Fisher's Equation and the Inflation Risk Premium in a Simple Endowment Economy

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One of the more important challenges facing policymakers is that of assessing inflation expectations. Goodfriend (1997) points out that one can interpret the meaning of a given interest rate policy action primarily in terms of its impact on the real rate of interest. However, evaluating this impact requires not only that one understands the various links between the nominal rate and expected inflation but also that one can quantify these relationships.

To find an approximate measure of expected inflation, one often turns to the behavior of long bond rates. Two key ideas explain why this approach might be appropriate. First, Fisher's theory holds that the real rate of interest is just the difference between the nominal rate of interest and the public's expected rate of inflation. Second, the long-term real rate is generally thought to exhibit very little variation.[1] Alternatively, and still based on Fisher's theory, one might use the yield spread between the ten-year Treasury note and its inflation-indexed counterpart as an estimate of expected inflation. In January 1997, the U.S. Treasury indeed began issuing ten-year inflation-indexed bonds.

While economic analysts typically attempt to capture inflation expectations using Fisher's equation, this method has its flaws. When inflation is stochastic, Fisher's relation may not actually hold. Barro (1976), Benninga and Protopapadakis (1983), as well as Cox, Ingersoll, and Ross (1985), show that the decomposition of the nominal rate into a real rate and expected inflation should

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[1] See Simon (1990).

include an additional component excluded from Fisher's equation: the inflation risk premium. This premium reflects the outcome of random movements in inflation that effectively cause nominal bonds to be risky assets relative to inflation-indexed bonds. As we shall see in this article, the sign of the premium may be positive or negative, depending on how unexpected movements in inflation co-vary with surprises in consumption growth.

Another reason the Fisher equation may not hold is that when one links the nominal rate to the real rate and expected inflation, one must consider the nonlinearity inherent in inflation when calculating expectations. Specifically, inflation is a ratio of prices. We shall see that this nonlinearity works through the variance of inflation surprises.

Since it is evident that Fisher's equation does not work in all situations, why should one consider the equation useful? (Note that if both the inflation risk premium and the variance of inflation surprises are negligible, then Fisher's equation holds precisely.) This article answers the question by building on earlier work by Labadie (1989, 1994). In particular, the analysis below relies upon three key building blocks. First, to study the effect of inflation risk on nominal rates, we formally incorporate uncertainty as part of the environment surrounding households' optimal bond purchasing decisions. Second, we assume that a bivariate vector autoregression (VAR) in the logs of consumption growth and inflation drives the model. This assumption makes it possible to work out exact analytical solutions for bond yields and expected inflation. Finally, we estimate the driving process empirically by using U.S. consumption-growth data to calibrate the model's analytical solutions. In contrast to Labadie (1989), we are able to derive solutions consistent with a general-order VAR process instead of a VAR(1). This allows us to better capture the joint time-series properties of consumption growth and inflation. Moreover, whereas Labadie's work focuses on the equity premium, we will concentrate mainly on the model's quantitative implications for the inflation risk premium.

Two important conclusions emerge from the analysis. One is that the model's quantitative estimates of the inflation risk premium are insignificant. This result occurs primarily because little covariation exists between shocks to consumption growth and unexpected movements in inflation in U.S. data. In other words, since inflation surprises are as likely to occur whether consumption growth is high or low, there is no reason why the inflation risk premium should be substantially positive or negative. This notion is unrelated to the fact that the equity premium tends to be very small in consumption-based asset pricing models. We will show that adopting a pricing kernel that helps explain the equity premium does not necessarily change the size of the inflation risk premium in any meaningful way. The implication is that, in practice, Fisher's equation may be a reasonable approximation even when inflation is stochastic.

The other important conclusion (for the sample period covering 1955 to 1996) is that the model's historical estimates of the yield on a one-year nominal bond match the actual yield on one-year Treasury notes relatively well. However, the model's estimates of the one-year nominal rate perform very poorly during the late 1970s. The model's inability to track the nominal rate during that period may reflect the unusual tightening by the Federal Reserve (the Fed) in an effort to bring down inflation at that time. Our benchmark model suggests a consumption-based real rate whose standard deviation is around 1 percent. Surprisingly, this is more than half the standard deviation of the ex post real rate despite the fact that consumption growth is relatively smooth. Using a different methodology, we find additional supporting evidence in favor of Fama (1990), who suggests that expected inflation and the real rate move in opposite directions. Finally, our model indicates that it is difficult to determine whether expected inflation is more or less volatile than the real rate at short horizons. Although conventional wisdom suggests that the real rate varies more than expected inflation in the short run, we find that the choice of preference specification is crucial for this result.

This article is organized as follows. Section 1 presents the basic framework used to price nominal and inflation-indexed bonds. Section 2 describes the joint driving process linking consumption growth and inflation. Sections 3 and 4 present the results which obtain under different preference specifications. Finally, Section 5 offers some concluding remarks.

1. PRICING NOMINAL AND INFLATION-INDEXED BONDS

The economy is populated by a continuum of infinitely lived households. These households are identical in terms of their preferences and endowments. The per capita endowment is nonstorable, exogenous, and stochastic. The typical household's wealth consists of currency, one-period inflation-indexed and one-period nominal discount bonds. Thus, an indexed bond purchased at time t pays one unit of the endowment good with certainty at time $t + 1$. As in Labadie (1989, 1994), this instrument provides a benchmark that helps isolate real from inflationary effects. Contrary to the indexed bond, the nominal bond is subject to inflation risk. That is, a nominal bond purchased at date t pays one unit of currency, say dollars, at date $t + 1$.

Each household maximizes its lifetime utility over an infinite horizon. The timing of trade follows that of the cash-in-advance economy described in Lucas (1982). Specifically, at the beginning of each period and before any trading takes place, a stochastic monetary transfer, $\nu_t M_{t-1}$, and a real endowment shock, y_t, are realized and observed publicly. After receiving the money transfer, as well as any payoffs on maturing bonds, the representative household decides on how to allocate its nominal wealth between money balances, M_t^d, indexed bonds, z_t, and nominal discount bonds, z_t^N. Once the asset market has closed, the household uses its money balances acquired at the beginning of the period M_t^d to finance its consumption purchases $p_t c_t$, where p_t is the price level at date t. The household then receives its nominal endowment income $p_t y_t$, which it cannot spend until the subsequent period. To summarize, the representative household solves

$$\max\ \mathcal{U} = E_t \sum_{s=t}^{\infty} \beta^{s-t} u(c_s), \quad 0 < \beta < 1, \tag{1}$$

subject to the constraints

$$\frac{p_{t-1}}{p_t} c_{t-1} + q_t z_t + \frac{x_t}{p_t} z_t^N + \frac{M_t^d}{p_t} = \frac{p_{t-1}}{p_t} y_{t-1} + \frac{M_{t-1} + \nu_t M_{t-1}}{p_t} + z_{t-1} + \frac{z_{t-1}}{p_t}, \tag{2}$$

and

$$c_t \le \frac{M_t^d}{p_t}. \tag{3}$$

We denote by q_t and x_t the real price of a one-period indexed bond and the price of a one-period nominal bond, respectively. E_t is the conditional expectations operator where the time t information set includes all variables dated t and earlier.

Appendix A contains the first-order conditions associated with the above problem. These optimality conditions yield the following Euler equation,

$$u'(c_t) = \beta E_t (1 + r_t) u'(c_{t+1}), \tag{4}$$

where we have defined $1/q_t$ as $(1 + r_t)$. Equation (4) states that in choosing how much to consume versus how much to save in the form of an indexed bond, the representative household explicitly compares marginal benefit and marginal cost. The marginal benefit, in utility terms, of consuming one additional unit of the endowment good today is given by $u'(c_t)$. Alternatively, the household could save that additional unit and use it to purchase an indexed bond that would yield $(1 + r_t)$ with certainty in the following period. Therefore the right-hand side of equation (4) captures the marginal cost of consuming one additional unit of the endowment good today in utility terms. As equation (4) indicates, the optimal consumption/savings allocation naturally equates marginal benefit and marginal cost.

Now, in this setup, the representative household also has the option of saving through a nominal pure discount bond. Optimality implies that

$$\frac{u'(c_t)}{p_t} = \beta E_t (1 + r_t^N) \frac{u'(c_{t+1})}{p_{t+1}}, \tag{5}$$

where $(1 + r_t^N)$ is defined as $1/x_t$. Analogous to the situation we have just described, the marginal benefit of consuming one additional dollar's worth of the endowment good today, where one dollar is worth $1/p_t$ units of the endowment good, is $u'(c_t)/p_t$. By instead saving this additional dollar in a nominal bond, the representative household would reap $(1 + r_t^N)/p_{t+1}$ units of the endowment good next period. The right-hand side of equation (5), therefore, represents the marginal cost of consuming an additional dollar's worth of the endowment good in the current period. As in equation (4), the optimal consumption/savings allocation still dictates equating marginal benefit to marginal cost.

Since equations (4) and (5) simply show different methods of how to best allocate income towards consumption and savings, one might naturally expect a precise link to emerge between the real rate and the nominal rate. Using equation (5) yields

$$\frac{1}{1 + r_t^N} = \beta E_t \frac{u'(c_{t+1})}{u'(c_t)} \frac{p_t}{p_{t+1}},$$

which may be rewritten[2] as

$$\frac{1}{1 + r_t^N} = \left(\frac{1}{1 + r_t}\right) E_t \left(\frac{p_t}{p_{t+1}}\right) + \beta \mathrm{cov}_t \left(\frac{u'(c_{t+1})}{u'(c_t)}, \frac{p_t}{p_{t+1}}\right). \tag{6}$$

Note that if inflation is deterministic, then the covariance term on the right-hand side of equation (6) disappears and the above equation reduces to Fisher's relation,

$$(1 + r_t^N) = (1 + r_t) \left(\frac{p_{t+1}}{p_t}\right). \tag{7}$$

To understand the nature of the differences between the modified Fisher equation and equation (7), let us first examine the covariance term in (6). This term is known as the inflation risk premium and already emerges in Benninga and Protopapadakis (1983) or Cox, Ingersoll, and Ross (1985). Recall that saving one additional dollar in period t yields $(1 + r_t^N)/p_{t+1}$ units of the endowment good in period $t+1$. However, the price level next period, p_{t+1}, is unknown at date t. Inflation, therefore, makes the nominal discount bond a risky asset; the premium in effect alters the nominal rate to account for this additional risk.

To make matters more concrete, let us temporarily suppose that momentary utility is given by the Constant Relative Risk Aversion (CRRA) function $u(c) = c^{1-\gamma} - 1/1 - \gamma$, $\gamma > 0$. Consequently, the ratio of marginal utilities in equation (6) is decreasing in consumption growth and given by $(c_{t+1}/c_t)^{-\gamma}$. Therefore, when the conditional covariance term is negative, inflation is likely to be high when consumption growth is low. In other words, the return on the nominal bond is adversely affected by inflation precisely when the household suffers from low consumption growth. Now observe that relative to a world without inflation uncertainty, a negative conditional covariance raises the nominal rate. We may, therefore, interpret this higher nominal yield as compensating

[2] Here we use the fact that for any two random variables x and y, $\mathrm{cov}(x, y) = E(xy) - E(x)E(y)$.

the household for the additional inflation risk associated with the nominal bond. The reverse is true when the conditional covariance term is positive.

Another reason equation (6) does not correspond to Fisher's relation when inflation is stochastic, even if the conditional covariance term were zero, has to do with Jensen's Inequality. In particular, $E_t(p_{t+1}/p_t)$ is generally not equal to $1/E_t(p_t/p_{t+1})$. As one might expect, we shall see below that the difference between $E_t(p_{t+1}/p_t)$ and $1/E_t(p_t/p_{t+1})$ rises with the volatility of inflation surprises. In a world without such surprises, the conditional expectations operator is irrelevant, so this difference would vanish.

To close the model, we simply note that in equilibrium, $c_t = y_t$, while $M_t^d = M_t$. In addition, since households are identical, indexed and nominal bonds are in zero net supply so that $z_t = z_t^N = 0$. In what follows, we assume for simplicity's sake that $\nu_t \geq \beta$ so that the cash-in-advance constraint always binds.

2. THE ENDOWMENT AND INFLATION PROCESSES

We now define a driving process for this economy. Let endowment growth and the inflation rate be denoted by $y_{t+1}/y_t = \zeta_{t+1}$ and $p_{t+1}/p_t = \phi_{t+1,}$ respectively. We assume that the joint time-series behavior of $\ln \zeta_{t+1}$ and $\ln \phi_{t+1}$ can be described by a covariance stationary bivariate VAR (p).[3] The law of motion for the endowment process is

$$\ln \zeta_{t+1} = \delta_{\zeta 0} + \sum_{j=0}^{p} \delta_{\zeta\zeta,j} \ln \zeta_{t-j} + \sum_{j=0}^{p} \delta_{\zeta\phi,j} \ln \phi_{t-j} + \varepsilon_{\zeta,t+1}. \qquad (8)$$

Similarly, the inflation rate follows a process that can be described by

$$\ln \phi_{t+1} = \delta_{\phi 0} + \sum_{j=0}^{p} \delta_{\phi\zeta,j} \ln \zeta_{t-j} + \sum_{j=0}^{p} \delta_{\phi\phi,j} \ln \phi_{t-j} + \varepsilon_{\phi,t+1}. \qquad (9)$$

Shocks to endowment growth and inflation, $(\varepsilon_{\zeta,t}, \varepsilon_{\phi,t})$, are assumed to be jointly distributed normal random variables such that $E(\varepsilon_{\zeta,t}) = E(\varepsilon_{\phi,t}) = 0$, $\text{var}(\varepsilon_{\zeta,t}) = \sigma_\zeta^2$, $\text{var}(\varepsilon_{\phi,t}) = \sigma_\phi^2$, and $\text{cov}(\varepsilon_{\zeta,t}, \varepsilon_{\phi,t}) = \sigma_{\zeta\phi}$. Moreover, as in Labadie (1989), the shocks satisfy $E(\varepsilon_{\zeta,t}, \varepsilon_{\phi,s}) = E(\varepsilon_{\zeta,s}, \varepsilon_{\phi,t}) = 0$, for $s \neq t$.

3. RESULTS WITH CRRA UTILITY

Analytical Solutions

In this section, we assume that momentary utility is of the CRRA form. Our main focus will be to derive and interpret solutions for bond prices or,

[3] Since the cash-in-advance constraint is assumed to bind, this bivariate system implicitly dictates the behavior of money growth.

alternatively, rates of return on the indexed and nominal discount bonds. The goal is to assess to what degree Fisher's equation approximates its generalized version in (6) in a calibrated consumption-based asset pricing model. With CRRA utility, equation (4) becomes

$$q_t = \beta E_t \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}, \tag{10}$$

which may also be written as

$$\ln q_t = \ln \beta + \ln E_t \zeta_{t+1}{}^{-\gamma}.$$

Using the properties of log-normal random variables described in Appendix B, as well as those of the driving process in Section 2, it immediately follows that

$$\ln q_t = \ln \beta - \gamma \delta_{\zeta 0} + \frac{\gamma^2 \sigma_\zeta^2}{2} - \gamma \sum_{j=0}^{p} \delta_{\zeta\zeta,j} \ln \zeta_{t-j} - \gamma \sum_{j=0}^{p} \delta_{\zeta\phi,j} \ln \phi_{t-j}.$$

The real price of the one-period inflation-indexed bond can therefore be expressed as

$$q_t = \beta \left[\exp(-\gamma \delta_{\zeta 0} + \frac{\gamma^2 \sigma_\zeta^2}{2}) \right] Q_t, \tag{11}$$

where $Q_t = \Pi_{j=0}^{p} \zeta_{t-j}^{-\gamma\delta_{\zeta\zeta,j}} \Pi_{j=0}^{p} \phi_{t-j}^{-\gamma\delta_{\zeta\phi,j}}$. Equation (11) suggests that the real rate, $1/q_t$, is not only a function of past endowment growth but also of past inflation rates. This result arises since, by equation (8), past inflation rates help forecast endowment growth next period, ζ_{t+1}. In addition, observe that greater volatility in unexpected endowment growth movements, as captured by σ_ζ^2, raises q_t and, therefore, lowers the real rate. Put another way, a more risky endowment growth process serves to lower the real rate of return. This latter effect, however, is only present to the degree that households care about risk so that $\gamma > 0$. When households are risk-neutral and $\gamma = 0$, q_t is independent of σ_ζ^2.

Turning our attention to the behavior of the nominal rate, equation (5) can be rewritten as

$$x_t = \beta E_t \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \frac{p_t}{p_{t+1}} \tag{12}$$

so that $\ln x_t = \ln \beta + \ln E_t (c_{t+1}/c_t)^{-\gamma} (p_t/p_{t+1})$. Again, using the properties of log-normal random variables yields

$$\begin{aligned} \ln E_t \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \frac{p_t}{p_{t+1}} = E_t \ln \zeta_{t+1}^{-\gamma} + E_t \ln \phi_{t+1}^{-1} + \\ \frac{\gamma^2}{2} \mathrm{var}_t \ln \zeta_{t+1} + \frac{1}{2} \mathrm{var}_t \ln \phi_{t+1} + \gamma \mathrm{cov}_t \left(\ln \zeta_{t+1}, \ln \phi_{t+1} \right). \end{aligned} \tag{13}$$

As before, we can use the properties of the driving process to obtain

$$x_t = \beta \left[\exp(-\gamma\delta_{\zeta 0} - \delta_{\phi 0} + \frac{\gamma^2\sigma_\zeta^2}{2} + \frac{\sigma_\phi^2}{2} + \gamma\sigma_{\zeta\phi}) \right] X_t, \qquad (14)$$

where $X_t = \Pi_{j=0}^{p} \zeta_{t-j}^{-(\gamma\delta_{\zeta\zeta,j}+\delta_{\phi\zeta,j})} \Pi_{j=0}^{p} \phi_{t-j}^{-(\gamma\delta_{\zeta\phi,j}+\delta_{\phi\phi,j})}$. As expected, the behavior of the nominal rate depends on the time-series characteristics of both endowment growth and the inflation rate. In particular, observe that the greater the unconditional variance of inflation surprises, the lower the nominal rate, since $1 + r_t^N = 1/x_t$. Furthermore, a larger negative covariance between unexpected movements in endowment growth and inflation surprises raises the nominal rate (so long as $\gamma > 0$). As mentioned earlier, this result reflects that when $\sigma_{\zeta\phi} < 0$, high inflation shocks tend to occur when endowment growth is unexpectedly low. In this case, the household, therefore, requires a higher yield on nominal bonds to account for the inflation risk. Alternatively, we can see this notion by tracing the effect of the covariance between endowment growth shocks and inflation shocks on the inflation risk premium directly. By using equation (6) and solving for $\ln 1/(1+r_t)E_t(p_t/p_{t+1})$ as we have done above, one sees that

$$\begin{aligned} \mathrm{cov}_t \left(\left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}, \frac{p_t}{p_{t+1}} \right) = \\ \left[\exp\left(-\gamma\delta_{\zeta 0} - \delta_{\phi 0} + \frac{\gamma^2\sigma_\zeta^2}{2} + \frac{\sigma_\phi^2}{2} \right) \right] \left[\exp(\gamma\sigma_{\zeta\phi}) - 1 \right] X_t. \qquad (15) \end{aligned}$$

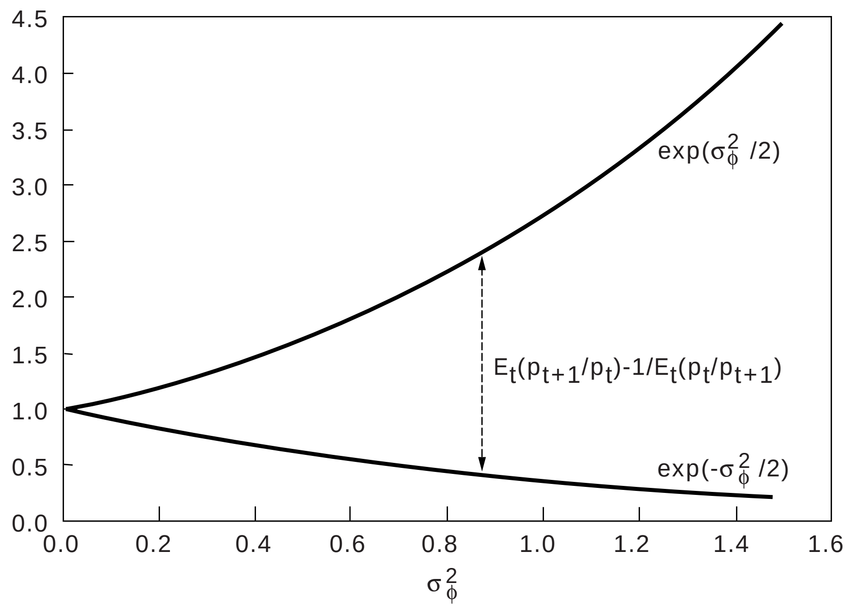
Hence, it is now clear that $\mathrm{cov}_t \left((c_{t+1}/c_t)^{-\gamma}, p_t/p_{t+1} \right) \le 0$ whenever $\sigma_{\zeta\phi} \le 0$, regardless of the other terms in equation (15). As suggested by equation (6), this effect raises the nominal rate over and above that implied by movements in the real rate and expected inflation alone. Finally, it should be intuitive that when households are risk-neutral and $\gamma = 0$, the inflation risk premium is identically zero irrespective of $\sigma_{\zeta\phi}$.

Earlier in the analysis, we hinted that even if the inflation risk premium were zero at all dates, equation (6) would not necessarily reduce to the Fisher equation when inflation is stochastic. We argued that, generally, $E_t(p_{t+1}/p_t) \neq 1/E_t(p_t/p_{t+1})$ and that this difference would rise with the volatility of inflation surprises. This result is shown formally in Appendix C and, in particular,

$$E_t \frac{p_{t+1}}{p_t} - \frac{1}{E_t(p_t/p_{t+1})} = \exp(\delta_{\phi 0}) \left[\exp\left(\frac{-\sigma_\phi^2}{2} \right) - \exp\left(\frac{\sigma_\phi^2}{2} \right) \right] P_t^{-1}, \qquad (16)$$

where $P_t = \Pi_{j=0}^{p} \zeta_{t-j}^{\delta_{\phi\zeta,j}} \Pi_{j=0}^{p} \phi_{t-j}^{\delta_{\phi\phi,j}}$. Figure 1 illustrates how the right-hand side of this last equation varies as a function of ϕ_ϕ^2. Since the result in equation (16) is essentially driven by Jensen's Inequality, the greater the variance of inflation

Figure 1 Effect of Jensen's Inequality on the Simple Fisher Equation



shocks, the more the convexity inherent in the price ratio matters. In a world without inflation surprises, $\sigma^2_\phi = 0$, and the right-hand side of (16) vanishes. Note that in the latter case, inflation is not necessarily constant but is deterministic and described by equation (9), without the $\varepsilon_{\phi,t+1}$ shock. Therefore the conditional expectations operator in (16) becomes, in some sense, irrelevant.

Thus far, we have been able to show that the discrepancy between the modified Fisher equation in (6) and the Fisher equation in (7) ultimately boils down to two crucial aspects of the environment; namely, the covariance between unexpected movements in endowment growth and inflation surprises, as well as the unconditional volatility of inflation surprises. However, whether this difference is quantitatively significant remains to be seen.

Quantitative Implications

To address the quantitative features of the model just presented, we must first tackle the issue of calibration. As a benchmark case we first fix the discount rate, β, to 0.996 and set the risk-aversion parameter, γ, to 0.75. The value of the discount rate is chosen so that, in the benchmark scenario, the mean of the model-implied ex post real rate matches its counterpart in the data at 2.32 percent. Note that since U.S. real consumption has generally been growing at about 2 percent over the sample, our discount factor is scaled up by a factor of

$(1.02)^{\gamma}$ relative to one that would be appropriate for stationary data. We then examine how the results vary with changes in the risk-aversion parameter. The only other necessary parameters of the model relate to the exogenous driving process. To this end, we estimate the bivariate VAR described by equations (8) and (9) using the following data:

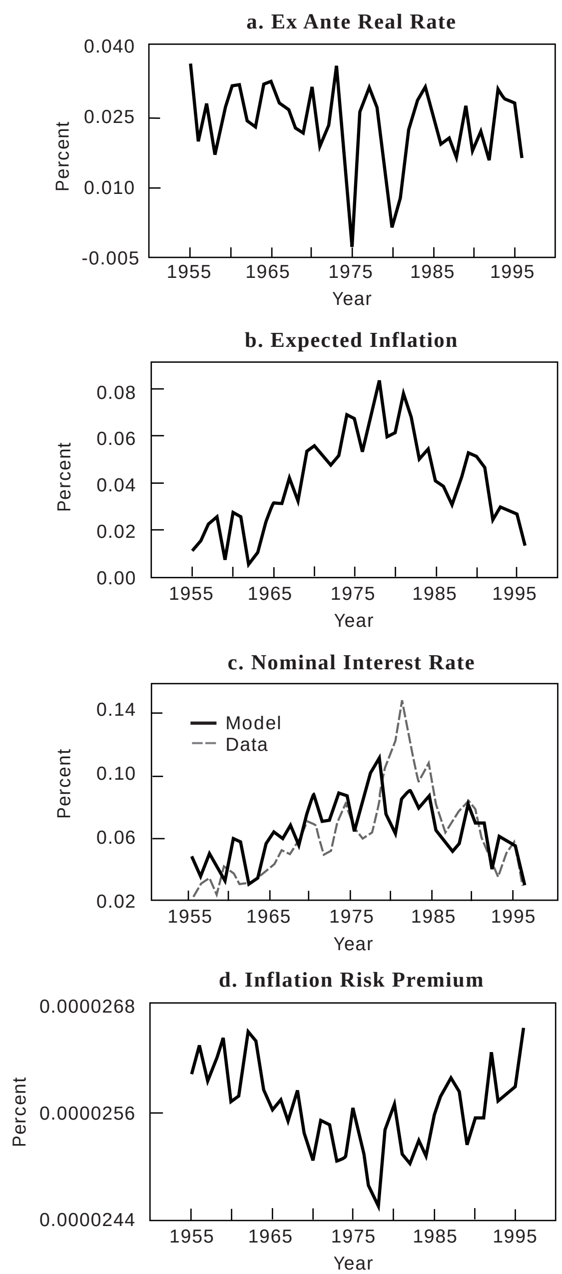
- Consumption refers to per capita annual U.S. consumption of non-durable goods, durable goods, and services, spanning 1955 to 1996 and expressed in 1992 dollars.
- Price level refers to the ratio of nominal consumption to real consumption.

Note that we are using annual data in order to avoid estimating equations (8) and (9) with variables averaged over extended periods. Using data averaged over a ten-year period, for instance, would result in a substantial loss of information. A VAR of order 4 is estimated with resulting R^2s of 0.75 and 0.90 for equations (8) and (9), respectively. The point estimates for $\sigma_{\zeta\phi}$ and σ_{ϕ}^2 are 3.20×10^{-5} and 2.43×10^{-4}. It directly follows that the inflation risk premium generated by this model is all but negligible. Observe that this result has little to do with the notion that the equity premium is typically small in this type of framework. Instead, it is driven almost exclusively by the fact that inflation surprises move in a way unrelated to unexpected changes in consumption growth. (We return to this point more fully in the next section.) Moreover, consistent with the high R^2 associated with the estimation of equation (9), the volatility of shocks to inflation also appears to be very small. Therefore, by equation (16), we may think of $E_t(p_{t+1}/p_t)$ as essentially equal to $1/E_t(p_t/p_{t+1})$.

Figure 2 presents the historical estimates generated from the model for the period 1955 to 1996 using the benchmark parameters. We chose this time span so that we could directly compare the model-implied nominal rate with the actual yield on one-year Treasury notes.

As we can see from Figure 2, panel c, except for the late 1970s and early 1980s, the model performs relatively well in matching the actual nominal rate. The model's inability to capture the sharp rise in nominal rates in the late 1970s can perhaps best be explained by the unusually aggressive disinflationary policy adopted by the Fed at that time. In response to strong inflationary pressures in the fall of 1980, Goodfriend (1993, pp. 11–12) notes that "the Fed began an unprecedented aggressive tightening. . . . Thus, the run-up of the funds rate to its 19 percent peak in January 1981 marked a deliberate return to the high interest rate policy." It may be, therefore, that the assumptions concerning the driving process described by equations (8) and (9) are not entirely justified. In particular, a specification for the driving process that included the possibility of a regime switch around 1980 might have been more appropriate.

Figure 2 Simulated Results with CRRA Utility



As shown in panel d, the inflation risk premium is insignificant over the entire period and since the variance of inflation surprises is small, the modified Fisher equation collapses almost exactly to the Fisher equation. To be specific, the gap that separates equation (6) from equation (7) is never more than 3 basis points over the entire period. Thus, while the Fisher equation does not hold in theory when inflation is stochastic, it may very well serve as a reasonable approximation in practice.

Panel a of Figure 2 also shows that the ex ante real rate can be quite volatile. Observe in particular the severe real rate drops that occur in 1975 and 1980. In the context of this model, recall that the real rate in equation (11) is in part a function of recent consumption growth. According to the driving process described in Section 2, past consumption growth helps predict future consumption growth in equation (10). Consequently, the sharp fall in real rates in 1975 and 1980 correspond respectively to the two recessions typically associated with the severe rise in oil prices and the credit controls imposed by the Carter Administration. Over the period under consideration, the one-year real rate fluctuates between 0.25 percent and 3.7 percent. This range is substantially greater than the 75-basis-point range found by Ireland (1996) for the ten-year real rate. Our findings therefore lend support to the stylized view that as maturity increases, variations in the nominal rate are more likely due to variations in expected inflation than variations in the real rate. Table 1 presents some key sample statistics concerning the time-series properties of the historical estimates generated by the model as we vary the risk-aversion parameter.

As suggested by the estimates in Table 1, the standard deviation of the real rate is about 1 percent in the benchmark case. This rate is more than half the standard deviation of the ex post real rate of 1.80 percent over the same period. Therefore, in spite of relatively smooth consumption growth, this framework generates a real rate with considerable volatility.

In Table 1, we also note that both the mean and the standard deviation of the real rate increase sharply with the risk-aversion parameter. This result emerges because a rise in the degree of risk aversion implies a fall in the elasticity of intertemporal substitution in consumption. Since the representative household is less willing to smooth consumption across periods, it generally requires a higher return on bonds in order to save. More importantly, this feature of the model is precisely that which makes it difficult to match the equity premium. As observed in Abel (1990), although the return on stocks typically rises with γ, the fact that the return on Treasury notes also rises with γ essentially leaves the difference between the stock return and the bond return unchanged, even for large increases in risk aversion. Ideally, to have a better chance of matching the equity premium without requiring extreme values of γ, one would like a framework in which increases in the degree of risk aversion do not necessarily yield increases in the real rate.

Table 1

	Ex Ante Real Rate: r_t	**Expected Inflation:** $E_t(p_{t+1}/p_t)$	**Nominal Rate:** r_t^N
$\gamma = 0.75$	mean: 2.14	mean: 4.03	mean: 6.23
	std: 1.00	std: 2.43	std: 2.39
	$\text{corr}(r_t, E_t(p_{t+1}/p_t))$: -0.29	$\text{var}(\frac{E_t(p_{t+1}/p_t)}{r_t})$: 5.99	
$\gamma = 1.75$	mean: 4.51	mean: 4.03	mean: 9.90
	std: 2.37	std: 2.43	std: 3.01
	$\text{corr}(r_t, E_t(p_{t+1}/p_t))$: -0.29	$\text{var}(\frac{E_t(p_{t+1}/p_t)}{r_t})$: 1.03	
$\gamma = 6$	mean: 15.10	mean: 4.03	mean: 19.64
	std: 9.02	std: 2.43	std: 8.98
	$\text{corr}(r_t, E_t(p_{t+1}/p_t))$: -0.28	$\text{var}(\frac{E_t(p_{t+1}/p_t)}{r_t})$: 0.07	

Finally, because the volatility of the real rate depends so crucially on γ in the above experiment, it is difficult to say whether the volatility of the real rate relative to that of expected inflation is greater or less than one. In addition, the model consistently generates a negative correlation between the real rate and expected inflation across all values of the risk-aversion parameter. The latter result supports earlier evidence to that effect by Fama (1990).

4. RESULTS WITH "KEEPING-UP-WITH-THE-JONESES" UTILITY

Analytical Solutions

Thus far, estimates of the inflation risk premium based on the above framework as well as U.S. consumption data appear to be quantitatively small. We have also suggested that this result is unrelated to the fact that the equity premium tends to be small in consumption-based asset pricing models. To see why this is true, we now adopt an alternative preference specification that we refer to as the "keeping-up-with-the-Joneses" (KUPJ) specification. Under this alternative way of modeling preferences, which defines utility as a function of *relative* consumption, Abel (1990) shows that while the return on stocks typically increases with the risk-aversion parameter, the real return on bonds generally remains constant. Therefore, when the degree of relative risk aversion is sufficiently high ($\gamma = 6$ in Abel [1990]), the author is able to generate an equity premium that is within the range of that observed in the data. We now formally show that even when utility is of the KUPJ form, the inflation risk premium remains small irrespective of the degree of risk aversion.

Following Abel (1990) and Gali (1994), momentary KUPJ utility is given by

$$u(c_t) = \frac{(c_t/C_{t-1})^{1-\gamma} - 1}{1-\gamma}, \tag{17}$$

where C_{t-1} denotes average consumption in the previous period. Thus, the specification in (17) captures the idea that it is not consumption per se but relative consumption that matters to households. Using $\widetilde{q}_t$ as the price of a one-period inflation-indexed bond under this alternative functional form for utility, equation (4) now reads as

$$\begin{aligned} \widetilde{q}_t &= \beta \left(\frac{C_t}{C_{t-1}} \right)^{\gamma-1} E_t \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \\ &= \left(\frac{C_t}{C_{t-1}} \right)^{\gamma-1} q_t, \end{aligned} \tag{18}$$

where q_t, given by equation (10), is the price of an inflation-indexed bond when utility is CRRA. Similarly, the inverse of the nominal rate in equation (12) is now given by

$$\begin{aligned} \widetilde{x}_t &= \beta \left(\frac{C_t}{C_{t-1}} \right)^{\gamma-1} E_t \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \frac{p_t}{p_{t+1}} \\ &= \left(\frac{C_t}{C_{t-1}} \right)^{\gamma-1} x_t. \end{aligned} \tag{19}$$

In equilibrium, $C_t = c_t$ when households are identical. Therefore, the solutions for $\widetilde{q}_t$ and $\widetilde{x}_t$ can simply be obtained by scaling up equations (11) and (14), respectively, by a power function of current consumption growth, $\zeta_t^{\gamma-1}$. More importantly, these results also indicate that the new inflation risk premium is now given by equation (15) multiplied by $\zeta_t^{\gamma-1}$. Since the inflation risk premium under KUPJ utility is simply the premium that emerges under CRRA utility scaled up by current consumption growth (to the power $\gamma - 1$), a value of $\sigma_{\zeta\phi} = 0$ still implies that the inflation risk premium is identically zero irrespective of γ. In other words, it is still true in this case that when unexpected movements in consumption growth and shocks to inflation are uncorrelated, the inflation risk premium is zero regardless of the degree of risk aversion. Given our estimate in the previous section of $\sigma_{\zeta\phi} = 3.20 \times 10^{-5}$, it follows that even when preferences follow the KUPJ specification, the simple Fisher equation in (7) remains a good approximation to the generalized Fisher equation in (6).

Quantitative Implications

Figure 3 presents the historical estimates from the benchmark case where utility is of the KUPJ form. The parameter values for the bivariate driving process are the same as those used in the previous section. A direct comparison with Figure 2 reveals little difference between the two sets of figures. In particular, observe that, as expected, the inflation risk premium continues to be negligible over the entire period under consideration. As in the earlier experiment, the model still fails to capture the behavior of the nominal rate at the end of the 1970s and beginning of the 1980s. However, it is interesting that both the ex ante real rate and the model-implied nominal rate seem to exhibit more variation relative to Figure 2. This result is consistent with the earlier work of Abel (1990) who finds that, while the mean return on bonds remains relatively constant as the degree of risk aversion rises with KUPJ preferences, the volatility of bond returns tends to exceed that which emerges with CRRA utility. The following table makes the last point more concretely.

When one compares Table 2 with Table 1, it is clear that under the alternative preference specification, the real rate is largely invariant with respect to the degree of risk aversion. This invariance property is precisely the mechanism that, for a high enough value of γ, allows Abel (1990) to generate an equity risk premium close to the one found in the data. Table 2 also clearly suggests that in all cases, the volatility of both the real rate and nominal rate is greater than its corresponding value in Table 1. As in the previous section, it remains that the volatility of the real rate increases sharply with the degree of risk aversion. Accordingly, whether the real rate varies more or less than expected inflation at short horizons still depends heavily on the particular preference specification adopted. In addition, as in Fama (1990), the model continues to suggest a consistent negative correlation between the real rate and expected inflation across different values of γ. Therefore, although we find that Fisher's equation holds relatively well in this framework, the nominal yield moves generally less than one-for-one with expected inflation at the one-year horizon.

5. CONCLUDING REMARKS

This article investigates the extent to which the simple Fisher equation can be interpreted as a reasonable approximation to its more complete counterpart in a dynamic endowment economy. The expanded Fisher equation, in addition to capturing movements in real rates and expected inflation, differs from its simpler version along two dimensions. First, it accounts for random movements in inflation through an inflation risk premium. Second, it acknowledges the inherent nonlinearity of inflation in drawing a link between the nominal rate and expected inflation.

Given U.S. consumption data, we find that the quantitative historical estimates of the inflation risk premium for the period 1955 to 1996 are small.

Figure 3 Simulated Results with KUPJ Utility

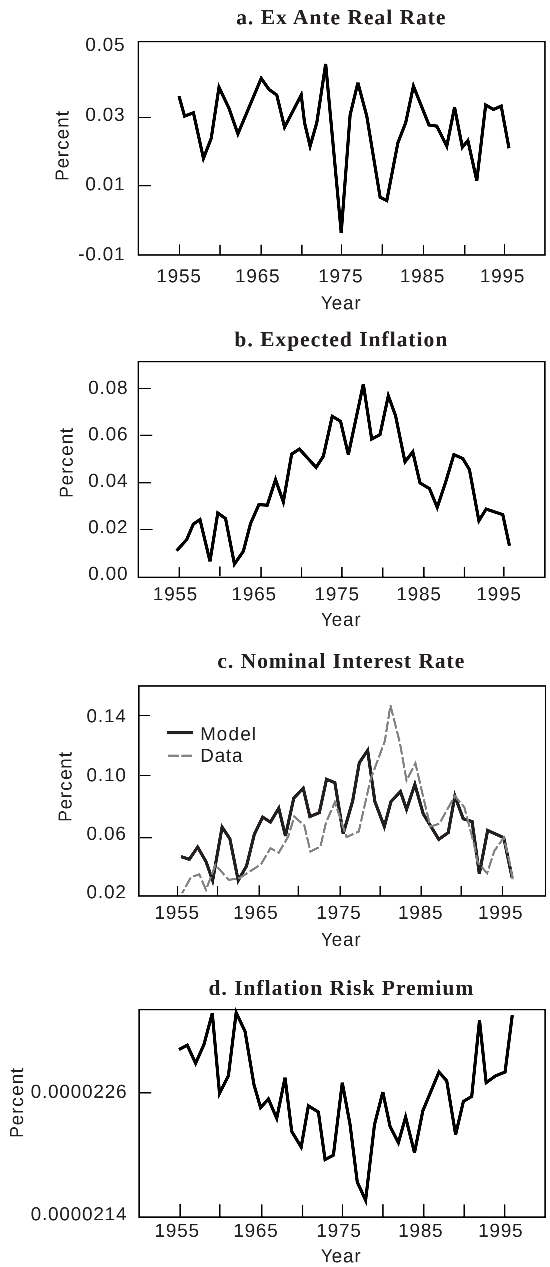


Table 2

	Ex Ante Real Rate: r_t	**Expected Inflation:** $E_t(p_{t+1}/p_t)$	**Nominal Rate:** r_t^N
$\gamma = 0.75$	mean: 2.75	mean: 4.03	mean: 6.86
	std: 1.24	std: 2.43	std: 2.62
	$\text{corr}(r_t, E_t(p_{t+1}/p_t))$: -0.16	$\text{var}(\frac{E_t(p_{t+1}/p_t)}{r_t})$: 3.80	
$\gamma = 1.75$	mean: 2.69	mean: 4.03	mean: 6.78
	std: 2.43	std: 2.43	std: 3.75
	$\text{corr}(r_t, E_t(p_{t+1}/p_t))$: -0.39	$\text{var}(\frac{E_t(p_{t+1}/p_t)}{r_t})$: 1.00	
$\gamma = 6$	mean: 2.65	mean: 4.03	mean: 6.65
	std: 11.06	std: 2.43	std: 10.75
	$\text{corr}(r_t, E_t(p_{t+1}/p_t))$: -0.37	$\text{var}(\frac{E_t(p_{t+1}/p_t)}{r_t})$: 0.04	

This result emerges primarily because unexpected movements in consumption and inflation surprises appear to have little covariation in U.S. data. In other words, since inflation surprises are largely unrelated to consumption growth, there is no reason why the inflation risk premium should be either positive or negative. Moreover, the latter notion was shown to have little to do with the equity premium being typically small in consumption-based asset pricing models. Therefore, although the Fisher equation does not theoretically apply in an environment with stochastic inflation, it may serve as an adequate approximation in practice.

Using two different preference structures, we also find that the model-implied nominal yield on one-year bonds matches the actual one-year yield on Treasury notes relatively well for most of the sample period. However, the model fails to track the nominal rate adequately in the late 1970s. We suspect that this latter result is partly driven by the singularly aggressive stance adopted by the Federal Reserve at that time in order to bring down very high inflation rates. In interpreting our results concerning the inflation risk premium, one needs to be cognizant of the model's failure along this dimension. Our benchmark cases also suggest a real rate whose volatility is more than half that of its U.S. ex post counterpart. Further, our framework in all cases provides additional evidence to support Fama's (1990) view that expected inflation and the real rate tend to move in opposite directions. Finally, we find that under both preference specifications, whether the real rate is more or less volatile than expected inflation depends heavily on households' degree of risk aversion. Taken together, these last two points suggest one should proceed with caution when interpreting movements in short-term nominal yields in terms of movements in expected inflation.

APPENDIX A: HOUSEHOLD OPTIMALITY CONDITIONS

Let λ_t and μ_t represent the Lagrange multipliers associated with constraints (2) and (3), respectively. Then, the first-order conditions associated with the household's problem are given by

$$u'(c_t) = \mu_t + \beta E_t \lambda_{t+1} \frac{p_t}{p_{t+1}}, \tag{20}$$

$$q_t \lambda_t = \beta E_t \lambda_{t+1}, \tag{21}$$

$$x_t \frac{\lambda_t}{p_t} = \beta E_t \frac{\lambda_{t+1}}{p_{t+1}}, \tag{22}$$

and

$$\lambda_t = \mu_t + \beta E_t \lambda_{t+1} \frac{p_t}{p_{t+1}}. \tag{23}$$

APPENDIX B: USEFUL PROPERTIES OF LOG-NORMAL RANDOM VARIABLES

This appendix describes properties of log-normal random variables that are useful in deriving the solution for bond prices described in Section 3. Let x be a log-normal random variable, then

- $\ln E(x) = E(\ln x) + (1/2)\mathrm{var}(\ln x)$ and
- $\ln E(x^a) = aE(\ln x) + (a^2/2)\mathrm{var}(\ln x)$ for $a \in \mathcal{R}$.

Furthermore, if y is a log-normal random variable, then so is $z = xy$. To see this, note that $\ln z = \ln x + \ln y$, which is the sum of two normal random variables and thus itself normally distributed. It directly follows from the first of the above properties that

- $\ln E(xy) = E(\ln x) + E(\ln y) + (1/2)\mathrm{var}(\ln x) + (1/2)\mathrm{var}(\ln y) + \mathrm{cov}(\ln x, \ln y).$

APPENDIX C: JENSEN'S INEQUALITY AND THE VARIANCE OF INFLATION SHOCKS

Since

$$\ln E_t \frac{p_{t+1}}{p_t} = E_t \ln \frac{p_{t+1}}{p_t} + \frac{1}{2}\mathrm{var}_t \ln \frac{p_{t+1}}{p_t}$$
$$= E_t \ln \phi_{t+1} + \frac{1}{2}\mathrm{var}_t \ln \phi_{t+1},$$

equation (9) directly implies that

$$E_t \frac{p_{t+1}}{p_t} = \exp(\delta_{\phi 0} + \frac{\sigma_\phi^2}{2})P_t, \tag{24}$$

where $P_t = \Pi_{j=0}^{p} \zeta_{t-j}^{\delta_{\phi\zeta,j}} \Pi_{j=0}^{p} \phi_{t-j}^{\delta_{\phi\phi,j}}$. Furthermore, since $\ln E_t p_t/p_{t+1}$ can simply be expressed as $\ln E_t(p_{t+1}/p_t)^{-1}$, we also have that

$$E_t \frac{p_t}{p_{t+1}} = \exp(-\delta_{\phi 0} + \frac{\sigma_\phi^2}{2})P_t^{-1}. \tag{25}$$

It then follows that

$$E_t \frac{p_{t+1}}{p_t} - \frac{1}{E_t(p_t/p_{t+1})} = \exp(\delta_{\phi 0}) \left[\exp(\frac{-\sigma_\phi^2}{2}) - \exp(\frac{\sigma_\phi^2}{2})\right] P_t^{-1}. \tag{26}$$

Hence, the difference on the left-hand side of equation (26) rises with σ_ϕ^2 as conjectured. This is shown in Figure 1.

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