# Potential Consequences of Linear Approximation in Economics

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The equilibrium of a dynamic macroeconomic model can usually be represented by a system of nonlinear difference equations, and in contemporary models these systems can be large and difficult to solve. Even for small models, it is common research practice to use approximations that allow for analytical statements about the model's behavior. The most popular form of approximation is linearization around a steady state.[1]

To study linear approximations, economists have access to the methods for solving dynamic linear models described in Sargent (1979) and Blanchard and Kahn (1980). Blanchard and Kahn provide conditions under which there exists a unique nonexplosive solution to a system of linear difference equations. In his analysis of first- and second-order linear difference equations, Sargent explains how, in some cases, models based on optimizing behavior justify the exclusion of explosive solutions as equilibria. Subsequently, attention has shifted toward explicitly nonlinear optimization-based models, but the methods described by Sargent and Blanchard and Kahn have been widely used to study the linear approximations to nonlinear models.

For as long as linear approximations have been used, economists have been aware of certain limitations. In particular, linear approximations may be quantitatively inaccurate unless one restricts attention to the model's behavior near the point around which the linearization was taken. Recent work by Benhabib, Schmitt-Grohé, and Uribe (2001) has highlighted an additional limitation of linearization that is potentially more severe: linearization may lead

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[1] A steady state is a point $\bar{x}$ such that if $\bar{x}$ is an equilibrium in any period $t$, then $\bar{x}$ is also an equilibrium in period $t+1$.

one to incorrect conclusions about the existence or uniqueness of equilibrium. These scholars have argued, based on this reasoning, that a monetary policy rule widely advocated for its stabilization properties may actually subject the economy to multiple equilibria. Our purpose in this article is to provide a simple exposition of the type of problems highlighted by Benhabib, Schmitt-Grohé, and Uribe. While we do not advocate that linearization be abandoned entirely, it is important for users to be aware of the risks.

We use two simple models to illustrate the risks of linearization. In both models, the dynamics boil down to one equation in one variable. This simplicity means that it is straightforward to compare the results based on linearization to the model's global properties.

In the first model, the single equation concerns the evolution of the stock of government debt. We will show that analysis of this model based on linearization can lead one to an erroneous conclusion about whether an equilibrium exists. A researcher might conclude, for example, that a particular tax policy rule leads to the nonexistence of equilibrium when, in fact, an equilibrium does exist for all but extreme initial values of debt. Or, linearization could suggest that an equilibrium always exists when actually there is none outside a narrow range of initial levels of debt.

In the second model, the single equation concerns the evolution of the inflation rate. There, an equilibrium always exists, but naive analysis based on linearization can lead one to erroneously conclude that there is only one equilibrium when in fact there are many. Thus, a researcher using linearization might advocate a particular policy rule based on its promise of delivering a unique equilibrium when in fact that rule is susceptible to multiple equilibria. This precise critique has been made by Benhabib, Schmitt-Grohé, and Uribe against recent work advocating "active Taylor rules" for monetary policy.

In more complicated models, it may not be possible to determine whether a linear approximation results in misleading conclusions about the uniqueness and existence of equilibrium. However, in closing we will offer some suggestions for minimizing the risk of being misled.

## 1. MACROECONOMIC EQUILIBRIUM

A typical macroeconomic model consists of a set of maximization problems and a set of market clearing conditions pertaining to a vector of variables. Solving a model involves two steps: The first step is to derive the optimality conditions that describe solutions to the maximization problems in isolation. The second step is to collect these conditions with the market clearing conditions and manipulate them so that the variables whose values are not known at the beginning of a period (for example, the price of a unit of capital) are expressed as functions of the variables whose values are known at the beginning of a period (for example, the capital stock). We refer to the former set of variables as nonpredetermined, and the latter set as predetermined. If at least one such vector-valued function exists, then an equilibrium of the model exists. If there is exactly one such function, equilibrium is unique, whereas multiple functions correspond to multiple equilibria. The second step can be difficult, especially for models with many variables, and it often requires some numerical approximations. The most popular approximation method is linearization around a steady state.

We will describe two simple models, which will then be used in our analysis of linearization. In both models, there is assumed to be an infinitely lived representative consumer who receives a constant endowment of consumption goods each period. The consumer discounts future utility at rate $\beta$ per period. In the first model, there is a government that purchases a constant amount of the consumption goods each period. The government issues debt and levies lump- sum taxes in order to pay for its consumption. In the second model, there is no government spending; however, the consumer derives utility from real money balances as well as consumption. The government issues nominal money by making lump-sum transfers to consumers.

### A Model with One Dynamic Variable, Predetermined

The representative consumer has preferences for current and future consumption ($c_t$) given by the maximization problem

$$\max \sum \beta^t u(c_t), \tag{1}$$

subject to the budget constraint

$$c_t + b_{t+1} + \tau_t \leq y + r_{t-1} b_t, \tag{2}$$

where $u$ () is an increasing and concave function; $b_t$ is the quantity of one-period, real government debt maturing in period $t$, paying a gross interest rate of $r_{t-1}$; $\tau_t$ is the lump-sum tax levied in period $t$; and $y$ is the constant endowment received each period. Denoting by $\lambda_t$ the Lagrange multiplier on the budget constraint at time $t$, the first order condition for consumption is

$$u'(c_t) = \lambda_t; \tag{3}$$

the marginal value to the consumer of an additional unit of income is equal to the marginal utility associated with using that income for consumption. The first order condition for bond holding is

$$\lambda_t = \beta r_t \lambda_{t+1}; \tag{4}$$

the marginal utility of income in the current period is equated to the present discounted utility of converting current income into future income at the given interest rate $r_t$. The transversality condition is

$$\lim_{t \longrightarrow \infty} \beta^t \lambda_t b_{t+1} = 0; \tag{5}$$

this condition can be viewed as the first order condition for bond purchases in the "final period." It would be suboptimal for a consumer to accumulate bonds so that the present utility value of consumption that could be realized by selling those bonds in the distant future did not go to zero.

The government budget constraint is

$$b_{t+1} + \tau_t \geq g + r_{t-1} b_t, \tag{6}$$

where $g$ is the constant level of per-period government purchases of goods. The left-hand side is the government's sources of revenue, and the right-hand side is the government's uses of revenue. In equilibrium the goods market clears, implying that government consumption plus private consumption equals the total endowment of goods:

$$c_t + g = y. \tag{7}$$

Because the endowment and government consumption are constant, private consumption must also be constant:

$$c_t = c \equiv y - g \qquad \forall t. \tag{8}$$

Since equilibrium consumption is constant, the marginal utility of consumption is constant, and thus, from (3) and (4), the real interest rate is constant in equilibrium:

$$r_t = \beta^{-1}. \tag{9}$$

It remains to solve for the equilibrium quantity of government debt ($b_{t+1}$) and the tax rate ($\tau_t$). The two equations left to determine these variables are the government budget constraint (6) and the transversality condition (5). One might think that the consumer's budget constraint (2) is an additional equation. However, if we substitute the market clearing condition (7) into the consumer's budget constraint, the consumer's budget constraint and the government budget constraint become identical:

$$b_{t+1} = (g - \tau_t) + \beta^{-1} b_t. \tag{10}$$

This result is an implication of *Walras's law* (see Varian [1992, 317]).

It is clear that the government budget constraint and the transversality condition are not sufficient to determine a unique equilibrium path, or even a finite number of equilibrium paths for $b_{t+1}$ and $\tau_t$. In order to narrow the set of equilibria, the standard research practice is to specify a policy rule for the quantity of debt issued or, more commonly, for the tax rate. Given a rule that sets the tax rate as a function of other variables, one can determine whether equilibrium exists and is unique.

Note that if the rule makes the tax rate a function of no variables other than $b_{t+1}$ or $b_t$, substituting the tax rule into (10) yields one equation that implicitly determines the current period debt as a function of the predetermined debt from the previous period. Henceforth, we will assume that the tax rule sets the lump-sum tax as a function of only the predetermined level of debt, $\tau_t = h(b_t)$. In this case, (10) explicitly describes the evolution of government debt:

$$b_{t+1} = g - h(b_t) + \beta^{-1} b_t. \tag{11}$$

Below we will assume a particular form for $h()$, and thus we will be able to determine whether equilibrium exists and is unique.

## A Model with One Dynamic Variable, Nonpredetermined

The second model we will consider is one in which we again end up with a single dynamic equation, although in this case the equation will not contain a predetermined variable. Here the representative consumer has preferences for current and future real money balances ($m_t$) as well as consumption ($c_t$), given by the maximization problem

$$\max \sum \beta^t [u(c_t) + v(m_t)], \tag{12}$$

where $u()$ and $v()$ are increasing, concave functions.[2] In this model, the government issues non-interest-bearing money by making lump-sum transfers to consumers.[3] The consumer maximizes utility subject to the budget constraint

$$c_t + \frac{M_t}{P_t} + \frac{B_{t+1}}{P_t} = y + \frac{M_{t-1}}{P_t} + \frac{R_{t-1} B_t}{P_t} + \frac{TR_t}{P_t}. \tag{13}$$

In (13), $c_t$ and $y$ are as defined above. The new variables in (13) are the nominal money supply ($M_t = m_t P_t$), the price level ($P_t$), the quantity of one-period nominal bonds maturing in periods $t+1$ and $t$ ($B_{t+1}$, $B_t$), the nominal interest rate on bonds maturing in the current period ($R_{t-1}$), and the quantity of nominal transfers from the government to the household ($TR_t$).[4]

Just as in the first model, the first order condition for consumption is given by (3), with $\lambda_t$ now the Lagrange multiplier on the budget constraint (13). The first order condition for real money balances is

$$v'(m_t) - \lambda_t + \beta \frac{P_t}{P_{t+1}} \lambda_{t+1} = 0. \tag{14}$$

If the consumer increases real balances marginally in period $t$, he or she gains current utility directly ($v'(m_t) > 0$) but sacrifices current period consumption

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[2] We also assume $\lim_{m \to 0} v'(m) < u'(y)$ and $v'(\breve{m}) = 0$, where $\breve{m} < \infty$. The former condition implies that, at a sufficiently high (finite) nominal interest rate, the economy will demonetize. The latter condition implies that, at a nominal interest rate of zero, individuals become satiated with a finite level of real balances.

[3] See Brock (1975) and Obstfeld and Rogoff (1983) for further discussion of related models.

[4] Although we have included nominal bonds in the consumer's budget constraint, their quantity will be zero in equilibrium (we assume that the government does not issue or purchase bonds, and since households are identical, the quantity of bonds must be zero).

valued at $\lambda_t$. The same *nominal* balances are available in the next period as a source of income to be used for consumption. However, the real value in the next period of those nominal balances is deflated by the inflation rate $P_{t+1}/P_t$, and the marginal utility is discounted back to the current period by the factor $\beta$. Condition (14) states that these effects are mutually offsetting: optimal behavior implies that a marginal change in real balances leaves utility unchanged.

The first order condition for holdings of nominal bonds is

$$\lambda_t = \beta \frac{P_t}{P_{t+1}} R_t \lambda_{t+1}. \tag{15}$$

The interpretation of (15) is similar to that of (4). However, because here the bonds pay off in dollars instead of goods, current real income is converted into future real income at rate $\frac{P_t}{P_{t+1}} R_t$. Finally, the transversality condition for money is[5]

$$\lim_{t \longrightarrow \infty} \beta^t \lambda_t m_t = 0. \tag{16}$$

This has a similar interpretation to the bond transversality condition in the first model.

The government budget constraint is

$$M_t/P_t + TR_t = M_{t-1}/P_t; \tag{17}$$

the left-hand side is the government's sources of revenue, and the right-hand side is the government's uses of revenue. Because we assume that any changes in the money supply are automatically accomplished by lump-sum transfers, the government budget constraint does not play any role in the determination of equilibrium.

In equilibrium the goods market clears, implying that private consumption equals the total endowment of goods:

$$c_t = y. \tag{18}$$

As before, constant consumption implies that the marginal utility of consumption is constant, and in this case, from (15), the equilibrium nominal interest rate is equal to expected inflation divided by the discount factor

$$R_t = \beta^{-1} \frac{P_{t+1}}{P_t} \tag{19}$$

(this is a version of the Fisher equation relating nominal and real interest rates; see Fisher [(1930) 1954]). Combining (15), (3), and (19), we see that there is a simple relationship between the nominal interest rate and the marginal

[5] There is also a transversality condition for bonds. However, since the quantity of bonds is zero, this condition is automatically satisfied.

utilities of consumption and real balances:

$$v'(m_t) = u'(c)\left(1 - \frac{1}{R_t}\right). \tag{20}$$

The marginal utility of consumption is known, so equation (20) then can be used to express $m_t$ as a function of $R_t$; it is a money demand function. In turn, equation (19) determines $R_t$ as a function of expected inflation. Without a specification of monetary policy, however, we cannot determine the price level or expected inflation. The standard research practice is to specify a policy rule for the quantity of money or the nominal interest rate. Given a rule that sets one of these variables as a function of other variables, one can determine whether equilibrium exists and is unique.

Note that if the rule makes the nominal interest rate depend only on $P_t$ or $P_{t+1}$, substituting the policy rule into (19) yields one forward-looking difference equation in the price level. Henceforth, we will assume that the monetary policy rule sets the nominal interest rate as a function of the current inflation rate, $R_t = \Re\left(P_t/P_{t-1}\right)$. In this case, the difference equation describes inflation, which we will denote by $\pi$ (that is, $\pi_t = P_t/P_{t-1}$):

$$\beta^{-1}\pi_{t+1} = \Re\left(\pi_t\right). \tag{21}$$

Below we will assume a particular form for $\Re\left(\right)$, and thus we will be able to determine whether there is a unique nonexplosive equilibrium.

The reader may be struck by the fact that the difference equation in (21) is independent of the preference specification in (12). It is a common feature of simple monetary models that one can derive a difference equation in either real balances or the price level (inflation is a transformation of the price level). However, in general, one must bring in information from the "other" part of the model in order to determine whether candidate paths are equilibria. Anticipating the discussion below, here the linearization ignores information from preferences, whereas the global analysis does not.

## 2. LINEARIZATION

The models we are working with contain just one dynamic variable and can be written in the form

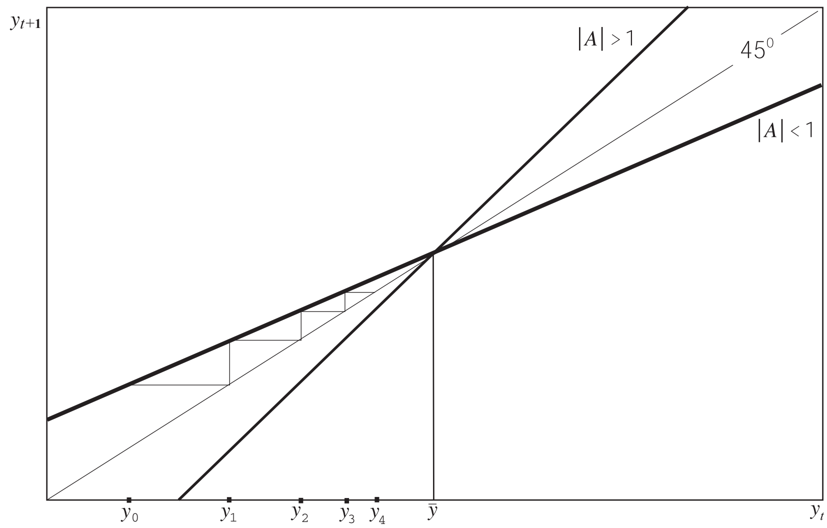
$$E_t y_{t+1} = f(y_t), \tag{22}$$

where $y_t$ is the endogenous dynamic variable.

Linearization involves first computing a steady state of this equation and then taking a first order Taylor-series approximation around that steady state. A steady state of the difference equation system is a point $\bar{y}$ such that $\bar{y} = f(\bar{y})$, and the linear approximation around this steady state is

$$(E_t y_{t+1} - \bar{y}) = f'(\bar{y})(y_t - \bar{y}). \tag{23}$$

**Figure 1 Dynamics of a Univariate Linear Difference Equation**




This approximation is guaranteed to be valid only for small deviations from the steady state. The univariate linear difference equation system (23) can be written

$$E_t \tilde{y}_{t+1} = A \tilde{y}_t, \tag{24}$$

where $\tilde{y}_t \equiv y_t - \bar{y}$.

Once we have the linearized equation, we can ask how many nonexplosive solutions there are in the neighborhood of the steady state. In general, when $y_t$ is predetermined, $|A|$ must be less than one for a unique nonexplosive solution to exist; when $y_t$ is nonpredetermined, $|A|$ must be greater than one.

The logic behind these conditions is easy to see when $A$ is positive.[6] Figure 1 plots two possible cases for this univariate linear difference equation: $A > 1$ and $0 < A < 1$. The interpretation of these two cases depends on whether the variable $y_t$ is predetermined.[7]

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[6] The case where $A < 0$ is similar. There, $-1 < A < 0$ produces dampened oscillations rather than monotone convergence; $A < -1$ produces explosive oscillations. The number of equilibria can then be determined in the same manner as in the case where $A > 0$.

[7] Using a plot of $y_{t+1}$ versus $y_t$ (such as Figure 1), it is simple to trace the time path for $y_t$ starting from an initial point $y_0$. Start with $y_0$ on the horizontal axis, and draw a vertical line up to the function $y_{t+1}$. Then draw a horizontal line to the 45-degree line and a vertical line back to the horizontal axis. This is $y_1$. Repeat to get $y_2$, etc. If $y_t$ is a predetermined variable, then the initial condition is known, and the process just described reveals the path of $y_t$ (if there

First, suppose $y_t$ is not predetermined, so that the initial condition $y_0$ is not known but instead needs to be determined in equilibrium. Then if $A > 1$, any initial value $y_0$ other than the steady state leads to $y_t$ exploding either upward or downward: the steady state $\bar{y}$ is the unique nonexplosive solution. If $A < 1$, then $y_t$ will converge back to the steady state regardless of the initial condition $y_0$: at any point in time there is a continuum of nonexplosive solutions, one of which is the steady state.

Now consider the case where $y_t$ is predetermined, so that at any point in time $y_t$ is known and $y_{t+1}$ can be read off the graph. Then if $A > 1$, unless $y_0$ happens to be equal to the steady state value, $y_t$ will explode over time: for most initial conditions, a nonexplosive solution does not exist. If $A < 1$, then $y_t$ will converge back to the steady state regardless of the initial condition $y_0$: at any point in time there does exist a unique nonexplosive solution.

For models containing more than one variable, there are related conditions involving the eigenvalues of a matrix $A$ (see Blanchard and Kahn [1980]). Sargent (1979, 177) describes the general principal as that of "solving stable roots backward and unstable roots forward."

Note that these conditions are concerned with the existence of nonexplosive solutions. It is common practice for researchers to restrict attention to nonexplosive solutions. Sometimes equilibrium must be nonexplosive because of a transversality condition. In other cases, nonexplosiveness is not a requirement of equilibrium, but researchers find other equilibria unappealing on a priori grounds. Our tax model falls into the former category: explosive behavior (at a rate greater than $\beta^{-1}$) cannot occur in equilibrium because of the transversality condition. In our monetary model, explosive behavior of the price level cannot be ruled out as an equilibrium per se, but we will nonetheless restrict attention to nonexplosive equilibria.[8]

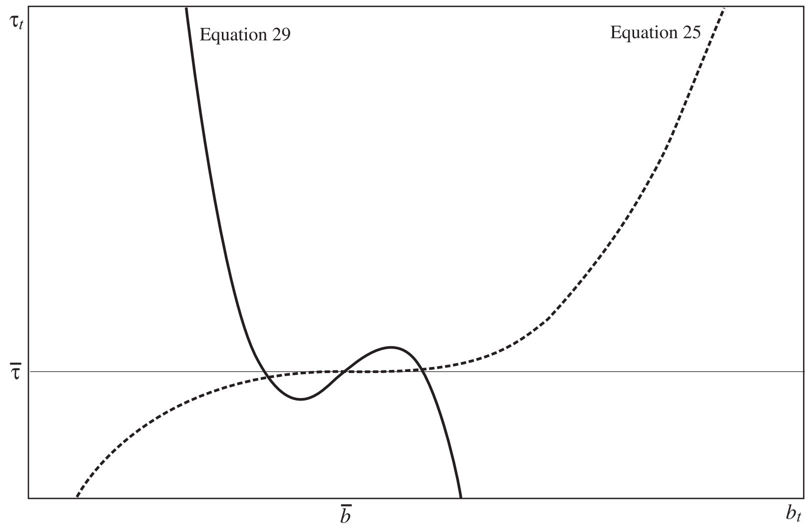
## 3. THREE PITFALLS OF EXCESSIVE RELIANCE ON LINEARIZATION

We have mentioned that linear approximations are less reliable far from the steady state. This fact typically motivates researchers who use linearization to study only examples in which there are small deviations around the steady state. However, linearization may even give incorrect answers near the steady state by suggesting an incorrect number of nonexplosive equilibria. This possibility exists because a linear approximation treats the local properties of the dynamic system as though they govern the model's global behavior, and

is an equilibrium path). If $y_t$ is not predetermined, then the process reveals whether there is a unique initial condition for which $y_t$ does not exhibit explosive behavior.

[8] As is clear from Obstfeld and Rogoff (1983), in monetary models, ruling out candidate equilibria based on simple explosiveness conditions is inappropriate. We use these conditions to make our points more clearly.

**Figure 2 Two Policy Rules for Lump-Sum Taxes**




the global behavior can be crucial in determining the number of nonexplosive equilibria. Locally the dynamics may imply that a variable explodes away from the steady state, whereas the global dynamics exhibit sufficient nonlinearity so that the explosiveness is shut off at some point. The opposite situation can also occur. Using the models described earlier, we illustrate three ways in which linear approximation can lead to an incorrect conclusion about the number of nonexplosive equilibria in a model.

### Spurious Nonexistence

It is possible that linearization suggests that there is no nonexplosive solution when global analysis reveals that one in fact exists. In the tax model above, the following tax policy gives such a result:

$$\tau_t = h(b_t) = \bar{\tau} + \tau_1 (b_t - \bar{b})^3, \tag{25}$$

where

$$\bar{b} = \frac{1}{1 - \beta^{-1}} (g - \bar{\tau}). \tag{26}$$

This policy rule is plotted in Figure 2 as the dashed line; it raises the lump-sum tax when the level of debt is above $\bar{b}$ and lowers the lump-sum-tax when the level of debt is below $\bar{b}$. The responsiveness of taxes to debt is nonlinear, rising in magnitude the further the stock of debt is from $\bar{b}$. This behavior appears reasonable, in that it might be expected to bring the level of debt back toward a steady state from any initial condition.

Combined with the government budget constraint (10), the tax rule yields an equation describing the evolution of the stock of government debt:

$$b_{t+1} = g - \bar{\tau} - \tau_1 (b_t - \bar{b})^3 + \beta^{-1} b_t. \tag{27}$$

Linearizing (27) around $\bar{b}$, which is a steady state, we get

$$b_{t+1} - \bar{b} = \beta^{-1} \left( b_t - \bar{b} \right). \tag{28}$$

Notice that in the linearized form of the tax policy, taxes do not respond to debt: $\tau_t = \bar{\tau}$. Given this nonresponsiveness, it is not surprising that an application of the conditions discussed in Section 2 indicates that an equilibrium does not exist unless the initial debt stock is equal to $\bar{b}$. According to the linearized model, for any initial debt level other than $\bar{b}$, the quantity of debt will grow at rate $1/\beta$, violating the transversality condition.

The global analysis of (27) tells a very different story. A plot of $b_{t+1}$ versus $b_t$ is given in Figure 3a.[9] It turns out that there are three steady states: $\bar{b}_1$, $\bar{b}$, and $\bar{b}_2$. If the initial debt happens to be equal to one of those steady state values, there is a unique equilibrium with constant debt. If the initial debt is not equal to one of the steady state values, but it lies in one of the intervals $(\bar{b}_1, \bar{b})$ or $(\bar{b}, \bar{b}_2)$, then there is a unique equilibrium in which the debt converges to $\bar{b}_1$ or $\bar{b}_2$, respectively. The debt levels $b_l$ and $b_u$ correspond to a unique equilibrium in which the debt cycles between those two levels. If the initial debt is between $b_l$ and the steady state $\bar{b}_1$ (or between $\bar{b}_2$ and $b_u$), then there is a unique equilibrium in which the debt converges to one of the steady states. Finally, if the initial debt is either below $b_l$ or above $b_u$, then there is no equilibrium, because the debt path implied by (27) violates the transversality condition.

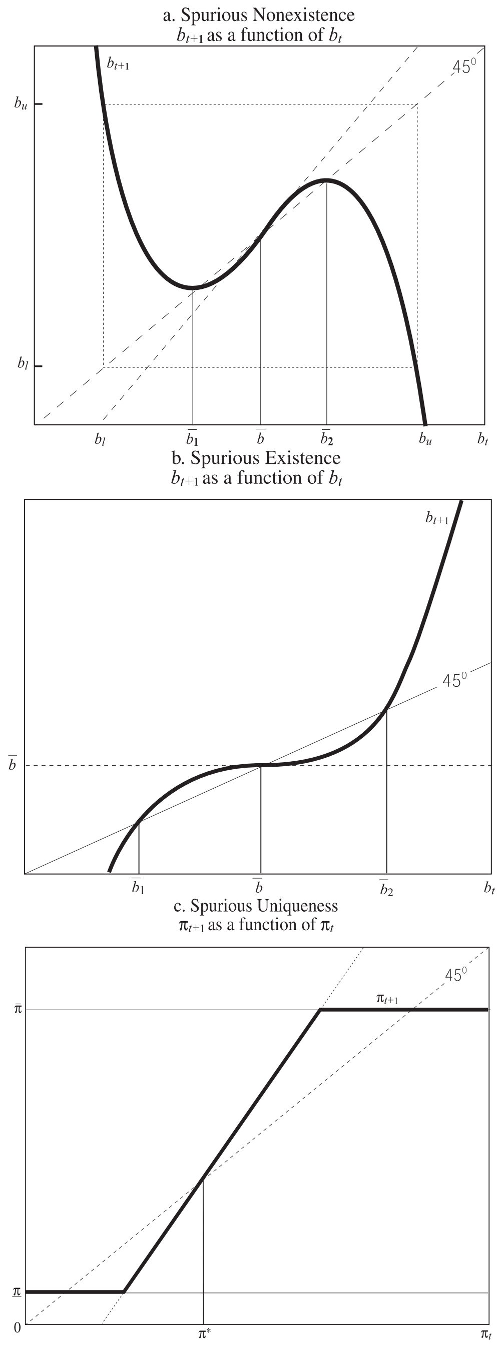
In this example, linearization leads us to conclude that an equilibrium does not exist when in fact our analysis of the global dynamics shows that there is an equilibrium for a wide range of initial conditions on the debt. Since the nonexistence of equilibrium suggests that there is a fundamental problem with a model, this possibility should lead to caution in interpreting linearization when it results in a finding of nonexistence.

## Spurious Existence

A second possibility is that there is a unique equilibrium to the linearized model, but global analysis shows that there are no equilibria for a wide range of initial conditions. Returning to the tax model, consider a tax policy given

[9] In each panel of Figure 3, the dynamics of the linear approximation are indicated by a dashed line.

**Figure 3 Three Nonlinear Difference Equations**



by

$$\tau_t = h(b_t) = -(b_t - a)(b_t - \bar{b})(b_t - c) + b_t. \tag{29}$$

This function is plotted as the solid line in Figure 2, for carefully chosen values of the parameters $a$, $\bar{b}$, and $c$. For this rule, the tax rate rises with the debt stock near the level $\bar{b}$ but decreases with the debt stock far away from $\bar{b}$. Substituting (29) into the government budget constraint (10), the equation describing the evolution of government debt is

$$b_{t+1} = g + (b_t - a)(b_t - \bar{b})(b_t - c) - b_t + \beta^{-1} b_t. \tag{30}$$

Linearizing (30), we find

$$b_{t+1} - \bar{b} = \overbrace{((\bar{b} - a)(\bar{b} - c) - 1 + \beta^{-1})}^{\gamma}(b_t - \bar{b}), \tag{31}$$

and we choose the parameters $a$, $\bar{b}$, and $c$ so that $(\bar{b} - a)(\bar{b} - c) = 1 - \beta^{-1}$, that is, $\gamma = 0$. For any starting value of $b_t$, the linearized version implies that government debt would converge immediately to the steady state $\bar{b}$. This is an example of the case where $|A| < 1$ and the one variable is predetermined. Therefore, there appears to be a unique nonexplosive solution to the linearized equations and thus a unique equilibrium.

The nonlinear difference equation (30) is graphed in Figure 3b. If the initial debt is between $\bar{b}_1$ and $\bar{b}_2$, then there is a unique equilibrium in which the debt converges to $\bar{b}$. However, if the initial debt is outside this range, no equilibrium exists; the debt path implied by (30) violates the transversality condition.

Here linearization leads us to conclude that there is a unique equilibrium, whereas global analysis reveals that existence depends on the initial debt stock. One could argue that if the initial debt stock is within a reasonable range, then there is a unique equilibrium and the linear dynamics give a good approximation to that equilibrium. However, one could choose the parameters of this example so that there is an arbitrarily small region in which the existence results from the linear analysis hold.

### Spurious Uniqueness

Finally, we can imagine a situation in which linearization suggests that there is a unique nonexplosive equilibrium when in fact there are multiple nonexplosive equilibria. This is the case that has been highlighted in the recent work by Benhabib, Schmitt-Grohe, and Uribe.[10] Turning to the monetary model,

[10] The research of Christiano and Rostagno (2001) is a related work that also uses global analysis.

consider the following interest rate rule:

$$R_t = \frac{1}{\beta} + \gamma \left( \pi_t - 1 \right), \tag{32}$$

with $\gamma > 1/\beta$. This rule represents a well-defined, feasible policy for setting the nominal interest rate, as long as the gross inflation rate is close to its *targeted* steady state value of 1. Furthermore, this type of rule has been studied in both empirical and theoretical contexts by authors such as Clarida, Gali, and Gertler (2000). It is known as an active Taylor rule, because it is a Taylor-style rule that raises the nominal interest rate more than one-for-one with the current inflation rate.

Combining the policy rule with the Fisher equation relating inflation to the nominal interest rate, we arrive at the following equation describing the evolution of inflation:

$$\pi_{t+1} = 1 + \beta\gamma \left( \pi_t - 1 \right). \tag{33}$$

This difference equation has a unique steady state $\bar{\pi} = 1$ (the targeted steady state). Furthermore, the equation is already linear, so we need merely note that the coefficient on $\pi_t$ is greater than one to see that any path for inflation other than the steady state will lead to inflation exploding upward (if $\pi_0 > 0$) or downward (if $\pi_0 < 0$). Thus, there appears to be a unique nonexplosive equilibrium.

The problem with the above reasoning is that along the explosive paths on either side of the steady state, the policy rule (33) eventually implies an infeasible choice of the nominal interest rate. First, consider an inflation path in which the initial inflation rate is positive ($\pi_0 > 1$). Along such a path, the inflation rate becomes arbitrarily high, and the path is hence ruled out as explosive. But if the inflation rate becomes arbitrarily high, at some point the gross nominal interest rate exceeds $R^* \equiv u'(y) / \left(u'(y) - \lim_{m \to 0} v'(m)\right)$. At a gross nominal interest rate of $R^*$, the model economy demonetizes; consumers will hold no money at interest rates equal to or greater than $R^*$, because the marginal benefit of real balances is bounded above by a number less than the marginal interest cost of holding real balances. The economy thus does not have a well-defined point-in-time equilibrium at a nominal interest rate above $R^*$. Similarly, if the initial inflation rate is negative ($\pi_0 < 1$), the dynamics in (33) indicate that the inflation rate will eventually become arbitrarily large with a negative sign. But then at some point the policy rule (32) requires that the gross nominal interest rate be less than unity. At a gross nominal interest rate equal to unity, consumers are satiated with real balances. The nominal interest rate cannot fall below unity because all agents would choose to hold negative quantities of nominal bonds (money would have a negative opportunity cost), and bonds are in zero net supply.

Because the interest rate rule given by (32) implies infeasible policy actions in certain situations, that rule is not a complete description of policy. A slightly modified rule that implies feasible policy actions in any situation is

$$R_t = \begin{cases} 1, \quad \text{if } \pi_t < 1 + \frac{1}{\gamma}\left(1 - \frac{1}{\beta}\right) \\ \frac{1}{\beta} + \gamma\left(\pi_t - 1\right), \quad \text{if } \frac{1}{\gamma}\left(1 - \frac{1}{\beta}\right) < \pi_t - 1 < \frac{1}{\gamma}\left(R^* - \frac{1}{\beta}\right) \\ R^*, \ \text{if } \frac{1}{\gamma}\left(R^* - \frac{1}{\beta}\right) < \pi_t - 1. \end{cases}$$

The difference equation describing equilibrium then becomes

$$\pi_{t+1} = \begin{cases} \beta, \quad \text{if } \pi_t < 1 + \frac{1}{\gamma}\left(1 - \frac{1}{\beta}\right) \\ 1 + \beta\gamma\left(\pi_t - 1\right), \quad \text{if } \frac{1}{\gamma}\left(1 - \frac{1}{\beta}\right) < \pi_t - 1 < \frac{1}{\gamma}\left(R^* - \frac{1}{\beta}\right) \\ \beta R^*, \ \text{if } \frac{1}{\gamma}\left(R^* - \frac{1}{\beta}\right) < \pi_t - 1. \end{cases}$$

This nonlinear difference equation is illustrated in Figure 3c. The consequences of modifying the policy rule so that it always delivers feasible policy actions are dramatic. In the linear difference equation (33), paths beginning from an initial inflation rate away from the steady state generated explosive behavior of inflation (see the dashed line in Figure 3c). By contrast, the modified policy rule implies that there are two steady states ($\underline{\pi} = \beta$ and $\bar{\pi} = \beta R^*$) in addition to the targeted steady state, and paths that begin away from the targeted steady state converge to one of these new steady states. Thus, instead of there being a unique nonexplosive equilibrium, there is a continuum, indexed by the initial inflation rate.

## 4. DISCUSSION

We have shown how approximating economic models by linearization around a steady state may lead to incorrect conclusions about the existence or uniqueness of equilibrium. In each of our examples, the misleading results implied by linearization were associated with a particular government policy rule. However, there is no reason to believe that it is only government policies that can lead to these problems with linear approximations. One should not assume that because a particular model has no role for government policy, a linear approximation will necessarily give the right answers about the existence and uniqueness of equilibrium.

In the simple models studied here, it was easy to see—and hence avoid—the problems associated with linearization. Unfortunately, with larger models it is harder to see the red flags signaling that linearization may be giving incorrect answers. Furthermore, global analysis (i.e., analysis of the model without any approximations) is infeasible with larger models.[11] There are,

[11] We should note that Benhabib, Schmitt-Grohé, and Uribe (2001) conduct global analysis of a two-variable system in continuous time. Kuznetsov (1998) discusses global analysis of a two-variable system in discrete time.

however, steps one can take to minimize the risk of falling victim to the problems described above. Before linearizing it is important to determine the number of steady states. If there is more than one steady state, it may not be advisable to work with a linear approximation unless one has a strong reason for believing that only one of the steady states is relevant. If there is a unique steady state, then in some models a check on the results of linear approximation can be provided by analyzing a simplified version of the model in which it is feasible to compare the local linear and global dynamics. In the case of a unique steady state, a promising approach currently receiving much attention involves taking a local *higher order* approximation to the model's system of difference equations.[12]

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[12] See Sims (2000) and Schmitt-Grohé and Uribe (2002).

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