The Early History of the Box Diagram

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Economists hail it as "a powerful tool," "a work of genius," and "one of the most ingenious geometrical constructions ever devised in economics." It graces the pages of countless textbooks on price theory, welfare economics, and international trade. It is associated with some of the greatest advances ever made in economic theory. It elegantly depicts the two fundamental welfare theorems that are absolutely central to modern economics. In short, it ranks with the preeminent schematic devices of economics since it illuminates the most important ideas economists have to offer. It is none other than the celebrated box diagram used to illustrate efficiency in exchange and resource allocation in hypothetical two-agent, two-good, two-factor models of general economic equilibrium.

The box comes in two variants. The exchange version has dimensions determined by total available stocks of the two goods (see Figure 1). It incorporates traders' indifference maps, one with origin sited in the southwest corner and the other in the northeast corner. The box depicts opportunities for mutually beneficial trade. Thus a movement from initial endowment point *E* to point *Z* on the contract curve—a movement accomplished through a trade of *ER* units of the second good for *RZ* units of the first—benefits both traders simultaneously by putting them on higher indifference curves. In general, so long as the straight trading line *EZ*, whose slope measures the price of the first good in terms of the second, cuts the indifference curves of both parties at point *E*, it pays each to move along that line to the contract curve. Once on the contract curve, however, the potential for further mutually advantageous trades is at an end. Since the contract curve is the locus of indifference-curve tangency points, it follows that movements along the contract curve improve the welfare of one trader only by reducing that of the other.

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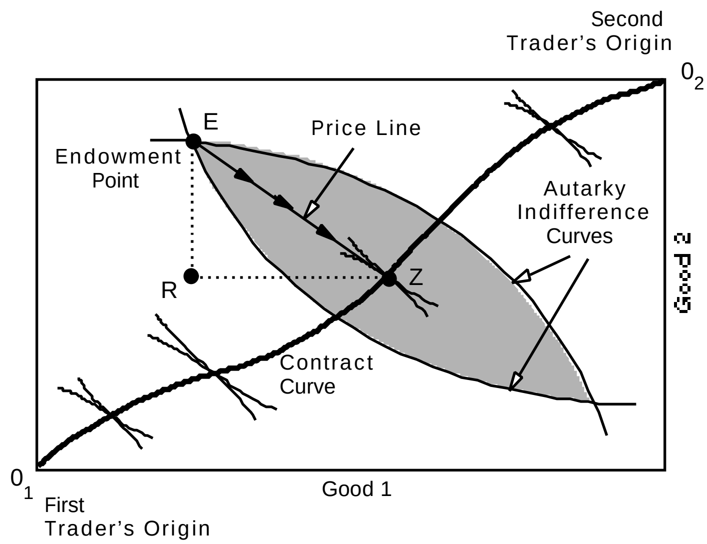
Figure 1 Exchange Box Diagram



Both traders prefer allocations in the shaded area to initial endowment *E*. Movement down the price vector connecting endowment point *E* to efficiency point *Z* on the contract curve puts both traders on higher indifference curves and thus makes them better off. At point *Z*, however, all potential mutually beneficial trades are at an end. Movements along the contract curve benefit one trader only at the cost of hurting the other.

The alternative production variant of the box depicts the fabrication of two goods from two factor inputs. It replaces indifference maps representing preference functions with isoquant maps representing production functions. It lets available factor quantities determine the dimensions of the box. Efficient factor allocations occur along the contract-curve efficiency locus. There, isoquants are tangent to each other such that the output of one good is maximized given the output of the other.

The chief appeal of the box diagram is its ability to explain much with little. A simple plane diagram, the box can, in Kelvin Lancaster's words, "show the interrelationships between no less than twelve economic variables" ([1957] 1969, p. 52). Moreover, it can do so without resort to algebra and calculus, techniques inaccessible to the mathematically untrained. Small wonder that economists extol the analytical and pedagogical properties of the box or that textbooks feature it as an expository device.

Where some textbooks go astray, however, is in their ahistoric presentation of the diagram. Typically, they say little or nothing about its origins and evolution. They simply present it as an accomplished fact without inquiring into its genealogy. A leading international trade theory textbook authored by Richard Caves and Ronald Jones (1981) provides a prime example. It attributes the box to no progenitor, not even to Francis Edgeworth or Arthur Bowley. The result is that the student is unaware of the circumstances prompting the diagram's development. He knows not who invented it, why it was invented, what problems it originally was designed to solve, or how it evolved under the impact of attempts to perfect it and extend its range of application. Nor can he appreciate the intellectual effort involved in its creation and refinement. Unaware of such matters, he may surmise that the diagram sprang fully developed from the brain of the latest theorist. Ahistoric textbooks indeed foster that very impression. Such are the hazards of disassociating an idea from its historical context and presenting it as a timeless truth.

Far from being timeless, the box diagram possesses a definite chronology. That chronology features some of the leading names in neoclassical and modern economics. Francis Edgeworth, Vilfredo Pareto, A.W. Bowley, Tibor Scitovsky, Wassily Leontief, Kenneth Arrow, Abba Lerner, Wolfgang Stolper, Paul Samuelson, T. M. Rybczynski, and Kelvin Lancaster all contributed to the diagram's development.

Edgeworth invented the exchange box in 1881. He used it to demonstrate the indeterminacy of isolated barter and the determinacy of competitive equilibrium. He showed that all final settlements are on the contract curve, that the competitive equilibrium is one such settlement, and that the contract curve shrinks to the competitive equilibrium as the number of traders increases. Pareto in 1906 demonstrated his celebrated optimality criterion with the aid of the box. Bowley in 1924 generalized Edgeworth's work with his notion of the bargaining locus. Scitovsky in 1941 employed the box to formulate his famous double-bribe test of increased efficiency. Leontief coordinated, consolidated, and clarified the earlier accomplishments in his 1946 rehabilitation of the exchange box. In so doing, he paved the way for the post-war popularity of the diagram. Following hard on Leontief's heels, Arrow in 1951 employed the concepts of convex sets and supporting hyperplanes to analyze the problem of corner solutions on the boundary of the box. And Samuelson in 1952 employed the box to investigate how international transfers affect the terms of trade.

When the foregoing contributions threatened to exhaust the analytical potential of the exchange box, economists turned to the alternative production version. Already, Lerner had presented the first production box in a pioneering 1933 paper whose publication unfortunately was delayed for nineteen years. In the meantime, Stolper and Samuelson published the first production box diagram to appear in print. As employed by them in 1941, by Rybczynski in 1954, and by Lancaster in 1957, the production diagram proved indispensable to the derivation and illumination of certain core propositions of the emerging Heckscher-Ohlin theory of international trade.

The paragraphs below attempt to trace this evolution and to identify specific contributions to it. Besides unearthing lost or forgotten insights, such an exercise may serve as a partial antidote to the textbooks' ahistorical treatment of the diagram. One conclusion emerges: namely that the box hardly developed autonomously. Rather it evolved in a two-way interaction with its applications. Thus an unsolved puzzle in microeconomics prompted the invention of the box—a prime example of a seemingly intractable problem inducing the very tool required for its solution. The resulting availability of the diagram then spurred economists to find new applications for it. These new uses in turn triggered modifications of the diagram. Applications were both cause and effect of the diagram's development.

1. FRANCIS Y. EDGEWORTH

The box diagram makes its first appearance on pages 28 and 113 of Francis Edgeworth's 1881 *Mathematical Psychics*. Motivated by a problem in microeconomic theory, Edgeworth invented the diagram and its constituent indifference-map and contract-curve components to solve the problem.

Edgeworth's predecessors had long known that equilibrium price in isolated, two-party exchange is indeterminate. They also understood that equilibrium between numerous buyers and sellers operating in competitive markets is determinate. But they had been unable to reconcile the two results. They could not show rigorously how increasing numbers lead to price determinacy.

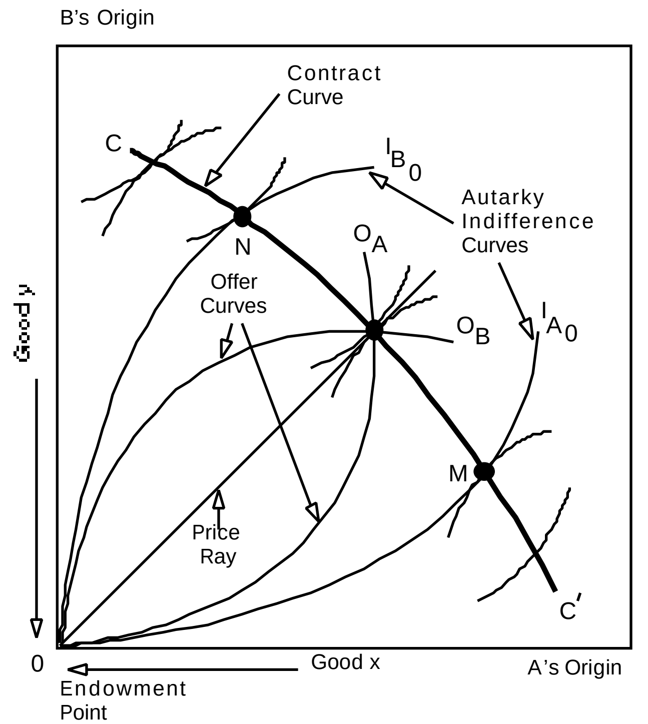
This task Edgeworth sought to accomplish. Using the box diagram, he established (1) that final outcomes must be on the contract curve, (2) that the contract curve shrinks as the number of competitors increases, (3) that competitive equilibrium is one point on the contract curve, and therefore (4) that as the number of competitors increases without limit the contract curve shrinks to a single point, namely the competitive equilibrium.[1] Here was his rationale for inventing the diagram.

Edgeworth's Invention and its Components

Edgeworth's diagram depicts two isolated individuals, *A* and *B*, trading fixed stocks of two goods, *x* and *y*, whose quantities determine the dimensions of the box (see Figure 2). Individual *A* initially holds the entire stock of good *x* and individual *B* the entire stock of good *y*. Superimposing indifference maps on the box, Edgeworth sites the origin of *A*'s map in the lower right corner and the origin of *B*'s map in the upper left corner. This arrangement fixes point 0

[1] In the case of multiple competitive equilibria, the contract curve shrinks not to one but to several points. Edgeworth recognized such a possibility. But he tended to focus on the case of singular rather than multiple equilibrium. See Newman (1990, p. 261).

Figure 2 Edgeworth's Version of the Box



Edgeworth's original diagram depicts three components: (1) autarky indifference curves going through the endowment point and defining the cigar-shaped area of mutually advantageous trades; (2) offer curves or sets of points of tangency of indifference curves and the price ray as it swings about the endowment point; and (3) the contract curve, whose relevant segment lies between the autarky indifference curves.

in the lower left corner as the endowment point of both individuals. That is, the length of the lower horizontal axis measured from right to left indicates the amount of good *x* held by *A* just as the length of the left vertical axis measured from top to bottom indicates the amount of good *y* held by *B*. Since *A* holds no *y* nor *B* any *x*, these axes establish the endowment point.

From the indifference curves radiating outward from their respective origins, Edgeworth selects one particular curve for each trader, namely the curves passing through the endowment point. These curves show alternative combinations of goods that yield the same satisfaction as the endowment bundle. They indicate the level of utility each person would enjoy if he consumed his endowment bundle and refrained from exchange. They also trace out the zone of mutually beneficial exchanges that make both traders better off than they would be under autarky.

Next, Edgeworth draws in the contract curve CC' along which indifference curves are tangent such that one trader cannot occupy a higher indifference curve unless the other is forced to occupy a lower one. Especially significant is the portion of the contract curve bounded by the autarky indifference curves. Since traders require that potential exchanges make them at least as well off as they would be under autarky, they will never voluntarily agree to trades outside those bounds. It follows that the relevant segment of the contract curve lies in the lens-shaped area between the indifference curves going through the endowment point.

Finally, Edgeworth sketches traders' reciprocal demand schedules or offer curves. These curves apply to the special case where the two traders act as representative price-takers operating on opposite sides of a competitive market. Offer curves show how much each trader is willing to exchange at all possible prices. Edgeworth of course did not invent such curves. That honor goes to Alfred Marshall. But he was the first to derive them as the locus of points of tangency of indifference curves and the price ray as it pivots about the endowment point. He likewise was the first to explain that each point on an offer curve represents an outcome of constrained utility maximization in which the commodity price ratio, or slope of the price ray, equals the ratio of marginal utilities, or slope of the indifference curves.

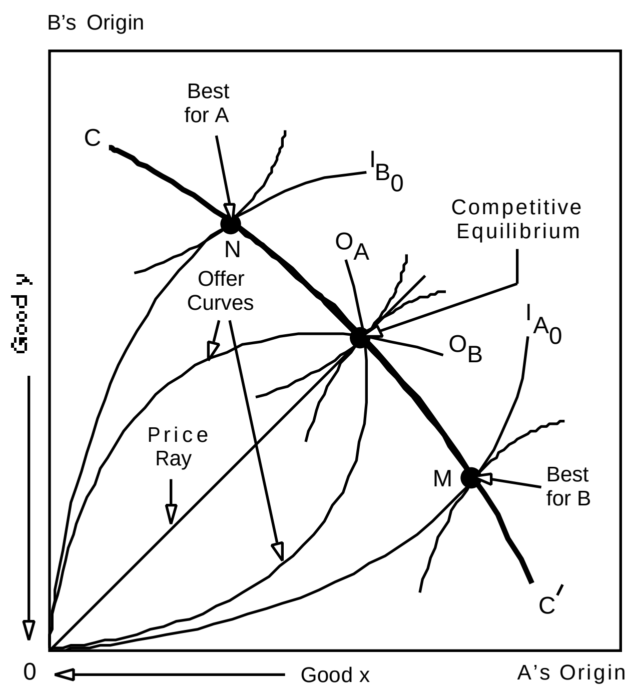
Exploiting Potential Mutual Gains from Exchange

Having derived the exchange box and its constituent components, Edgeworth employed it to illuminate five basic propositions. His first proposition states that final settlements must be on the contract curve. At any other point, both parties could make themselves better off by renegotiation. Consider any point lying off the contract curve. Going through that point are intersecting indifference curves enclosing a cigar-shaped area that spells unexploited potential mutual gains from exchange. Traders will not let such opportunities go unrealized. Instead, they will exploit them until they reach the contract curve where indifference curves are tangent and further mutual gains are at an end.

Efficiency of Competitive Equilibrium

Edgeworth's second proposition refers to the efficiency of competitive equilibrium. It states that such equilibrium is always on the contract curve. The reason? Competition establishes a common, market-clearing price ratio. Competitive price-takers independently respond to that ratio by trading at the point where each supplies the quantity the other demands and vice versa. That is, price-takers operate at the point where their offer curves intersect (see

Figure 3 Competitive Equilibrium versus the Indeterminacy of Two-Party Barter



Bargaining between two isolated traders can lead to an outcome anywhere on the segment of the contract curve between the autarky indifference curves. By contrast, the competitive equilibrium is uniquely determined at the intersection of the price ray and offer curves. There the traders' indifference curves are tangent to the common price ray and thus to each other.

Figure 3). At this point, indifference curves are tangent to the common price ray emanating from the endowment point and thus are tangent to each other. Since such tangencies occur only on the contract curve, it follows that competitive equilibrium is on that same curve.

Indeterminacy of Isolated Two-Party Barter

Edgeworth's third proposition refers to the indeterminacy of isolated two-party exchange. In such bilateral monopoly situations, it is impossible to determine, from indifference maps and endowments alone, the precise price-quantity equilibrium that will emerge. All one can say is that equilibrium must lie on the segment of the contract curve between the autarky indifference curves. But which one of the infinity of possible equilibria will prevail will depend upon considerations external to Edgeworth's model, namely the relative bargaining skills and strengths of the traders as well as the strategies and tactics they employ. Economists traditionally have had little to say about such matters.[2] They cannot confidently predict any unique outcome. The precise ingredients of shrewd, effective bargaining remain subtle, elusive, and obscure. Still, economists can note that the gain one bargainer gets from exchange is limited only by the other's effort to get the best for himself. Final settlement will be near point *N* if *A* is the superior bargainer. It will be nearer to point *M* if *B* has the bargaining advantage.

Neither outcome, Edgeworth noted, necessarily coincides with the point of maximum aggregate welfare on the contract curve. There the sum of the traders' satisfactions is at its peak. Identifying this unique maximum point of course requires that utility be cardinally measurable and comparable across individuals—properties Edgeworth thought utility possessed. It was on these grounds that he advanced his famous principle of arbitration. Compulsory arbitration, he argued, could do what unrestricted bargaining could not do. By imposing the utilitarian sum-of-satisfactions solution on the bilateral monopolists, arbitration would yield a determinate, socially optimum outcome. Conversely, in the absence of such arbitration indeterminacy would continue to characterize the isolated two-party case.

Recontracting and the Role of Numbers

Edgeworth's fourth proposition, his recontracting theorem, refers to the role of numbers in reducing indeterminacy. It states that as the number of traders gets large, the contract curve shrinks to a single point, the competitive equilibrium.

Edgeworth sketches a proof on pages 35–37 of his *Mathematical Psychics* (see Creedy [1992], pp. 158–65, for a particularly clear and insightful interpretation). He starts with the two-person case in which party *A* provisionally contracts with party *B* to reach point *C* on *A*'s indifference curve I_{A_0} (see Figure 4a). He then introduces a new pair of traders identical to the first pair. This

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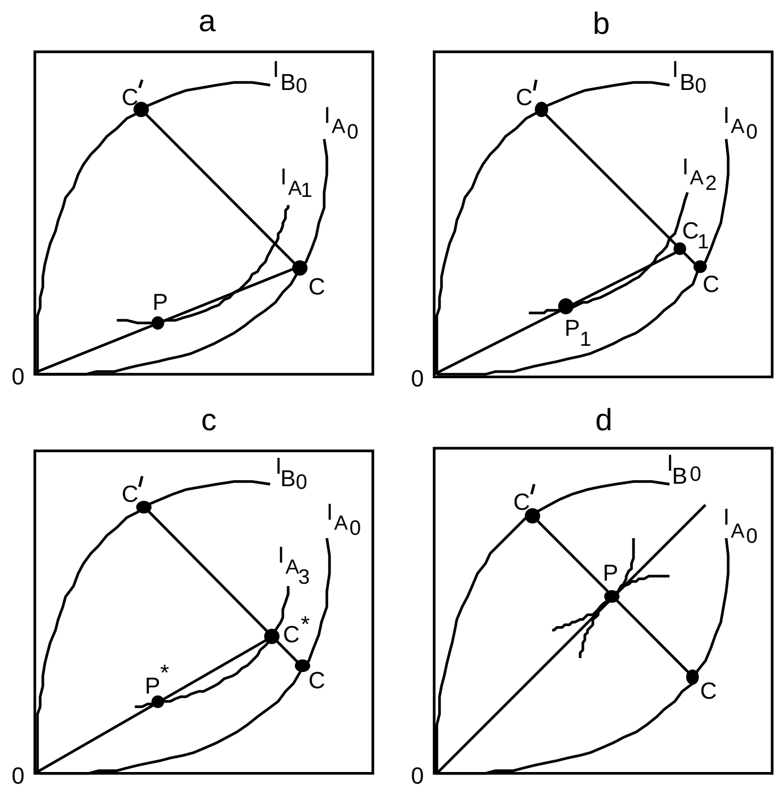
[2] They have said something, however. John Nash (1950) thought the bargainers might agree to maximize the multiplicative product of their respective utility gains from trade (the excess of post-trade over autarky levels of satisfaction). John Harsanyi (1956) showed that the Danish economist Frederick Zeuthen (1930) had proposed essentially the same solution two decades before Nash. Ariel Rubinstein (1982) showed that the Nash solution is the outcome of a non-cooperative, offer-counteroffer game. John Creedy (1992, pp. 193–99) suggested a variant of the Nash solution, namely the maximization of a geometrical weighted average of the traders' utility gains, with the weights measuring the relative bargaining powers of the two parties. These solutions establish unique potential agreement points on the contract curve. Since it is unlikely that the bargainers would always agree to go to such proposed points, however, indeterminacy remains.

Figure 4 Edgeworth's Recontracting Process



Recontracting plus convexity of indifference curves implies that the contract curve shrinks as traders become more numerous. With a single pair of traders, the contract curve is *CC*' as shown in panel a. Adding another pair shrinks the curve by the amount *CC** at both ends (panel c). When the number of pairs gets very large, the curve shrinks to the competitive equilibrium (panel d).

maneuver allows him to use the same box diagram to deal with four parties. It permits him to represent the preferences of each of the *A*s (and the *B*s) with a single indifference map.

It also means that the agreement reached at point *C* cannot be final. For the two *A*s can now ignore one of the *B*s and deal with the other at point *C*. When they split the resulting bundle equally among themselves, they will each reach the half-way point *P* on the trade vector 0*C*. That point, because of the convexity of indifference curves, is on a higher such curve than before. Thus the *A*s are better off, their trading partner *B* is just as well off as initially, and the excluded *B* is at his endowment point.

In retaliation, the excluded B then underbids his competitor by offering the As a trade on better terms at point C_1 (see Figure 4b). The As, by accepting, can share the resulting bundle among themselves to each attain point P_1 on a still higher indifference curve than before. Repeated recontracting brings the parties to point C^* (see Figure 4c). There the As are indifferent between (1) trading with both Bs at C^* and (2) dealing with just one B at C^* and splitting the resulting bundle at point P^*. Either option puts them on the same indifference curve.

There being no advantage to choosing option 2 over option 1, the As will trade with the two Bs at point C^*. The result is that recontracting, which began at point C, ends at point C^*. Adding a trader to each side of the market shrinks the contract curve by the amount CC^*. The same logic of course applies to point C' at the other end of the contract curve. Recontracting initiated there shrinks the contract curve inward as each of the As continually underbids the other to attract the business of the Bs. In other words, the contract curve shrinks at both ends.

Although two pairs of traders shrink the range of indeterminacy, they hardly eliminate it. To reduce it further, Edgeworth adds a third pair. Doing so gives room for two Bs to underbid the third for the patronage of the As. Dealing with the two Bs, the three As each can reach a point P two-thirds the distance from the origin to any point C on the contract curve. Final settlement occurs when point C shrinks inward sufficiently to lie on the same A-indifference curve as point P. The same reasoning holds for the other end of the contract curve, which of course shrinks too.

Let the number of pairs of traders N grow without limit. Then point P, which according to Edgeworth is $(N - 1)/N$ times the distance from the origin to point C, converges on that latter point. Expressed geometrically, final settlement in the large-numbers case occurs where an A-indifference curve is tangent to a ray from the origin (see Figure 4d). The same holds true for an indifference curve of the Bs. The result is that both indifference curves are tangent to the same ray and thus to each other just as in the competitive equilibrium. Large numbers shrink the contract curve to the point of competitive equilibrium.

Monopoly Pricing—An Exception to Edgeworth's Rule?

Finally, Edgeworth considered a case that apparently violated his postulate that final settlements lie on the contract curve. That case has two bargainers agreeing on price but making no agreement on the quantities to be traded. An extreme example confronts a representative competitive price-taker with a monopolistic price-maker. The monopolist is of the simple, or non-price-discriminating, variety. He sets a single price for all units exchanged and leaves the competitor free to determine how much he (the competitor) wants to trade at that price along his offer curve.

Let A be the representative competitor and B the monopolist. If B's monopoly power is absolute, he will set the single price that puts him on his highest attainable indifference curve given *A*'s offer curve (see Figure 5). That is, he chooses the price that takes him to point *Q*, where his indifference curve just touches *A*'s offer curve. Of course, if his monopoly power is less than absolute, his fear of losing *A*'s patronage to potential rival traders may induce him to charge the slightly lower price shown by the slope of ray 0*q*. In any case, the result is that trade takes place at a point like *Q* (or *q*) on *A*'s offer curve rather than on the contract curve. Here is an apparent exception to the rule that final settlements tend to be efficient.

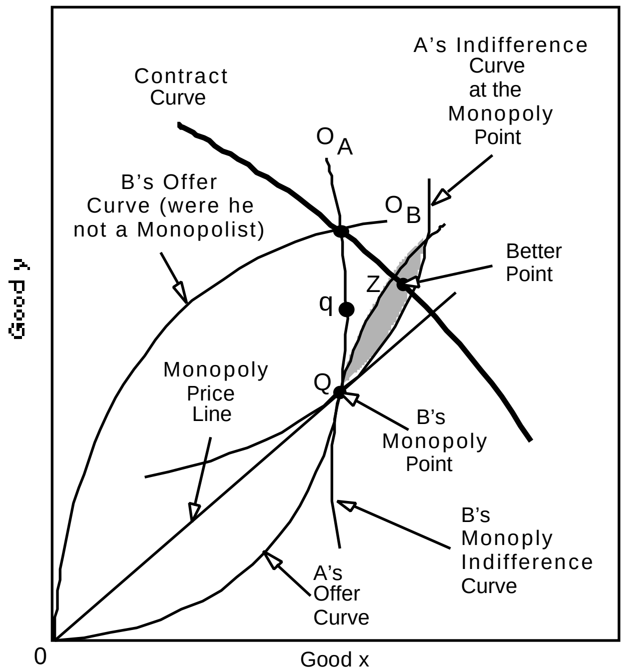
Edgeworth was quick to point out, however, that the exception stems from the assumption that the parties contract over price alone. Were they to contract over quantity as well, they both could move advantageously to the contract curve. Thus Edgeworth questioned the validity of the assumption. To him, rational behavior required that parties bargain over both price and quantity dimensions of a deal, especially when it was to their mutual advantage to do so.

In illustration, Edgeworth referred again to monopoly point *Q* reached through a price-only contract. From that point, superior outcomes are possible in the sense that both parties can move to higher indifference curves than those crossing through *Q*. Edgeworth realized, however, that such improved positions would never be attained by new price settings alone. For, given that the monopolist is constrained by the competitor's offer curve, any change in the slope of the price line 0*Q* would make him (the monopolist) worse off than he is at *Q* and for that reason would be resisted. But mutually beneficial positions could be reached if the competitor somehow could be induced to leave his offer curve. Such an inducement could take the form of a new contract specifying quantity as well as price.

For example, monopolist *B* might dictate terms corresponding to point *Z*, thus improving his own welfare. He would lower the price against himself in exchange for a more-than-compensating rise in quantity traded. And he would do so confident that *A* would gladly agree to the larger trade volume in return for the guarantee of a lower price. In other words, *A* would concur with any price-quantity package moving him to an indifference curve higher than the one he would otherwise occupy at point *Q*. And if such a negotiated package fell short of the contract curve, the parties could renegotiate other packages until they finally arrived there.[3]

[3] Tibor Scitovsky, in his classic 1942 article "A Reconsideration of the Theory of Tariffs," showed that the parties could reach point *Z* by an alternative route. Competitor *A* could bribe monopolist *B* to act as a competitor operating on his own offer curve. The bribe, paid in *A*'s own good, would result in a rightward shift of the endowment point and its attendant offer curves by the amount of the payment. So shifted, the offer curves would intersect at point *Z*. The monopolist would gain from the bribe and the price-taker would gain from the lower, competitive price. Edgeworth, however, said nothing of this scheme.

Figure 5 Monopoly Outcomes



Monopolist *B* sets the price that puts him on his highest attainable indifference curve given the offer curve of the competitive price-taker *A*. The monopolist goes to tangency point *Q*. There, however, *A*'s indifference curve is tangent to the price ray and not to *B*'s indifference curve. The resulting intersecting indifference curves create a lens-shaped area of unexploited mutually beneficial exchanges. If the parties could agree to let monopolist *B* set both price and quantity, they could move to efficient point *Z* where both are better off than at point *Q*.

In short, Edgeworth showed that agreements fixing both price and quantity inevitably lead to the contract curve. By contrast, agreements limited to establishing price alone may, under certain circumstances, lead only to the offer curve. But he insisted that rational agents have an incentive to choose the former agreements over the latter. Thus all final settlements tend to be on the contract curve.

Appraisal

Edgeworth's contribution must be judged one of the greatest virtuoso performances in the history of economics. Going beyond the mere creation of the box diagram itself, he invented its principal components, the indifference map and the contract curve. True, he did not invent offer curves. But he did give the earliest demonstration of their derivation from the underlying indifference contours and price ray. Moreover, in showing that offer curves intersect at the contract curve, he was the first to use them to demonstrate the efficiency of competitive equilibrium.

Edgeworth's work is remarkable in another respect. His five propositions essentially point the way to all of modern economics. One finds in them both a treatment of competitive equilibrium and its efficiency in exhausting the gains from trade and a framing of the problems that arise when perfect competition ceases to prevail. These problems arguably constitute the fundamental motivation for the development of game theory.

Indeed, Edgeworth himself contributed to this development by anticipating key game-theoretic ideas. He demonstrated that final allocations must lie on the segment of the contract curve spanning the indifference curves going through the endowment point. In so doing, he identified what game theorists some seventy-five years later were to call the *core* of the economy. And, in illustrating that the contract curve shrinks to a single point, he showed how the core behaves as its agents increase in number. Finally, his recontracting theory foreshadowed the game-theoretic notion that no coalition of traders can block the emergence of competitive equilibrium. Mark Blaug (1986, p. 70) said it all when he described Edgeworth's theory of the core as "his most beautiful contribution."

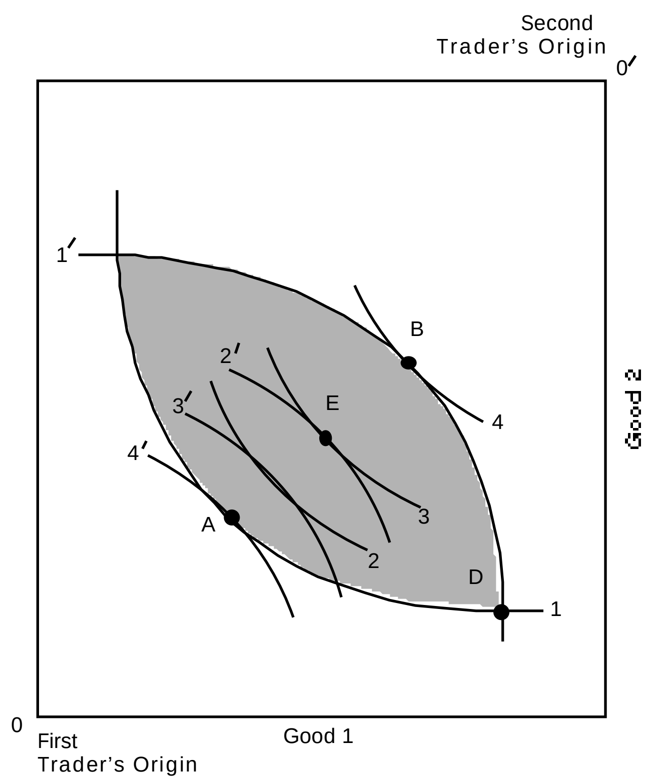
2. VILFREDO PARETO

The next to present the box diagram was Vilfredo Pareto, who did so in his 1906 *Manuale d'economie politica*. Pareto's work obviously owes much to Edgeworth. Indeed, commentators including Maffeo Pantaleoni (1923, p. 584) and John Creedy (1980, p. 272) have stressed that very point. But Pareto also modified Edgeworth's work in at least two key respects.

For one thing, he presented the box in its now-conventional form. That is, he located the origins of the indifference maps in the southwest and northeast corners, respectively, rather than in the other two corners as Edgeworth had done (see Figure 6). The result was that the succession of indifference-curve tangency points—Pareto did not draw the efficiency locus—sloped upward from left to right rather than downward as in Edgeworth's version.

Second and more important was Pareto's interpretation of the welfare implications of the box. Unlike Edgeworth, who believed that interpersonal comparisons of utility make it possible in principle to identify a unique point of maximum aggregate welfare on the contract curve, Pareto denied that such comparisons could be made and indeed refused to make them.

Figure 6 Pareto's Diagram



Movements from point *D* to any point within the lens-shaped area are unambiguously welfare-improving since both parties gain and neither lose. But movements across tangency points *A*, *E*, and *B* are welfare-ambiguous and defy comparison since one party gains while the other loses.

Pareto Optimality

Accordingly, he held that only outcomes involving gains for some and losses for none are unambiguously welfare-improving just as outcomes involving gains for none and losses for some are unambiguously welfare-decreasing. By contrast, outcomes involving gains for some and losses for others are ambiguous. They cannot be judged in terms of quantitative utility comparisons. The inadmissibility of interpersonal comparisons of utility (or "ophelimity" as Pareto termed the utility concept) foils their evaluation.

It follows that movements from points like *D*, where indifference curves cross, to points like *A*, *E*, and *B*, where the curves are tangent, constitute

Pareto-superior moves. They put at least one party on a higher indifference curve and none on a lower one. But movements across successive tangency points like *A*, *E*, and *B*, involving as they do higher curves for one person and lower curves for the other, defy comparison. An infinity of such Pareto-optimal points exists, none of which can be judged superior to the others.

In short, there is no single point of maximum welfare, Edgeworth's claim to the contrary notwithstanding. All one can say is that points off the tangency locus are economically inefficient since everyone could gain by moving to a point at which no mutually advantageous reallocations are possible. Likewise, points on the locus are economically efficient in the sense that no reallocation could improve the position of both parties. Edgeworth's notion of a unique welfare optimum gave way to Pareto's notion of an infinity of noncomparable optima.

3. ARTHUR W. BOWLEY

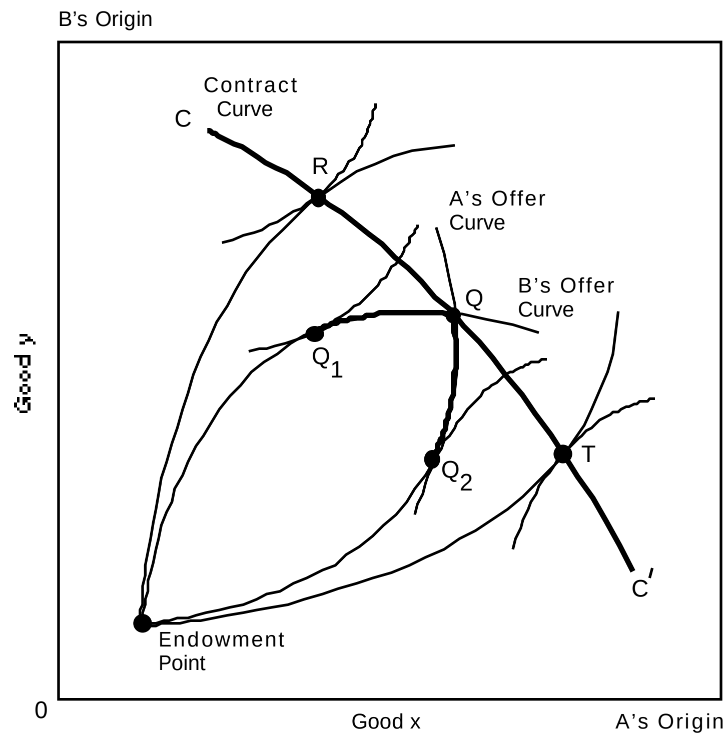
After Pareto's *Manuale*, fully eighteen years elapsed before the box diagram made its next appearance in A.W. Bowley's famous 1924 *Mathematical Groundwork of Economics*. Inspired by Edgeworth and Pareto, Bowley generalized and extended their work in three ways. First, he replaced their assumption that each hypothetical trader initially holds the entire stock of one good and none of the other. He replaced it with the alternative assumption that each trader initially holds some of both goods. The result was to fix the endowment point in the interior of the box rather that at one of its corners (see Figure 7). Bowley's innovation is conventional practice today.

Bargaining Locus

Second, he supplemented Edgeworth's analysis of bilateral monopoly with his concept of the bargaining locus. In defining that locus, which consists of the offer-curve segments Q_1QQ_2, Bowley argued as follows. If the two parties contract over price alone, equilibrium may well be on the offer curves rather than on the contract curve. The party possessing the superior bargaining power will set the price and leave the other free to determine the trade volume at that price along his offer curve. Accordingly, the outcome will be somewhere on the price-taker's offer curve.

Suppose *B* is the price-maker whose bargaining superiority is absolute. He will set the price to reach point Q_2 where his highest attainable indifference curve just touches *A*'s offer curve. But if his bargaining superiority is somewhat weakened by the countervailing bargaining skills of *A*, he will be forced to shade his price downward and occupy a position on *A*'s offer curve in the direction of competitive point *Q*. These considerations trace out the lower Q_2Q segment of the bargaining locus.

Figure 7 Bowley's Version of the Box



Bargaining between price-making, price-taking traders establishes the curve Q_1QQ_2 as the locus of final outcomes. Final settlement occurs on the upper or lower segment depending upon whether *A* or *B* is the dominant bargainer. If both parties possess equal and offsetting monopoly power, final settlement occurs at *Q*.

Similarly, if *A* is the price-maker, trade will occur at the point of intersection of the price ray he sets and *B*'s offer curve. Trader *A* will aim at reaching point Q_1, where the offer curve is tangent to his highest attainable indifference curve. But if *A*'s bargaining power is less than absolute, he may be forced to lower the price against himself and thus move to a point on *B*'s offer curve to the right of point Q_1. These considerations establish the upper Q_1Q segment of the bargaining locus.

The upshot is that if either one trader or the other sets the price, trade occurs at some point on the combined upper and lower segments of the offer curves between points Q_1 and Q_2.[4] With the single exception of point Q, where equal and offsetting bargaining power yields the competitive equilibrium, all these points are off the contract curve. Thus Bowley confirms Edgeworth's contention that when price-maker confronts price-taker over price alone the outcome is rarely efficient.

Trading at Disequilibrium Prices

Finally, Bowley advanced an alternative to Edgeworth's treatment of how the economy converges to its core. As mentioned above, Edgeworth, in considering such convergence, ruled out trading at disequilibrium prices. For him, contracts become binding and exchanges occur only at final equilibrium prices corresponding to points on the contract curve. Disequilibrium contracts he treated as tentative, provisional, non-binding, and subject to revision until the equilibrium contract emerged.

By contrast, Bowley permitted exchanges to take place at disequilibrium prices. He envisioned traders moving across a succession of intermediate positions in the lens-shaped area enclosed by indifference curves emanating from the endowment point. From each such intermediate trading position, they would move to a subsequent, Pareto-improving one changing the price as they went. They would continue in this fashion until they reached the core. The resulting path to equilibrium is described by a broken, or segmented, price line and final settlement can occur anywhere on the section *RT* of the contract curve.

For all its apparent realism, however, Bowley's analysis comes at a high cost. It greatly complicates the diagram. Each disequilibrium trade means a new allocation of goods such that the endowment point shifts continually. Since offer curves emanate from such endowment points, a new set of offer curves has to be drawn at each stage of the process. The result is to clutter the diagram unduly. For this reason, Edgeworth's simplification seems superior pedagogically to Bowley's treatment.

4. TIBOR SCITOVSKY

In the twenty-two years following the publication of Bowley's *Mathematical Groundwork*, the exchange box virtually disappeared from the literature. It surfaced briefly in 1941 when Tibor Scitovsky employed it to expose a flaw

[4] This result holds even when both bargainers, after agreeing on a noncompetitive price, treat it as given and act as price-takers operating on their respective offer curves. In this special case, the price ray will cut the respective offer curves at different points. One party, in other words, will wish to trade a larger quantity at the bargained price than will the other. Here, the smaller quantity will be the one actually traded. The outcome will be exactly the same as if one party unilaterally set the price (see Scitovsky 1951, p. 418).

in compensation tests of increased efficiency. Nicholas Kaldor and John R. Hicks had proposed such tests to circumvent Pareto's prohibition banning the evaluation of changes favoring some people while hurting others. Applied to such situations, the compensation test was supposed to reveal whether a change from one non-optimal state to another was, on balance, welfare-improving if some gained and some lost. The change was said to pass the test if the gainers could fully compensate the losers and still be better off.

But Scitovsky noted a paradox. The test might reveal both states to be superior to each other. Observe a change-induced reallocation from goods-bundle A to bundle B (see Figure 8). Let points A and A' have the same vertical height with the same being true of points B and B'. Then compensation can be represented as a quantity of the horizontally measured good alone. Gainer J (whose indifference map originates at the lower left) could fully compensate loser I (whose indifference map originates in the upper right) by an amount $B'B$ and still be better off. The hypothetical transfer would leave him occupying a higher indifference curve than the one going through his initial position A. Similarly, in a reverse transition from B to A, individual I could bribe individual J by an amount AA' and still be better off than at B. The test would reveal allocation B as preferred to allocation A. Once at B, however, the same test would reveal A as the superior allocation.

Double-Bribe Criterion

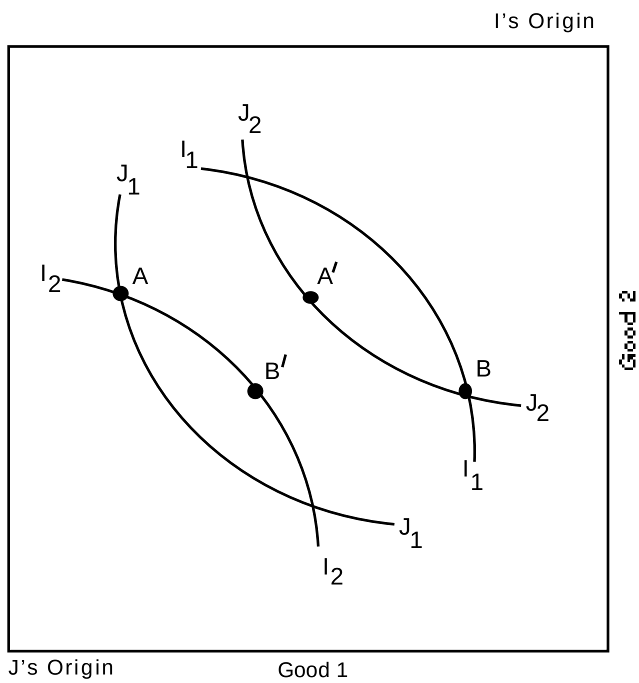
To avoid such contradictions, Scitovsky proposed a double test. Situation B is preferred to situation A if the gainers from the change can profitably compensate the losers, or bribe them to accept it, while the potential losers cannot profitably bribe the gainers to oppose the change.

Scitovsky's double-bribe criterion impressed economists far more than did the box diagram he used to exposit it. For that reason, his paper served merely to interrupt rather than to halt the diagram's pre-World War II lapse into obscurity. That lapse persisted for five more years.

5. WASSILY LEONTIEF

Then came Wassily Leontief's 1946 *Journal of Political Economy* article on "The Pure Theory of the Guaranteed Annual Wage Contract." Employing perhaps the most elaborate version of the exchange box to be found in the scholarly literature of the time, Leontief summarized, consolidated, and clarified all earlier work. He spelled out such notions as the lens-shaped zone of mutually advantageous trades, the contract curve, offer curves, the competitive and simple monopoly (price-maker, price-taker) outcomes and their welfare implications with a lucidity and elegance unmatched in earlier work. In so doing, he reawakened economists to the power and subtlety of the diagram and thus initiated its post-war revival.

Figure 8 Scitovsky's Paradox

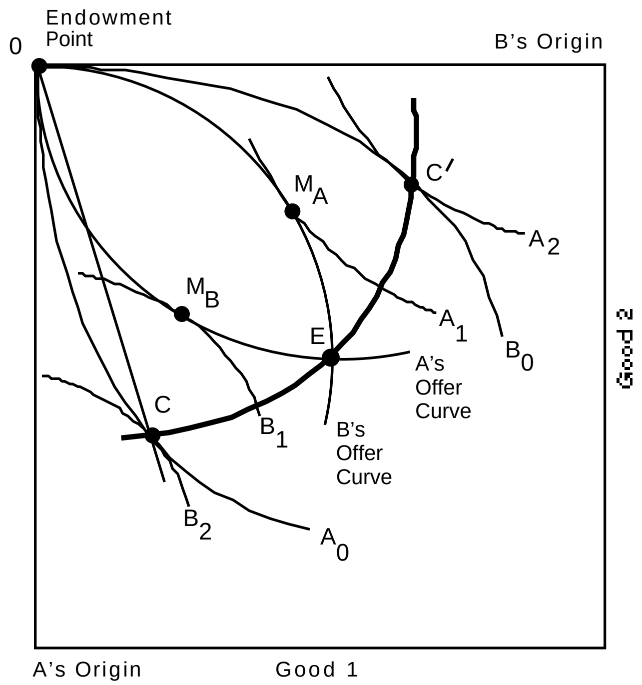


Paradoxically, the Kaldor-Hicks compensation test may justify both a move from situation *A* to situation *B* and a reverse move from *B* back to *A*. In the move from *A* to *B*, agent *J* could compensate agent *I* by the amount *BB'* and still be better off. He would still occupy a higher indifference curve than at *A*. Contrariwise, in the reverse move from *B* to *A*, agent *I* could compensate agent *J* by the amount *AA'* and still be better off than at *B*.

Perfectly Discriminating Monopoly

Leontief's main contribution, however, was to specify exactly how a dominant bargainer might extract for himself all the potential gains from trade. Let that bargainer present his passive counterpart with an all-or-nothing, take-it-or-leave-it option to trade the entire fixed bundle C at a fixed price equal to the slope of ray $0C$ (see Figure 9). The passive party either accepts the option or rejects it and remains at his endowment point. Since the option leaves him no worse off than does the autarky outcome, he accepts it. The resulting settlement is at one end of the core, namely at the extreme that yields the dominant party all the gains from exchange.

Figure 9 Perfectly Discriminating Monopolist



Trader *B*, a perfectly discriminating monopolist, captures all the potential gains from trade for himself by going to point *C* on his trading partner *A*'s autarky indifference curve A_0. He presents *A* with an all-or-nothing, take-it-or-leave-it option to trade the fixed bundle *C* at the fixed price denoted by the slope of ray 0*C*. Alternatively, *B* achieves the same result by moving down *A*'s autarky indifference curve, charging the highest price he can get for each successive unit of trade until he arrives at point *C*. Unlike the simple monopoly outcome M_B, the discriminating monopoly outcome is on the contract curve and therefore is efficient.

Leontief further noted that all-or-nothing option contracts are equivalent to perfect price discrimination. With price discrimination, the dominant trader moves along the autarky indifference curve of the passive trader. He does so by charging the highest price he can get for each successive unit of trade—that is, the highest price his partner is willing to pay rather than do without the unit—until he (the dominant trader) reaches the core at a point most favorable to himself. The result, in terms of the distribution of the gains from trade, is clearly the same as that achieved by the take-it-or-leave-it option.

In stressing this point, Leontief also emphasized that price discrimination, because it leads to the contract curve, is economically efficient. Like perfect competition, it wastes no resources. In this respect, the discriminating monopoly outcome is preferable to the simple monopoly one.

The significance of Leontief's contribution was this. Edgeworth and Bowley had stated that final settlement might occur at either extreme of the core. But they had failed to identify such outcomes with all-or-nothing options and discriminatory pricing. Leontief did so and established once and for all the exact price-quantity agreements that produce such outcomes.

6. OTHER POST-WAR CONTRIBUTIONS: KENNETH ARROW AND PAUL SAMUELSON

Leontief's rehabilitation of the exchange box contributed greatly to its popularity in the late 1940s and early 1950s. Extensions and generalizations followed when Kenneth Arrow and Paul Samuelson found imaginative new uses for the box.

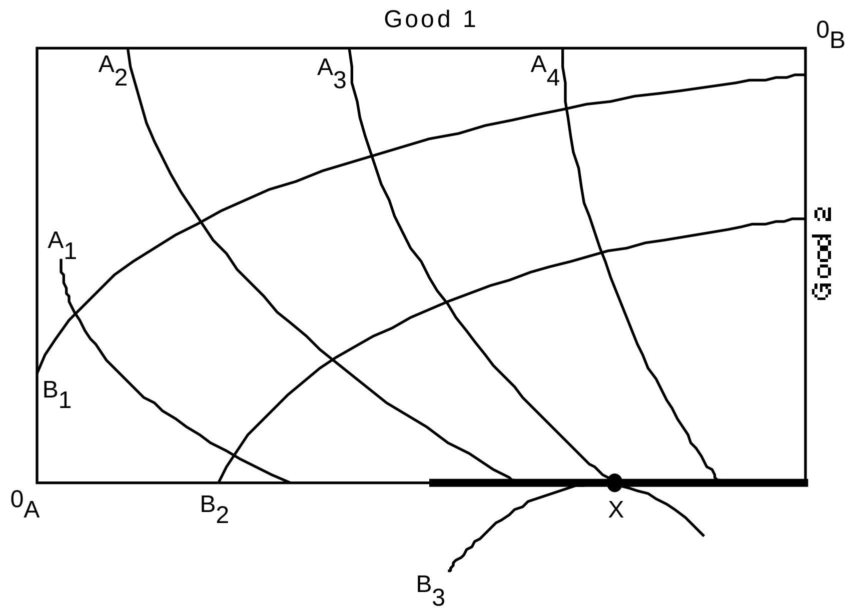
Arrow, in his 1951 essay "An Extension of the Basic Theorems of Classical Welfare Economics," did at least three things. First, he introduced modern set-theoretic concepts into the box. He interpreted the relevant regions of indifference maps as convex consumption sets and price or budget lines as their supporting hyperplanes. Doing so allowed him to replace local or first-order optimality criteria—the familiar marginal conditions—with global criteria.

Second, he employed the foregoing concepts to establish the two fundamental theorems of welfare economics. Theorem one states that every competitive equilibrium, because it occurs at a point where each agent maximizes his satisfaction given the level of satisfaction of the other, is a Pareto optimum. Theorem two states that every Pareto optimum, because it can be supported by a price vector that equates supply and demand, is a competitive equilibrium. Arrow demonstrated that both theorems hold for the standard case where indifference-curve tangencies occur in the interior of the box.

Third, he analyzed boundary optima in which interior tangencies give way to corner solutions on the edges of the box. His analysis yielded a positive and a negative result. The positive result was that competitive equilibria retain their optimality properties even when they occur on the borders of the box. His negative result was that, without extra assumptions, there may be Pareto optimal points on the boundaries that cannot possibly be equilibrium allocations (see Figure 10).

Consider point *X*. There agent *A*'s downward-sloping indifference curve meets the corresponding curve of agent *B* at its peak. Clearly this is a Pareto-efficient allocation since each agent is on his highest attainable indifference curve given the curve of the other. Nevertheless, this optimum cannot sustain an equilibrium. For given the flatness of *B*'s curve at its peak, the tangent price vector that separates the two indifference curves at *X* is necessarily a

Figure 10 Arrow's Exceptional Case



Pareto-efficient solution *X* contradicts the notion that optimality guarantees a competitive equilibrium. For the horizontal, tangent price line separating the indifference curves at *X* induces agent *B* to maximize his satisfaction by remaining at that point. Contrariwise, it induces agent *A* to maximize his satisfaction by moving as far to the right as possible. The upshot is that *A* and *B* seek incompatible allocations and the market fails to clear.

horizontal line coinciding with the lower edge of the box. Its slope implies a zero relative price that induces the agents to register incompatible claims. Given the zero price, *B* maximizes his utility by remaining at point *X*. By contrast, *A* maximizes his utility by moving rightward as far as possible along the price line, reaching ever-higher indifference curves as he goes.

The upshot is that *A* and *B* seek inconsistent allocations at the prices implied by corner-solution *X* and so the market fails to clear. Students refer to this curiosum as Arrow's Exceptional Case. It violates the theorem that every Pareto optimum guarantees a competitive equilibrium.[5]

[5] The theorem holds, however, when both indifference curves possess negative slopes at boundary optima. In such cases, a downward-sloping, tangent price line can always be fitted between the curves. Its slope represents the market-clearing price ratio that induces both parties to go to the optimum point.

Samuelson, in his classic 1952 *Economic Journal* article on “The Transfer Problem and Transport Costs,” used the exchange box to determine if a transfer payment made by Europe to America would worsen or improve Europe’s terms of trade. According to him, the transfer shifts the endowment point to the left and with it the offer curves and terms-of-trade ray that intersect at world trade equilibrium (see Figure 11). But whether the new ray is less or more steeply sloped than the old depends on the relative marginal propensities to consume Europe’s export good, clothing, in both countries. If the transfer reduces Europe’s clothing consumption more than it expands America’s, the result is an excess world supply of clothing whose relative price must therefore fall. The terms of trade will turn against Europe. On the other hand, if the transfer-induced fall in Europe’s demand for its exportable good, clothing, is less than the rise in America’s demand for that same good, the resulting excess world demand for clothing will bid up its relative price. Europe’s terms of trade will improve. The slope of the terms-of-trade ray can become either flatter or steeper. It all depends on the relative propensities to consume.

These extensions, however, brought the evolution of the exchange box to a halt. For the combined contributions of Leontief, Arrow, and Samuelson had virtually exhausted the analytical potential of the diagram and left it with little new to do. True, it maintained its popularity in the textbooks. But it was clear to all that the exchange box had seen its heyday. By the mid-1950s, its main use was to illustrate established ideas rather than to generate new ones.[6] Not so the alternative production variant, however. Economists were increasingly finding new applications for that version of the box.

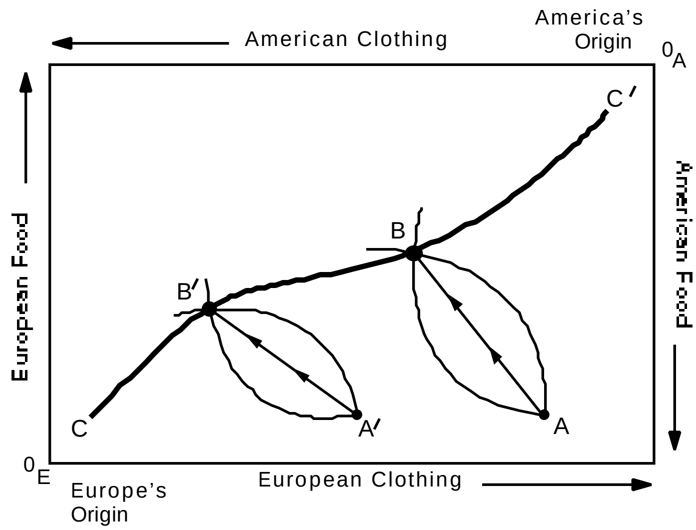
7. ABBA LERNER

Already, in December 1933, Abba Lerner had drawn perhaps the earliest version of the production box. He presented it in a term paper on factor-price equalization which he wrote for Lionel Robbins’s seminar at the London School of Economics.

Lerner’s diagram superimposes isoquant, or production indifference, maps of two industries fully employing two factor inputs whose fixed quantities determine the dimensions of the box (see Figure 12). Each isoquant shows alternative factor combinations capable of producing a given level of output. Any point in the box represents a particular allocation of the two factors between

[6] This situation, however, proved to be temporary. Unforeseen at the time was the post-1970 resurrection of the box to depict Kenneth Arrow’s notion of insurance as trade in state-contingent commodities (see Duffie and Sonnenschein [1989], pp. 584–86, and Niehans [1990], pp. 493–95). By exchanging such commodities, agents could in principle profitably insure themselves against the risks of unfavorable states of the world. In so doing, they could reach the contract curve where the allocation of risk-bearing is optimal.

Figure 11 The Transfer Problem



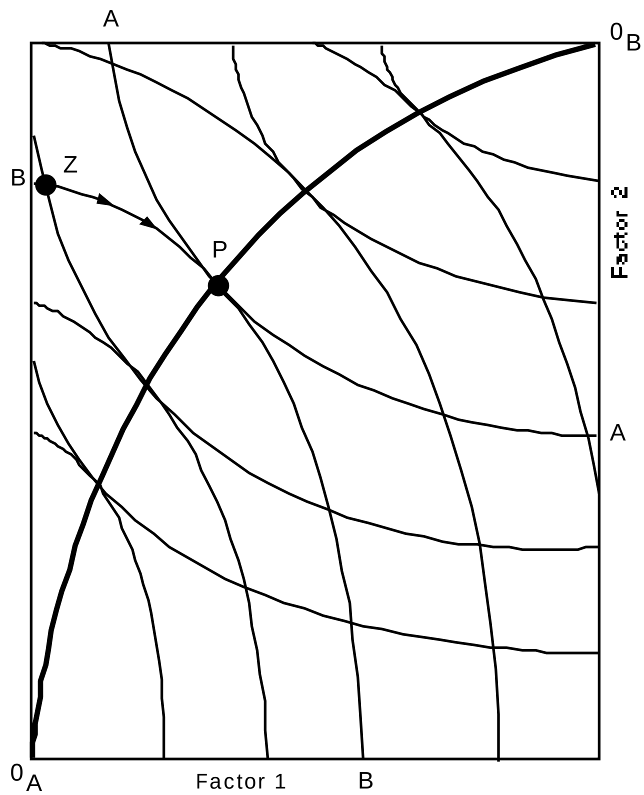
Before transfer, Europe and America trade along the terms-of-trade vector *AB* linking the autarky endowment point *A* to the free-trade equilibrium point *B*. When Europe pays to America a tribute of *AA*' units of cloth, the endowment point shifts horizontally leftward from *A* to *A*'. The corresponding new trade equilibrium point is *B*'. Whether the post-transfer terms-of-trade vector *A*'*B*' is steeper or flatter than the old one depends on the countries' marginal propensities to spend the proceeds of the tribute on cloth versus food. Either result is possible. Thus a transfer can cause the paying country's terms of trade to deteriorate, improve, or remain unchanged.

the production of the two goods. Isoquants going through such points show quantities of both goods produced with this factor allocation.

Regarding factor allocations off the locus of isoquant tangency points, Lerner notes that they are technologically inefficient. They squander scarce resources. They leave room for at least one industry, via mere reallocation of existing inputs, to increase its output with no loss of output of the other. Thus, starting from point *Z*, one can, by moving along isoquant *B* until it reaches tangency with isoquant *A*, increase the output of good *A* with no decrease in the output of good *B*. No such feat is possible on the efficiency locus itself, however. There, one industry's expansion spells the other's contraction. There, factor allocations are technologically efficient in the sense that they maximize the output of one good given the output of the other.

Efficiency, then, requires producers to operate on the contract curve. And, according to Lerner, competition and factor mobility together ensure that they

Figure 12 Lerner's Production Box Diagram



At inefficient point *Z*, good *A*'s output can be increased with no loss in good *B*'s output. Producers simply reallocate factor inputs to *A*-production so as to move along the given *B* isoquant cutting successively higher *A* isoquants in the process. At efficient point *P* on the contract curve, however, the output of one good can be increased only by decreasing the output of the other.

will do so. Competition forces producers to hire both factors until the ratio of their marginal products, represented by the slopes of isoquants, equals the ratio of their prices, represented by the slope of a relative factor-price line. And factor mobility dictates that resource prices, and so their ratio, are the same in both industries. Consequently, both industries operate at a point where their isoquants are tangent to a common factor-price-ratio line and thus are tangent to each other. Such points lie on the contract curve.

Delayed Publication

Unfortunately, economists had to wait for nineteen years to see Lerner's pioneering diagram. Tibor Scitovsky, in his essay on "Lerner's Contributions to Economics," tells why. In 1948 and 1949, Paul Samuelson published his celebrated proof of the factor-price-equalization theorem. Robbins, upon reading Samuelson's papers, recalled Lerner's 1933 term paper on the same subject. Robbins still had a copy of the paper in his files. Upon his urging, Lerner published the manuscript without alteration as the 1952 *Economica* piece "Factor Prices and International Trade."

As to why Lerner neglected to publish the paper in 1934, Tibor Scitovsky recounted a story he heard in 1935 when he was one of Lerner's students. Evidently, Lerner had given his only corrected copy to another student to be typewritten for submission to a scholarly journal. But the student lost the paper on a London bus and was unable to retrieve it. Lerner, who was busy writing other papers at the time, could not find the time to reproduce the lost manuscript. The resulting delay made Lerner's pathbreaking work and its innovative diagram seem less-than-novel when they finally appeared. In any case, it was not from Lerner but from Wolfgang Stolper and Paul Samuelson that the economics profession first learned of the production box.

8. WOLFGANG STOLPER AND PAUL SAMUELSON

Wolfgang Stolper and Paul Samuelson published the first production box diagram to appear in print. It features prominently in their 1941 *Review of Economic Studies* article on "Protection and Real Wages." Of the two authors, Stolper (1994, p. 339) credits Samuelson with the idea of using the box. In any case, they applied it to derive their famous theorem according to which free trade benefits the relatively plentiful factor and hurts the relatively scarce one while protective tariffs do the opposite.

Stolper-Samuelson Theorem

The Stolper-Samuelson theorem rests on two propositions. First, compared with autarky, free trade raises the price of the relatively abundant factor and lowers the price of the relatively scarce one. Conversely, trade restriction raises the scarce factor's price and lowers the plentiful factor's. Second, it follows that a tariff-induced restriction of trade may benefit labor in countries where labor is the scarcer factor. In such countries, a tariff may raise real wages and increase labor's real income both absolutely and relatively as a percent of the national income.

Stolper and Samuelson reached these conclusions via the following route. Suppose in the absence of trade a country produces wheat and watches with a fixed factor endowment consisting of much capital and little labor (see Figure 13). Measure capital on the horizontal axes of the box and labor on the vertical ones. The box, being wider than it is tall, indicates a high ratio of capital to labor and thus identifies capital as the relatively plentiful factor and labor as the relatively scarce one.

Next assume that, at any given factor-price ratio, wheat production requires a higher ratio of capital to labor than does watch production. Wheat, in other words, is capital intensive and watches are labor intensive. The slopes of labor-to-capital factor-proportion rays going through any point on the contract curve show as much. Those rays are steeper for watches than for wheat. Moreover, the contract curve lies everywhere below the diagonal of the box. Were factor intensities the same in both industries, the contract curve would coincide with the diagonal. And were factor-intensity reversals to occur, the contract curve would cross the diagonal. Neither possibility is allowed. Both are ruled out by assumption.

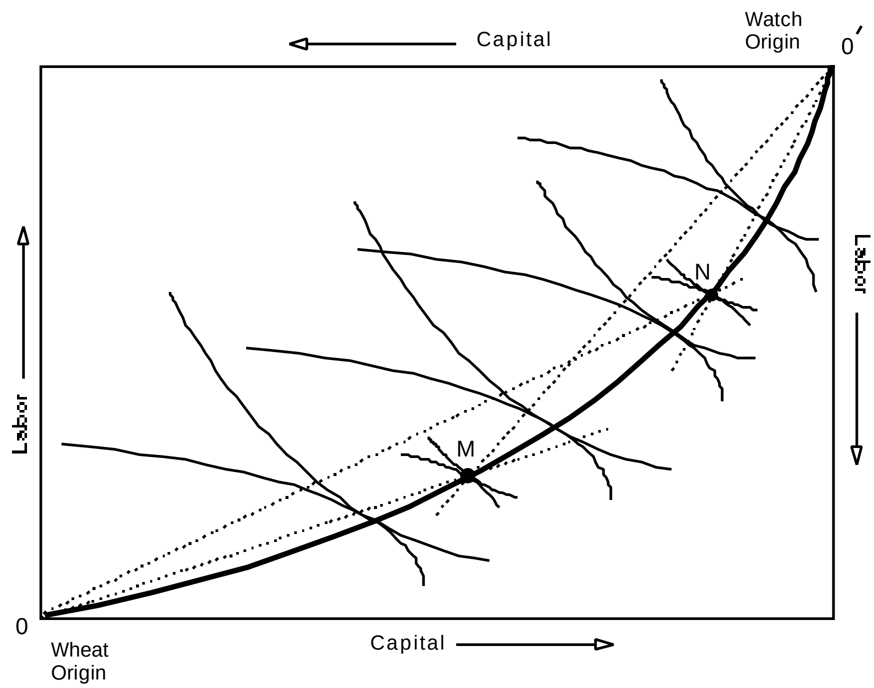
Initially, in the absence of trade, the country produces and consumes at point *M* on the contract curve. Wheat, embodying relatively large amounts of relatively cheap and plentiful capital, is the low-cost good. Conversely, watches, embodying much scarce and hence relatively dear labor, constitute the high-cost good. Given that the opposite conditions prevail in the rest of the world, the result is that wheat is cheaper in terms of watches at home than abroad.

Free Trade Helps the Plentiful Factor

When trade opens up, foreigners will import the home country's cheap wheat and home residents will import foreigners' cheap watches. The consequent increased demand for the home country's wheat and the decreased demand for its watches bids up the domestic price of wheat relative to the price of watches. The resulting price rise induces wheat producers to expand by hiring capital and labor from watch producers so as to move to free-trade point *N*. But the contracting watch industry, being labor-intensive, releases relatively little capital and relatively much labor compared to the ratio in which the capital-intensive wheat industry wants to absorb those factors. The ensuing labor surplus and capital shortage bids wages down and capital rentals up. The lower wages and higher rentals in turn induce both industries to substitute cheaper labor for dearer capital. The upshot is that the labor-to-capital ratio rises in both industries. And it does so even as the overall economy-wide endowment ratio shown by the slope of the diagonal stays unchanged. In terms of the diagram, both factor-proportion rays through point *N* are steeper than those through point *M*.

With less capital working with each unit of labor in both industries, the marginal product of labor falls and the marginal product of capital rises. Under competitive conditions, those marginal products constitute factor real rewards

Figure 13 Stolper-Samuelson Theorem



Free trade moves the capital-rich economy from autarky point *M* to free-trade point *N*. The dashed rays show that the labor-to-capital ratio rises in both industries. With more labor working with each unit of capital in each industry, the marginal productivity of capital and hence its real return rises while the marginal productivity of labor and hence its real wage falls. Free trade helps the plentiful factor and hurts the scarce one. Conversely, a protectionist move from point *N* to point *M* helps the scarce factor and hurts the plentiful one.

which factor mobility equalizes across industries. It therefore follows that real wages fall and real rentals rise expressed in terms of either good. Indeed, the flatter common slope of the isoquants at point *N* than at point *M* signifies as much. Those slopes indicate the rise in capital's and the fall in labor's real return. They show that free trade benefits the country's abundant capital factor and hurts its scarce labor one.

Protection Helps the Scarce Factor

Conversely, protection does the opposite. It raises the relative price and thus stimulates the output of import-competing watches at the expense of wheat production. In so doing, protection moves the domestic product-mix and its associated interindustry factor allocation from free-trade point *N* toward autarky point *M*. To induce the expanding watch industry to absorb factors in the proportion released by the contracting wheat industry, rentals must fall relative to wages. The consequent fall in capital's relative price encourages both sectors to adopt more capital-intensive techniques. The result is a rise in the capital-to-labor ratio in both industries as shown by the flatter slope of the rays going through *M* than through *N*. With more capital working with each unit of labor in both sectors, the marginal product of labor rises and the marginal product of capital falls. With factor real rewards equal to marginal products, the real wage of scarce labor rises while the real rental of abundant capital falls, as shown by the steeper common slope of the isoquants at *M* than at *N*. In short, import tariffs raise real wages and lower real rentals when the import-competing sector is more labor-intensive than the export sector. Protection benefits the scarce factor and hurts the plentiful one.

Evaluation

The Stolper-Samuelson paper is a milestone in the history of the box diagram and the evolution of trade theory. It crystallized certain components of the emerging Heckscher-Ohlin theory of international trade into a two-good, two-factor general equilibrium model. It then condensed that model into a simple box diagram capable of showing how commercial policy affects distributive shares. In so doing, it demonstrated the box's power in handling a large number of interrelated variables and thus established it as the standard tool of trade theory. Once established, the box proved indispensable in the derivation of such key trade propositions as the factor-price-equalization, Heckscher-Ohlin, and Rybczynski theorems.

Most important, the box diagram, in Stolper's and Samuelson's hands, taught that informal intuition on trade issues could be misleading. Before Stolper and Samuelson, most economists believed instinctively that free trade benefits all factor inputs. In demonstrating rigorously that such was not necessarily the case, Stolper and Samuelson made economists more cautious in discussing the benefits of trade. Thereafter, economists would acknowledge possible losses to the scarce factor in movements to free trade. But they would insist, on the grounds that trade benefits the country as a whole, that the gains of the abundant factor exceed the scarce factor's losses. Citing Scitovsky, they would argue that the abundant factor could in principle compensate the scarce factor for its losses and still be better off whereas the scarce factor would be unable to profitably bribe the abundant factor to oppose free trade.

9. T. M. RYBCZYNSKI

Stolper and Samuelson had used the box to link trade- or tariff-induced changes in commodity prices to changes in factor prices. They had shown how a product price increase causes a more-than-proportional rise in one factor's real reward while lowering the reward of the other. By contrast, T. M. Rybczynski in 1955 used the box to link changes in factor endowments to changes in commodity outputs. He showed that when one factor increases in quantity (product prices held constant), it causes a more-than-proportional increase in the output of one good and an absolute fall in the output of the other. Here was a startling revelation. Before Rybczynski, most economists felt that an increase in the endowment of one non-specific factor would lead to a rise in the output of all goods.

Rybczynski's demonstration goes as follows (see Figure 14). Let the country's initial factor endowment be that indicated by the dimensions of box *ABCD*. The economy initially produces at point *P* on the contract curve. The slope of the factor-intensity ray emanating from the wheat origin *A*, being flatter than its counterpart originating from the watch origin *C*, identifies wheat as the capital-intensive good and watches as the labor-intensive one.

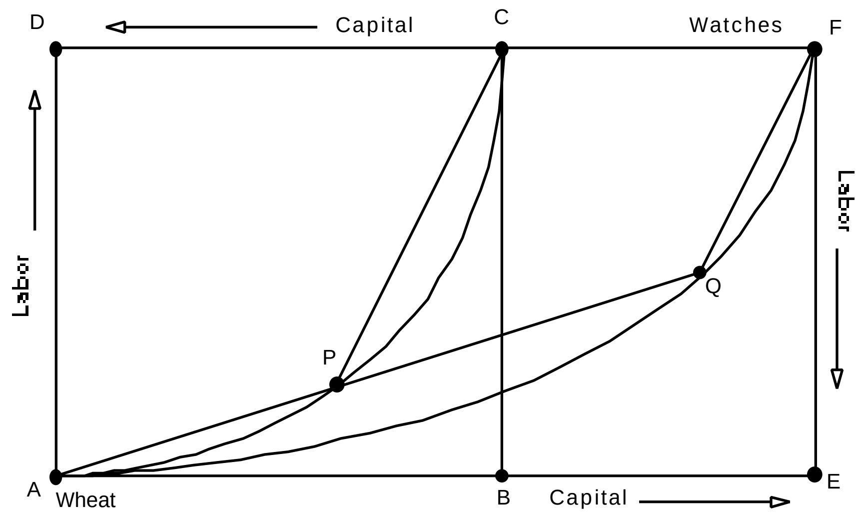
Rybczynski Theorem

Now assume that the economy's capital endowment expands by the amount *BE* while its labor endowment remains unchanged. The result is that the box annexes the new rectangle *BEFC*. How does the capital accumulation and the corresponding expansion of the box affect the output-mix of wheat and watches? Rybczynski's assumption of constant commodity prices provides the answer. Such constancy holds for small open economies taking their prices as given exogenously from the closed world economy.

Constant commodity prices imply constant factor prices. And with linear homogeneous production functions, constant factor prices imply unchanged factor proportions in both industries. Point *Q* in the new box satisfies that latter criterion. Only at that point are the capital-to-labor ratios (as shown by the slopes of the factor-intensity rays) the same as they were at point *P* in the old box. Thus the new equilibrium factor allocation must be *Q*. This new allocation, however, sees more labor and capital devoted to wheat production and less of both to watch production. The result is that wheat production expands and watch production contracts. Here is the famous Rybczynski theorem: Let one factor increase while the other stays constant. Then output of the good intensive in the increased factor will, at constant commodity prices, increase in absolute amount. Conversely, output of the other decreases absolutely.

The reasoning is straightforward. The expanding factor must be absorbed in producing the good using it intensively. To keep factor proportions fixed, as implied by the assumption of constant commodity prices, the expanding

Figure 14 Rybczynski Theorem



At constant commodity prices, a rise in the country's capital stock with no growth in its labor force raises the output of capital-intensive wheat and lowers the output of labor-intensive watches. Why? Because fixed commodity prices imply fixed factor prices which imply unchanged factor proportions in both industries. Thus as wheat output expands to absorb the extra capital, it requires extra labor to keep factor proportions unchanged. The only source of this extra labor is the watch industry, which therefore must contract. We go from point *P* to point *Q* with the slopes of the factor proportion rays remaining unchanged throughout.

industry must hire the non-increasing factor too. The only source of this factor is the other industry, which therefore must contract. Once again, the box diagram had rendered a seemingly counterintuitive proposition transparent.

10. KELVIN LANCASTER

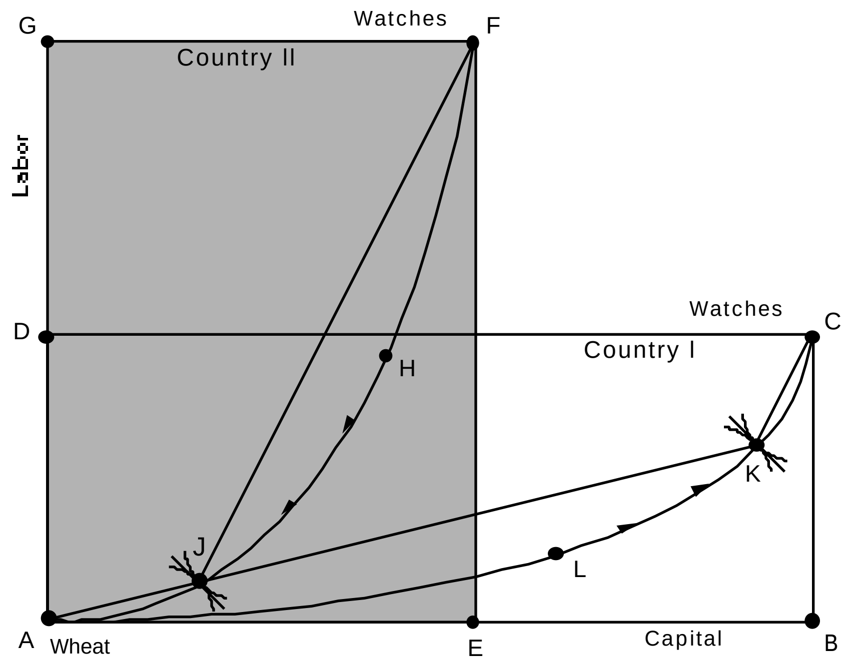
The box diagrams of Lerner, Stolper-Samuelson, and Rybczynski referred to a single country only. As such, they were hardly equipped to accommodate two-country models of international trade. The emerging Heckscher-Ohlin model was a prime example of such a model. True, the above-mentioned writers had introduced some Heckscher-Ohlin components into their single-country diagrams. But the list of included components was incomplete. Exposition of the full model required boxes referring to at least two countries.

Credit for developing the two-country box in its Heckscher-Ohlin form goes to Kelvin Lancaster.[7] His diagram, as presented in his 1957 *Economica* article on "The Heckscher-Ohlin Trade Model: A Geometric Treatment," embodies the standard features of that two-by-two-by-two model. Two countries produce two goods from two factor inputs. The countries are incompletely specialized. They produce both goods before and after trade. One good is always more capital-intensive than the other. Factor endowments differ across countries. Full employment prevails as does perfect competition in product and factor markets. Both countries share the same linear homogeneous production technology exhibiting constant returns to scale. Such technology ensures that factor marginal productivities are determined by factor-input ratios and not by scale of output.

The diagram incorporating these assumptions superimposes production boxes representing the countries' different factor endowments (see Figure 15). Capital is measured horizontally, labor vertically. The wide box *ABCD* identifies country *I* as the relatively capital-abundant nation. Similarly, the tall box *AEFG* specifies country *II* as the relatively labor-plentiful nation. Lancaster assumes that wheat production is always capital-intensive and watch production labor-intensive in both countries. The contract curves indicate as much. They lie below the diagonals of the boxes. Thus as one moves along a contract curve from left to right, the capital-to-labor ratio declines in response to a rising rental-to-wage ratio. But, at any given factor- price ratio, the capital-to-labor ratio is always higher in wheat than in watches. Were such not the case, the contract curves would either coincide with the diagonal or cross it.

[7] Even before Lancaster, Jan Tinbergen (1954, p. 137) had presented an alternative version of the two-country box. But his diagram, unlike Lancaster's, maps production possibility curves into commodity space. Depicting global trade equilibrium, he fits an equilibrium world price line between the production transformation curves and consumption indifference maps of the two countries. Both countries produce at the common point of tangency of their respective transformation curves and the price line. Then they trade along that line, each exporting its comparative advantage good and importing its comparative disadvantage one, until they reach the point of maximum satisfaction on their highest attainable indifference curves. In this way, trade enables both to consume beyond their transformation curves.

Figure 15 Heckscher-Ohlin Theorem and Factor-Price Equalization



Free trade moves countries *I* and *II* from autarky production points *L* and *H* to post-trade points *K* and *J*. Capital-rich country *I* produces more capital-intensive wheat and fewer labor-intensive watches. Labor-rich country *II* does the opposite. The slopes of the isoquants are the same at both post-trade points. This implies equal labor marginal productivities and equal capital marginal productivities in both countries. Since trade equalizes product prices worldwide and factor prices equal product prices times factor marginal productivities, it follows that trade equalizes factor prices too.

Heckscher-Ohlin Theorem

Having constructed the diagram, Lancaster used it to demonstrate the celebrated Heckscher-Ohlin and factor-price-equalization theorems. These theorems, together with their companion Stolper-Samuelson and Rybczynski postulates, constitute the core propositions of Heckscher-Ohlin trade theory. The Heckscher-Ohlin theorem predicts that each country will export the good intensive in its abundant factor and import the good intensive in its scarce factor. And the factor-price-equalization theorem says that free trade in commodities equalizes factor prices worldwide just as unrestricted factor mobility would do. The box diagram clarifies the underlying logic.

Initially, in the absence of trade, the countries operate in isolation at points *L* and *H* on their respective contract curves. At those autarky points, factor prices and factor combinations used to produce each good differ across the two countries as do product prices. Wheat, the capital-intensive good, is cheapest in terms of watches in capital-rich country *I*. Conversely, watches, the labor-intensive good, are cheapest in terms of wheat in labor-abundant country *II*.

When trade opens up, country *I* produces more of its export good, wheat, and fewer import-competing watches. The country moves along its contract curve to the free-trade point *K*. There, *I*'s relative commodity prices, or terms of trade, are the same as those abroad such that no incentive remains for further expansion of trade. At point *K* the rays *AK* and *CK*, whose slopes represent the factor proportions employed in *I*'s wheat and watch industries, respectively, intersect as required by the full-employment assumption.

Lancaster proves that the corresponding free-trade point for country *II* is *J*. The reason is simple. Free trade equalizes the ratio of commodity prices worldwide. In equilibrium, that ratio equals the marginal rate of factor substitution which equals the ratio of factor prices. Relative factor prices in turn uniquely determine factor input ratios in production functions exhibiting constant returns to scale. With both countries facing the same relative factor prices and sharing the same production functions, it follows that both must use the same factor input ratios too. Geometrically, the capital-to-labor factor-proportion rays going through country *II*'s free-trade point must have the same slopes as those intersecting at country *I*'s free-trade point. Such indeed is the case. Ray *AJ* is identical to ray *AK*. And ray *FJ* is parallel to ray *CK*. So country *II*'s free-trade point *J* corresponds to country *I*'s free-trade point *K*. Point *J* is the only point on *II*'s contract curve cut by factor-intensity rays of the same slope as those going through point *K* on *I*'s contract curve.

Having established the corresponding free-trade points, Lancaster required one final step to complete his demonstration. He took advantage of the property that linear homogeneous production functions allow output to be measured as the distance along any ray from the origin. He employed this property to compare the post-trade product mixes in the two countries. Country *I* produces more wheat ($AK > AJ$) and fewer watches ($CK < FJ$) than does country *II*. Thus country *I*'s product mix is heavily weighted toward wheat and country *II*'s toward watches. Here is Lancaster's demonstration of the Heckscher-Ohlin theorem: each country produces (and exports) relatively more of the good intensive in its abundant factor.

Factor-Price-Equalization Theorem

As for absolute factor-price equalization, Lancaster offered the following demonstration. Observe the tangent isoquants at the free-trade equilibrium points *J* and *K*. Constant-returns-to-scale considerations dictate that these isoquants, lying as they do on identical or parallel factor-proportion rays, possess the same slopes at one equilibrium point as they do at the other. But these slopes represent the ratios of factor marginal productivities which, as noted above, free trade equalizes across countries. Indeed, Lancaster shows that a stronger condition holds. When the two countries share the same linear production technology, free trade equalizes absolute as well as relative marginal productivities. Each factor's individual marginal productivity is the same in both nations.

Two additional steps complete the argument. The first cites the law-of-one-price notion that free trade renders the price of any traded commodity everywhere the same. The second refers to the competitive equilibrium condition that the price of any factor equals its marginal productivity multiplied by commodity price. Since trade equalizes commodity prices and marginal productivities worldwide, it equalizes their multiplicative product, factor prices, as well.

Lancaster's demonstration appeared at a time when other scholars were contributing to production-box analysis. Complementing his work were Kurt Savosnick's 1958 derivation of production possibility curves within the confines of the box and Ronald Jones's 1956 use of the diagram to examine the effects of factor-intensity reversals. These applications would have delighted Edgeworth. Apparently there was no end to what his invention could accomplish.

11. CONCLUSION: HOW THE BOX EVOLVED

The history of the box diagram reveals how analytical tools evolve in a complex interaction with their uses (see Koopmans [1957] 1991, pp. 169–71, for the definitive statement of this thesis). In this interaction, tools play a double role of servant and guide. As servants, they help solve problems motivating their invention. As guides, they alert their users to other problems solvable with their aid. Certainly Edgeworth regarded the box as servant when he invented it to demonstrate gains from exchange and to resolve the puzzle of how increasing numbers lead to the competitive equilibrium.

Once invented, however, the exchange box took on the status of guide. Its very existence made economists aware of other phenomena potentially seeking its application. In short order, it was employed to explain the rationale of such things as Pareto optimality, simple and discriminatory monopoly pricing, compensation tests, the transfer problem, and corner solutions. All were manifestations of the drive to generalize the exchange box and extend its range of application.

This same drive produced the alternative production box. Here a simple analogy sufficed. Economists saw how the exchange box depicted the allocation of fixed stocks of goods between the utilities of two individuals. They quickly realized that an analogous version could depict the allocation of a fixed stock of factor inputs between the production of two goods. Thus was born the production box, whose initial use was to devise rules for optimal factor allocation. Once available, however, the production box spurred economists to find new applications for it. Chief among these applications was the Heckscher-Ohlin theory of international trade. Accordingly, the box was deployed to derive, prove, or illustrate the key propositions of that theory.

The above experience contradicts Tjalling Koopmans's ([1957] 1991, p. 175) contention that diagrams, though a powerful aid to intuition and exposition, are nevertheless no match for higher mathematics in rigorous economic analysis. For the box diagram is surely an exception to that rule. In support of his allegation, Koopmans cites (1) the unreliability of diagrams as guides to reasoning, (2) their confining effect on the choice of problems studied, and (3) their inability to handle problems of more than two dimensions.

While these charges may stand up in a general comparison of diagrammatic and mathematical techniques, the box diagram itself pleads innocent to them. Far from being unreliable, it proved to be a highly accurate tool in the hands of economists ranging from Edgeworth to Samuelson. So accurate was it, in fact, that successive users found little need to modify it substantially. Far from being confining, it freed its users to attack long-unsolved problems such as how free trade affects the absolute and relative income shares of factor inputs. Its limited dimensionality likewise proved to be no handicap. On the contrary, Lancaster noted that the diagram could show the interrelationships between no less than twelve economic variables. Edgeworth likewise found the diagram's two-dimensionality no bar to analyzing what happens when unlimited pairs of traders are introduced into the model. In short, the history shows that this simple geometrical diagram, in terms of its ability to yield penetrating insights into problems of economic theory, has been a powerful mathematical tool in its own right.

Of course the box, like any diagram, cannot handle all problems. Far from it. Nobody would deny, for example, the diagram's insufficiency to represent infinite-horizon models involving infinite-dimensional commodity space. Such complex models are beyond the capacity of the box. Rather the box's strength lies in depicting simple general equilibrium models. As these models are extremely useful, so too is the diagram that embodies them.

In any case, it was the box diagram itself, more than any accompanying mathematics, that captured the attention of the economics profession. The result was that the box became a fixture of trade and welfare theory and a commonplace of textbooks. The survival of the concept testifies to its continued usefulness. Even today, if one wishes to understand the sources of and gains from exchange as well as the optimality of competitive equilibrium and the logic of efficient resource allocation, one can do no better than study the diagram.

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