Productivity and the Welfare of Countries*

Susanto Basu
Boston College and NBER

Luigi Pascali
Pompeu Fabra University

Fabio Schiantarelli
Boston College and IZA

Luis Serven†
World Bank

October 17, 2011

Abstract

We prove that in a closed economy without distortionary taxation, the welfare of a representative consumer is summarized to a first order by the current and expected future values of the log level Solow productivity residual and by the initial endowment of capital. The equivalence holds if the representative household maximizes utility while taking prices parametrically. This result justifies TFP as the right summary measure of welfare, even in situations where it does not properly measure technology. We show how the welfare measure must be modified if the economy is open or if taxes are distortionary. We illustrate our results by computing indexes of welfare change over time and welfare gaps across countries.

JEL: D24, D90, E20, O47
Keywords: Productivity, Welfare, Reallocation, Technology, TFP

*This paper builds on Basu, Pascali, Schiantarelli, Serven (2009). We are grateful to M. Dimitriev for very useful comments.

†Basu: Susanto.Basu@bc.edu; Pascali: Luigi.Pascali@bc.edu; Schiantarelli: Fabio.Schiantarelli@bc.edu; Serven: Lserven@worldbank.org

1 Introduction

Standard models in many fields of economics posit the existence of a representative household in either a static or a dynamic setting, and then seek to relate the welfare of that household to observable aggregate data. A separate, large, literature examines the productivity residual defined by Solow (1957) and interprets it as a measure of technical change or policy effectiveness. Yet a third literature, often termed "development accounting," studies productivity differences across countries, and interprets them as measures of technology gaps or institutional quality. To our knowledge, no one has suggested that these three literatures are intimately related.[1] We show that they are.

In the simplest case of a closed economy with no distortionary taxes we show that to a first-order approximation the welfare change of the representative household depends on three objects: the expected present discounted value of total factor productivity (TFP) growth as defined by Solow, the change in expectations of the same quantity, and the growth in the stock of capital per person. The result sounds similar to one that is proven in the context of a competitive optimal growth model, which might lead one to ask what assumptions on technology and product market competition are required to obtain the results. The answer is, None. The result holds for all types of technology and market behavior, as long as consumers take prices as given and are not constrained in the amount they can buy or sell at those prices. Thus, for example, the same result holds whether the TFP growth is generated by exogenous technological change, as in the Ramsey-Cass-Koopmans model; by changes in the size of the economy combined with increasing returns to scale, as in the "semi-endogenous growth" models of Arrow (1962) and Jones (1995); or by externalities or public policy in fully-endogenous growth models, such as Romer (1986) or Rebelo (1991).

Our results propose a very different interpretation of TFP from the usual one. Usually one argues that TFP growth is interesting because it provides information on the change or diffusion of technology, or measures improvement in institutional quality, the returns to scale in the production function, or the markup or price over marginal cost. We show that whether all or none of these things is true, TFP is interesting for a very different reason. Using only the first-order conditions for optimization of the representative household, we can show that welfare depends on TFP growth. We interpret TFP purely from the household side, producing what one might call "the household-centric Solow residual."[2] Here we follow the intuition of Basu and Fernald (2002), and supply a general proof of their basic proposition that TFP is relevant for welfare.

The intuition for our result comes from noting that TFP is output growth minus share-weighted input growth. The representative household receives all output, which *ceteris paribus* increases its welfare. But at the same time it supplies some inputs: labor input, which reduces leisure, and capital input, which involves deferring consumption (and perhaps losing some capital to deprecia-

[1] Basu and Fernald (2002) showed a link between the first and second, but not the third. We build on their insight in the time-series context, as we discuss later.

[2] The term is due to Miles Kimball.

tion). The household measures the cost of the inputs supplied relative to the output gained by real factor prices–the real wage and the real rental rate of capital. TFP also subtracts inputs supplied from output gained, and uses exactly the same prices to construct the input shares. The welfare result holds in a very general setting because relative prices measure the consumer's marginal rate of substitution even in many situations when they do not measure the economy's marginal rate of transformation–for example, if there are externalities, increasing returns or imperfect competition.

This intuition suggests that in cases where prices faced by households differ from those facing firms, it is the former that matter for welfare. We show that this intuition is correct, and here our household-centric Solow residual differs from Solow's original measure, which uses the prices faced by firms. Proportional taxes are an important source of price wedges in actual economies. We show that the shares in the household-centric Solow residual need to be constructed using the factor prices faced by households. Since average marginal tax rates can be substantial, especially in rich countries, this modification is quantitatively important, as we show in empirical implementations of our results.

We then move to showing analogous results for open economies. Here we show that our previous results need to be modified substantially if we construct TFP using the standard output measure, real GDP. To the three terms discussed above we need to add the present discounted value of expected changes in the terms of trade, the present discounted value of expected changes in the rate of return on foreign assets, and the growth rate of net foreign asset holdings. Intuitively, both the terms of trade and the rate of return on foreign assets affect the consumer's ability to obtain welfare-relevant consumption and investment for a given level of factor supply. Holdings of net foreign assets are analogous to domestic physical capital in that both can be transformed into consumption at a future date.

While these results connect to and extend the existing literature, as we discuss below, they are difficult to take to the data. It is very hard to get good measures of changes in asset holdings by country for a large sample of countries.[3] Furthermore, measuring asset returns in a comparable way across countries would require us to adjust for differences in the risk of country portfolios, which is a formidable undertaking. Fortunately, we are able to show that these difficulties disappear if we switch to using real absorption rather than GDP as the measure of output.[4] In this case, exactly the same three terms that summarize welfare in the closed economy are also sufficient statistics in the open economy. Thus, we can measure welfare change empirically in ways that are invariant to the degree of openness of the economy.

These results pertain to the evolution of welfare in individual economies over time. The indexes we obtain are not comparable across countries. Thus it is natural to ask whether our methods shed any light on a pressing and long-standing question, the measurement of relative welfare across countries using a method firmly grounded in economic theory. It turns out that they do if one is

[3] The important work of Lane and Milesi-Ferretti (2001, 2007) has shed much light on this subject, but the measurement errors that are inevitable in constructing national asset stocks lead to very noisy estimates of net asset growth rates.

[4] We are indebted to Mikhail Dmitriev for pointing out this result.

willing to assume a common utility function. Then, the welfare gap between two countries at a point in time depends not only upon steady state differences in consumption per capita and leisure, but also upon the TFP gaps between countries and the difference in their per-capita capital stocks (both in deviation from their steady state levels). The TFP gap is the "development accounting" residual defined by Hall and Jones (1999) and used by a large subsequent literature in cross country comparisons. We show that this gap, which has been interpreted as a measure of technological and institutional differences, and its evolution over time is a determinant of the welfare differences across countries and of their evolution.

However, the same insights that apply to the time series are relevant for the cross section: TFP needs to be defined using the prices perceived by households, and if the economy is open then other terms become relevant. Thus, differences in tax rates, terms of trade, and foreign asset holdings also matter for cross-country welfare comparisons.

These results show that we can perform interesting welfare comparisons using readily-available national income accounts data. We illustrate our methods using data for several industrialized countries for which excellent data are available: Canada, France, Germany, Great Britain, Italy, Japan, Spain and the United States. We show the importance of fiscal considerations in constructing measures of welfare change over time. For example, ignoring government spending and taxation, Britain has the largest welfare gain among our group of countries over our sample period, 1985-2005, while Spain lags far behind due to its low TFP growth rate. Allowing for distortionary taxation and assuming that government expenditures are chosen optimally, the United States's growth rate exceed's Britain, but Spain's performance is very close to that of these two leading economies.

We then apply our methodology to cross country comparisons and show how these comparisons evolve over time. At the start of our sample, we find that Italy and France have a considerable welfare advantage over the United States, in the sense that a representative US consumer would experience a gain equivalent to 10-20 percent of consumption by moving to one of these countries. Italy and France led the US at this time because they were catching up to the US in TFP levels while maintaining higher rates of TFP growth and/or had fast capital accumulation. 20 years later the situation was reversed: a US consumer would give up 5-10 percent of consumption in order to stay in the United States instead of moving to France or Italy. The low TFP growth rates in these countries and slow capital accumulation, especially Italy, combined with the post-1995 productivity acceleration in the US make the US the welfare leader by the end of the sample period.

These measures have a clear interpretation because they are derived from a well-posed optimization problem. Starting from a precise statement of the household's optimization problem also forces us to confront two issues in national income and welfare measurement.[5] First, our derivation shows that "consumption" should be defined as any good or service that consumers value, whether or not it is included in GDP. Similarly, "capital" should include all consumption that is foregone

[5] The relationship between welfare, productivity and national income measurement is also explored in Nordhaus and Tobin (1973), Weitzman (1976, 2003), Hulten (1978), Hulten and Schreyer (2009), Baker and Rosnick (2007).

now in order to raise consumption possibilities for the future. These items include, for example, environmental quality and intangible capital. Of course, both are hard to measure and even harder to value, since there is usually no explicit market price for either good. But our derivation is quite clear on the principle that the environment, intangibles and other non-market goods should be included in our measure of "welfare TFP." We follow conventional practice in restricting the output measure for our TFP variable to market output (and the inputs to measured physical capital and labor), but in so doing we, and almost everyone else, are mismeasuring real GDP and TFP. Second, our starting point of a representative-consumer framework implies that we automatically ignore issues of distribution that intuition says should matter for social welfare. We believe that distributional issues are very important. However, our objective of constructing a welfare measure from aggregate data alone implies that we cannot incorporate measures of distribution into our framework. Thus, we maintain the representative-consumer framework, but without in any way minimizing the importance of issues that cannot be handled within that framework.

Our results contribute to unifying and extending several disparate literatures. A recent line of work in international trade argues that output growth computed using real GDP does not capture all the gains from trade, and suggests that GDP growth needs to be supplemented with other variables, such as the terms of trade. However, since these results are generally obtained in a full general-equilibrium setting, it is not clear whether they survive if one changes assumptions about technology, trade institutions or market competition. We prove that similar results hold generally, under much weaker assumptions, although the right growth statistic to examine is the growth of TFP, not GDP. Furthermore, we extend the welfare results in the trade literature to a dynamic environment, and show that in such a setting the standard intratemporal terms of trade need to be supplemented with the rate of return on foreign asset holdings, which one might term the intertemporal terms of trade.

Our results are also relevant for a large literature on reallocation and allocative efficiency. This literature suggests that for some countries and industries a substantial fraction of output growth comes from faster input growth at firms with higher marginal product. However, it is not clear how to measure the welfare gains from this reallocation. Our work suggests that the right metric is the increment to aggregate TFP growth from reallocation. Furthermore, it shows that TFP is what matters, not technical efficiency. Sometimes the question of allocative efficiency is framed by asking whether firms with higher levels of technology produce more output. But TFP contributions can come from either higher technical efficiency or by exploiting increasing returns to scale. When TFP and technical efficiency diverge, it is TFP that matters for welfare. We draw out the implications of these points in our companion paper.[6]

Finally, our work leads to a reinterpretation of a substantial literature on the efficacy of economic policy. Much of this literature regresses TFP growth rates on measures of economic policy or institutions, and interprets the result as informative about the ability of policies to spur inno-

[6] See Basu, Pascali, Schiantarelli and Serven (2009). In so doing we follow Basu and Fernald (2002), who also wished to draw a link between micro measures of reallocation and aggregate welfare. See also Petrin and Levinsohn (2009) and Petrin, White and Reiter (2009).

vation or technology transfer. These papers usually include a long list of caveats regarding the shortcomings of TFP as a measure of technical change. We suggest that, with some changes, these regressions can be informative about the ability of policies to improve a country's welfare growth. Thus, rather than investigating the effects of policy on an intermediate target, our results allow researchers to estimate the effects of policies on welfare directly.

The paper is structured as follows. The next section presents our main theoretical results, with the full derivation presented in an appendix. We discuss the data in Section 3, and implement the theory to produce indexes of welfare change and cross-country comparisons in Section 4. We draw conclusions in Section 5, and suggest directions for future research.

2 The Productivity Residual and Welfare

Both intuition and formal empirical work link TFP growth to increases in the standard of living, at least as measured by GDP per capita.[7] The usual justification for studying the Solow productivity residual is that, under perfect competition and constant returns to scale, it measures technological change. However, should we care about the Solow residual in an economy with non-competitive output markets, non-constant returns to scale, and possibly other distortions where the Solow residual is no longer a good measure of technological progress? Here we build on the intuition of Basu and Fernald (2002) that a slightly modified form of the Solow residual is welfare relevant even in those circumstances and derive rigorously the relationship between a modified version of the productivity residual (in growth rates or log levels) and the intertemporal utility of the representative household. The fundamental result we obtain is that, to a first-order approximation, utility reflects the present discounted values of productivity residuals (plus possibly other terms).

Our results are complementary to those in Solow's classic (1957) paper. Solow established that if there was an aggregate production function then his index measured its rate of change. We now show that under a very different set of assumptions, which are disjoint from Solow's, the familiar TFP index is also the correct welfare measure. The results are parallel to one another. Solow did not need to assume anything about the consumer side of the economy to give a technical interpretation to his index, but he had to make assumptions about technology and firm behavior. We do not need to assume anything about the firm side (which includes technology, but also firm behavior and industrial organization) in order to give a welfare interpretation, but we do need to assume the existence of a representative consumer. Both results assume the existence of a potential function (Hulten, 1973), and show that TFP is the rate of change of that function. Which result is more useful depends on the application, and the trade-off that one is willing to make between having a result that is very general on the consumer side but requires very precise assumptions on technology and firm behavior, and a result that is just the opposite.

[7] For a review of the literature linking TFP to GDP per worker, in both levels and growth rates, see Weil (2008).

2.1 Approximating Around the Steady State

More precisely, assume that the representative household maximizes intertemporal utility:

$$V_t = E_t \sum_{s=0}^{\infty} \frac{1}{(1+\rho)^s} \frac{N_{t+s}}{H} U(C_{1,t+s}, .., C_{Z,t+s}; \overline{L} - L_{t+s}) \tag{1}$$

where $C_{i,t}$ is the capita consumption of good i at time t, L_t are hours of work per capita, $\overline{L}$ is the time endowment, and N_t population. H is the number of households, assumed to be fixed and normalized to one from now on. Population grows at constant rate n an per capita variables at a common rate g. X_t is and index for per capita variables in the sense that their level at time t is proportional to X_t.For a well defined steady state in which hours of work are constant while consumption and real wage share a common trend, we assume that the utility function has the King, Plosser and Rebelo (1988) form:

$$U(C_{1,t+s}, .., C_{Z,t+s}; L - \overline{L}_s) = \frac{1}{1-\sigma} C(C_{1,t+s}, .., C_{Z,t+s})^{1-\sigma} \nu(\overline{L} - L_{t+s})$$

with $0 < \sigma < 1$ or $\sigma > 1$.[8] We assume that $C()$ has constant returns to scale. Define $c_{i,t+s} = \frac{C_{i,t+s}}{X_{t+s}}$. We can rewrite the utility function in a normalized form as follow:

$$v_t = \frac{V_t}{N_t X_t^{(1-\sigma)}} = E_t \sum_{s=0}^{\infty} \beta^s U(c_{1,t+s}, .., c_{Z,t+s}; \overline{L} - L_{t+s}) \tag{2}$$

where $\beta = \frac{(1+n)(1+g)^{1-\sigma}}{(1+\rho)}$ is assumed to be less than one. The budget constraint (with variables scaled by $N_t X_t$) is:

$$k_t + b_t = \frac{(1-\delta) + p_t^K}{(1+g)(1+n)} k_{t-1} + \frac{(1+r_t)}{(1+g)(1+n)} b_{t-1} + p_t^L L_t + \pi_t - \sum_{i=1}^{Z} p_{i,t}^C c_{i,t} \tag{3}$$

New capital goods are the numeraire, $k_t = \frac{K_t}{X_t N_t}$ denotes capital per effective worker, $b_t = \frac{B_t}{P_t^I X_t N_t}$are real bonds. $p_t^K = \frac{P_t^K}{P_t^I}$, $p_t^L = \frac{P_t^L}{P_t^I X_t}$, $p_{i,t}^C = \frac{P_{i,t}^C}{P_t^I}$ denote, respectively, the user cost of capital, the wage per hour of effective worker, and the price of consumption goods. $(1 + r_t)$ is the real interest rate (again in terms of new capital goods) and $\pi_t = \frac{\Pi_t}{P_t^I X_t N_t}$ denotes profits.

Log linearizing around the non stochastic steady state, intertemporal household utility can be written (to a first order approximation) as:

$$v_t = v + E_t \sum_{s=0}^{\infty} \beta^s \lambda \left[\sum_{i=1}^{Z} p_i^C c_i \widehat{c}_{i,t+s} + i\widehat{i}_{t+s} - p^L L \ \widehat{L}_{t+s} - \frac{p^K k}{(1+g)(1+n)} \ \widehat{k}_{t+s-1} \right] + \lambda \frac{(1-\delta) + p^K}{(1+g)(1+n)} k \widehat{k}_{t-1} \tag{4}$$

where v is the steady state value of utility, $\widehat{x} = \log x_t - \log x$ denote log deviation from the steady

[8] If $\sigma = 1$, then the utility function must be $U(C_1, .., C_G; \overline{L} - L) = \log(C) - \nu(\overline{L} - L)$. See King, Plosser and Rebelo (1988).

state. In obtaining this result we have used the FOCs of the household maximization problem; for time t these consist of:

$$U_{c_{i,t}} - \lambda_t p_{i,t}^C = 0 \tag{5}$$

$$U_{L_t} + \lambda_t p_t^L = 0 \tag{6}$$

$$-\lambda_t + \beta E_t \frac{(1-\delta) + p_{t+1}^K}{(1+g)(1+n)} \lambda_{t+1} = 0 \tag{7}$$

$$-\lambda_t + \beta_t \frac{1}{(1+g)(1+n)} E_t (1 + r_{t+1}) \lambda_{t+1} = 0 \tag{8}$$

plus the two trasversality conditions, one for bonds and one for capital. We also make use of the log linear approximation of the budget constraint around the steady state:

$$k\,\widehat{k}_t + b\widehat{b}_t - \frac{(1-\delta) + p^K}{(1+g)(1+n)} k\widehat{k}_{t-1} - \frac{(1+r)}{(1+g)(1+n)} b\widehat{b}_{t-1} - p^L L\widehat{L}_t - p^L L\widehat{p}_t^L - \frac{p^K k}{(1+g)(1+n)} \widehat{p}_t^K$$
$$-\pi\widehat{\pi}_t + \sum_{i=1}^{Z} p_i^C c_i \widehat{c}_{i,t} + \sum_{i=1}^{Z} p_i^C c_i \widehat{p}_{i,t} = 0$$

Equation (4) says that intertemporal utility (in log deviation from the steady state) equals the expected present discounted value of terms that represent the sum of the components of final demand (in log deviation from the steady state), weighted by their steady state contribution to demand, minus primary inputs (in log deviation from the steady state) times their respective steady state factor prices.

2.2 Connecting With the Productivity Residual

We are now close to relating utility to a modified version of the productivity residual. There are two options here. The first one is to obtain a first order approximation for the level of utility in terms of the log level productivity residual. The second one focuses instead on approximating the change in utility over time as a function of the productivity residual in growth rates, as in Solow (1957).

To establish a relationship with the (log) level of productivity, we will, use the fact that, to a first order approximation, the level of value added (in terms of normalized variables) is given by:

$$\widehat{y}_t = \log y_t - \log y = \sum_{i=1}^{Z} \frac{P_i^C C_i N}{P^Y Y} \widehat{c}_{it} + \frac{P^I I}{P^Y Y} \widehat{i}_t = \sum_{i=1}^{Z} s_{c_i} \widehat{c}_{it} + s_i \widehat{i}_t \tag{9}$$

Using (9), intertemporal utility in (4) can be written as:

$$v_t = v + E_t \sum_{s=0}^{\infty} \beta^s \lambda p^Y y \left[\widehat{y}_t - \frac{p^L L}{p^Y y} \widehat{L}_{t+s} - \frac{p^K k}{p^Y y (1+g)(1+n)} \widehat{k}_{t+s-1} \right] + \lambda \frac{(1-\delta) + p^K}{(1+g)(1+n)} k \widehat{k}_{t-1} \tag{10}$$

which, after some manipulations detailed in the appendix can be rewritten as:

$$\frac{v_t - v}{\lambda p^Y y} = E_t \sum_{s=0}^{\infty} \beta^s \log PR_{t+s} + \frac{(1-\delta) + p^K}{(1+g)(1+n)} \frac{k}{p^Y y} \log K_{t-1} - f(t) \tag{11}$$

where:

$$\log PR_{t+s} = \log Y_t - s_L \log N_t L_t - s_K \log K_{t-1} \tag{12}$$

is the log level of aggregate value added, $\log Y_t$, minus aggregate factor inputs, $\log N_t L_t$ and $\log K_t$ multiplied by their respective distributional shares, s_L and s_K. $s_C = \frac{\sum_{i=1}^{Z} p_i^C c_i}{p^y y}$ is the share of consumption goods in value added. $f(t)$ is a deterministic function of time and it captures the value along the steady growth path of the first two terms on the right hand side of equation (11), respectively the present discounted value of the log level productivity residual and the initial condition (see Appendix for details).

Utility, therefore, is an increasing function of the sequence of (log) level aggregate productivity residuals, appropriately discounted.[9] It also depends upon the initial level of the capital stock, K_{t-1}, since for any sequence of productivity, welfare is higher if the consumer starts with an higher initial endowment of capital.

In order to illustrate the relationship between the change in welfare and the Solow residual, we return to (11) and take its difference through time ($\Delta v_t = v_t - v_{t-1}$). In this case, we will rely on the following (Divisia) definition of growth in normalized value added:

$$\Delta \log y_t = \sum_{i=1}^{Z} \frac{p_i^C c_i}{p^Y y} \Delta \log c_{i,t+s} + \frac{i}{p^Y y} \Delta \log i_t \tag{13}$$

Using the fact that nominal value added $P_t^Y Y_t = \sum_{i=1}^{Z} P_{i,t}^C C_{i,t} N_t + P_t^I I_t$, it is also true that non-normalized value added growth, $\Delta \log Y_t$, equals:

$$\Delta \log Y_t = \sum_{i=1}^{Z} \frac{P_i^C C_i N}{P^Y Y} \Delta \log (C_{i,t} N_t) + \frac{P^I I}{P^Y Y} \Delta \log I_t \tag{14}$$

where the growth rate of each demand component is aggregated using constant steady state shares.[10]Using this definition of value added in growth terms, equation (13), the growth rate of normalized utility can be written as follows:

[9]Note that the utility index v is positive for $0 < \sigma < 1$ and negative for $\sigma > 1$.

[10]Here we are departing slightly from convention, as value added is usually calculated with time varying shares.

$$
\begin{aligned}
\frac{\Delta v_t}{\lambda p^y y} &= E_t \sum_{s=0}^{\infty} \beta^s \Delta \log PR_{t+s} \\
&+ \sum_{s=0}^{\infty} \beta^s \left[E_t \log PR_{t+s} - E_{t-1} \log PR_{t+s} \right] \\
&+ \frac{(1-\delta) + p^K}{(1+g)(1+n)} \frac{k}{p^y y} \Delta \log K_{t-1} - f_1
\end{aligned} \tag{15}
$$

where $\Delta \log PR_{t+s}$ denotes the "modified" Solow productivity residual:

$$
\Delta \log PR_{t+s} = \Delta \log Y_{t+s} - s_L \Delta \log N_{t+s} L_{t+s} - s_K \Delta \log \ K_{t+s-1} \tag{16}
$$

We use the word "modified," for two reasons. First, we do not assume that the distributional shares of capital and labor add to one, as they would if there were zero economic profits. Zero profits are guaranteed in the benchmark case with perfect competition and constant returns to scale, but can also arise with imperfect competition and increasing returns to scale, as long as there is free entry, as in the standard Chamberlinian model of imperfect competition. Second, the distributional shares are calculated at their steady state values and, hence, are not time varying. Rotemberg and Woodford (1991) argue that in a consistent first-order log-linearization of the production function the shares of capital and labor should be taken to be constant, and Solow's (1957) use of time-varying shares amounts to keeping some second-order terms while ignoring others.

The term $E_t \log PR_{t+s} - E_{t-1} \log PR_{t+s}$ represents the revision in expectations of the log level of the productivity residual, based on the new information received between t-1 and t. In addition, $\Delta \log K_{t-1}$ captures the change in the initial endowment of capital. Finally, the constant f_1 represents the steady growth rate of the first two terms on the right hand side of equation (11), respectively the present discounted value of the log level productivity residual and the initial condition (see Appendix for details).

Note that the revision term in the second summation will reduce to a linear combination of the innovations in the stochastic shocks affecting the economy at time t. Moreover, if we assume that the modified Solow residual follows a simple stable first order autoregressive process, then the current Solow residual, $\Delta \log PR_t$, is a sufficient statistic for all the terms in the first summation. In this case, the growth in expected per capita utility is a linear function of today's actual Solow residual, of innovations at time t in the stochastic processes driving the economy and of the change in initial endowment of capital.

It may be more convenient and informative to express the left hand side of equation (15) in terms of utility per capita. Consider first, for ease of exposition, the case of $\sigma < 1$, so that $v_t > 0$. Using the fact that, approximately, $\frac{\Delta v_t}{v}$ equals $\Delta \log v_t$, we can write (omitting higher order terms):

$$
\frac{\Delta v_t}{\lambda p^y y} = \frac{v}{\lambda p^y y} \Delta \log v_t = \frac{v}{\lambda p^y y} \left(\Delta \log \frac{V_t}{N_t} - (1-\sigma) g \right)
$$

Using the FOC for consumption, and assumptions about the shape of the utility function imply that in the steady state $\frac{v}{\lambda} = \frac{cp^c}{(1-\beta)(1-\sigma)}$. This and (15) imply that:

$$\begin{aligned} \frac{v}{\lambda p^y y}\Delta \log \frac{V_t}{N_t} &= E_t \sum_{s=0}^{\infty} \beta^s \Delta \log PR_{t+s} + \sum_{s=0}^{\infty} \beta^s \left[E_t \log PR_{t+s} - E_{t-1} \log PR_{t+s}\right] \\ &+ \frac{(1-\delta)+p^K}{(1+g)(1+n)} \frac{k}{p^y y} \Delta \log K_{t-1} + \frac{cp^c g}{(1-\beta)p^y y} - f_1 \end{aligned} \tag{17}$$

The right hand side of the last equation equals the right hand side of (15) plus the constant $\frac{cp^c g}{(1-\beta)p^y y}$ to reflect the steady state growth in per person utility due to the increase in per capita consumption. Since λ represent the marginal utility of wealth, Equation (17) says that this quantity represents the money value (as a proportion of steady state GDP) of the logarithmic growth rate of per person utility.

When $\sigma > 1$, so that $v_t < 0$, the interpretation remains essentially the same but one must recognize that the growth rate in utility now equals $-\Delta \log \frac{-V_t}{N_t}$ and so the left hand side of equation (17) becomes: $\frac{-v}{\lambda p^y y}\left(-\Delta \log \frac{-V_t}{N_t}\right)$.

To aid the interpretation of our the results, we can express them in terms of "equivalent consumption" per capita, denoted by C_t^*. C_t^* is defined as the level of consumption per capita at time t that, if it grows at the steady state rate g from t onward, with leisure set at its steady state level, delivers the same intertemporal utility per capita as the actual stream of consumption and leisure, More precisely, C_t^* satisfies:

$$\begin{aligned} \frac{V_t}{N_t} &= \sum_{s=0}^{\infty} \frac{(1+n)^s}{(1+\rho)^s} \left(C_t^*(1+g)^s\right)^{1-\sigma} \nu(\overline{L}-L)) \\ &= \frac{1}{(1-\sigma)(1-\beta)} C_t^{*1-\sigma} \nu(\overline{L}-L) \end{aligned} \tag{18}$$

Taking log differences of equation (18) together with equation (17), one obtains:

$$\Delta \log (C_t)^* = \frac{(1-\beta)}{s_C}\left[E_t \sum_{s=0}^{\infty} \beta^s \Delta \log PR_{t+s} + \sum_{s=0}^{\infty} \beta^s \Delta E_t \log PR_{t+s} + \frac{(1-\delta)+p^K}{(1+g)(1+n)} \frac{k}{p^y y} \Delta \log K_{t-1} - f_1\right] + g \tag{19}$$

It is easily seen that this final expression holds for σ smaller or greater than one. This means that if we multiply the right hand side of (17) by $\frac{(1-\beta)}{s_C}$, we can interpret it as the growth rate of per-capita equivalent consumption between t and $t-1$.

3 Implications for Cross Country Analysis

We can use the framework developed above to make cross country comparisons in welfare and to show how they are affected by differences in productivity and capital accumulation. Welfare

comparisons across countries have been investigated recently by Jones and Klenow (2010). Instead of focusing on instantaneous (single period) utility, as they do, we use our intertemporal utility function in the comparison and considers out of steady state dynamics and how they are related to capital accumulation and productivity. This allows us to properly take into account the dynamic welfare effects associated with the fact that lower consumption or leisure today may raise capital accumulation and support greater consumption in the future. We do not, however, allow for cross country differences in life expectancy or in inequality as in their paper.

In order to proceed, however, additional assumptions are required that are not needed when we consider the evolution of welfare within each country. In particular one has to assume a common specification of preferences across countries. We will also assume that the rate of growth of technological progress, g, is identical across countries. Moreover, we will conduct the analysis assuming a common population growth rate in order not to have to to focus on differences in utility generated by differences in the growth rate of household size. All these assumptions suggest that it is more reasonable to focus on a subset of countries that are relatively homogenous and this is exactly what we will do by using core OECD countries in our empirical illustration. Finally, contrary to what is needed in the within country analysis of welfare changes, we now need to parametrize the part of the utility function containing leisure. We will assume that[11]:

$$\nu(\overline{L} - L_t) = \left(\overline{L} - L_t\right)^{\frac{1-\gamma}{\gamma}(1-\sigma)}$$

Consider a US individual inserted in the household of country i.Given the assumption of a common utility function, the intertemporal utility function of such individual experiencing the consumption per person and leisure of a household of country i will equal the utility of a resident of country i, $\frac{V_t^i}{N_t^i}$. We can now ask by what factor, $\theta_{t,i}$, we should multiply the equivalent consumption of an individual in the US, given the steady state level of leisure in the US, in order to deliver the same level of utility that he would enjoy in country i. $\theta_{t,i}$ is defined by the following equation:

$$\frac{V_t^i}{N_t^i} = \frac{\left(\theta_{i,t} C_{t,US}^*\right)^{(1-\sigma)} \left(\overline{L} - L_{US}\right)^{\frac{1-\gamma}{\gamma}(1-\sigma)}}{(1-\sigma)(1-\beta)} \tag{20}$$

where $C_{t,US}^*$ and L_{US} denote respectively the per-capita equivalent consumption and the per-capita steady-state hours worked in the US. Equations (18) for country i and (20) imply that:

$$\log \theta_{i,t} = \log C_{t,i}^* - \log C_{t,US}^* + \frac{1-\gamma}{\gamma}(\log\left(\overline{L} - L_i\right) - \log\left(\overline{L} - L_{US}\right)) \tag{21}$$

Using a logic similar to that used in deriving (19), the log deviation of equivalent per capita consumption from the steady state can be written as (see appendix for details):

[11] For concavity we must assume that $(1-\gamma)(1-\sigma) < \gamma$.

$$\log C_{jt}^{*} = \log(C_{j,t})_{SS} + \frac{(1-\beta)}{s_{C_j}}\left[E_t \sum_{s=0}^{\infty} \beta_j^s \log PR_{t+s,j} + \frac{(1-\delta_j)+p_j^K}{(1+g_j)(1+n_j)} \frac{k_j}{p_j^Y y_j} \log K_{t-1,j} + f(t)_j\right] \tag{22}$$

where the subscript SS denotes the value of a time varying variable along the steady state growth path and $j = i, US$. Using the fact that $f(t)_j$ contains the value of productivity and capital along the balanced growth path in country j, we can write:

$$\begin{aligned}
\log \theta_{i,t} \quad = \quad & \log(C_{i,t})_{SS} - \log(C_{US})_{SS} + \frac{1-\gamma}{\gamma}(\log\left(\overline{L} - L_i\right) - \log\left(\overline{L} - L_{US}\right)) \\
& + \frac{(1-\beta_i)}{s_{C,i}} E_t \sum_{s=0}^{\infty} \beta_i^s \left[\log PR_{t+s,i} - (\log PR_{t+s,i})_{SS}\right] \\
& + \frac{(1-\beta_i)}{s_{C,i}} \frac{\left[(1-\delta_i)+p_i^K\right] k_i}{(1+g)(1+n)\, p_i^Y y_i} \left(\log \frac{K_{t-1,i}}{N_{t-1,i}} - \log\left(\frac{K_{t-1,i}}{N_{t-1,i}}\right)_{SS}\right) \\
& - \frac{(1-\beta_{US})}{s_{C,U\,\mathrm{Si}}} E_t \sum_{s=0}^{\infty} \beta^s \left[\log PR_{t+s,US} - (\log PR_{t+s,US})_{SS}\right] \\
& - \frac{(1-\beta_{US})}{s_{C,U\,\mathrm{Si}}} \frac{\left[(1-\delta_{US})+p_{US}^K\right] k_{US}}{(1+g)(1+n)\, p_{US}^Y y_{US}} \left(\log \frac{K_{t-1,US}}{N_{t-1,US}} - \log\left(\frac{K_{t-1,US}}{N_{t-1,US}}\right)_{SS}\right)
\end{aligned} \tag{23}$$

This equation can be a basis for welfare comparisons across countries.[12] Note that $\theta_{i,t}$ is increasing in the consumption per capita and leisure along the steady steady state growth path of country i relative to the US. Moreover,it is also related to the present discounted value of the relative deviation of productivity relative to the steady state growth path: countries with greater deviations relative to others experience greater relative welfare. Finally it increases with the deviation from the steady state growth path of the per-capita capital stock.

4 Extensions

We now show that our method of using TFP to measure welfare can be extended to cover multiple types of capital and labor, taxes, and government expenditure. This extension also modifies in obvious ways the formulas for within and across countries welfare comparisons. The first extension modifies our baseline results in only a trivial way, but the others all require more substantial changes to the formulas above. These results show that the basic idea of using TFP to measure welfare holds in a variety of economic environments, but also demonstrate the advantage of deriving the

[12] Note that FOC's for consumption and leisure, given common utility across countries, imply that:

$$\frac{1-\gamma}{\gamma} = \frac{p_j^L\left(\overline{L} - L_j\right)}{p_j^C c_j} \tag{24}$$

with $j = i, US$, i.e. the ratio of expenditure on leisure relative to consumption is the same across countries, which emphasizes the tight restrictions imposed by the assumption of a common utility.

welfare measure from an explicit dynamic model of the household. The model shows exactly what modifications to the basic framework are required in each case, and demonstrates that some of these modifications are quantitatively significant.

4.1 Multiple Types of Capital and Labor

The extension to the case of multiple types of labor and capital is immediate. For simplicity, we could assume that each individual is endowed with the ability to provide different types of labor services, $L_{h,t}$ and that the utility function can be written as:

$$U(C_{1,t+s},..,C_{Z,t+s},\overline{L},L_{1,t+s},...,L_{H_L,t+s}) = \frac{1}{1-\sigma}C(C_{1,t+s},..,C_{Z,t+s})^{1-\sigma}\nu\left[\overline{L}-L(L_{1,t+s},...,L_{H_L,t+s})\right] \tag{25}$$

where $L(.)$ is an homogenous function of degree 1, H_L is the number of types of labor and $P_t^{L_h}$denotes the payment to a unit of $L_{h,t}$.[13] Similarly consumers can accumulate different types of capitals $K_{h,t}$ and rent them out at $P_t^{K_h}$. Take capital good 1 as the numeraire. Equation (4) now becomes:

$$\begin{aligned} v_t - v &= E_t\sum_{s=0}^{\infty}\beta^s\lambda\left[\sum_{i=1}^{Z}p_i^C c_i\widehat{c}_{i,t+s}+\sum_{h=1}^{H_K}p_h^I i_h\widehat{i}_{h,t+s}-\sum_{h=1}^{H_K}p_h^L L_{h,t+s}\ \widehat{L}_{h,t+s}-\sum_{h=1}^{H_K}\frac{p_h^K k_h}{(1+g)(1+n)}\ \widehat{k}_{h,t+s-1}\right] \\ &\quad +\sum_{h=1}^{H_K}\lambda\frac{(1-\delta_h)+p_h^K}{(1+g)(1+n)}k_h\widehat{k}_{h,t-1} \end{aligned} \tag{26}$$

Redefine the normalized real GDP in deviation from SS as:

$$\widehat{y}_t = \sum_{i=1}^{Z}s_{c_i}\widehat{c}_{i,t+s}+\sum_{h=1}^{H_K}s_{i_h}\widehat{i}_{h,t+s} \tag{27}$$

Using the two equations above, we get:

$$\begin{aligned} \frac{v_t - v}{\lambda p^Y y} &= E_t\sum_{s=0}^{\infty}\beta^s\left[\widehat{y}_t-\sum_{h=1}^{H_K}\frac{p_h^L L_{h,t+s}}{p^Y y}\ \widehat{L}_{h,t+s}-\sum_{h=1}^{H_K}\frac{p_h^K k_h}{(1+g)(1+n)p^Y y}\ \widehat{k}_{h,t+s-1}\right] \\ &\quad +\sum_{h=1}^{H_K}\frac{(1-\delta_h)+p_h^K}{(1+g)(1+n)}\frac{k_h}{p^Y y}\widehat{k}_{h,t-1} \end{aligned} \tag{28}$$

Proceeding exactly as in the previous section , the same equations will characterize the relationship between utility and the productivity residual, with the only difference that the latter is

[13] We assume that the nature of the utility function is such that positive quantities of all types of labors are supplied.

defined now as:

$$\log PR_t = \log Y_t - \sum_{h=1}^{H_K} s_{L_h} \log N_t L_{h,t} - \sum_{h}^{H_L} s_{K_h} \log K_{h,t-1} \tag{29}$$

4.2 Taxes

Our derivation of section 2.2 requires only reinterpretation to apply exactly to an environment with either distortionary and/or lump-sum taxes. The reason is that all prices in the budget constraint, equation (4), are from the point of view of the consumer. Thus, if there are taxes, the prices should all be interpreted as after-tax prices. Therefore our derivation implicitly allows for proportional taxes on capital and labor income as well as sales or value-added taxes levied on consumption and/or investment goods. The variable that we have been calling "profits," π, is really any transfer of income that the consumer takes as exogenous. Thus, it can be interpreted to include lump-sum taxes or rebates.

However, for the sake of exposition, we shall interpret all prices in equation (4) as being from the point of view of a firm, and thus before all taxes. To modify (4) to allow for taxes, we define some notation. Let τ^K be the tax rate on capital income, τ^L be the tax rate on labor income, τ_i^C be the *ad valorem* tax on consumption goods of type i, and τ^I be the corresponding tax on investment goods[14]. We assume that the revenue so raised is distributed back to individuals using lump-sum transfers. (We consider government expenditures in the next sub-section.) Then it is apparent that we arrive at the following modified version of equation (4):

$$\begin{aligned} v_t - v \;=\; & \lambda E_t \sum_{s=0}^{\infty} \beta^s [\sum_{i=1}^{Z} \left(1+\tau_i^C\right) p_i^C c_i \widehat{c}_{i,t+s} + \left(1+\tau^I\right) i\widehat{i}_{t+s} - \left(1-\tau^L\right)\; p^L L \widehat{L}_{t+s} \\ & - \left(1-\tau^K\right)\; \frac{p^K k}{(1+g)(1+n)} \widehat{k}_{t+s-1}] + \lambda \frac{(1-\delta) + p^K \left(1-\tau^K\right)}{(1+g)(1+n)} k\widehat{k}_{t-1} \end{aligned} \tag{30}$$

To make contact with the data, note that the national accounts define nominal expenditure using prices as perceived from the demand side. Thus, equation (9) can be written exactly as before and still be consistent with standard national accounts data:

$$\widehat{y}_{t+s} = \sum_{i=1}^{Z} s_{c_i} \widehat{c}_{it} + s_i \widehat{i}_t \tag{31}$$

where s_{c_i} and s_i are inclusive of indirect taxes (subsidies) on consumption and investment. On the other hand, the national accounts define factor prices as perceived by firms, before income taxes. Thus, the data-consistent definition of the welfare residual with taxes needs to be based on a new definition of $\log PR_t$. Rewrite equation (12) as:

[14] For simplicity, we are assuming no capital gains taxes and no expensing for depreciation. These could obviously be added at the cost of extra notation.

$$\begin{aligned}\log PR_{t+s} &= \log Y_{t+s} - \frac{\left(1-\tau^L\right)p^L LN}{p^Y y}\log N_{t+s}L_{t+s} - \frac{\left(1-\tau^K\right)p^K k}{p^Y y\,(1+g)\,(1+n)}\log K_{t+s-1} \quad (32)\\ &= \log Y_{t+s} - \left(1-\tau^L\right)s_L \log N_{t+s}L_{t+s} - \left(1-\tau^K\right)s_K \log K_{t+s-1}\end{aligned}$$

This new definition of $\log PR_t$ replaces equation (12) and the distributional shares must be adjusted accordingly in equation (13) in section 2.2. These are the only adjustments that are needed.

While it is easy to incorporate taxes into the analysis—as noted above, they are present implicitly in the basic expressions derived in section 2.2—the quantitative impact of modeling taxes explicitly can be large. Suppose that output is produced using an aggregate, constant-returns-to-scale production function of capital, labor and technology, as in Solow's classic (1957) paper. Then, without distortionary taxes, only changes in technology change welfare.

Now suppose the average marginal tax rate on both capital and labor income is 30 percent, and the share of consumption in output is 0.60. Suppose the government manages to raise aggregate capital and labor inputs by 1 percent permanently without a change in technology (perhaps via a small cut in tax rates). Then the flow increase in utility is equivalent to an increase in steady-state consumption of 0.5 percent. If the discount factor is 0.95 on an annual basis, the present value of this policy change is equivalent to a one-year increase in consumption of 10 percent of the steady-state level!

4.3 Government Expenditure

With some minor modification, our framework can be extended to allow for the provision of public goods and services. We illustrate this under the assumption that government activity is financed with lump-sum taxes. Using the results from the previous subsection, it is straightforward to extend the argument to the case of distortionary taxes.

Assume that government spending takes the form of public consumption valued by consumers. We rewrite the instantaneous utility function as

$$U(C_{1,t+s},..,C_{Z,t+s},\overline{L},L_{1,t+s},...,L_{H_L,t+s}) = \frac{1}{1-\sigma}C(C_{1,t+s},..,C_{Z,t+s};C_{G,t+s})^{1-\sigma}\nu(\overline{L}-L_{t+s}) \quad (33)$$

where C_G denotes per-capita public consumption, and we continue to assume that $C(.)$ is homogenous of degree one in its arguments. Equation (4) now becomes:

$$\begin{aligned}v_t - v &= \lambda E_t \sum_{s=0}^{\infty}\beta^s\left[\frac{U_{c_G}c_G\widehat{c}_{G,t+s}}{\lambda} + \sum_{i=1}^{Z}p_i^C c_i\widehat{c}_{i,t+s} + i\widehat{i}_{t+s} - p^L L\widehat{L}_{t+s} - \frac{p^K k}{(1+g)\,(1+n)}\widehat{k}_{t+s-1}\right]\\ &\quad +\lambda\frac{(1-\delta)+p^K}{(1+g)\,(1+n)}k\widehat{k}_{t-1} \quad (34)\end{aligned}$$

where $c_{G,t} = \frac{C_{G,t}}{X_t}$. The definition of GDP in deviation from steady state is now:

$$\widehat{y}_t = \sum_{i=1}^{Z} s_{c_i} \widehat{c}_{it} + s_i \widehat{i}_t + s_{c_G} \widehat{c}_{G,t}$$

where $s_{c_G} = \frac{P^G C_G}{P^Y Y}$ and P^G is the public consumption deflator. Let $s^*_{c_G} = \frac{U_{c_G} c_G \widehat{c}_{G,t+s}}{\lambda}$. Then we can write:

$$\frac{v_t - v}{\lambda p^Y y} = E_t \sum_{s=0}^{\infty} \beta^s \left[\widehat{y}_{t+s} - s_L \widehat{L}_{t+s} - s_K \widehat{k}_{t+s-1} + \left(s^*_{c_G} - s_{c_G} \right) \widehat{c}_{G,t+s} \right] + \frac{(1-\delta) + p^K}{(1+g)(1+n)} \frac{k}{p^Y y} \widehat{k}_{t-1} \tag{35}$$

Hence in the presence of public consumption the Solow residual needs to be adjusted up or down depending on whether public consumption is under- or over-provided (i.e., $s^*_{c_G} > s_{c_G}$ or $s^*_{c_G} < s_{c_G}$ respectively). If the government sets public consumption exactly at the utility-maximizing level, $s^*_{c_G} = s_{c_G}$ and no correction is necessary. In turn, in the standard neoclassical case in which public consumption is pure waste $s^*_{c_G} = 0$, the welfare residual is computed on the basis of private final demand – i.e., GDP minus government purchases.

What if government purchases also yield productive services to private agents? This could be the case if, for example, the government provides education or health services, or public infrastructure, which may be directly valued by consumers and may also raise private-sector productivity. In such case, the above expression remains valid, but it is important to note that the net contribution of public expenditure to welfare would not be fully by captured by $\left(s^*_{c_G} - s_{c_G} \right) \widehat{c}_{G,t+s}$. To this term we would need to add a measure of the productivity of public services, which in the expression is implicitly included in the productivity residual $\widehat{y}_{t+s} - s_L \widehat{L}_{t+s} - s_K \widehat{k}_{t+s}$.

4.4 Open Economy

In a closed economy without government b_t represent the net stock of domestic bonds. In deriving our basic equation (10), however, we have not made used of the fact that net bond holdings equal zero in equilibrium. Therefore (10) applies also to an open economy. In the latter case, we must interpret b_t as net foreign assets, and replace GDP in (10) with domestic absorption as a measure of output. However, we can still write:

$$\frac{v_t - v}{\lambda p^a a} = E_t \sum_{s=0}^{\infty} \beta^s \left[\widehat{a}_t - p^L L \ \widehat{L}_{t+s} - p^K k \ \widehat{k}_{t+s} \right] + \frac{(1-\delta)}{(1+g)(1+n)} \frac{k}{\lambda p^a a} \widehat{k}_{t-1}$$

where a denotes domestic absorption and is defined as:

$$\widehat{a}_t = \sum_{i=1}^{Z} s_c \widehat{c}_{it} + s_i \widehat{i}_t$$

Suppose one wants to use a standard measure of output, real GDP, defined as consumption, plus investment, plus net exports. Then it is possible to show that utility can be written as a function of a more conventionally defined productivity residual and of additional components that capture terms of trade and capital gains effect. Moreover the initial conditions should also include the initial value of net foreign assets. We can show this by starting from the definition of a country's current account:

$$CA_t = B_t - B_{t-1} = i_t B_{t-1} + P_t^{EXP} EXP_t - P_t^{IMP} IMP_t \tag{36}$$

where B_t is now the value of the net foreign assets, EXP_t and IMP_t are total exports and total imports and P_t^X and P_t^M are their respective prices. In a normalized form (36) becomes:

$$b_t = \frac{(1+r_t)}{(1+g)(1+n)} b_{t-1} + p_t^{EXP} \exp_t - p_t^{IMP} imp_t \tag{37}$$

Loglinearizing we obtain:

$$\widehat{bb}_t = \frac{(1+r)\,b}{(1+g)(1+n)} \widehat{b}_{t-1} + \frac{br}{(1+g)(1+n)} \widehat{r}_t + p^{EXP} \exp \widehat{p}_t^{EXP} - p^{IMP} imp \widehat{p}_t^{IMP} + p^{EXP} \exp \widehat{\exp}_t - p^{IMP} imp \widehat{imp}_t \tag{38}$$

where $\exp_t = \frac{EXP_t}{P_t^I X_t N_t}$; $imp_t = \frac{IMP_t}{P_t^I X_t N_t}$; $p_t^{EXP} = \frac{P_t^{EXP}}{P_t^I}$ and $p_t^{IMP} = \frac{P_t^{IMP}}{P_t^I}$. Equation (4) can now be rewritten as:

$$\begin{aligned} v_t &= v + E_t \sum_{s=0}^{\infty} \beta^s \lambda \left[\sum_{i=1}^{Z} p_i^C c_i \widehat{c}_{i,t+s} + i\widehat{i}_{t+s} + b\widehat{b}_{t+s} - \frac{(1+r)\,b\widehat{b}_{t+s-1}}{(1+g)(1+n)} - p^L L\ \widehat{L}_{t+s} - \frac{p^K k\widehat{k}_{t+s-1}}{(1+g)(1+n)} \right] \\ &\quad + \lambda \frac{(1-\delta)+p^K}{(1+g)(1+n)} k\widehat{k}_{t-1} + \lambda \frac{(1+r)}{(1+g)(1+n)} b\widehat{b}_{t-1} \end{aligned} \tag{39}$$

Define the normalized real GDP in deviation from the SS as:

$$\widehat{y}_t = \sum_{i=1}^{Z} s_{c_i} \widehat{c}_{it} + s_i \widehat{i}_t + s_{\exp} \widehat{\exp}_t - s_{imp} \widehat{imp}_t \tag{40}$$

where s_x and s_m are respectively the share of exports and imports out of total value added. Using the equations (38) and (40) into (39), we get:

$$\begin{aligned} \frac{v_t - v}{\lambda p^Y y} &= E_t \sum_{s=0}^{\infty} \beta^s \left[\widehat{y}_{t+s} - s_L \widehat{L}_{t+s} - s_K\ \widehat{k}_{t+s-1} + \left(\frac{br/p^Y y}{(1+g)(1+n)} \widehat{r}_t + s_x \widehat{p}_t^X - s_m \widehat{p}_t^M \right) \right] \\ &\quad + \frac{(1-\delta)+p^K}{(1+g)(1+n)} \frac{k}{p^Y y} \widehat{k}_{t-1} + \frac{(1+r)}{(1+g)(1+n)} \frac{b}{p^Y y} \widehat{b}_{t-1} \end{aligned} \tag{41}$$

Hence, in an open economy the standard Solow residual needs to be adjusted for the returns on net foreign assets, $\frac{br/p^Y y}{(1+g)(1+n)}\widehat{r}_t$, and for terms capturing the terms of trade, $s_x\widehat{p}_t^X - s_m\widehat{p}_t^M$. An improvement in the terms of trade has effects analogous to an increase in TFP - both give the consumer higher consumption for the same input of capital and labor (and therefore higher welfare). See Kohli (2004) for a static version of this result.

The terms in $\widehat{r}_{t+s}$ also capture present and expected future capital gains and losses on net foreign assets due either to exchange rate movements or to changes in the foreign currency prices of the assets. Finally, the initial conditions include not only the (domestic) capital stock, but also the net stock of foreign assets.

5 Data and Measurement

Throughout the empirical part, we use yearly data on consumption, investment, capital and labor services and leisure for the years 1985-2005 and for seven countries: USA, Great Britain, Japan, Germany, France, Italy and Spain.

To construct the "modified Solow residual" we combine data coming from the World Bank National Account (WNA) and the EU-KLEMS dataset.

Our basic index of value added is constructed from the WBN dataset as the weighted growth of household final consumption, gross investment and government expenditure, using as weights their respective shares of value added. According to our theory, these shares should be kept constant at their steady state level; in practice, we use shares averaged across the twenty years in our sample.

Data on capital (physical capital services, compensation to capital and depreciation) comes from EU-KLEMS. As documented by O'Malony and Timmer (2009), the stock of a certain asset is constructed using the perpetual inventory method on investment time series. Data are then aggregated over the different asset types using a translog function, where weights are given by the average share of each asset in the total capital compensation.

Data on labor (labor service index and compensation) are also provided by EU-KLEMS. The labor service index is a translog function of types of persons engaged (classified by skill, gender, age and sex), where weights are given by the average share of each type of worker in the value of total labor compensation.

The tax data is an updated version of that presented in Boscá et al. (2005). It relates realized tax revenues, from the OECD Revenue Statistics, to estimates of the associated tax bases derived from the OECD National Accounts. The underlying methodology adds some technical improvements to Mendoza et al. (1994).

For the cross-country welfare comparison, we also need steady state measures of per-capita consumption and leisure rate. We assume that countries are on their steady state growth path in 1996, although we experiment with other options. The PPP-adjusted measure of consumption per capita is taken from the World Penn Table. Leisure is constructed along the lines of Jones and Klenow (2010) who assume a time endowment of 5840 hours per year (16 hours*365 days), and

measure the leisure rate as $1 - \frac{Hours\ Worked}{5840 * Adult\ Population}$.

Data on total hours worked come from EU-KLEMS, while data on the total adult population (age 15-64) are provided by the World Bank's World Development Indicator (WDI) dataset.

6 Sources of Welfare Changes and Differences

We first calculate welfare change over time for each country in our data set. These are "within" results that cannot be compared across countries; effectively, each is an index that is specific to the country in question. We cannot compare these indexes across countries because the representative consumers in the various countries are allowed to have different utility functions. We calculate cross-country welfare differences later in this section, under the assumption that utility functions are the same across countries.

Since the change in welfare over time depends on the expected present discounted value of TFP growth, as shown by equation (15), we need to investigate the time-series property of TFP. We do so using annual data for the eight countries in our data set: Canada, France, Germany, Great Britain, Italy, Japan, Spain and the United States. We use data from 1985-2005 for all countries except Germany. The German sample is 1995-2005, since EU-KLEMS data for unified Germany start only in 1995. We use the (log of the) various aggregate TFP measure suggested by our theory, and estimate simple AR processes for each country. The persistence of TFP growth is a key statistic, since it shows how the entire summation of expected productivity residuals changes as a function of the innovation in the log level of TFP. For most countries the various measures for the log level of the TFP index are well described by an AR(1) stationary process around a linear trend. In Table (1) we report, as an example, the estimation results obtained using the definition of TFP for the case of no government. Additional lags of log TFP beyond the first one are not significant and the residual is white noise, as suggested by the Lagrange Multiplier test for residual serial correlation (shown in the last line of the table). The exceptions are Britain, Japan and Spain, where the data demand an AR(2) specification. In all cases, we can comfortably reject the null of a unit root in the log TFP process (after allowing for a time trend). We use the results in this table to form expectations of future levels or differences of TFP, which are required to construct our welfare change indexes.

Given the time series processes for TFP in each country, we can readily construct the first two terms in equation (19), the present value of expected TFP growth, and the change in expectations of that quantity. The third term, which depends on the change in the capital stock can also be constructed using data from EU-KLEMS (again, for the entire private economy). All the terms are multiplied by functions of constants. The constants we use are listed in Appendix B. We assume that $\beta = 0.96$ for all countries. We also assume the countries in our sample share the same values for the growth rates of population and per-capita consumption, n and g. We set these parameters equal to their averages across the countries in our sample. Recall that g is the steady-state growth rate of all per-capita variables. We think it sensible to assume that in the very

long run, per-capita variables do not diverge infinitely across countries. Thus, we allow countries to differ in the steady state in levels, but not in growth rates. For most countries in the world, and certainly for the countries in our sample, the assumption of a common growth rate is consistent with the econometric results of Quah (1996) and Kremer, Onatski and Stock (2001).

We use equation (19) to express the average welfare change per year in each country in terms of permanent changes in consumption, expressed as a percent of consumption in the steady state. (For this thought experiment, we assume that the level of leisure is fixed.) The results are in Table 2. In Figures 1 and 2 we report the evolution over time of our welfare index for each country, in log deviation from its value in 1985 (for the case of no government and for the case of optimal spending with distortionary taxes). We see that assumptions about fiscal policy affect the results in significant ways. We focus first on results for the US, which are given in the first row. In the first column, we construct the output data and the capital and labor shares under the assumption that there is no government expenditure and no distortionary taxes. Thus, output consists of consumption and investment, aggregated using weights that sum to one. Without distortionary taxes, the capital and labor shares also sum to one (we assume that profits are approximately zero in the steady state). In this case, the average annual growth rate of welfare is equivalent to a permanent increase in consumption of about 2.2 percent. Recall from Section 4.4 that this result continues to apply even when we allow for the fact that the US is an open economy. The same is true for all the other results in Table 2–all apply to open as well as closed economies.

Now we allow for the existence of a government that consumes real resources, still under the assumption that taxes are not distortionary. In the second column we assume that government expenditures are complete waste, in the sense that they provide no utility to the representative household. In this case "utility-relevant output" again comprises just consumption and investment, but now aggregated using weights that sum to less than one.[15] By contrast, in the third column we assume that expenditures are determined optimally, so that at the margin the consumer is indifferent between an additional unit of private consumption and an additional unit of government expenditures. In this case, output consists of consumption, investment and government purchases, aggregated using nominal expenditure shares which sum to one. Note that in all cases our output concepts correspond to different measures of absorption; this is why they are relevant for both closed and open economies.

Of course welfare growth is lower when we assume that expenditures are wasteful (1.3 versus 2.2 percent for the no-government case). But even in the case with optimal spending, average welfare growth is only 1.7 percent, reflecting the fact that real government expenditure has grown more slowly than private expenditure in the US.

We repeat our welfare calculations under the assumption that the government raises revenue via distortionary taxes. The results are in columns 3-6 of Table 2. As shown above, if taxes are distortionary we need to construct the shares in the Solow residual using the after-tax wage

[15] The weight on consumption is the nominal value of consumption divided by nominal expenditures on consumption, investment and government purchases. The weight on investment is the nominal value of investment, divided by the same denominator.

and capital rental rate perceived by the household, implying that the shares will sum to less than one. We construct the new shares using the tax rates provided described in the previous section. The quantitative effect of this change is significant. In all three cases (no government expenditure, wasteful spending and optimal spending) per capita welfare growth expressed in terms of consumption growth rates is *higher* by almost one full percentage point per year.[16] Intuitively, if taxes are distortionary then steady-state output is too low; thus, any increase in output, even with unchanged technology, is a welfare improvement. We see that it is quantitatively very important to allow for the fact that taxes are distortionary and not lump-sum. For example, it matters significantly more for the final result than whether we assume that government spending is wasteful or optimal. We take as our benchmark the case shown in the last column, where spending is optimally chosen and taxes are distortionary. In this case, average US welfare growth is equivalent to a growth rate of per-capita consumption of 2.6 percent per year.

We now turn to the results for other countries. In our benchmark case, two other countries have average welfare growth rates nearly equal to that of the US. They are Great Britain and, surprisingly, Spain. The treatment of fiscal variables is very important for the Spanish results. In the case where we assume there is no government expenditure or taxation, the Spanish welfare growth rate is less than half the corresponding value for the US. Since Spanish tax rates are high, Spain's welfare growth is boosted significantly by taking distortionary taxes into account. Since Spain's government expenditures grew at a fast rate over our sample period, the results also change significantly depending on whether we assume that government expenditures are wasteful or optimal. For Spain, the difference between these two assumptions is equivalent to about 1 percent consumption growth per year, as opposed to the US, where the difference is only about 0.4 percentage points. In Great Britain, by contrast, under all assumptions the welfare growth rates are very similar to their US counterparts.

Ranking countries by descending order of welfare growth rates in our benchmark case of optimal government spending and distortionary taxes, the next group of countries comprises France (welfare growth equivalent to a 2 percent growth rate of per-capita consumption per year), closely followed by Canada (1.9 percent), and, with larger gaps, by Japan (1.5 percent) and Italy (1.3 percent). For many of these countries, the assumptions regarding fiscal variables matter significantly more than they do for the US. For example, Japanese welfare growth would be cut in half if we assumed that Japan's government spending is wasteful rather than optimally chosen.

Another surprising result is that in our sample Germany clearly has the worst welfare performance, a growth rate of essentially zero over 10 years. Given the well-known strength of the German economy, this result appears anomalous. However, an examination of the data shows that the 10 years starting in 1995 were a dismal period for the newly-unified German state: TFP growth was non-existent, and investment was weak. Thus, the results are correct for the sample period, but this period captures neither the traditional strength of the West German economy, nor

[16] In the case of no government expenditure, we assume that the revenues raised by distortionary taxes are rebated lump-sum to the household.

the eventual rise of Germany after reunification. The German example thus serves as a cautionary reminder that univariate forecasts of TFP growth based on just a few years of data may not be reliable when a country is undergoing a massive structural transformation that has large short-run costs.

We now investigate which components of per-capita welfare change are important for our results. We break down the components of equation (19) by country in Table 3. To keep the results manageable, we focus on just two of the six cases we reported in Table 2, the no-government case and our benchmark results with optimal government spending and distortionary taxes. In both cases and for almost all countries, we find that the average expectations-revision term is close to zero, as it should be. (Italy is the exception to this rule.) Of the two remaining terms (abstracting from the constant), for most countries the present value of current and expected future productivity growth is 3-4 times as important a contributor to welfare growth as the change in the capital stock. Unsurprisingly, the productivity growth term is uniformly more important when allowing for distortionary taxes. In some cases the change is dramatic, especially in Germany, Japan, Spain and Canada. The common factor uniting these countries is that all have slow growth in conventionally-measured TFP over our sample period, so changes in measuring input growth have a large effect.

We now turn to measuring welfare differences across the countries in our sample. For each country and time period, we estimate θ_{it} as defined in equation (23). Recall that θ_{it} is the extra "equivalent permanent consumption" that a representative US consumer would need in order to be indifferent between staying in the US and moving permanently to country i starting at time t. In this hypothetical move, the consumer loses the per-capita capital stock of the US, but gains the equivalent capital stock of country i. From time t on, the consumer faces the same product and factor prices and tax rates, and receives the same lump-sum transfers and government expenditure benefits as all the other consumers in country i. In a slight abuse of language, we often use refer to the incremental equivalent permanent consumption as "the welfare difference" or "the welfare gap."

Note that in order to compute θ_{it}, we need to know the gap in steady-state consumption and leisure between country i and the US. In order to estimate this steady-state difference, we assume that all countries were in the steady state in 1996, which is a time of moderate growth for most countries in our sample. We then use the gaps between observed consumption and leisure in each country and the US in 1996 as our measure of the steady-state gap.

To this steady-state gap, we add the net effect of the expected transitional dynamics in TFP and capital that were the focus of our earlier within-country analysis. It turns out that the net steady-state difference in welfare between the US and the other countries in our sample is relatively small. For example, relative to the European countries, on average the US consumes more but enjoys less leisure. These differences have offsetting effects on the steady-state welfare gap, and in many cases the two nearly cancel one another. Thus, the differences in welfare between countries are heavily influenced by differences in expected future TFP growth and do not reflect just differences

in steady-state consumption and leisure. Another way of making this point is to note that the relative rankings of the countries change over time, often dramatically so. Steady-state differences would only explain constant welfare gaps between countries.

These results are easiest to understand in graphical form. We plot the log of θ_{it} for the countries and time periods in our sample in Figures 3 and 4. Note that by definition log $\theta_{it} = 0$ for the US, since the US consumer neither gains nor loses by moving to the US at any point in time. The verical axis shows, therefore, the approximate deviation from one of θ_{it}.Figure 3 shows the results for the "no-government" case, where we ignore both government expenditure and taxes. Figure 4 shows the results for our benchmark case, where we allow for distortionary taxes and assume that government expenditure is optimally chosen. Since both figures show qualitatively similar results, for brevity we discuss only the benchmark case.

It is instructive to begin by focusing on the beginning and end of the sample. At the beginning of the sample, expected lifetime welfare was higher in both France and Italy than in the US. An US consumer would have needed an increase in equivalent consumption of nearly 10 percent per year to give up the opportunity of moving to France, and nearly 20 percent to forego emigrating to Italy. This gap reflects both the long-run European advantage in leisure and the fact that in the mid-1980s the US was still struggling with its productivity slowdown, while TFP in the leading European economies was growing faster than in the US. Capital accumulation was also proceeding briskly in those countries. For France and Italy the higher level of leisure slightly more than compensated the lower steady state level of consumption, however the bulk of the gap reflected gaps in TFP and/or capital accumulation.[17] Although our framework imposes the assumption that such gaps are transitory–in the long run, TFP growth in all countries is expected to converge to a common value–the transitional dynamics prevailing in 1985 made Italy and France appear a better bet than the US. By the end of the sample, the continental European economies and Canada are both falling behind the US, because they had not matched the pickup in TFP growth and investment experienced in the US after 1995. Italy experiences the greatest relative "reversal of fortune," going from being nearly 20 percent above the US to more than 10 percent below, but the results for France are very similar, albeit less extreme. Spain fares better in terms of changes: its welfare gap is about 10 percent relative to the US, but it maintains this gap rather than falling further behind over time. As we have already noted, the results for Germany are anomalous, showing Germany falling steadily behind the US and ending the sample with a gap of more than 20 percent, the second-lowest in our sample.

The only economy in our sample that keeps pace with the US throughout is Great Britain. Its welfare level is always slightly below that of the US, but the two countries exhibit almost identical dynamics. This result is interesting, because the UK experienced much the same lack of TFP growth in the late 1990s and early 2000s as the major continental European economies.[18] According to EU-KLEMS, in the US TFP grew nearly 8 percent from 1995 to 2005 but only about

[17] In 1996, where by construction the TFP gaps are zero,

[18] For discussion and a suggested explanation, see Basu, Fernald, Oulton and Srinivasan (2004).

3 percent in the UK over the same period. The other "Anglo-Saxon" country in our sample, Canada, had a welfare level fairly close to that of the US and the UK in 1985, but ended nearly 20 percent below these countries at the end of the sample. This result is due primarily to the differential productivity performance of the two countries: TFP in Canada actually fell during the 1990s, and rose only slowly in the early 2000s.

Perhaps the most striking comparison is between the US and Japan. Even in 1985, when its economic performance was the envy of much of the world, Japan was the least attractive country in our sample to an US consumer contemplating emigration; such a consumer would give up nearly 20 percent of his consumption permanently in order to stay in the US instead of moving to Japan. The gap remains approximately constant until the bursting of Japan's real estate bubble in 1991, and then widens steadily at a nearly constant rate. At the end of our sample, nearly 15 years into its "lost decades," the Japanese welfare gap relative to the US exceeds 40 percent. This cautionary history suggests that it would be interesting to see what the same calculations will show for the US in another 10 or 15 years, after the bursting of the US real estate bubble and the associated financial crisis.

7 Conclusions

We show that the present value of aggregate TFP growth, for a given initial endowment of capital, is a complete welfare measure for a representative consumer, up to a first-order approximation. This result rigorously justifies TFP, rather than technical change or labor productivity, as the central statistic of interest in any exploration of productivity, at all levels of aggregation. Importantly, the result holds even when TFP is not a correct measure of technical change, for example due to increasing returns, externalities, or imperfect competition. Our theoretical results point to a key role for the persistence of aggregate TFP growth, since welfare change is related to the entire expected time path of productivity growth in addition to the current growth rate. Moreover, our derivation shows that in order to create a proper welfare measure, TFP has to be calculated using prices faced by households rather than prices facing firms. In modern, developed economies with high rates of income and indirect taxation, the gap between household and firm TFP can be considerable. Finally, in an open economy, the change in welfare will also reflect present and future changes in the returns on net foreign assets and in the terms of trade.

We illustrate our methods using data for several industrialized countries We show the importance of fiscal considerations in constructing measures of welfare change over time within each country. For example, ignoring government spending and taxation, Britain has the largest welfare gain among our group of countries over our sample period, 1985-2005, while Spain lags far behind due to its low TFP growth rate. Allowing for distortionary taxation and assuming that government expenditures are chosen optimally, the United States's growth rate exceed's Britain, but Spain's performance is very close to that of these two leading economies.

We then apply our methodology to cross country comparisons and show how these comparisons

evolve over time. At the start of our sample, we find that Italy and France have a considerable welfare advantage over the United States, in the sense that a representative US consumer would have to be compensated with higher consumption in order to achieve the welfare of one of these countries. Italy and France led the US at this time not only because of the higher level of leisure, but also because they were catching up to the US in TFP levels while maintaining higher rates of TFP growth and/or where experiencing faster capital accumulation. 20 years later the situation was reversed: a US consumer would prefer to give up consumption in order to stay in the United States instead of moving to France or Italy. The low TFP growth rates and/or lower capital accumulation in these countries, especially Italy, combined with the post-1995 productivity and investment acceleration in the US make the US the welfare leader by the end of the sample period.

References

[1] Arrow, K. J. 1962. The Economic Implications of Learning by Doing. Review of Economic Studies 29: 155-73.

[2] Baker, D., Rosnick, D., 2007. "Usable productivity" growth in the US: an international comparison, 1980-2005. International Productivity Monitor

[3] Basu, S., Fernald, J.G., 2002. Aggregate productivity and aggregate technology. European Economic Review 46, 963-991

[4] Basu, S., Fernald J. G., Oulton, N., Srinivasan, S., 2003. The Case of the Missing Productivity Growth: Or, Does Information Technology Explain why Productivity Accelerated in the US but not the UK? NBER Working Papers 10010.

[5] Basu, S., L. Pascali, F. Schiantarelli and L. Serven, 2009. Productivity, Welfare and Reallocation: Theory and Firm Level Evidence. National Bureau of Economic Reasearch, Working Paper 15579.

[6] Basu, S., Pascali, L., Schiantarelli F., Serven, L., 2009. Productivity, Welfare and Reallocation: Theory and Firm-Level Evidence. NBER Working Papers 15579.

[7] Bosca, J. E., Garcia, J., Taguas, D., 2005. Tipos Efectivos de Gravamen y Convergencia Fiscal en la OCDE: 1965-2001. Revista de Economia Publica 174, 119-141

[8] Jones, C.I., 1995. R&D-Based Models of Economic Growth, Journal of political Economy, 103, 759-83.

[9] Jones, C. I., Klenow, P. J., 2010. Beyond GDP? Welfare across Countries and Time. NBER Working Papers 16352.

[10] Hall, R., Jones, C. I., 1999. Why Do Some Countries Produce So Much More Output Per Worker Than Others? The Quarterly Journal of Economics 114(1), 83-116.

[11] Hulten, C. R., 1973. Divisia Index Numbers. Econometrica 41, 1017-1026.

[12] Hulten, C.R, 1978. Growth accounting with intermediate inputs, The Review of Economic Studies 45, 511-518

[13] Hulten, C.R., Schreyer, P., 2009. GDP, Technical Change and the Measurement of Net Income: the Weitzman

[14] King, R.G., Plosser, C.I., Rebelo, S.T., 1988. Production, growth and business cycles. Journal of Monetary Economics 21, 195-232

[15] Kohli, U., 2004. Real GDP, Real Domestic Income, and Terms-of-Trade Changes. Journal of International Economics 62, 83-106

[16] Kremer, M., Onatski, A., Stock J., 2001. Searching for Prosperity. NBER Working Papers 8250.

[17] Lane, P. R., Milesi-Ferretti, G. M., 2001. The external wealth of nations: measures of foreign assets and liabilities for industrial and developing countries. Journal of International Economics 55(2), 263-294.

[18] Lane, P. R., Milesi-Ferretti, G. M., 2007. The external wealth of nations mark II: Revised and extended estimates of foreign assets and liabilities, 1970–2004. Journal of International Economics 73, 223-250.

[19] Mendoza, E.G., Razin, A., Tesar L. L., 1994. Effective Tax Rates in Macroeconomics, Cross0Country Estimates of Tax Rates on Factor Incomes and Consumption. Journal of Monetary Economics, 34(3), 297-323.

[20] Nordhaus, W., Tobin, J., 1972. Is growth obsolete? NBER General Series 96

[21] O'Mahony, M., Timmer, M.P., 2009. Output, Input and productivity measures at the industry level: the EU KLEMS database. Economic Journal 119, pp.F374-F403.

[22] Petrin, A., Levinsohn, J., 2009. Measuring Aggregate Productivity Growth Using Plant-Level Data. Unpublished.

[23] Petrin, A., White, K., Reiter, J.P., 2009. The impact of plant-level resource reallocations and technical progress on U.S. macroeconomic growth. Unpublished.

[24] Quah, D. T, 1996. Twin Peaks: Growth and Convergence in Models of Distribution Dynamics. Economic Journal 106, 1045-55.

[25] Rebelo, S. , 1991. Long-Run Policy Analysis and Long-Run Growth. Journal of Political Economy 99 (3): 500.

[26] Romer, P. M., 1986. Increasing Returns and Long Run Growth. Journal of Political Economy, 94, 1002–37.

[27] Rotemberg, J.J., Woodford, M., 1991. Markups and the Business Cycle in NBER Macroeconomics Annual Volume 6 , Olivier Jean Blanchard and Stanley Fischer, editors (p. 63 - 140)

[28] Solow, R., M., 1957, "Technical Change and the Aggregate Production Function", Review of Economics and Statistics, 39, 312-320.

[29] Weil, D. , 2008. Economic Growth. Pearson.

[30] Weitzman, M., 1976. On the welfare significance of national product in a dynamic economy. Quarterly Journal of Economics 90, 156-162.

[31] Weitzman, M., 2003. Income capital and the maximum principle. Harvard University Press, Cambridge.

A Appendix A: Derivations

A.1 Making the problem stationary

The representative household maximizes intertemporal utility:

$$V_t = \sum_{s=0}^{\infty} \frac{1}{(1+\rho)^s} \frac{N_{t+s}}{H} U(C_{1,t+s}, .., C_{Z,t+s}; \overline{L} - L_{t+s}) \tag{A.1}$$

where $C_{i,t+s}$ is the capita consumption of good i at time t+s, L_{t+s} are hours of work per capita, $\overline{L}$ is the time endowment, and N_{t+s} population. H is the number of households, assumed to be fixed and normalized to one from now on. Population grows at constant rate n and per capita variables at a common rate g. X_t is and index for per capita variables in the sense that their level at time t is proportional to X_t.Consider the laws of motion for N_t and for X_t:

$$N_t = N_0(1+n)^t \tag{A.2}$$

$$X_t = X_0(1+g)^t \tag{A.3}$$

and normalize $H = 1$.

We can rewrite the utility function as:

$$V_t = N_t \sum_{s=0}^{\infty} \frac{(1+n)^s}{(1+\rho)^s} U(C_{1,t+s}, .., C_{Z,t+s}; \overline{L} - L_{t+s}) \tag{A.4}$$

For a well defined steady state in which hours of work are constant we assume that the utility function has the King Plosser and Rebelo form (1988):

$$U(C_{1,t+s}, .., C_{Z,t+s}; L - \overline{L}_s) = \frac{1}{1-\sigma} C(C_{1,t+s}, .., C_{Z,t+s})^{1-\sigma} \nu(\overline{L} - L_{t+s})$$

We assume that $C()$ is homogenous of degree 1 and $\nu(\overline{L} - L_{t+s})$ is an increasing and concave function of leisure, and assumed to be positive. Define $c_{i,t+s} = \frac{C_{i,t+s}}{X_{t+s}}$. We can rewrite the utility function in the following form:

$$U(C_{1,t+s}, .., C_{Z,t+s}; L - \overline{L}_{t+s}) = \frac{1}{1-\sigma} X_{t+s}^{(1-\sigma)} C(c_{1,t+s}, .., c_{Z,t+s})^{1-\sigma} \nu(\overline{L} - L_{t+s})$$

or

$$U(C_{1,t+s}, .., C_{Z,t+s}; L - \overline{L}_s) = (1+g)^{s(1-\sigma)} X_t^{(1-\sigma)} \frac{1}{1-\sigma} C(c_{1,t+s}, .., c_{Z,t+s})^{1-\sigma} \nu(\overline{L} - L_{t+s})$$

Inserting this into V_t, we get:

$$V_t = N_t X_t^{(1-\sigma)} \sum_{s=0}^{\infty} \beta^s U(c_{1,t+s}, .., c_{Z,t+s}; \overline{L} - L_{t+s}) \tag{A.5}$$

where: $\beta = \frac{(1+n)(1+g)^{1-\sigma}}{(1+\rho)}$.

A.2 Budget constraint

Start from the usual budget constraint:

$$P_t^I K_t + B_t = (1-\delta) P_t^I K_{t-1} + (1+i_t) B_{t-1} + P_t^L L_t N_t + P_t^K K_{t-1} + \Pi_t - \sum_{i=1}^{Z} P_{i,t}^C C_{i,t} N_t \tag{A.6}$$

Divide both sides by $P_t^I X_t N_t$ to get:

$$\begin{aligned} \frac{K_t}{X_t N_t} + \frac{B_t}{P_t^I X_t N_t} &= (1-\delta) \frac{K_{t-1}}{X_{t-1} N_{t-1}} \frac{X_{t-1} N_{t-1}}{X_t N_t} + (1+i_t) \frac{B_{t-1}}{P_{t-1}^I X_{t-1} N_{t-1}} \frac{P_{t-1}^I}{P_t^I} \frac{X_{t-1} N_{t-1}}{X_t N_t} \\ &+ \frac{P_t^L}{P_t^I} \frac{L_t N_t}{X_t N_t} + \frac{P_t^K}{P_t^I} \frac{K_{t-1}}{X_{t-1} N_{t-1}} \frac{X_{t-1} N_{t-1}}{X_t N_t} + \frac{\Pi_t}{P_t^I X_t N_t} - \sum_{i=1}^{Z} \frac{P_{i,t}^C}{P_t^I} \frac{C_{i,t} N_t}{X_t N_t} \end{aligned}$$

Define: $k_t = \frac{K_t}{X_t N_t}$, $b_t = \frac{B_t}{P_t^I X_t N_t}$, $p_t^K = \frac{P_t^K}{P_t^I}$, $p_t^L = \frac{P_t^L}{P_t^I X_t}$, $p_{i,t}^C = \frac{P_{i,t}^C}{P_t^I}$, $(1+r_t) = \frac{(1+i_t)}{(1+\pi_t)}$, $\pi_t = \frac{\Pi_t}{P_t^I X_t N_t}$. The budget constraint can be rewritten as:

$$k_t + b_t = \frac{(1-\delta) + p_t^K}{(1+g)(1+n)} k_{t-1} + \frac{(1+r_t)}{(1+g)(1+n)} b_{t-1} + p_t^L L_t + \pi_t - \sum_{i=1}^{Z} p_{i,t}^C c_{i,t} \tag{A.7}$$

A.3 Optimality conditions

The representative household maximizes normalized intertemporal utility $v_t = \frac{V_t}{N_t X_t^{(1-\sigma)}}$. The Lagrangean for this problem is:

$$\begin{aligned} \Lambda_t &= E_t \sum_{s=0}^{\infty} \beta^s \{ U(c_{1,t+s}, .., c_{Z,t+s}; \overline{L} - L_{t+s}) \\ &+ \lambda_{t+s} \left(-k_{t+s} - b_{t+s} + \frac{(1-\delta) + p_{t+s}^K}{(1+g)(1+n)} k_{t+s-1} + \frac{(1+r_{t+s})}{(1+g)(1+n)} b_{t+s-1} + p_{t+s}^L L_{t+s} + \pi_{t+s} - \sum_{i=1}^{Z} p_{i,t+s}^C c_{i,t+s} \right) \} \end{aligned}$$

The FOCs are:

$$U_{c_{i,t}} - \lambda_t p_{i,t}^C = 0 \tag{A.8}$$

$$U_{L_t} + \lambda_t p_t^L = 0 \tag{A.9}$$

$$-\lambda_t + \beta E_t \frac{(1-\delta) + p_{t+1}^K}{(1+g)(1+n)} \lambda_{t+1} = 0 \tag{A.10}$$

$$-\lambda_t + \beta_t \frac{1}{(1+g)(1+n)} E_t (1 + r_{t+1}) \lambda_{t+1} = 0 \tag{A.11}$$

A.4 Approximation around SS

Define with $\widehat{x} = \log x_t - \log x$ the log deviation from the steady state of a variable (x is the steady state value of x_t). Taking a first order approximation in logs of the Lagrangean (which equals the value function along the optimal path), one obtains:

$$\begin{aligned}
v_t - v \quad = \quad & E_t[\sum_{s=0}^{\infty} \beta^s (\sum_{i=1}^{Z} U_{c_i} c_i \widehat{c}_{i,t+s} + U_L L \widehat{L}_{t+s} \\
& + \lambda p^L L \widehat{L}_{i,t+s} - \lambda \sum_{i=1}^{Z} p_i^C c_i \widehat{c}_{i,t+s} - \lambda k \widehat{k}_{t+s} - \lambda b \widehat{b}_{t+s}) \\
& + \sum_{s=0}^{\infty} \beta^{s+1} (\lambda \frac{(1-\delta) + p^K}{(1+g)(1+n)} k \widehat{k}_{t+s} + \lambda \frac{(1+r)}{(1+g)(1+n)} b \widehat{b}_{t+s}) \\
& + \sum_{s=0}^{\infty} \beta^s \widehat{\lambda}_{t+s} \left(-k - b + \frac{(1-\delta) + p^K}{(1+g)(1+n)} k + \frac{(1+r)}{(1+g)(1+n)} b \right. \\
& \left. + p^L L + \pi - \sum_{i=1}^{Z} p_i^C c_i \right)] \\
& + \sum_{s=0}^{\infty} \beta^s (\lambda p^L L \widehat{p}_{t+s}^L + \frac{p^K k}{(1+g)(1+n)} \widehat{p}_{t+s}^K - \sum_{i:1}^{Z} p_{i,t}^C c_i \widehat{p}_{i,t+s} + \pi \widehat{\pi}_{t+s} + \frac{rb}{(1+g)(1+n)} \widehat{r}_{t+s}) \\
& + \lambda \frac{(1-\delta) + p^K}{(1+g)(1+n)} k \widehat{k}_{t-1} + \lambda \frac{(1+r)}{(1+g)(1+n)} b \widehat{b}_{t-1}
\end{aligned}$$

Using the first order conditions, the first four lines equal zero *and, therefore,* we get:

$$\begin{aligned}
v_t \quad = \quad & v + E_t \sum_{s=0}^{\infty} \beta^s \lambda \left[p^L L \widehat{p}_{t+s}^L + \frac{p^K k}{(1+g)(1+n)} \widehat{p}_{t+s}^K + \pi \widehat{\pi}_{t+s} - \sum_{i=1}^{Z} p_{i,t}^C c_i \widehat{p}_{i,t+s} \right] \\
& + \lambda \frac{(1-\delta) + p^K}{(1+g)(1+n)} k \widehat{k}_{t-1} + \lambda \frac{(1+r)}{(1+g)(1+n)} b \widehat{b}_{t-1}
\end{aligned} \tag{A.12}$$

Now log linearize the budget constraint:

$$k\ \widehat{k}_t + b\widehat{b}_t - \frac{(1-\delta)+p^K}{(1+g)(1+n)}k\widehat{k}_{t-1} - \frac{(1+r)}{(1+g)(1+n)}b\widehat{b}_{t-1} - p^L L\widehat{L}_t - p^L L\widehat{p}_t^L - \frac{p^K k}{(1+g)(1+n)}\widehat{p}_t^K$$
$$-\pi\widehat{\pi}_t + \sum_{i=1}^{Z} p_i^C c_i \widehat{c}_{i,t} + \sum_{i=1}^{Z} p_i^C c_i \widehat{p}_{i,t} = 0$$

Using this result in (A.12) gives us:

$$\begin{aligned} v_t &= v + E_t \sum_{s=0}^{\infty} \beta^s \lambda [\sum_{i=1}^{Z} p_i^C c_i \widehat{c}_{i,t+s} + k\widehat{k}_{t+s} - \frac{(1-\delta)+p^K}{(1+g)(1+n)} k\widehat{k}_{t+s-1} + b\widehat{b}_{t+s} - \frac{(1+r)}{(1+g)(1+n)} b\widehat{b}_{t+s-1} \\ &\quad -p^L L\widehat{L}_{t+s}] + \lambda \frac{(1-\delta)+p^K}{(1+g)(1+n)} k\widehat{k}_{t-1} + \lambda \frac{(1+r)}{(1+g)(1+n)} b\widehat{b}_{t-1} \end{aligned} \tag{A.13}$$

Rearranging the terms, we get:

$$\begin{aligned} v_t &= v + E_t \sum_{s=0}^{\infty} \beta^s \lambda \left[\sum_{i=1}^{Z} p_i^C c_i \widehat{c}_{i,t+s} + k\widehat{k}_{t+s} - \frac{(1-\delta)+p^K}{(1+g)(1+n)} k\widehat{k}_{t+s-1} - p^L L\widehat{L}_{t+s} \right] \\ &\quad +\lambda \frac{(1-\delta)+p^K}{(1+g)(1+n)} k\widehat{k}_{t-1} + \lambda \sum_{s=0}^{\infty} \beta^s \left[b\widehat{b}_{t+s} - \beta \frac{(1+r)}{(1+g)(1+n)} b\widehat{b}_{t+s} \right] \end{aligned} \tag{A.14}$$

Using the FOC and the trasversality condition for bonds, the equation above becomes:

$$\begin{aligned} v_t &= v + E_t \sum_{s=0}^{\infty} \beta^s \lambda \left[\sum_{i=1}^{Z} p_i^C c_i \widehat{c}_{i,t+s} + k\widehat{k}_{t+s} - \frac{(1-\delta)+p^K}{(1+g)(1+n)} k\widehat{k}_{t+s-1} - p^L L\widehat{L}_{t+s} \right] \\ &\quad +\lambda \frac{(1-\delta)+p^K}{(1+g)(1+n)} k\widehat{k}_{t-1} \end{aligned} \tag{A.15}$$

Notice that the law of motion of capital: $K_t = (1-\delta)K_{t-1} + I_t$, can be rewritten as: $\frac{K_t}{X_t N_t} = (1-\delta)\frac{K_{t-1}}{X_{t-1}N_{t-1}}\frac{X_{t-1}N_{t-1}}{X_t N_t} + \frac{I_t}{X_t N_t}$ which after some algebra becomes:

$$k_t = \frac{(1-\delta)}{(1+g)(1+n)} k_{t-1} + i_t$$

Differentiating it around the steady state, we get:

$$k\widehat{k}_t = \frac{(1-\delta)}{(1+g)(1+n)} k\widehat{k}_{t-1} + i\widehat{i}_t$$

Inserting this equation into equation (A.14) we get:

$$v_t = v + E_t \sum_{s=0}^{\infty} \beta^s \lambda \left[\sum_{i=1}^{Z} p_i^C c_i \widehat{c}_{i,t+s} + i\widehat{i}_{t+s} - p^L L \ \widehat{L}_{t+s} - \frac{p^K k}{(1+g)(1+n)} \ \widehat{k}_{t+s-1} \right] + \lambda \frac{(1-\delta)+p^K}{(1+g)(1+n)} k\widehat{k}_{t-1} \tag{A.16}$$

A.5 Connecting the level of productivity to the level of welfare

Define value added (for normalized variables in deviation from steady state) as:

$$\widehat{y}_t = \log y_t - \log y = \sum_{i=1}^{Z} \frac{P_i^C C_i N}{P^Y Y} \widehat{c}_{it} + \frac{P^I I}{P^Y Y} \widehat{i}_t = \sum_{i=1}^{Z} s_{c_i} \widehat{c}_{it} + s_i \widehat{i}_t \tag{A.17}$$

Inserting this equation into (A.16), and noticing that $\frac{p^K k}{p^Y y(1+g)(1+n)}$is the SS value of $s_{K,t} \equiv \frac{P_t^K K_{t-1}}{P_t^Y Y_t}$ we get:

$$v_t = v + E_t \sum_{s=0}^{\infty} \beta^s \lambda p^Y y \left[\widehat{y}_t - s_L \ \widehat{L}_{t+s} - s_K \ \widehat{k}_{t+s-1} \right] + \lambda \frac{(1-\delta)+p^K}{(1+g)(1+n)} k\widehat{k}_{t-1} \tag{A.18}$$

Using the definition of the normalized variable, this can be rewritten as:

$$\begin{aligned} v_t &= v + \left(\lambda p^Y y\right) E_t \sum_{s=0}^{\infty} \beta^s \left[(\log \frac{Y_{t+s}}{N_{t+s} X_{t+s}} - \log y) - s_L (\log L_{t+s} - \log L) - s_K (\log \frac{K_{t+s-1}}{N_{t+s-1} X_{t+s-1}} - \log \ k) \right] \\ &\quad + \lambda \frac{(1-\delta)+p^K}{(1+g)(1+n)} k\widehat{k}_{t-1} \end{aligned} \tag{A.19}$$

or:

$$\frac{v_t - v}{\lambda p^Y y} = E_t \sum_{s=0}^{\infty} \beta^s \left[\log Y_{t+s} - s_L \log \ N_{t+s} L_{t+s} - s_K \log K_{t+s-1} \right] + \frac{(1-\delta)+p^K}{(1+g)(1+n)} \frac{k}{p^Y y} \log K_{t-1} - f(t) \tag{A.20}$$

where:

$$\begin{aligned} f(t) &= \frac{1}{1-\beta} \left[\log y - s_L \log L - s_K \log k + \frac{\beta}{(1-\beta)} \left[g(1-s_K) + n(1-s_L-s_K) \right] \right] \\ &\quad + \frac{1}{1-\beta} \left[(1-s_K) \log X_t + (1-s_L-s_K) \log N_t \right] + \frac{s_K}{1-\beta}(n+g) \\ &\quad + \frac{(1-\delta)+p^K}{(1+g)(1+n)} \frac{k}{p^Y y} (\log k + \log X_{t-1} + \log N_{t-1}) \end{aligned}$$

Define aggregate productivity (in log level) as: $\log PR_t = \log Y_t - s_L \log N_t L_t - s_K \log K_{t-1}$. Notice that we are taking a definition with constant shares. Inserting this definition into the equation

above, we get:

$$\frac{v_t - v}{\lambda p^Y y} = E_t \sum_{s=0}^{\infty} \beta^s \log PR_{t+s} + \frac{(1-\delta) + p^K}{(1+g)(1+n)} \frac{k}{p^Y y} \log K_{t-1} + f(t) \tag{A.21}$$

Assume $v_t > 0$ ($\sigma < 1$). Since $\frac{v_t - v}{v} \simeq \log v_t - \log v$, this result can be also re-written, to a first order approximation, as:

$$\frac{v}{\lambda p^Y y} (\log v_t - \log v) = E_t \sum_{s=0}^{\infty} \beta^s \log PR_{t+s} + \frac{(1-\delta) + p^K}{(1+g)(1+n)} \frac{k}{p^Y y} \log K_{t-1} + f(t) \tag{A.22}$$

which can be also expressed in terms of log-deviation from the steady state of intertemporal per-capita utility as:

$$\frac{v}{\lambda p^Y y} (\log \frac{V_t}{N_t} - \log \left(\frac{V_t}{N_t}\right)_{SS}) = E_t \sum_{s=0}^{\infty} \beta^s \log PR_{t+s} + \frac{(1-\delta) + p^K}{(1+g)(1+n)} \frac{k}{p^Y y} \log K_{t-1} + f(t) \tag{A.23}$$

where $\log \left(\frac{V_t}{N_t}\right)_{SS}$ denotes the log value of per-capita intertemporal utility along the balanced-growth path.

If $\sigma > 1$ so that $v_t < 0$ (recall we are assuming $\nu(\overline{L} - L) > 0$), the R.H.S. of (A.23) should equal instead $\frac{v}{\lambda p^Y y}(\log \left(-\frac{V_t}{N_t}\right) - \log \left(-\frac{V_t}{N_t}\right)_{SS})$, which is positive for positive deviations from the steady state.

Alternatively, welfare can expressed in terms of "equivalent consumption" per capita, denoted by C_t^*. C_t^* is defined as the level of consumption per capita at time t that, if it grows at the steady state rate g from t onward, with leisure set at its steady state level, delivers the same intertemporal utility per capita as the actual stream of consumption and leisure, More precisely, C_t^* satisfies:

$$\begin{aligned} \frac{V_t}{N_t} &= \sum_{s=0}^{\infty} \frac{(1+n)^s}{(1+\rho)^s} \left(C_t^*(1+g)^s\right)^{1-\sigma} \nu(\overline{L} - L)) \\ &= \frac{1}{(1-\sigma)(1-\beta)} C_t^{*1-\sigma} \nu(\overline{L} - L) \end{aligned} \tag{A.24}$$

Per-capita utility on the steady state growth path can be written as:

$$\begin{aligned} \left(\frac{V_t}{N_t}\right)_{SS} &= \sum_{s=0}^{\infty} \frac{(1+n)^s}{(1+\rho)^s} \left((C_t)_{SS}(1+g)^s\right)^{1-\sigma} \nu(\overline{L} - L)) \\ &= \frac{1}{(1-\sigma)(1-\beta)} (C_t)_{SS}^{1-\sigma} \nu(\overline{L} - L) \end{aligned} \tag{A.25}$$

The subscript SS denotes the steady state values of time varying variables. By taking the log-difference between equation (A.24) and (A.25) (for $\sigma < 1$), we get:

$$(\log \frac{V_t}{N_t}_t - \log \left(\frac{V_t}{N_t}\right)_{SS}) = (1-\sigma)\left(\log C_t^* - \log(C_t)_{SS}\right) \tag{A.26}$$

Using the definition of v and the F.O.C. for consumption, it follows that $\frac{v}{\lambda} = \frac{cp^c}{(1-\beta)(1-\sigma)}$. This, together with equation (A.22) implies that:

$$\log C_t^* = \log\left(C_t\right)_{SS} + \frac{(1-\beta)}{s_C}\left[E_t \sum_{s=0}^{\infty} \beta^s \log PR_{t+s} + \frac{(1-\delta)+p^K}{(1+g)(1+n)} \frac{k}{p^Y y} \log K_{t-1} + f(t)\right] \tag{A.27}$$

Rearranging the terms in $f(t)$, we can rewrite:

$$\begin{aligned} \log C_t^* &= \log\left(C_t\right)_{SS} + \frac{(1-\beta)}{s_C} E_t \sum_{s=0}^{\infty} \beta^s \left[\log PR_{t+s} - (\log PR_{t+s})_{SS}\right] \\ &\quad + \frac{(1-\beta)}{s_C} \frac{(1-\delta)+p^K}{(1+g)(1+n)} \frac{k}{p^Y y}\left[\log \frac{K_{t-1}}{N_{t-1}} - \log\left(\frac{K_{t-1}}{N_{t-1}}\right)_{SS}\right] \end{aligned} \tag{A.28}$$

This holds for σ smaller or greater than one.

A.6 Connecting the aggregate Solow residual with the change in welfare

Take the difference between the expected level of intertemporal utility v_t defined in (A.16) and v_{t-1}.

$$\begin{aligned} \Delta v_t &= E_t \sum_{s=0}^{\infty} \beta^s \lambda \left[\sum_{i=1}^{Z} p_i^C c_i \log c_{i,t+s} + i \log i_{t+s} - p^L L \log L_{t+s} - \frac{p^K k}{(1+g)(1+n)} \log\ k_{t+s-1}\right] \\ &\quad - E_{t-1} \sum_{s=0}^{\infty} \beta^s \lambda \left[\sum_{i=1}^{Z} p_i^C c_i \log c_{i,t+s-1} + i \log i_{t+s-1} - p^L L \log L_{t+s-1} - \frac{p^K k}{(1+g)(1+n)} \log\ k_{t+s-2}\right] \\ &\quad + \lambda \frac{(1-\delta)+p^K}{(1+g)(1+n)} k \Delta \widehat{k}_{t-1} \end{aligned}$$

The right hand side, after adding and subtracting, for each variable x_{t+s}, $E_t x_{t+s}$, can be written as:

$$
\begin{aligned}
\Delta v_t &= E_t \sum_{s=0}^{\infty} \beta^s \lambda [\sum_{i=1}^{Z} p_i^C c_i \Delta \log c_{i,t+s} + i \Delta \log i_t - p^L L \Delta \log L_{t+s} - \frac{p^K k}{(1+g)(1+n)} \Delta \log \ k_{t+s-1}] \\
&+ \sum_{s=0}^{\infty} \beta^s \lambda [\sum_{i=1}^{Z} p_i^C c_i \left(E_t \log c_{i,t+s} - E_{t-1} \log c_{i,t+s}\right) + i \left(E_t \log i_{t+s} - E_{t-1} \log i_{t+s}\right) \\
&- p^L L E_t (\log L_{t+s} - E_{t-1} \log L_{t+s}) - \frac{p^K k}{(1+g)(1+n)} \left(E_t \log k_{t+s-1} - E_{t-1} \log k_{t+s-1}\right)] \\
&+ \lambda \frac{(1-\delta) + p^K}{(1+g)(1+n)} k \Delta \widehat{k}_{t-1} \qquad \text{(A.29)}
\end{aligned}
$$

Define value added growth (at constant shares) as:[19]

$$
\Delta \log y_t = \sum_{i=1}^{Z} \frac{p_i^C c_i}{p^Y y} \Delta \log c_{i,t+s} + \frac{i}{p^Y y} \Delta \log i_t \qquad \text{(A.30)}
$$

Using the fact that nominal value added $P_t Y_t = \sum_{i:0}^{Z} P_{i,t}^C C_{i,t} N_t + P_t^I I_t$, it is also true that:

$$
\Delta \log Y_t = \sum_{i=1}^{Z} \frac{P_i^C C_i N}{P^Y Y} \Delta \log (C_{i,t} N_t) + \frac{P^I I}{P^Y Y} \Delta \log I_t \qquad \text{(A.31)}
$$

Now, insert this into equation (A.29) and factor out $p^Y y$ to obtain:

$$
\begin{aligned}
\Delta v_t &= \lambda p^Y y E_t \sum_{s=0}^{\infty} \beta^s [\Delta \log y_t - s_L \Delta \log L_{t+s} - s_K \Delta \log \ k_{t+s-1}] \qquad \text{(A.32)} \\
&+ \sum_{s=0}^{\infty} \beta^s \lambda [(E_t \log y_{t+s} - E_{t-1} \log y_{t+s}) \\
&- s_L E_t (\log L_{t+s} - E_{t-1} \log L_{t+s}) - s_K \left(E_t \log k_{t+s-1} - E_{t-1} \log k_{t+s-1}\right)] \\
&+ \lambda \frac{(1-\delta) + p^K}{(1+g)(1+n)} k \Delta \widehat{k}_{t-1}
\end{aligned}
$$

Using the fact that:

$$
\Delta \log y_t = \Delta \log (\frac{Y_t}{X_t N_t}) = \Delta \log Y_t - g - n
$$

$$
\Delta \log L_t = \Delta \log (N_t L_t \frac{1}{N_t}) = \Delta \log N_t L_t - n
$$

[19] Here we are departing slightly from convention, as value added is usually calculated with time varying shares.

$$\Delta \log k_t = \Delta \log(\frac{K_t}{X_t N_t}) = \Delta \log K_t - g - n$$

and dividing both terms by $\lambda p^Y y$ we can rewrite equation (A.32) as:

$$\begin{aligned}
\frac{\Delta v_t}{\lambda p^Y y} &= E_t \sum_{s=0}^{\infty} \beta^s \Delta \log PR_{t+s} \\
&+ \sum_{s=0}^{\infty} \beta^s \left[E_t \log PR_{t+s} - E_{t-1} \log PR_{t+s}\right] \\
&+ \frac{(1-\delta) + p^K}{(1+g)(1+n)} \frac{k}{p^Y y} \Delta \log K_{t-1} - f_1
\end{aligned} \tag{A.33}$$

where $E_t \log PR_{t+s} - E_{t-1} \log PR_{t+s}$ represents the revision in expectations of the level of the productivity residual (normalized by population and Harrod neutral technological progress) based on the new information received between t-1 and t and:

$$f_1 = \frac{1}{(1-\beta)} \left[g(1-s_K) + n(1-s_K-s_L)\right] + \frac{(1-\delta)+p^K}{(1+g)(1+n)} \frac{k}{p^Y y}(n+g)$$

Since $\frac{\Delta v_t}{v} \simeq \Delta \ln v_t$ (for $v_t > 0$ ($\sigma > 1$)), *equation* (A.33) *can be re-written, to a first order approximation, as:*

$$\begin{aligned}
\frac{v}{\lambda p^Y y} \Delta \log v_t &= E_t \sum_{s=0}^{\infty} \beta^s \Delta \log PR_{t+s} \\
&+ \sum_{s=0}^{\infty} \beta^s \left[E_t \log PR_{t+s} - E_{t-1} \log PR_{t+s}\right] \\
&+ \frac{(1-\delta) + p^K}{(1+g)(1+n)} \frac{k}{p^Y y} \Delta \log K_{t-1} - f_1
\end{aligned} \tag{A.34}$$

Using the fact that $\frac{v}{\lambda} = \frac{cp^c}{(1-\beta)(1-\sigma)}$, this result can be also expressed in terms of log-change of per capita intertemporal utility as following:

$$\begin{aligned}
\frac{v}{\lambda p^Y y} \Delta \log \frac{V_t}{N_t} &= E_t \sum_{s=0}^{\infty} \beta^s \Delta \log PR_{t+s} \\
&+ \sum_{s=0}^{\infty} \beta^s \left[E_t \log PR_{t+s} - E_{t-1} \log PR_{t+s}\right] \\
&+ \frac{(1-\delta) + p^K}{(1+g)(1+n)} \frac{k}{p^Y y} \Delta \log K_{t-1} - f_1 \\
&+ \frac{s_C}{(1-\beta)} g
\end{aligned} \tag{A.35}$$

For $\sigma > 1$ so that $v_t < 0$ the R.H.S. of (A.35) should equal, instead, $\frac{v}{\lambda p^Y y}\Delta \ln\left(-\frac{V_t}{N_t}\right)$

Alternatively, we can measure changes in welfare in terms of changes in equivalent per-capita consumption. Taking the log-change over time of equation (A.24), we have:

$$\Delta \log C_t^* = \frac{1-\beta}{s_C}\frac{v}{\lambda p^Y y}\Delta \ln \frac{V_t}{N_t} \tag{A.36}$$

where, again, we used the fact that: $\frac{v}{\lambda} = \frac{cp^c}{(1-\beta)(1-\sigma)}$. Using (A.35), we obtain:

$$\begin{aligned}
\Delta \log C_t^* \quad = \quad & \frac{1-\beta}{s_C} E_t \sum_{s=0}^{\infty} \beta^s \Delta \log PR_{t+s} \\
& + \frac{1-\beta}{s_C} \sum_{s=0}^{\infty} \beta^s \left[E_t \log PR_{t+s} - E_{t-1} \log PR_{t+s}\right] \\
& + \frac{1-\beta}{s_C} \frac{(1-\delta)+p^K}{(1+g)(1+n)} \frac{k}{p^Y y} \Delta \log K_{t-1} - \frac{1-\beta}{s_C} f_1 + g
\end{aligned} \tag{A.37}$$

Table 1: VAR - Log Productivity

Dependent Variable: $\log PR_t$								
	(USA)	(JPN)	(ITA)	(GBR)	(ESP)	(FRA)	(CAN)	(GER)
$\log PR_{t-1}$	0.6577	1.4331	0.9559	1.0510	1.1892	0.7892	0.5738	0.6898
	(0.1632)	(0.1850)	(0.1330)	(0.1916)	(0.1879)	(0.1335)	(0.1826)	(1.9230)
$\log PR_{t-2}$		-0.5952		-0.4144	-0.4352			
		(0.1902)		(0.1702)	(0.1853)			
LM1(Prob>chi2)	0.0801	0.9256	0.4272	0.2697	0.3380	0.0922	0.2601	0.8480

Notes: Time period: 1985-2005 (for Germany: 1995-2005).

Table 2: Change in per-capita log permanent consumption

	$\text{dlog}\left(\frac{C_t}{N_t}\right)^*$					
	No government	Wasteful spending, lump-sum taxes	Optimal spending, lump-sum taxes	No spending, distortionary taxes	Wasfeful spending, distortionary taxes	Optimal spending, distortionary taxes
USA	0.0218	0.0132	0.0171	0.0296	0.0225	0.0264
GBR	0.0234	0.0133	0.0185	0.0295	0.0211	0.0261
ESP	0.0108	-0.0007	0.0096	0.0242	0.0158	0.0260
FRA	0.0149	0.0040	0.0104	0.0225	0.0137	0.0198
CAN	0.0114	-0.0028	0.0031	0.0239	0.0129	0.0190
JPN	0.0062	-0.0006	0.0072	0.0128	0.0076	0.0151
ITA	0.0098	0.0013	0.0036	0.0170	0.0104	0.0130
GER	0.0003	-0.0065	-0.0045	0.0033	-0.0030	-0.0007

Notes: Beta=0.96 for all countries. Time period: 1985-2005 (for Germany: 1995-2005).

Table 3: Components of per-capita welfare change

	$\frac{s_C}{(1-\beta)} E_t \sum_{s=0}^{\infty} \beta^s \Delta \log pr_{t+s}$	ER	$\frac{s_C}{(1-\beta)} \quad \frac{(1-\delta)+p^K}{(1+g)(1+n)} \frac{k}{p^y y} \Delta K_{t-1}$	Constant	dlog $\left(\frac{C_t}{N_t}\right)^*$
No government					
USA	0.802	0.003	0.215	-0.020	0.0218
GBR	0.889	0.016	0.188	-0.092	0.0234
ESP	0.562	0.031	0.775	-0.369	0.0108
FRA	1.138	-0.005	0.289	-0.422	0.0149
CAN	0.504	0.011	0.524	-0.039	0.0114
JPN	0.383	0.087	1.187	-0.657	0.0062
ITA	0.918	0.129	0.463	-0.511	0.0098
GER	-0.286	0.052	13.349	-12.116	0.0003
Optimal spending and distortionary taxation					
USA	0.975	0.004	0.206	-0.184	0.0264
GBR	1.065	0.009	0.208	-0.281	0.0261
ESP	0.904	0.012	0.376	-0.291	0.0260
FRA	1.321	0.002	0.276	-0.599	0.0198
CAN	0.964	0.009	0.387	-0.360	0.0190
JPN	1.005	0.053	0.555	-0.612	0.0151
ITA	1.190	0.088	0.423	-0.701	0.0130
GER	-2.938	0.031	-7.722	11.630	-0.0007

Notes: Beta=0.96 for all countries. ER is the expectation revision term. Time period: 1985-2005 (for Germany: 1995-2005).

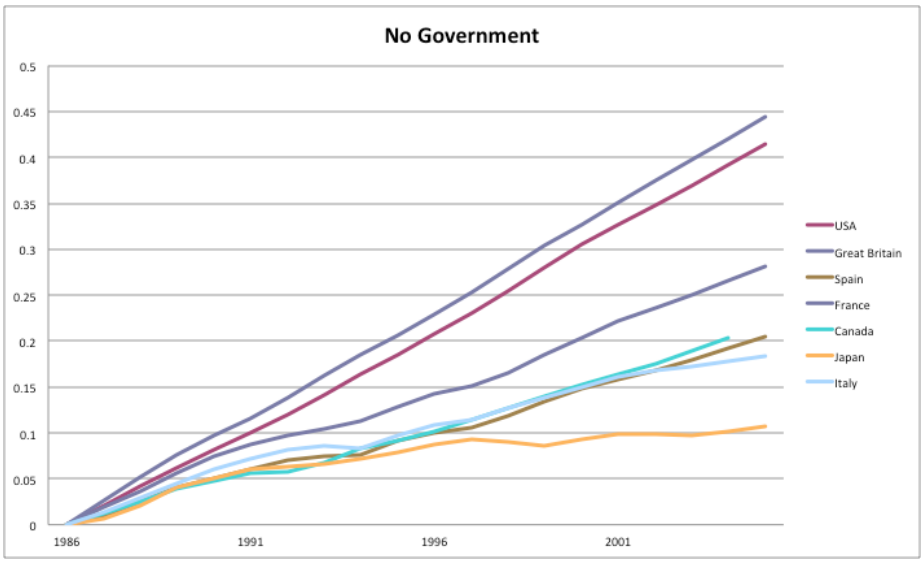

Figure 1: Within-country Welfare Comparison

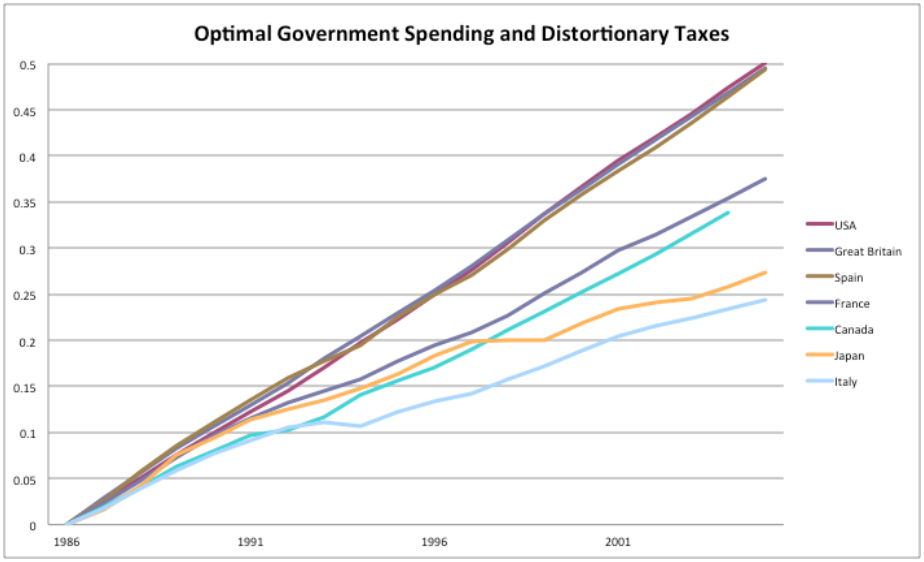

Figure 2: Within-country Welfare Comparison

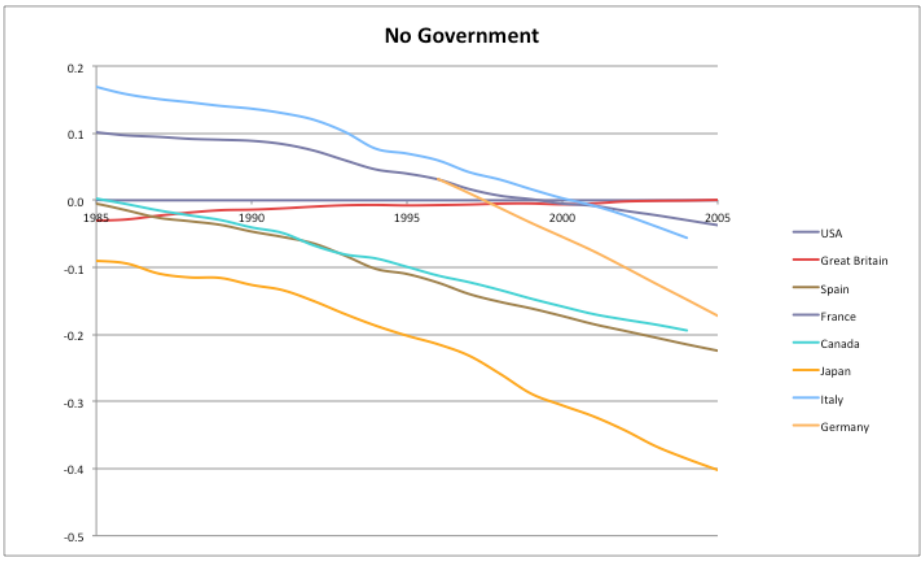

Figure 3: Cross-country Welfare Comparison

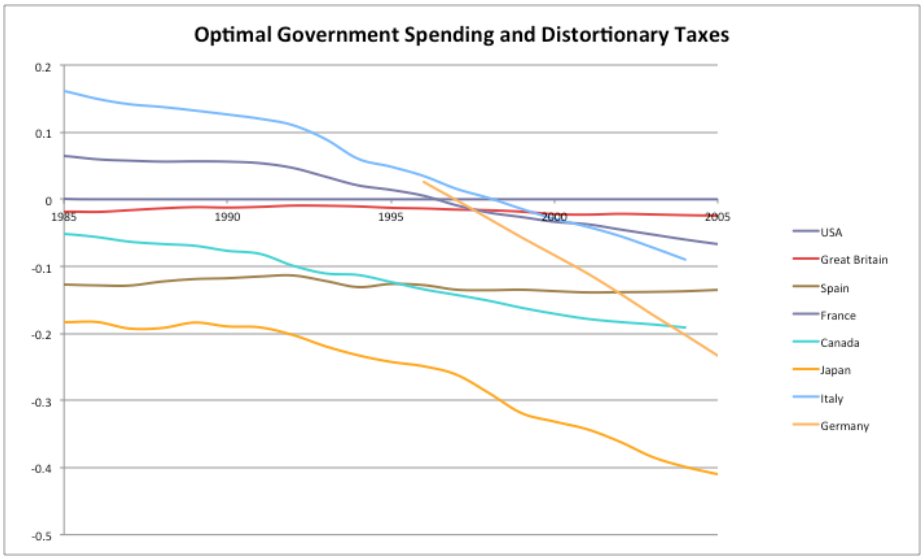

Figure 4: Cross-country Welfare Comparison

Table B.1: Constants

Shares and Tax Rates

	$\mathbf{s}_c$	$\mathbf{s}_I$	s_{cG}	$\mathbf{s}_K$	$\mathbf{s}_L$	$\boldsymbol{\tau}^L$	$\boldsymbol{\tau}^K$
USA	.6499899	.188674	.1618266	.3494443	.6505557	.2340667	.334381
GBR	.6224762	.1756063	.2018266	.2781275	.7218725	.2438476	.2957333
ESP	.6009379	.2462851	.1522132	.3607018	.6392982	.3288714	.2075571
FRA	.5666239	.2140578	.21937	.2964148	.7035852	.433081	.2642905
CAN	.5681292	.2183389	.2134744	.3947859	.6052116	.29076	.387615
JPN	.5477534	.3026225	.1494968	.3838601	.6161399	.2543619	.3827476
ITA	.5897042	.2252629	.1848636	.2960806	.7039194	.4137095	.2851429
GER	.5880985	.2230822	.196314	.3069767	.6930233	.3840909	.1957364

Notes: Beta=0.96 for all countries. Time period: 1985-2005 (for Germany: 1995-2005)

Table B.2: Growth Rates

Growth Rates

	$\Delta \log I_t$	$\Delta \log C_t$	$\Delta \log G_t$	$\Delta \log Y_t$ (NG)	$\Delta \log Y_t$ (OS)	$\Delta \log N_t L_t$	$\Delta \log K_{t-1}$
USA	.0372949	.0337453	.0191593	.0345639	.0320712	.0156969	.0324886
GBR	.0366512	.0331362	.0150795	.0339066	.0301061	.0110708	.0303731
ESP	.0544522	.0324773	.0431055	.0388377	.0394889	.0284803	.035191
FRA	.0304011	.0219307	.0193742	.0242497	.0231842	.0083518	.0157778
CAN	.0415289	.0287682	.0187347	.0323071	.0294108	.0194258	.0411084
FRA	.0304011	.0219307	.0193742	.0242497	.0231842	.0083518	.0157778
JPN	.0190816	.0220017	.0279597	.0209624	.0220059	.0017071	.0384726
ITA	.0237829	.0193079	.0144735	.0205392	.0194189	.0070979	.0208868

Notes: Beta=0.96 for all countries. Time period: 1985-2005 (for Germany: 1995-2005).