrium** from an initial position (k_0, b_0) consists of prices $\{r_t, w_t, r_t^k\}$, taxes $\{\phi_t^k = 0, \tau_t^n, \tau_t^k\}$ with $\tau_0^k = 0$, non-negative lump-sum transfers $\{T_t\}$, and quantities $\{c_t, n_t, k_t\}$ such that* **(i)** *r_t, r_t^k, p_t, q_t satisfy equations and (2), (3), (10);* **(ii)** *households optimize given prices and taxes subject to their budget constraint (BC);* **(iii)** *firms maximize profits given prices and taxes; that is, equations (8) and (9) hold;* **(iv)** *the government budget constraint (11) holds;* **(v)** *the sequence of closed-economy aggregate resource constraints (RC) hold; and* **(vi)** *the sovereign debt constraints (12) hold for all t.*

As before, we use the primal approach. The closed economy version of Proposition 1 is:

Proposition 5. *An allocation is consistent with a closed economy competitive equilibrium if and only if it satisfies the implementability condition*

$$\sum_{t=0}^{\infty} \beta^t \left(u_c(c_t, n_t)c_t + u_n(c_t, n_t)n_t\right) \geq u_c(c_0, n_0)\Big(b_0 + \big(F_k(k_0, n_0) + 1 - \delta\big)k_0\Big) \qquad \text{(IC')}$$

and the sequence of closed-economy resource constraints (RC) hold.

Equation (12) continues to define the additional sovereign constraints on the problem. The natural interpretation of (12) in a closed economy is one of a lower bound on aggregate savings, rather than an upper bound on aggregate debt. In this regard, the dynastic model with insufficient private altruism (relative to the government's Pareto weights) is perhaps the most relevant interpretation. For the closed economy, we add an additional assumption on the functions W_t:

Assumption 5. $\partial W_s / \partial k_t \leq 0 \forall s, t.$

This assumption insures that additional capital (weakly) tightens the constraint. This is consistent, for example, with more capital raising the incentive of the government to renege on its tax promises.

The definition for an efficient allocation given in Definition 2 continues to hold with the relevant notion of equilibrium being a closed economy competitive equilibrium. To characterize (interior) efficient allocations, let $\beta^t \psi_t$ denote the multiplier on the time t aggregate resource constraint.[17] The first order conditions for $t \geq 1$ are:

$$\frac{1}{\psi_t}\left[1+\eta+\eta\left(\frac{u_{cc}(c_t,n_t)}{u_c(c_t,n_t)}c_t+\frac{u_{cn}(c_t,n_t)}{u_c(c_t,n_t)}n_t\right)+\sum_{s=0}^{\infty}\beta^{s-t}\lambda_s\frac{\partial W_s}{\partial u_t}\right]=\frac{1}{u_c(c_t,n_t)} \tag{30}$$

$$\frac{1}{\psi_t}\left[1+\eta+\eta\left(\frac{u_{cn}(c_t,n_t)}{u_n(c_t,n_t)}c_t+\frac{u_{nn}(c_t,n_t)}{u_n(c_t,n_t)}n_t\right)+\sum_{s=0}^{\infty}\beta^{s-t}\lambda_s\frac{\partial W_s}{\partial u_t}\right]=-\frac{F_n(k_t,n_t)}{u_n(c_t,n_t)} \tag{31}$$

$$F_k(k_t,n_t)+1-\delta+\frac{1}{\psi_t}\sum_{s=0}^{\infty}\beta^{s-t}\lambda_s\frac{\partial W_s}{\partial k_t}=\frac{\psi_{t-1}}{\beta\psi_t}. \tag{32}$$

The initial period first order conditions ($t = 0$) are adjusted in the same way as they were in the open economy formulation.

We can now state the closed economy version of Proposition 2:

Proposition 6. *[Zero Labor Tax in the Long-run – Closed Economy] Suppose that Assumptions 1, 3 and 5 hold. If an (interior) efficient allocation satisfies no-immiseration and if:*

(a) there exist $M > 0$ and T such that $1 > M > \beta(F_k(k_t,n_t)+1-\delta)$ for all $t > T$;

then $\lim_{t\to\infty}\tau_t^n = 0$.

The impatience condition (a) replaces Assumption 2 in Proposition 2. Intuitively, an impatience condition for a close economy involves $\beta\left(F_k(k_t,n_t)+1-\delta\right)$, as the relevant marginal rate of transformation in a closed economy is the marginal product of capital. Differently from the small open economy case, impatience now imposes a restriction on the allocation rather than the exogenous parameters. In particular, the new impatience condition implies that capital is above the modified golden rule from the perspective of the private agents. That is, capital is "over-accumulated" relative to the private agents' discount factor. In the closed economy, we impose that $\partial W_s/\partial k_t \leq 0$, which places some restrictions on the reasons why capital is above the modified golden rule. In particular, it implies that all else equal, reducing capital (weakly) relaxes the constraint, so the over-accumulation of capital is not hard-wired.[18] Rather, the over-accumulation of capital is required to ensure a certain level of future generations' utility.

[17] As in the open economy case, we leave to the appendix discussion of the existence of Lagrange multipliers.

[18] This rules out such constraints as $k \geq \underline{k} > k^\star$, where $k^\star$ is the modified golden rule capital. It allows

In the open economy, we proposed the intuition that the efficient allocation maximizes steady state aggregate debt (see proposition 3). In a closed economy, aggregate debt is zero by definition. Nevertheless, a similar intuition carries through to the closed economy case. The constraints in the closed economy ensure that sufficient resources are provided to future agents. From the perspective of the private agents, the efficient allocation in a closed economy is distorted by inducing savings, which in turn is necessary to sustain a relatively high level of future utility. The closed-economy equivalent of maximizing steady state aggregate debt is to minimizing the amount of capital left for future generations.

Specifically, suppose that the efficient allocation converges to the steady state allocation $(c_\infty, n_\infty, k_\infty)$, with steady state fiscal expenditure g. Then the efficient allocation minimizes the steady state capital necessary to sustain steady state utility:

Proposition 7. *[Minimal Steady State Capital] Suppose the efficient allocation converges to a steady state. Under the conditions of Proposition 6, the efficient allocation converges to the minimal steady state capital allocation, where the latter allocation is the solution to:*

$$\min_{c,n,k} k,$$

subject to

$$\begin{aligned} u(c,n) &\geq u(c_\infty, n_\infty), \\ c &\leq F(k,n) - \delta k - g. \end{aligned}$$

This proposition highlights the link between the closed and open economy problems. In both cases, there is a floor on savings (or a ceiling on debt). Fiscal policy in the presence of impatience is designed to exhaust these constraints in order to maximize front loading. This requires efficient production in the steady state, or zero tax on labor inputs.

6 Conclusion

This paper characterized the optimal fiscal policy when agents are relatively impatient. A defining feature of the optimal policy is that labor taxes are front loaded. A consequence

the standard limited commitment constraints in which more capital makes deviation more profitable. Such a restriction on W was unnecessary in the open economy case as any over-accumulation of capital did not affect inter-temporal prices, which were pinned down by international financial markets.

of such a policy is that the transition to the steady involves a conservative fiscal policy in which the government accumulates enough assets to finance expenditures in the absence of labor tax revenue. While the fiscal authority may run surpluses, the economy as a whole is accumulating foreign liabilities. As noted in the introduction, this normative description does not resemble the observed policy in many indebted economies. Such economies frequently run fiscal deficits and back load taxes. Nevertheless, the mechanism rationalizes why taxes are front loaded rather than smoothed in response to an unanticipated fiscal shock. In particular, front loading allows the economy to aggressively access international debt markets, which is the relevant concern in our benchmark scenario.

One potential drawback of our deterministic environment (or an extension with state-contingent debt) is the inability to speak to spikes in sovereign risk premia as a country's borrowing capacity becomes saturated. In our framework, limited commitment is manifested through quantity rationing. The increase in the cost of borrowing is reflected in the accumulating Lagrange multipliers on the debt constraints; that is, via the shadow cost of debt rather than a change in the market price. Given this symmetry, the desire to lower the cost of borrowing in a model with explicit default may also call for front loading of labor taxes. Similarly, the model does not encompass other potential frictions in the asset markets, such as illiquidity of public or private debt, that may play a role in fiscal crises. Such additional complications do not a priori challenge the argument for front loading tax revenues, but we leave for future research a full analysis. Nevertheless, the intuition behind the result is likely to play a role in other environments. Namely, an economy that wishes to front load consumption and leisure should operate as efficiently as possible in the long run to maximize its debt servicing capacity. As private indebtedness does not undermine efficiency to the same extent as public indebtedness, this requires the fiscal authority to "make room" by front loading its labor tax revenue.

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Appendix: Proofs

Proof of Proposition 1

Proof. **The only if part:** Take the budget constraint of the consumer (BC) and substitute p_t by $\beta^t u_c(c_t, n_t)/u_c(c_0, n_0)$ and substitute w_t by $-u_n(c_t, n_t)/u_c(c_t, n_t)$. Then you get that:

$$\sum_{t=0}^{\infty} \beta^t \Big(u_c(c_t, n_t)c_t + u_n(c_t, n_t)n_t \Big) - u_c(c_0, n_0)a_0 = u_c(c_0, n_0)T \geq 0$$

where the last inequality follows from $T \geq 0$ and that $u_c \geq 0$. The necessity that the resource constraint and the sovereign debt constraints hold follows from the definition of equilibria.

The if part: Given an allocation, let us define the following objects:

$$\begin{aligned}
w_t &= -u_n(c_t, n_t)/u_c(c_t, n_t), \\
r_t^k &= r_t^* - \delta, \\
\tau_t^n &= -\frac{F_n(k_t, n_t)u_c(c_t, n_t)}{u_n(c_t, n_t)} - 1, \\
\tau_t^k &= F_k(k_t, n_t)/r_t^k - 1,
\end{aligned}$$

for all $t \geq 0$. Define as well:

$$\begin{aligned}
\phi_t^k &= 1 - \frac{1}{r_t^*}\left(\frac{u_c(c_{t-1}, n_{t-1})}{\beta u_c(c_t, n_t)} - 1\right), \\
p_t &= \beta^t u_c(c_t, n_t)/u_c(c_0, n_0), \\
T_t &= 0,
\end{aligned}$$

for all $t \geq 1$; and

$$\begin{aligned}
T_0 &= \sum_{t=0}^{\infty} \beta^t \frac{u_c(c_t, n_t)c_t + u_n(c_t, n_t)n_t}{u_c(c_0, n_0)} - a_0 \geq 0, \\
\theta &= u_c(c_0, n_0).
\end{aligned}$$

Now note that p_t satisfies (3) by construction. So part (i) of the equilibrium definition is satisfied. Given the definition of $\theta \geq 0$, w_t and p_t, we get that conditions (16) and (17)

are satisfied as well. Note that the budget constraint of the consumer is satisfied with T_t as defined above. So part (ii) is satisfied.

For part (iii), note that from the definition of r_t^k and τ_t^n and τ_t^k it follows that equations (8) and (9) hold. So part (iii) is satisfied. Part (v) holds as well, by hypothesis of the Proposition. We now show that part (iv) holds. We don't appeal directly to Walras' law because the budget constraint of the government is not necessarily holding with equality. However, rewriting equation (RC*), we get:

$$\sum_{t=0}^{\infty} q_t \left(g_t + T_t - \tau_t^n w_t n_t - \tau_t^k r_t^k k_t - \frac{\phi_t^k r_t}{1+(1-\phi_t^k)r_t} a_t \right) + \sum_{t=0}^{\infty} q_t \left(c_t - w_t n_t - T_t + \frac{\phi_t^k r_t}{1+(1-\phi_t^k)r_t} a_t \right) \leq A_0 \quad (33)$$

where we used the first order conditions of the firm together with $F = F_n n + F_k k$. Now, note that:

$$\begin{aligned}
&\sum_{t=0}^{\infty} q_t \left(c_t - w_t n_t - T_t + \frac{\phi_t^k r_t}{1+(1-\phi_t^k)r_t} a_t \right) \\
&= \sum_{t=0}^{\infty} q_t \left(c_t - w_t n_t - T_t - a_t + \frac{1+r_t}{1+(1-\phi_t^k)r_t} a_t \right) \\
&= \sum_{t=0}^{\infty} q_t \left(c_t - w_t n_t - T_t - a_t + \frac{1}{1+(1-\phi_t^k)r_t} a_{t+1} \right) + a_0 = a_0
\end{aligned}$$

where we have used the definition of a_t. Then, plugging back into (34):

$$\sum_{t=0}^{\infty} q_t \left(g_t + T_t - \tau_t^n w_t n_t - \tau_t^k r_t^k k_t - \frac{\phi_t^k r_t}{1+(1-\phi_t^k)r_t} a_t \right) \leq A_0 - a_0 = -b_0 \quad (34)$$

And thus part (iv) holds.

Taken together, the above implies that the sequences of prices, quantities and taxes we have constructed is a competitive equilibrium. So the allocation is consistent with an open economy competitive equilibrium.

□

Proof of Lemma 1

Proof. Consider a possible efficient allocation $\{c_t^\star, n_t^\star, k_t^\star\}_{t=0}^\infty$. Let us first prove that consumption is bounded above zero in the limit: $\liminf_{t\to\infty} c_t^\star > 0$. Towards a contradiction, suppose that this is not true. Hence, we can find a $t_0 > T$ such that $c_{T_0}^\star < \epsilon_c$. But then we know that:

$$W_{t_0}(\{u(c_s^\star, n_s^\star)\}_{s=t_0}^\infty, k_{t_0}) < W_{t_0}(\{u(c_s, n_s^\star)\}_{s=t_0}^\infty, k_{t_0}) < 0$$

with $c_{t_0} = \epsilon_c$ and $c_t = c_t^\star$ for $t \geq t_0$. The first inequality follows from strict monotonicity of u_{t_0} with respect to c_{t_0} and of W_{t_0} with respect to u_{t_0}, while the second follows from the hypothesis of Lemma 1. But then this implies that the sovereign debt constraint is violated at time t_0, generating a contradiction.

The proof of $\limsup_{t\to\infty} n_t^\star < \bar{n}$ is similar, and we omit it. □

Proof of Proposition 2

Let $\{c^\star, n^\star, k^\star\}$ be an interior efficient allocation. To eliminate the need to work in an infinite dimensional space,[19] consider the subproblem (P^T) which equals problem (P) with the additional restriction that $\{c_t, n_t\} = \{c_t^\star, n_t^\star\}$ for all $t > T$ and $k_t = k_t^\star$ for all $t \geq 0$. Clearly, $\{c^\star, n^\star, k^\star\}$ is a solution to subproblem (P^T). Note that subproblem (P^T) has a finite number of constraints, and to avoid dealing with the regularity conditions,[20] we use a version of the Lagrangian theorem stated in Luenberger (1969) chapter 9.4 problem 3. Applied to our setting, it implies that there exist a non-negative vector $\{r^T, \mu^T, \eta^T, \lambda_0^T, ..., \lambda_T^T\}$, *not identically zero*, where $\{\mu^T, \eta^T, \lambda_0^T, ..., \lambda_T^T\}$ represent the multipliers associated to constraints (P.1), (P.2), and (P.3) respectively; and are such that the standard complementary conditions hold, together with the following first order conditions:

[19] Working directly in infinite dimensional space is not an issue if we restrict attention to bounded utility functions (that is, $u(0, n) > -\infty$ and $u(c, \bar{n}) > -\infty$). In this case, the constraint set maps our commodity space into the space of bounded sequences (ℓ_∞), and it can be shown that the associated multipliers are summable sequences of non-negative numbers. In order to extend our results to unbounded utility functions, we pursue an alternative approach.

[20] The difficulty with the standard regularity conditions stems from the nature of the sovereign debt constraint. In particular, the optimal allocation may be such that the gradient of the constraint set does not have full rank. For example, an extreme case that cannot be ruled out in our setting is if there is only one allocation that satisfies all the constraints. Adding a multiplier r^T in front of the objective function addresses this issue.

$$\beta^t \left[r^T u_c(c_t, n_t) + \eta^T \left(u_c(c_t, n_t) + u_{cc}(c_t, n_t)(c_t - \mathbb{1}_{\{0\}}(t) a_0) + u_{cn}(c_t, n_t) n_t \right) \right] + \left[\sum_{s=0}^{t} \beta^s \lambda_s^T \frac{\partial W_s}{\partial u_t} \right] u_c(c_t, n_t) = q_t \mu^T, \tag{35}$$

$$\beta^t \left[r^T u_n(c_t, n_t) + \eta^T \left(u_{cn}(c_t, n_t)(c_t - \mathbb{1}_{\{0\}}(t) a_0) + u_n(c_t, n_t) + u_{nn}(c_t, n_t) n_t \right) \right] + \left[\sum_{s=0}^{t} \beta^s \lambda_s^T \frac{\partial W_s}{\partial u_t} \right] u_n(c_t, n_t) = -F_n(k_t, n_t) q_t \mu^T, \tag{36}$$

for $t \in \{0, 1, .., T\}$ and where $\mathbb{1}_{\{0\}}$ is an indicator function taking value of 1 when evaluated at 0. The first equation is the first order condition with respect to consumption and the second with respect to labor. Note that the difference with the usual necessity theorem is the presence of $r^T \geq 0$ that multiplies the objective function's contribution to the first order conditions (r^T is a multiplier of the objective function).

Using (35) and (36), together with interiority of the efficient allocation, we get that:

$$\eta^T \left(\frac{\beta^t}{q_t} \right) \left[\left(u_{cc}(t) - \frac{u_{cn}(t) u_c(t)}{u_n(t)} \right) (c_t^\star - \mathbb{1}_{\{0\}}(t) a_0) + \left(u_{cn}(t) - \frac{u_{nn}(t) u_c(t)}{u_n(t)} \right) n_t^\star \right] = \mu^T \left(1 + \frac{F_n(t) u_c(t)}{u_n(t)} \right) \tag{37}$$

for all $t \in \{0, .., T\}$.

We first proof the following simple lemma:

Lemma A3. *Suppose that assumption 1 holds. Then*

$$c \left(u_{cc} - \frac{u_{cn} u_c}{u_n} \right) + n \left(u_{cn} - \frac{u_{nn} u_c}{u_n} \right) < 0$$

as long as $c > 0$ and $n > 0$.

Proof. Normality guarantees that the terms inside the brackets are non-positive. Our strict concavity assumptions guarantee that at least one of them is strictly negative. To see this, suppose not. Then we have that $u_c / u_n = u_{cn} / u_{nn}$ from the second term, and from the first we get $u_{cc} - \frac{u_{cn}^2}{u_{nn}} = 0$ which is a contradiction of our strict concavity assumption. □

We can now state that the multiplier in the resource constraint is always strictly positive:

Lemma A4. *Under assumptions 1 and 3, in any subproblem (P^T), with $T \geq 1$, the resource constraint binds (i.e., $\mu^T > 0$).*

Proof. Suppose that $\mu^T = 0$. Then, using (37) for some $t \geq 1$, we get:

$$\frac{\eta^T}{u_c(t)}\left[c_t\left(u_{cc}(t) - \frac{u_{cn}(t)u_c(t)}{u_n(t)}\right) + n_t\left(u_{cn}(t) - \frac{u_{nn}(t)u_c(t)}{u_n(t)}\right)\right] = 0 \tag{38}$$

Using Lemma A3, it follows that, for the above equation to hold, η^T must be zero. Using (35) and interiority, we get

$$r^T = -\sum_{s=0}^{t} \beta^{s-t}\lambda_s^T \frac{\partial W_s}{\partial u_t}$$

for all $t \geq 0$. Given that $\lambda_t^T \geq 0$ and that $\partial W_t/\partial u_t > 0$ by Assumption 3, it follows then that $r^T = \lambda_t^T = 0$ for all $t \leq T$. But this is a contradiction, as $\{r^T, \mu^T, \eta^T, \lambda_0^T, ..., \lambda_T^T\}$ cannot be identically zero. □

Then, we have the following Lemma:

Lemma A5. *There exists a non-negative constant C_1 such that $\eta^T/\mu^T = C_1$ for any subproblem (P^T) with $T \geq 1$.*

Proof. Note that for any subproblem with $T \geq 1$ we have that:

$$\frac{\eta^T}{\mu^T} = \left(\frac{\beta}{q_1}\right)\left[\left(u_{cc}(1) - \frac{u_{cn}(1)u_c(1)}{u_n(1)}\right)c_1^\star + \left(u_{cn}(1) - \frac{u_{nn}(1)u_c(1)}{u_n(1)}\right)n_1^\star\right]^{-1} \times \left(1 + \frac{F_n(1)u_c(1)}{u_n(1)}\right)$$

which follows from equation (37), Lemma A3 and Lemma A4. The left hand side of the above equation is a constant for all $T \geq 1$, as it is only a function of the allocation at time $t = 1$: $\{c_1^\star, n_1^\star, k_1^\star\}$. Hence, η^T/μ^T is constant. Non-negativity follows from the non-negativity of the Lagrange multipliers. □

We can now prove Proposition 2:

Note that $\frac{\beta^t}{p_t^\star} = \Pi_{s=0}^t \beta\,(1+r_s^\star)$. Remember that $\beta(1+r_t^\star) > 0$ for all t. And then, $\beta(1+r_t^\star) < M_1 < 1$ for sufficiently large t implies that $\lim_{t\to\infty} \beta^t/q_t = 0$. Using Lemma A5 and equation (37), we have that:

$$C_1\left(\frac{\beta^t}{q_t}\right)\left[\left(u_{cc}(t) - \frac{u_{cn}(t)u_c(t)}{u_n(t)}\right)c_t + \left(u_{cn}(t) - \frac{u_{nn}(t)u_c(t)}{u_n(t)}\right)n_t\right] = 1 + \frac{F_n(t)u_c(t)}{u_n(t)} = -\tau_t^n$$

Now note that for sufficiently large t, $c_t > \epsilon_c$ for some $\epsilon_c > 0$ and $n_t < \epsilon_n$ for some $\epsilon_n < \bar{n}$, by the no-immiseration condition. Assumption 1 (iv) then guarantees that the terms inside the square brackets are bounded. Taking the limits as $t \to \infty$, it follows then that $\lim_{t\to\infty} \tau_t^n = 0$. □

Proof of Proposition 3

Proof. The maximal steady state debt problem in the proposition is a convex programming problem. Let ψ_u and ψ_k denote the multipliers on the utility and capital constraints. The first order conditions are

$$\begin{aligned}\psi_u u_c(c,n) &= 1\\ \psi_u u_n(c,n) &= -F_n(k,n)\\ \psi_k &= F_k(k,n) - r^\star - \delta\end{aligned}$$

where we have omitted the normalization term $(1+r^\star)/r^\star$ from the objective function as this term does not affect choices. These conditions, the constraints, plus the complementary slackness conditions are necessary and sufficient for an optimum. The conditions of proposition 2 ensure that $\psi_u > 0$ (that is, u_c and u_n are finite for any allocation that yields weakly greater utility than the steady state).

Let $x = (c, n, k, \psi_u, \psi_k) \in \Re_+^5$ represent an arbitrary allocation and multipliers, and $x^\star$ denote the x that satisfies the maximal debt first order conditions. Define the function

$H(x) : \Re^5_+ \to \Re^5$ by:

$$H(x) = \begin{cases} \psi_u u_c(c,n) - 1, \\ \psi_u u_n(c,n) + F_n(k,n), \\ \psi_k - F_k(k,n) + r^\star + \delta, \\ \psi_u \left(u(c,n) - u(c_\infty, n_\infty)\right), \\ \psi_k \left(k - k_\infty\right). \end{cases}$$

Note that the strict concavity of u and F ensures that $x^\star$ is the unique zero of this function such that the constraints are satisfied.[21]

Now let $x^t = (c_t, n_t, k_t, \psi^t_u, \psi^t_k)$ denote the efficient allocation at time t, where $\psi^t_u = (q_t\mu)^{-1} \sum_{s=0}^t \beta^s \lambda_s \left(\frac{\partial W_s}{\partial u_t} + \frac{\partial V_s}{\partial u_t}\right)$ and $\psi^t_k = -\sum_{s=0}^t \beta^s \lambda_s \frac{\partial W_s}{\partial k_t}$ from the efficient allocation first order conditions (equations 16 - 18). Note that our assumptions on V and W ensure these are non-negative. From the efficient allocation first order conditions, we have:

$$H(x^t) = \begin{cases} \left(\frac{\beta^t}{q_t\mu}\right) \left[1 + \eta + \eta \left(\frac{u_{cc}(c_t,n_t)}{u_c(c_t,n_t)} c_t + \frac{u_{cn}(c_t,n_t)}{u_c(c_t,n_t)} n_t\right)\right] u_c(c_t,n_t), \\ \left(\frac{\beta^t}{q_t\mu}\right) \left[1 + \eta + \eta \left(\frac{u_{cn}(c_t,n_t)}{u_n(c_t,n_t)} c_t + \frac{u_{nn}(c_t,n_t)}{u_n(c_t,n_t)} n_t\right)\right] u_n(c_t,n_t), \\ 0, \\ \psi^t_u \left(u(c_t,n_t) - u(c_\infty,n_\infty)\right), \\ \psi^t_k \left(k_t - k_\infty\right). \end{cases}$$

From the conditions of Proposition 2, the first two elements of $H(x^t)$ are converging to zero as t becomes large. The other terms converge to zero by definition as the allocation converges to the steady state. In particular, $H(x^\infty) = H(\lim_{t\to\infty} x^t) = 0$. As $x^\star$ is the unique zero of $H(x)$, we have that $x^\infty = x^\star$. □

Proof of Lemma 2

Proof. Let's proceed by contradiction. Suppose that for some $t > 0$, $W_t = \underline{U}$ and $W_{t+1} > \underline{U}$. Then, from the first order conditions we know that $c_t \geq c_{t+1}$ and $n_t \leq n_{t+1}$. This implies that $u_t \equiv u(c_t, n_t) \geq u(c_{t+1}, n_{t+1}) \equiv u_{t+1}$. However note that $u_t + \beta(u_{t+1} + \beta u_{t+2} + ..) = \underline{U}$ and that $u_{t+1} + \beta u_{t+2} + \beta^2 u_{t+3} + .. > \underline{U}$ by the hypothesis of the lemma. Then, it must be that $u_t + \beta\underline{U} < \underline{U}$, and thus $u_t < (1-\beta)\underline{U}$. From above, we know then that $u_{t+1} \leq u_t < (1-\beta)\underline{U}$. Using that $u_{t+1} + \beta u_{t+2} + ... > \underline{U}$, we have that $u_{t+2} + \beta u_{t+3} +$

[21] We omit the constraints themselves as elements of H, as the steady state efficient allocation will satisfy the constraints by definition.

.. $> (\underline{U} - u_{t+1})/\beta > (\underline{U} - (1-\beta)\underline{U})/\beta = \underline{U}$. It follows then that at $t+2$ the borrowing constraint is not binding and thus, $u_{t+2} \leq u_{t+1}$. Proceeding in this fashion we can show that $u_{t+s} \leq (1-\beta)\underline{U}$ for all $s > 1$. But this violates the borrowing constraint as it implies that $\sum_{s=0}^{\infty} \beta^s u_{t+s+1} < \underline{U}$. □

Proof of Proposition 4

We solve the household's problem in the presence of a sequence of debt constraints using Lagrangian techniques. If u is bounded, then the constraint set maps the commodity space into ℓ_∞, the space of bounded sequences. Standard results (see Luenberger, 1969) imply that necessary conditions for an optimum is that the associated Lagrangian is at a stationary point. One technical detail is that the multipliers are from the dual of ℓ_∞, which is larger than ℓ_1, the space of summable sequences. However, it can be shown that for the private agent's problem, the multipliers are indeed sequences of summable, non-negative numbers (see Dechert, 1982 and Rustichini, 1998). For environments in which u is not bounded over the relevant commodity space (e.g., $u(0,n) = u(c,\bar{n}) = -\infty$), we cannot ensure the validity of the Lagrangian with summable multipliers. We proceed assuming that if u is unbounded, such multipliers do indeed exist and the usual first order and complementary slackness conditions are necessary at an interior optimum. This assumption is only necessary for the case in which household's face debt constraints. The main propositions in the case with only sovereign debt constraints are proven without these additional assumptions.

Proof. **The if part:** The "if part" follows from the proof of Proposition 1 above. With $\gamma_t = 0$ for all t, the household's first order conditions are satisfied at the defined prices and allocations, as well as the complementary slackness on the household debt constraint.

The only if part: For part (a), the necessity of the resource constraint follows from the definition of equilibrium. It follows from 26 that the budget constraint of the household must hold with equality in a competitive equilibrium. The first order conditions plus the household's budget constraint imply (7) holds for a sequence of transfers $T_t \geq 0$ and a sequence of $\gamma_t \geq 0$, with $\gamma_0 = 0$ and $\gamma_t = 0$ whenever $V_t > 0$. We need to show that this allocation must satisfy the household's first order and complementary slackness conditions at $\gamma_t = 0$ for all t. We begin with the following claim:

Lemma A6. *Suppose that $\{c_t, n_t\}$ solves the household's problem for a sequence of prices (p_t, w_t),*

punishments χ_t, and transfers T_t. Let Γ_t denote the associated multipliers. If there exists a t_0 such that $V_{t_0} = 0$, then

$$\sum_{t=t_0}^{\infty} \beta^t \Gamma_t \left[u_c(c_t, n_t)(c_t - T_t) + u_n(c_t, n_t) n_t \right] \leq 0. \tag{39}$$

Proof. We proceed by contradiction. Suppose there exists t_0 such that $V_{t_0} = 0$ but the left hand side of (39) is strictly positive. From the household's first order conditions, we can substitute in prices and write this as:

$$\sum_{t=t_0}^{\infty} p_t \left[c_t - T_t - w_t n_t \right] > 0. \tag{40}$$

As $V_{t_0} = 0$, we have

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0} u(c_t, n_t) = \underline{V}_{t_0} \leq \sum_{t=t_0}^{\infty} \beta^{t-t_0} u(w_t \tilde{n}_t + T_t, \tilde{n}_t),$$

where $\tilde{n}_t \equiv \text{argmax}_{\hat{n}_t} u(w_t \hat{n}_t + T_t, \hat{n}_t)$. The last inequality follows from the upper bound assumption on deviation utility and that $\chi_t \in [0, 1]$. Moreover, defining $\tilde{c}_t \equiv w_t \tilde{n}_t + T_t$, the same inequality implies that

$$\sum_{t=t_0}^{\infty} p_t (c_t - T_t - w_t n_t) \leq \sum_{t=t_0}^{\infty} p_t (\tilde{c}_t - T_t - w_t \tilde{n}_t) = 0,$$

a contradiction of (40). □

We now show that any equilibrium allocation must be decentralizable with $\gamma_t = 0$ for all t:

Lemma A7. *If $\{c_t, n_t, k_t, T_t, p_t, w_t, \tau_t^k, \tau_t^n, \phi_t^k, \chi_t\}$ is a competitive equilibrium, then $\{c_t, n_t, k_t, \tilde{T}_t, \tilde{p}_t, w_t, \tau_t^k, \tau_t^n, \tilde{\phi}_t^k, \chi_t\}$ is also a competitive equilibrium, where*

$$\tilde{p}_t = \beta^t \frac{u_c(c_t, n_t)}{u_c(c_0, n_0)}, \forall t. \tag{41}$$

That is, prices are such that $\Gamma_t = 1$ for all t.

Proof. Define $\Delta \equiv \sum_{t=0}^{\infty} (p_t - \tilde{p}_t)(c_t - w_t n_t - T_t)$, as the impact on the budget constraint of the alternative price sequence. The household's budget constraint in the original equi-

librium implies:

$$a_0 = \sum_{t=0}^{\infty} \tilde{p}_t \left(c_t - w_t n_t - T_t\right) + \Delta.$$

If $\Delta \leq 0$, then the allocation satisfies the household's budget constraint under the new prices. Note that by definition, $\tilde{p}_t = p_t/\Gamma_t$. We have

$$\Delta = \frac{1}{u_c(c_0, n_0)} \sum_{t=0}^{\infty} \left(1 - \frac{1}{\Gamma_t}\right) \beta^t \Gamma_t \left[u_c(c_t, n_t)(c_t - T_t) + u_n(c_t, n_t) n_t\right].$$

Let $\lambda_t = \left(1 - \Gamma_t^{-1}\right)$. Note that $\lambda_0 = 0$, $\lambda_t = \lambda_{t-1}$ if $\gamma_t = 0$ and $\lambda_t > \lambda_{t-1}$ if $\gamma_t > 0$. Let $\{t_0, t_1, ...\}$ denote the elements of the set $T = \{t | \gamma_t > 0\}$. We can then write:

$$\begin{aligned}
\Delta = & \sum_{t=t_0}^{\infty} \lambda_t \beta^t \Gamma_t \left[u_c(c_t, n_t)(c_t - T_t) + u_n(c_t, n_t) n_t\right] \\
= & \lambda_{t_0} \sum_{t=t_0}^{\infty} \beta^t \Gamma_t \left[u_c(c_t, n_t)(c_t - T_t) + u_n(c_t, n_t) n_t\right] \\
& (\lambda_{t_1} - \lambda_{t_0}) \sum_{t=t_1}^{\infty} \beta^t \Gamma_t \left[u_c(c_t, n_t)(c_t - T_t) + u_n(c_t, n_t) n_t\right] \\
& (\lambda_{t_2} - \lambda_{t_1}) \sum_{t=t_1}^{\infty} \beta^t \Gamma_t \left[u_c(c_t, n_t)(c_t - T_t) + u_n(c_t, n_t) n_t\right] \\
& + ...
\end{aligned}$$

From lemma A6, $\sum_{t=t_k}^{\infty} \beta^t \Gamma_t \left[u_c(c_t, n_t)(c_t - T_t) + u_n(c_t, n_t) n_t\right] \leq 0$ for all $t_k \in T$, and so $\Delta \leq 0$. To ensure that the household's budget constraint holds with equality at the alternative prices, define $\tilde{T}_0 = T_0 - \Delta \geq 0$. The sequence of $\tilde{\phi}_t$ can be recovered from $\tilde{p}_t$ as usual. The wages and source-based taxes can be left as in the original allocation and satisfy the conditions of equilibrium. □

We now show that any equilibrium with positive transfers can be supported with zero transfers:

Lemma A8. *If $\{c_t, n_t, k_t, T_t, p_t, w_t, \tau_t^k, \tau_t^n, \phi_t^k, \chi_t\}$ is a competitive equilibrium, then there exists an alternative equilibrium with $\tilde{T}_t = 0$ for all $t \geq 1$.*

Proof. From the preceding lemma, we only need consider equilibria for which $\Gamma_t = 1$ for all t, as any alternative equilibrium can be supported as a $\Gamma_t = 1$ equilibrium. As transfer

only affect the household's problem, we need to check that the original allocation satisfies the household's problem at the original prices but with new sequence of transfers. Consider $\tilde{T}_0 = \sum_{t=0}^{\infty} p_t T_t$, and $\tilde{T}_t = 0$ for all $t \geq 1$. Substituting $\tilde{T}_t$ for T_t has no impact on the household's budget constraint, by construction. Note that $\underline{V}_t(\{w_s, T_s\}_{s\geq t}) \geq \underline{V}_t(\{w_s, 0\}_{s\geq t})$ for all t by monotonicity of deviation utility with respect to transfers. Therefore the new allocation satisfies the household debt constraints. The fact that $\Gamma_t = 1$ implies that complementary slackness continues to hold. Therefore, the original allocation with the new sequences of transfers continues to satisfy the household's problem at the original prices. □

Lemmas A6 through A8 prove that any competitive equilibrium can be implemented with zero transfers after the first period and $\Gamma_t = 1$ for all t. Using the fact that $T_0 \geq 0$, we can rewrite (7) as (IC). □

Proof of Corollary 1

Proof. Given the result of Proposition 4, the efficient allocation problem with household debt constraints is the same as the benchmark problem (P) with the addition of constraint (29) and the additional choice sequence $\{\chi_t\}_{t=0}^{\infty}$. The policy of setting $\chi_t = 0$ for all t minimizes the sequence of $\underline{V}_t$'s, and thus maximizes the set of allocations that satisfy the household's debt constraint without affecting any other aspects of the problem. Any efficient allocation must therefore satisfy this constraint, and we can thus confine attention to the problem subject to $V(c_t, n_t|0,0,0) \geq 0$. Note that setting $\chi_t = 0$ is full garnishment of wages and thus removes equilibrium prices from the constraint set, making the constraint identical to a sovereign debt constraint. This renders the problem isomorphic to the benchmark model, and so the result of Proposition 2 holds. □

Proof of Proposition 5

The proof of the closed economy case follows that of the open economy (Proposition 1), and we omit the details.

Proof of Proposition 6

Proof. The proof of the closed economy result follows the open economy proof (Proposition 2). The only difference is we know face a sequence of resource constraints, rather than a single present value constraint. As before, let $\{c^\star, n^\star, k^\star\}$ be an interior efficient allocation, and we consider the subproblem (P^T). It will be useful to define a closed economy counterpart to q_t:

$$\tilde{q}_t \equiv \prod_{s=0}^{t} \left(F_k(k_s^\star, n_s^\star) + 1 - \delta\right)^{-1}.$$

Note that $F_k(k_s^\star, n_s^\star) + 1 - \delta > 0$ for $\delta \in (0,1)$. Let η^T be the multipliers on (IC'), $\beta^t \psi_t^T$ be the multiplier on the sequence of resource constraints (RC), and $\beta^t \lambda_t^T$ be the multiplier on the sequence of debt constraints. As in the proof of Proposition 2, we let $r^T \geq 0$ be the multiplier on the objective function. The first order conditions are:

$$\begin{aligned}
r^T + \eta^T + \eta^T \left(\frac{u_{cc}(c_t, n_t)}{u_c(c_t, n_t)} (c_t - \mathbb{1}_{\{0\}}(t) a_0) + \frac{u_{cn}(c_t, n_t)}{u_c(c_t, n_t)} n_t \right) + \sum_{s=0}^{t} \beta^{s-t} \lambda_s^T \frac{\partial W_s}{\partial u_t} \\
= \psi_t^T \frac{1}{u_c(c_t, n_t)}, \\
r^T + \eta^T + \eta^T \left(\frac{u_{cn}(c_t, n_t)}{u_n(c_t, n_t)} (c_t - \mathbb{1}_{\{0\}}(t) a_0) + \frac{u_{nn}(c_t, n_t)}{u_n(c_t, n_t)} n_t \right) + \sum_{s=0}^{t} \beta^{s-t} \lambda_s^T \frac{\partial W_s}{\partial u_t} \\
= -\psi_t^T \frac{F_n(k_t, n_t)}{u_n(c_t, n_t)},
\end{aligned}$$

for all $t \in \{0, ..., T\}$, and

$$\psi_t^T \left(F_k(k_t, n_t) + 1 - \delta\right) + \sum_{s=0}^{\infty} \beta^{s-t} \lambda_s^T \frac{\partial W_s}{\partial k_t} = \frac{\psi_{t-1}^T}{\beta},$$

for all $t \geq \{1, ..., T\}$.

As in the open economy case, in any subproblem (P^T), with $T \geq 1$, the resource constraint binds for every $1 \leq t \leq T$ (i.e., $\psi_t^T > 0$). The proof follows that of lemma A4 and we omit it.

Define $\tilde{C}_1$ to be the closed economy counterpart of C_1 from lemma A5. Specifically:

$$\begin{aligned}\tilde{C}_1 &= \frac{\tilde{q}_1 \eta^T}{\beta \psi_1^T} \\ &= \frac{\tilde{q}_1}{\beta}\left[\left(u_{cc}(1) - \frac{u_{cn}(1)u_c(1)}{u_n(1)}\right)c_1^\star + \left(u_{cn}(1) - \frac{u_{nn}(1)u_c(1)}{u_n(1)}\right)n_1^\star\right]^{-1} \\ &\times \left(1 + \frac{F_n(1)u_c(1)}{u_n(1)}\right).\end{aligned}$$

From the first order condition for capital, we have for $1 \leq t \leq T$:

$$\begin{aligned}\psi_t^T &= \frac{\psi_{t-1}^T}{\beta}\left(F_k(k_t, n_t) + 1 - \delta + \frac{1}{\psi_t^T}\sum_{s=0}^{\infty}\beta^{s-t}\lambda_s^T\frac{\partial W_s}{\partial k_t}\right)^{-1} \\ &\geq \frac{\psi_{t-1}^T}{\beta}\frac{\tilde{q}_t}{\tilde{q}_{t-1}},\end{aligned}$$

where the last line follows from the definition of $\tilde{q}$, the equivalence of (c_t, n_t, k_t) with $(c_t^\star, n_t^\star, k_t^\star)$, and the fact that $\frac{1}{\psi_t^T}\sum_{s=0}^{\infty}\beta^{s-t}\lambda_s^T\frac{\partial W_s}{\partial k_t} \leq 0$. Iterating on this last inequality backwards in time until period 1, we have

$$\psi_t^T \geq \frac{\tilde{q}_t}{\beta^t}\frac{\psi_1^T\beta}{\tilde{q}_1}.$$

Returning to the labor tax, subtracting the first two first-order conditions, we have for $t \in \{1, ..., T\}$:

$$-\tau_t^n = \frac{\eta^T}{\psi_t^T}\left[\left(u_{cc}(t) - \frac{u_{cn}(t)u_c(t)}{u_n(t)}\right)c_t + \left(u_{cn}(t) - \frac{u_{nn}(t)u_c(t)}{u_n(t)}\right)n_t\right],$$

which implies

$$
\begin{aligned}
|\tau_t^n| = & \\
& \frac{\eta^T}{\psi_t^T} \left| \left(u_{cc}(t) - \frac{u_{cn}(t)u_c(t)}{u_n(t)} \right) c_t + \left(u_{cn}(t) - \frac{u_{nn}(t)u_c(t)}{u_n(t)} \right) n_t \right| \\
& \leq \frac{\beta^t}{\tilde{q}_t} \frac{\eta^T \tilde{q}_1}{\psi_1^T \beta} \left| \left(u_{cc}(t) - \frac{u_{cn}(t)u_c(t)}{u_n(t)} \right) c_t + \left(u_{cn}(t) - \frac{u_{nn}(t)u_c(t)}{u_n(t)} \right) n_t \right| \\
& = \frac{\beta^t}{\tilde{q}_t} \tilde{C}_1 \left| \left(u_{cc}(t) - \frac{u_{cn}(t)u_c(t)}{u_n(t)} \right) c_t + \left(u_{cn}(t) - \frac{u_{nn}(t)u_c(t)}{u_n(t)} \right) n_t \right|.
\end{aligned}
$$

The assumptions listed in Proposition 6 imply $\beta^t / \tilde{q}_t \to 0$, and that the final term is finite. Therefore $\tau_t^n \to 0$. □

Proof of Proposition 7

Proof. Write the Lagrangian for the minimal capital problem:

$$q\mathcal{L} = k + \xi \left(u(c_\infty, n_\infty) - u(c,n) \right) + \rho \left(c + g + \delta k - F(k,n) \right).$$

where ψ and ρ are the associated Lagrange multipliers. As in the open economy case (Proposition 3), we have strictly convex programming problem. We can write the first order conditions as:

$$
\begin{aligned}
F_k(k,n) - \delta - \frac{1}{\rho} &= 0, \\
\frac{1}{u_c} - \frac{\xi}{\rho} &= 0, \\
\frac{-F_n}{u_n} - \frac{\xi}{\rho} &= 0,
\end{aligned}
$$

where the first condition ensures that $\rho > 0$. The conditions of proposition 6 ensure that c is bounded away from zero and n away from $\bar{n}$. This ensures that $\xi > 0$ as well. Define $x = (c, n, k, \xi, \rho)$ and $H^c(x)$ as the left hand side of the above conditions plus the complementary slackness expressions. Let $x^\star$ denote the (unique) zero of $H^c(x)$ such that the constraints are satisfied.

Define $x^t = (c_t, n_t, k_t, \xi_t, \rho_t)$ as the efficient closed economy allocation at time t to-

gether with

$$\rho^t \equiv \left[\frac{1}{\psi_t}\sum_{s=0}^{t}\beta^{s-t}\lambda_s\frac{\partial W_s}{\partial k_t} - \frac{\psi_{t-1}}{\beta\psi_t}\right]^{-1}$$
$$\xi^t/\rho^t \equiv \frac{1}{\psi_t}\sum_{s=0}^{t}\beta^{s-t}\lambda_s\left(\frac{\partial W_s}{\partial u_t} + \frac{\partial V_s}{\partial u_t}\right),$$

where we use the fact that the bracketed term in the first expression is equal to $F_k(k_t, n_t) - \delta$, which is strictly greater than zero in the efficient allocation. Then from equations (30)–(32), we have:

$$H^c(x^t) = \begin{cases} \frac{1}{\psi_t}\left[1+\eta+\eta\left(\frac{u_{cc}(c_t,n_t)}{u_c(c_t,n_t)}c_t + \frac{u_{cn}(c_t,n_t)}{u_c(c_t,n_t)}n_t\right)\right], \\ \frac{1}{\psi_t}\left[1+\eta+\eta\left(\frac{u_{cn}(c_t,n_t)}{u_n(c_t,n_t)}c_t + \frac{u_{nn}(c_t,n_t)}{u_n(c_t,n_t)}n_t\right)\right], \\ 0, \\ \xi^t\left(u(c_t,n_t) - u(c_\infty,n_\infty)\right), \\ \rho^t\left(c+g+\delta k_t - F(k_t,n_t)\right). \end{cases}$$

The first two terms converge to zero under the conditions for proposition 6, and the remaining terms converge to zero by definition of the steady state and the closed economy aggregate resource constraint (RC). The constraints are satisfied by definition. Therefore, $H^c(x^\infty) = 0 = H^c(x^\star)$, and the fact that H^c has a unique zero implies that x^∞ coincides with the minimal steady state capital allocation. □