

Indeterminacy from Inflation Forecast Targeting: Problem or Pseudo-Problem?

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Monetary economists have been rather proud about developments in their subject over the past two decades. There has been great progress in formal analysis and also in the actual conduct of monetary policy. Analytically, the profession has developed an approach to policy analysis that centers around a somewhat standardized dynamic model framework that is designed to be structural—respectful of both theory and evidence—and therefore usable in principle for policy analysis. This framework includes a policy instrument that agrees with the one typically used in practice and, in fact, models of this type are being used (in similar ways) by economists in both academia and in central banks, where several economic researchers have gained leading policymaking positions. Meanwhile, in terms of practice, most central banks have been much more successful than in previous decades in keeping inflation low while avoiding major recessions (with a few exceptions) prior to 2008. Furthermore, these improvements have been interrelated: The “inflation targeting” style of policy practice that has been adopted by numerous important central banks—and that arguably has been practiced unofficially by the Federal Reserve¹—is strongly related in principle to the prevailing framework for analysis. For a recent exposition discussing

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¹ For an argument to this effect, see Goodfriend (2005).

this development, by an author who has participated both as researcher and policymaker, see Goodfriend (2007).

There are, nevertheless, reasons for concern about current analysis including ongoing disputes about the empirical performance of key relationships in the semi-standard model; about communication and commitment mechanisms in theory and especially in practice; about the relationship of monetary policy to credit, fiscal, and foreign exchange policies; and about a myriad of technical details.² Also, there is much uneasiness about current policy approaches in the face of major credit market difficulties and indications of rising inflation.

In this context, the present article will be devoted to one specific problematic feature of the recent analytical literature, namely, a lack of agreement concerning the importance of multiple-solution indeterminacies in the analysis of monetary policy rules. References to “indeterminacy,” in the sense of more than one dynamically stable solution, or “determinacy” appear on about 75 different pages of the hugely influential treatise on monetary policy analysis by Woodford (2003a) and are ubiquitous in the literature,³ with a substantial majority of references expressed from the point of view that takes indeterminacy per se to be a matter of serious concern, e.g., implying that policies leading to model equilibria with that property should be rigorously avoided. The motivation is that indeterminacy should be avoided because it implies both that the policymaker cannot know which candidate equilibrium will prevail and also the possibility that “sunspot” effects may be created so as to greatly increase the volatility of crucial variables.⁴ Several writers, however, have expressed the view that indeterminacy per se is not necessarily a problem—that a more appropriate criterion would be based on the concept of learnability of potential equilibria. This latter position has been taken overtly by McCallum (2001, 2003), Bullard and Mitra (2002), and Bullard (2006), and is stated or indirectly implied in a large number of writings by Evans and Honkapohja, including their influential and authoritative treatise (2001).

In the present article I wish to develop the position that indeterminacy is not necessarily problematic in the context of one particular application, namely, inflation forecast targeting in the sense of Taylor-style policy rules that respond, not to current (or past) inflation rates, but to currently expected values of inflation in future periods. That such indeterminacies might be brought about, and be undesirable, by strong responses of this type was first suggested

² In some quarters there is substantial concern over the neglect, in current mainstream analysis, of monetary aggregates. That topic is, however, distinct from those to be discussed in the present paper. For recent perspectives, see Woodford (2008) and McCallum (2008).

³ Using the EBSCOhost search engine, one finds that, over the time span January 1995–June 2008, the number of papers and books with both “indeterminacy” and “monetary policy” appearing in their abstract is 78, while the number with these two terms appearing in their text is 166. There is some double-counting in these figures as both journal articles and working papers are included in the database.

⁴ Some additional discussion is provided toward the end of Section 1.

by Woodford (1994)⁵, and the argument was further developed by Bernanke and Woodford (1997), Clarida, Galí, and Gertler (2000), and (most thoroughly) Woodford (2003a, 256–61). Subsequently, many other authors have adopted this point of view, which is briefly mentioned in the textbooks of Walsh (2003, 247) and Galí (2008, 79–80). Indeed, it is apparently the prevailing point of view among analysts, despite the positive actual experience of the Bank of England over (say) 1996–2006.⁶ I have briefly taken the opposing line of argument, that strong responses to expected future inflation rates will not be problematic, in McCallum (2001) and (2003), but those papers were primarily occupied with more general topics, which prevented a full development of this particular issue.⁷

In what follows, I will begin in Section 1 with an exposition of the nature of the indeterminacy problem in the context of inflation forecast targeting. Section 2 will then be devoted to the concept of learnability of a rational expectations (RE) solution. The position taken here is that the learnability of any particular RE solution should be considered a necessary condition for that solution to be plausible and, therefore, an equilibrium appropriate as a basis for thinking about real-world policy. In Section 3, numerical examples are developed to illustrate the points that have been made more generally, but also more abstractly, in Sections 1 and 2 and in previous writings. Section 4 then takes up the somewhat esoteric topic of “sunspot” solutions, i.e., solutions that include random components that are entirely unrelated to the specified model, including its exogenous variables. Finally, Section 5 provides a brief conclusion.

1. BASIC ANALYSIS

For concreteness, let us now adopt a simple model, representative of the recent literature, in which to discuss the issues at hand. It can be expressed in terms of a familiar three-equation structure as follows:

$$y_t = E_t y_{t+1} + b (R_t - E_t \pi_{t+1}) + v_t \quad b < 0, \quad (1)$$

⁵ It has been, accordingly, referred to by Svensson (1997) as “the Woodford warning.”

⁶ It is well known that the Bank of England’s policy during these years was to make adjustments in their policy rate in a Taylor-rule manner that responded to discrepancies between expected inflation two years ahead and the target rate.

⁷ Woodford (2003b) criticized my 2003 paper primarily with regard to aspects concerning its subsidiary position with respect to a specific MSV (minimum state variable) solution. He did express an objection in the case of the application at hand, from a perspective that will be discussed below.

$$\pi_t = \beta E_t \pi_{t+1} + \kappa (y_t - \bar{y}_t) + u_t \quad \kappa > 0; 1 > \beta > 0. \quad (2)$$

$$R_t = (1 - \mu_3) \left[(1 + \mu_1) \pi_t + \frac{\mu_2}{4} (y_t - \bar{y}_t) \right] + \mu_3 R_{t-1} + e_t \quad \mu_1 \geq 0. \quad (3)$$

Here, y_t and π_t are output and inflation expressed as fractional deviations from steady state and R_t is a one-period nominal interest rate that serves as the policy instrument. Thus, (1) is an IS-type relation consisting of a consumption Euler equation in combination with the overall resource constraint, (2) is a Calvo-style price adjustment equation, and (3) is the monetary policy rule.⁸ Also, \bar{y}_t is the flexible-price, natural rate of output, assumed to be generated exogenously by an AR(1) process⁹ with AR coefficient ρ_a and innovation standard deviation SD(a). In the policy rule, the implicit target rate of inflation is zero. We will take the shock processes for u_t and e_t to be white noise (with standard deviations SD(u) and SD(e)) and the process for v_t to be AR(1) with AR coefficient ρ and standard deviation SD(v).

In the policy rule, the policy parameters μ_1 and μ_2 govern the strength of the central bank's policy response to deviations of inflation and output, respectively, from their target values, while μ_3 reflects the extent of interest rate smoothing. We begin with the central bank's policy responding to current observed inflation, π_t , and subsequently consider rules with a response to expected future inflation. In what follows, I will, for clarity, typically take μ_2 to be zero so that policy is responding only to inflation (usually with considerable smoothing). This practice (i.e., setting $\mu_2 = 0$) changes the numerical values at which effects such as indeterminacy occur but does not alter the arguments to be made in any essential manner.

Let us begin the analysis by also setting $\mu_3 = 0$ and $u_t = 0$ so that there is no smoothing and no price-setting shock. Then, substitution of equation (3) into (1) yields

$$y_t = E_t y_{t+1} + b \left[(1 + \mu_1) \pi_t + e_t - E_t \pi_{t+1} \right] + v_t, \quad (4)$$

and we can consider (2) and (4) as a two-equation system determining the evolution of π_t and y_t . The "fundamentals" or minimum-state-variable (MSV) solution will have these variables determined as linear functions of v_t , e_t , and \bar{y}_t .¹⁰ As one final simplification we take the latter (\bar{y}_t) to be constant and

⁸ Constant terms are omitted from (1) and (3) only for expositional simplicity. The analysis that follows implicitly presumes that nonzero constants are present in both of these relationships.

⁹ That is, autoregressive of order one.

¹⁰ Here I am using MSV as an alternative name for the fundamentals solution in the manner of Evans and Honkapohja (2001, 193–4), not in the manner proposed by McCallum (1983, 2003)

normalize it at zero. Then a fundamentals solution to the model (2)(4) will be of the form

$$y_t = \phi_{11}v_t + \phi_{12}e_t \text{ and} \quad (5)$$

$$\pi_t = \phi_{21}v_t + \phi_{22}e_t, \quad (6)$$

with constant terms again omitted only for simplicity.

In this case, the expected values one period ahead are $E_t y_{t+1} = \phi_{11}\rho v_t$ and $E_t \pi_{t+1} = \phi_{21}\rho v_t$. Then we can substitute these two expressions plus (5) and (6) into (2) and (4) to obtain undetermined-coefficient relations that express the ϕ_{ij} coefficients of the solution expressions in terms of the parameters of the structural equations (2) and (4). The results of that (tedious but straightforward) exercise are as follows:

$$\phi_{11} = \frac{1 - \beta\rho}{(1 - \rho)(1 - \beta\rho) - b\kappa(1 + \mu_1 - \rho)}, \quad (7)$$

$$\phi_{12} = \frac{b}{1 - \kappa b(1 + \mu_1)}, \quad (8)$$

$$\phi_{21} = \frac{\kappa}{(1 - \rho)(1 - \beta\rho) - b\kappa(1 + \mu_1 - \rho)}, \text{ and} \quad (9)$$

$$\phi_{22} = \frac{\kappa b}{1 - \kappa b(1 + \mu_1)}. \quad (10)$$

Here, the specified signs of the basic parameters imply that $\phi_{11} > 0$, $\phi_{12} < 0$, $\phi_{21} > 0$, and $\phi_{22} < 0$, so the solution equations show that a positive shock to demand (v_t) increases both inflation and output while a positive shock to the policy rule (e_t) decreases both inflation and output. From the expressions it can also be seen that (since $b < 0$) an increase in μ_1 increases the values of the positive denominators in all four expressions, implying that the variances of both inflation and output are decreased by a stronger positive policy response to observed inflation.

Are there other solutions, i.e., other expressions in addition to (5) and (6), that give values of the jointly dependent variables y_t and π_t in terms of exogenous and/or predetermined variables while satisfying the structural equations (1)–(3)? Without attempting an exhaustive search, let us (for comparison below) consider whether π_{t-1} , the lagged inflation rate, might be an additional

or Evans (1986). The latter, but not the former, involves a concept that is, in all cases, unique by construction.

variable that should appear in the solution equations. Thus, we add π_{t-1} to (5) and (6) with coefficients ϕ_{13} and ϕ_{23} , and then repeat the steps leading to (7)–(10). Upon doing so, we find that ϕ_{23} must satisfy the cubic equation

$$(\phi_{23} - \beta\phi_{23}^2)(1 - \phi_{23}) = \kappa b(1 + \mu_1)\phi_{23} - \kappa b\phi_{23}^2. \quad (11)$$

One root of the latter is clearly 0 and the other two must satisfy the quadratic $(1 - \beta\phi_{23})(1 - \phi_{23}) = \kappa b(1 + \mu_1) - \kappa b\phi_{23}$, which can be written as

$$\beta\phi_{23}^2 - [\beta + 1 - \kappa b]\phi_{23} + [1 - \kappa b(1 + \mu_1)] = 0. \quad (12)$$

For $\mu_1 = 0$, (12) can be written as $[(1 - \kappa b) - \beta\phi_{23}](1 - \phi_{23}) = 0$, from which we see that the two roots are 1 and $(1 - \kappa b)/\beta$. Since $b < 0$ and $\kappa > 0$, the latter is unambiguously greater than 1. With $\mu_1 > 0$ but very small, there are two positive real roots that both exceed 1, and for larger μ_1 there are conjugate complex roots with modulus greater than 1. Thus, with positive μ_1 , there is no stable root to the quadratic (12) and therefore no stable root to the cubic except 0. With negative μ_1 , by contrast, one root of (12) would equal 1 and the other would be positive and smaller than 1. Thus, there is an additional stable solution when μ_1 is negative but no additional stable solution—that is, additional to the fundamentals solution given by (5)–(10)—when it is positive. This finding, of course, represents the Taylor principle for the model at hand, in the case with $\mu_2 = 0$. This conclusion agrees exactly with that of Woodford (2003a, 254), which is obtained by an alternative procedure.¹¹

With that background, we now turn to the case of inflation forecast targeting, in which the central bank's policy rule responds not to current inflation, but to $E_t\pi_{t+1}$, the current expectation of inflation one period into the future.¹² Going through steps to obtain the fundamentals solution as before, we find the equations comparable to (7)–(10) to be:

$$\phi_{11} = \frac{1 - \beta\rho}{(1 - \rho)(1 - \beta\rho) - b\kappa\rho\mu_1}, \quad (13)$$

$$\phi_{12} = b, \quad (14)$$

$$\phi_{21} = \frac{\kappa}{(1 - \rho)(1 - \beta\rho) - b\kappa\rho\mu_1}, \text{ and} \quad (15)$$

¹¹Note that Woodford's condition (2.7, Ch. 4) is more general in that it permits responses to the output gap. His analytical procedure is more amenable to generalization than the one given here, but the latter is more elementary in terms of concepts utilized.

¹²This terminology does not agree with that of Svensson (1997) but is, I think, consistent with general usage.

$$\phi_{22} = \kappa b. \quad (16)$$

By inspection we can see that the signs are as before. We can also see that in this case a stronger policy response (larger μ_1) has no effect on the variability of inflation or output that occurs in response to a policy shock. Also, it is easy to determine that the denominators in (13) and (15) are smaller than in (7) and (9), so stabilization with respect to demand shocks is less effective than when policy responds to the current inflation rate.

Our present interest in the contrast, however, concerns the multiplicity of stable solutions that is possible under the policy rule that responds to expected inflation, $E_t \pi_{t+1}$. One way to demonstrate the existence of this multiplicity is to again go through the steps leading to solution expressions while including lagged inflation as an additional state variable in solutions such as (5) and (6), i.e., by using

$$y_t = \phi_{11} v_t + \phi_{12} e_t + \phi_{13} \pi_{t-1} \quad (5')$$

$$\pi_t = \phi_{21} v_t + \phi_{22} e_t + \phi_{23} \pi_{t-1}. \quad (6')$$

That change implies that $E_t y_{t+1} = \phi_{11} \rho v_t + \phi_{13} (\phi_{21} v_t + \phi_{22} e_t + \phi_{23} \pi_{t-1})$ and $E_t \pi_{t+1} = \phi_{21} \rho v_t + \phi_{23} (\phi_{21} v_t + \phi_{22} e_t + \phi_{23} \pi_{t-1})$. Then, undetermined coefficient reasoning implies that the values for the ϕ_{ij} are given by six relations analogous to those used in deriving (11) and (12), among which are

$$\phi_{13} = b\mu_1 \phi_{23}^2 + \phi_{13} \phi_{23} \text{ and} \quad (17)$$

$$\phi_{23} = \beta \phi_{23}^2 + \kappa \phi_{13}. \quad (18)$$

From these, ϕ_{13} can be solved out, yielding the cubic equation

$$\phi_{23} = \beta \phi_{23}^2 + \kappa b \mu_1 \phi_{23}^2 / (1 - \phi_{23}). \quad (19)$$

Clearly, one solution to the latter is provided by $\phi_{23} = 0$, which then by (18) implies $\phi_{13} = 0$. This eliminates the π_{t-1} variable and leads back to the fundamentals solution obtained previously. But (19) is also satisfied by roots of the quadratic

$$\beta \phi_{23}^2 - [1 + \beta + \kappa b \mu_1] \phi_{23} + 1 = 0, \quad (20)$$

i.e., by

$$\phi_{23} = \frac{d \pm [d^2 - 4\beta]^{0.5}}{2\beta}, \quad (21)$$

where d is the term in square brackets in (20). Therefore, for some values of the parameters κ , β , b , and μ_1 there may be other real solutions, in addition to the fundamentals solution, that are stable.¹³ (If such solutions are dynamically explosive, they do not create indeterminacy.) Tedious but simple algebra shows that with $\kappa > 0$, $b < 0$, and $0 < \beta < 1$, the region of determinacy includes all values of μ_1 between 0 and $\mu_1^c = -2(1 + \beta)/\kappa b$, which is positive. For $\mu_1 < 0$ there are two positive real solutions with one of them stable (smaller than 1.0 in absolute value) so there is a second stable solution (indeterminacy), just as in the case with current inflation in the policy rule. But now it is the case that for $\mu_1 > \mu_1^c$ there are two real solutions to the quadratic, both negative, and one of them is stable. Thus, in this case we have a nonfundamentals solution for which there is no transversality condition to rule out the implied dynamic behavior as a rational expectations equilibrium. Instead, there is an infinite multiplicity of stable RE solutions indexed by the initial start-up value of π_{t-1} . In such cases, moreover, “sunspot” solutions are also possible in the sense of not being ruled out by the conditions of RE equilibria.¹⁴ This is the problem suggested by the “Woodford warning” and presented in Woodford (2003a, 252–61, 2.11) where he generalizes our expression for μ_1^c .¹⁵ Similar results are developed by King (2000, 78–82). The danger in question is made less likely, it should be mentioned, when values of μ_2 exceed zero.¹⁶

In what follows, it will be of considerable importance to take account of interest rate smoothing, that is, cases in which μ_3 in policy rule (3) is positive. For these cases, the analogous critical value will be denoted μ_1^{cc} and is given by

$$1 + \mu_1^{cc} = \left[1 + \frac{2(1 + \beta)}{-b\kappa} \right] \left(\frac{1 + \rho}{1 - \rho} \right). \quad (22)$$

This expression is the special case, with no response to the output gap, of Woodford’s expression (2003a, 258, 2.13) of Chapter 4.

¹³ Analyses are provided by Bullard and Mitra (2002; 1,121–3) and Woodford (2003a, 256–60).

¹⁴ A sunspot solution is one that includes random variables (of a martingale difference variety) that have no connection with other elements of the model. Such solutions will be considered in Section 5.

¹⁵ Ours is a special case of Woodford’s formula, as the latter formula admits the possibility of responses to the current output gap, as well as to expected inflation.

¹⁶ See, e.g., Bullard and Mitra (2002) and Woodford (2003a, 257–8).

Thus far, I have discussed inflation forecast targeting only for the case in which the expected inflation rate, to which the central bank responds, is one period into the future, i.e., $E_t\pi_{t+1}$. But clearly it could instead be $E_t\pi_{t+j}$ with $j > 1$.¹⁷ Since algebraic analysis of such cases is tedious, and since some additional concreteness to the discussion might in any case be useful, I will proceed by way of numerical calculations pertaining to a specific quantitative version of the model at hand. For our basic results, let us adopt the following calibrated parameter values, which have been chosen to be representative of semi-realistic specifications: $\beta = 0.99$, $b = -0.6$, $\kappa = 0.05$, $\rho = 0.5$, and $\rho_a = 0.95$.¹⁸ Also, we have a policy rule that responds to expected inflation j periods into the future, with $j = 0, 1, 2, 3$, or 4 , that involves no response to the output gap and includes interest rate smoothing of a realistic magnitude. Thus, we use equation (3) with $\mu_2 = 0$ and $\mu_3 = 0.8$. The object now will be to look for critical values of μ_1 at which determinacy is lost, and multiple solutions begin, as μ_1 is increased. Results are shown in the next-to-last column of the following:

j	Inflation Variable in Policy Rule	Critical Value, μ_1^{cc} with $\mu_2 = 0$	Critical Value with $\mu_2 = 0.5$
0	Δp_t	None	None
1	$E_t \Delta p_{t+1}$	1,202–1,203	1,221–1,222
2	$E_t \Delta p_{t+2}$	105–106	117.8–117.9
3	$E_t \Delta p_{t+3}$	16–17	25.8–25.9
4	$E_t \Delta p_{t+4}$	3.5–3.6	10.4–10.5

Thus, with $j = 0$, we have the familiar result that the Taylor principle holds in the sense that, with current inflation in the Taylor-style rule, determinacy obtains for all values of $1 + \mu_1 > 1$. With $E_t \Delta p_{t+1}$ in the rule, determinacy holds for all μ_1 up through 1,202, but indeterminacy sets in before μ_1 reaches 1,203. And for expectations of inflation farther in the future, the determinacy region becomes progressively smaller until it disappears entirely at $j = 5$.¹⁹ The importance of cases with j greater than 1 or 2 is underlined by the aforementioned example of the Bank of England, which in recent years conducted policy so as to bring the expected inflation rate average over $j = 5, 6, 7$, and 8 into equality with its target value of 2.5 percent per annum.²⁰

¹⁷ Of course averages or other combinations would also be possible.

¹⁸ The values of β and ρ_a are quite standard in the literature, while the value for b is representative of most analysts (though not Woodford [2003a]). For κ , 0.05 is a bit larger than Woodford's 0.024 and, finally, my choice of ρ_v is quite arbitrary but designed to reflect a moderate degree of positive autocorrelation.

¹⁹ Complete disappearance might not occur, depending on specificational details, if the model's equations were enriched so as to imply more persistence. One reader has asked how to interpret the magnitudes of μ_1 . One relevant point is that the original Taylor rule value is 0.5, but the literature seems to suggest that values up to 5.0, at least, are practical. More implausibly, but still of analytical relevance, some writers have implicitly suggested that no finite value is large enough.

²⁰ See, for example, Bean and Jenkinson (2001) and U.K. Treasury (2003).

As mentioned above, policy-rule responses to the output gap can also be helpful in creating determinacy. To illustrate this possibility, the final column in the table shows the critical value for μ_1 when the output response coefficient, μ_2 , is set equal to 0.5. It will be seen that for each j , the critical value is increased, thereby reflecting a larger range of μ_1 values over which determinacy prevails. The quantitative magnitude of this improvement is not great, however.

What are the undesirable consequences of indeterminacy? Since I am doubtful that there are any, I will quote other writers. Woodford (2003a, 45) states that

...even if one restricts one's attention to bounded solutions ...there is an extremely large set of equally possible equilibria. These include equilibria in which endogenous variables such as inflation and output respond to random events that are completely unrelated to economic "fundamentals" (i.e., to the exogenous disturbances that affect the structural relations ...) and also equilibria in which "fundamental" disturbances cause fluctuations in equilibrium inflation and output that are arbitrarily large relative to the degree to which the structural relations are perturbed. Thus, in such a case, macroeconomic instability can occur owing purely to self-fulfilling expectations.

More compactly, Lubik and Schorfheide (2004, 190) state that "broadly speaking, indeterminacy has two consequences. First, the propagation of fundamental shocks, such as technology or monetary policy shocks, [through] the system is not uniquely determined. Second, sunspot shocks can influence equilibrium allocations and induce business-cycle fluctuations that would not be present under determinacy." From a more specific perspective, several writers have attributed poor performance of U.S. monetary policy during the 1970s to a policy rule that permitted indeterminacy.²¹

2. LEARNABILITY, NOT DETERMINACY

Having posed the indeterminacy problem, I now proceed to argue that it is in fact a pseudo-problem, i.e., one that should not be considered as relevant for policymaking in actual economies. The basic argument is that for any RE solution to be considered plausible, and therefore potentially relevant for policy analysis, it should be *learnable*, in the sense of Evans and Honkapohja (2001).²² Then this requirement, which pertains to a specific least-squares

²¹ For this interpretation, see, e.g., Clarida, Galí, and Gertler (2000) and Lubik and Schorfheide (2004).

²² The Evans and Honkapohja (E&H) learning procedure has been described by various authors, including Bullard and Mitra (2002), Bullard (2006), and McCallum (2001, 2003, 2007), as well as a long list of papers by E&H.

learning procedure, eliminates the multiplicity of stable solutions described above, that is, for cases with $\mu_1 > \mu_1^{cc}$. Furthermore, from a broader and more general perspective, there is a strong tendency for the learnability requirement to rule out, as implausible, RE solutions other than a single, learnable, fundamentals solution.

The essential rationale for the learnability requirement is as follows: In any dynamic market economy, individual agents seeking optimal (or even desirable) outcomes for themselves will need to form expectations regarding future values of some endogenous variables. To do this in a manner consistent with the RE hypothesis, the agents must have a considerable amount of quantitative information regarding the time series properties of these variables, i.e., knowledge beyond a listing of relevant variables and functional forms. That quantitative knowledge cannot be gained by introspection or divine intervention or magic; instead, it must be obtained from information generated by the economy itself. A plausible model of this economy should then be one in which the agents can learn accurately about the quantitative properties of the model economy on the basis of data generated by that model. More specifically, any RE solution in the model must, to be plausible, be one that is *learnable* in the sense of there existing the possibility of the model's agents learning about the properties of that solution from data generated by the model economy. Of course, there are many conceivable learning schemes that an analyst could specify. For that reason, some economists have objected to results based on this particular learning mechanism, arguing that many others would be possible. I would argue, however, that the least-squares (LS) learning process is strongly "slanted" or "biased" toward a positive finding, i.e., toward generating a finding that the potential equilibrium in question is learnable. Specifically, the process is such that agents are depicted as forecasting on the basis of least-squares estimates of a vector-autoregression model that is correctly specified, i.e., includes the relevant lagged variables and the proper number of lags, and is re-estimated each period using data generated up through the previous period. Any particular RE solution is regarded as learnable if, as time passes with agents basing their expectations (forecasts) each period on these regressions, the system approaches this RE equilibrium. Thus, it is the case that "the LS learning process in question assumes that (i) agents are collecting an ever-increasing number of observations on all relevant variables while (ii) the structure is remaining unchanged. Furthermore, (iii) the agents are estimating the relevant unknown parameters (iv) with an appropriate estimator (v) in a properly specified model. Thus if a proposed RE solution is not learnable by the process in question—the one to which the Evans and Honkapohja (E&H) results pertain—then it would seem highly implausible that it could prevail in practice" (McCallum 2007; 1,378).

What is the relationship between a model's determinacy (or indeterminacy) and the learnability of its equilibria? McCallum (2007) demonstrates

that, for a very broad class of linear models, determinacy implies learnability (of the single stable solution) under the assumption that the economy's agents have access to current-period values of endogenous variables in the learning process. If only lagged values are available in that process, however, determinacy does not imply learnability. More importantly for the topic at hand, it is shown that models with indeterminacy (more than one stable solution) may have one or more learnable equilibria.²³ If in fact they have one, and the others are not learnable, then there would seem to be only one equilibrium that is plausible and therefore relevant for policy analysis.

It may be useful to provide a brief summary of the formulation and results developed in McCallum (2007). Accordingly, we consider a model of the form

$$y_t = A E_t y_{t+1} + C y_{t-1} + D u_t, \quad (23)$$

where y_t is a $m \times 1$ vector of endogenous variables, A and C are $m \times m$ matrices of real numbers, D is $m \times n$, and u_t is a $n \times 1$ vector of exogenous variables generated by a dynamically stable process,

$$u_t = P u_{t-1} + \varepsilon_t, \quad (24)$$

with ε_t a white noise vector and P a matrix with all eigenvalues less than 1.0 in modulus. It will not be assumed that A is invertible. This specification is useful in part because it is the one utilized in Section 10.3 of Evans and Honkapohja (2001), for which conditions relevant for learnability are reported on their p. 238.²⁴ Furthermore, the specification is very broad; in particular, any model satisfying the formulations of King and Watson (1998) or Klein (2000) can (with the use of auxiliary variables) be written in this form—which will accommodate any number of lags, expectational leads, and lags of expectational leads. In this setting, we consider solutions to model (1)–(2) of the form

$$y_t = \Omega y_{t-1} + \Gamma u_t, \quad (25)$$

in which Ω is required to be real. Then we have that $E_t y_{t+1} = \Omega(\Omega y_{t-1} + \Gamma u_t) + \Gamma P u_t$, and straightforward undetermined-coefficient reasoning shows that Ω and Γ must satisfy

²³ Cases with more than one learnable equilibrium seem to be quite rare but are possible in principle.

²⁴ Constant terms can be included in the equations of (1) by including an exogenous variable in u_t that is a random walk whose innovation has variance zero. In this case there is a borderline departure from process stability. The conditions on E&H (2001, 238) actually pertain to E-stability; see the discussion below.

$$A\Omega^2 - \Omega + C = 0, \text{ and} \quad (26)$$

$$\Gamma = A\Omega\Gamma + A\Gamma P + D. \quad (27)$$

For any given Ω , (27) yields a unique Γ generically,²⁵ but there are many $m \times m$ matrices that solve (26) for Ω . These result from different orderings of the generalized eigenvalues of the matrix pencil $B - \lambda A$. If more than one of the Ω s that satisfies (26) has all its eigenvalues less than 1 in modulus, there are multiple stable solutions, i.e., indeterminacy.

Let us then turn to conditions for learnability of specific solutions. First we review the main results outlined in McCallum (2007) with details of the argument relegated to Appendix A. We begin with the assumption that agents have full information on current values of endogenous variables during the learning process, and then we will mention a second assumption, namely, that only lagged values of endogenous variables are known during the learning process. The manner in which learning takes place in the E&H analysis is as follows. Agents are assumed to know the structure of the economy as specified in equations (1) and (2), in the sense that they know what variables are included, but do not know the numerical values of the parameters. What they need to know, to form expectations, are values of the parameters of the solution equations (25). In each period t they form forecasts on the basis of least-squares regression of the variables in y_{t-1} on previous values of y_{t-2} and any exogenous observables. Given those regression estimates, however, expectations of y_{t+1} may be calculated assuming knowledge of y_t or, alternatively, assuming that y_{t-1} is the most recent observation that is usable in the forecasting process. In the former case, the conditions reported by E&H (2001, 238) are that the following three matrices must have all eigenvalues with real parts less than 1.0:

$$F \equiv (I - A\Omega)^{-1} A, \quad (28a)$$

$$[(I - A\Omega)^{-1} C]' \otimes F, \text{ and} \quad (28b)$$

$$P' \otimes F. \quad (28c)$$

In the second case, however, the analogous conditions (E&H 2001, 245) are that the following matrices must have all eigenvalues with real parts less than 1.0:

²⁵ Generically, $I - P' \otimes [(I - A\Omega)^{-1} A]$ will be invertible, permitting solution for $\text{vec}(\Gamma)$.

$$A(I + \Omega), \quad (29a)$$

$$\Omega' \otimes A + I \otimes A\Omega, \text{ and} \quad (29b)$$

$$P' \otimes A + I \otimes A\Omega. \quad (29c)$$

Except in the case that $\Omega = 0$, which will obtain when $C = 0$, these conditions are not equivalent to those in (28).

It is important to note that use of the first information assumption is not inconsistent with a model specification in which supply and demand decisions in period t are based on expectations formed in the past, such as $E_{t-1}y_{t+j}$ or $E_{t-2}y_{t+j}$. It might also be mentioned parenthetically that conditions (28) and (29) literally pertain to the *E-stability* of the model (23)(24)—see Evans (1986) and, for a heuristic introduction, McCallum (2003; 1,157–9)—under the two information assumptions, not its learnability. Under quite broad conditions, however, E-stability is necessary and sufficient for LS learnability. This near-equivalence is referred to by E&H as the “E-stability principle.” Since E-stability is technically easier to verify, applied analysis typically focuses on it rather than on direct exploration of learnability.

Given the foregoing discussion, it is a simple matter to verify that if a model of form (23)(24) is determinate, then it satisfies conditions (28). First, determinacy requires that all eigenvalues of F must have moduli less than 1.0, so their real parts must all be less than 1.0, thereby satisfying (28a). Second, from equation (26) it can be seen that $(I - A\Omega)^{-1}C = \Omega$. Therefore, matrix (28b) can be written as $\Omega' \otimes F$. Furthermore, it is a standard result (E&H 2001, 116) that the eigenvalues of a Kronecker product are the products of the eigenvalues of the relevant matrices (e.g., the eigenvalues of $\Omega' \otimes F$ are the products $\lambda_{\Omega}\lambda_F$). Therefore, condition (28b) holds. Finally, since $|\lambda_F| < 1$, condition (28c) holds provided that all $|\lambda_P| \leq 1$, which we have assumed by specifying that (24) is dynamically stable.

3. NUMERICAL ANALYSIS

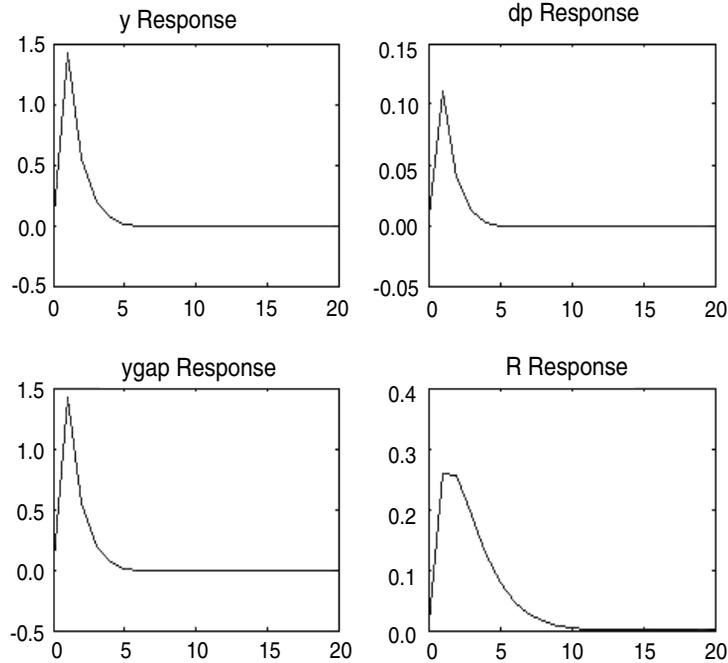
We now continue with the numerical example introduced in Section 1, where it was shown that the critical value, at which strong responses of monetary policy to expected inflation j periods into the future creates indeterminacy, decreases as j is increased—thereby making the problem (if it is one) more serious. We now look beyond that familiar finding, however, in a manner suggested by the discussion provided above. For specificity, let’s focus on one particular case—that in which Δp_{t+2} is the inflation variable in the rule. Specifically, we inspect the system eigenvalues for a policy feedback value of $\mu_1 = 105$, just below the

critical value. There the nonzero and finite eigenvalues are $0.6187 + 0.7857i$, $0.6187 - 0.7857i$, and 0.4920 . Thus, the modulus of each of the complex values is 1.0000418 , whereas the eigenvalue pertaining to the single relevant predetermined variable, R_{t-1} , is 0.4920 . If we increase μ_1 to 106 , however, the three nontrivial eigenvalues become $0.6159 + 0.7864i$, $0.6159 - 0.7864i$, and 0.4913 . These are very little changed, but now the modulus of the two complex values is 0.998903 . Thus, there are three stable eigenvalues but still only one predetermined variable, so there are three stable solutions. Among these, consider first the MOD solution, which has the same ordering as in the previous case. Then for E-stability we need, from (28a) and (A6) of Appendix A, the inverses of the two complex eigenvalues to have real parts less than 1. In fact, these inverses are $0.617236 - 0.78817i$ and $0.617236 + 0.78817i$, so both have real parts less than 1, meeting this criterion. The criteria (28b) and (28c) are both clearly met, as well. Thus, the MOD solution is E-stable and learnable.

What about the other stable solutions? One cannot exchange the place of the eigenvalue 0.4920 with either one of the complex numbers because that would give a solution expression that assigns complex values to reduced-form coefficients (thereby implying that numerical observations on interest rates, inflation rates, etc., are complex!). The only way to get a real solution is to reorder by shifting both complex eigenvalues to below the line, with both 0.4920 and 0 shifting to above the line.²⁶ But that implies that both of the latter have inverses—therefore eigenvalues of F —with real parts greater than 1. Accordingly, this solution violates the criterion (28a) necessary for E-stability and learnability. There is, therefore, only a single RE solution of form (25) for the calibration at hand. It features only a single RE solution that is plausible.

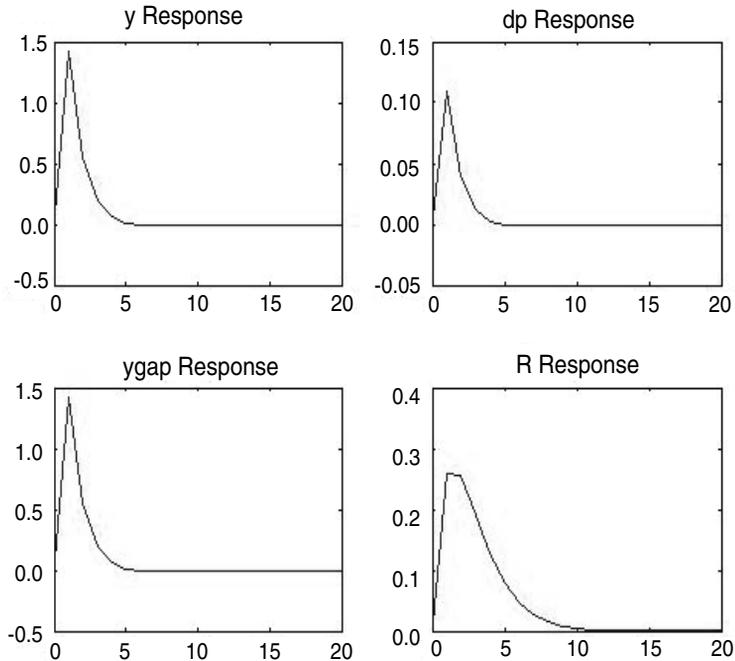
It is of some interest to continue with this example by examining impulse response functions (IRFs) pertaining to these alternative solutions. In Figure 1, we have impulse responses for the endogenous variables y_t , Δp_t , \tilde{y}_t , and R_t generated by a unit shock to the innovation in the policy rule when the μ_1 policy parameter is set at 104 , just below the critical value at which indeterminacy arises. In Figure 2, we show the IRFs for the MOD solution when $\mu_1 = 106$, just above the critical value. It will be readily observed that the responses are very similar in these two cases—indeed, are visually identical. Next, consider the implications of the hypothesis that one of the alternative (non-MOD) stable solutions is relevant for values of $\mu_1 > 105$, even though the MOD solution is relevant at $\mu_1 = 105$. Thus, in Figure 3 we report the

²⁶ The line, that is, that separates eigenvalues associated with predetermined variables from the others.

Figure 1 Responses to Unit Shock to IS, $\mu_1 = 104$ 

IRFs at $\mu_1 = 106$ for the alternative stable solution,²⁷ and in this case there is clearly no similarity to Figure 1. Indeed, the initial-period response is in the opposite direction from that in Figure 1 for both the inflation and interest rate variables. In sum, we find that there is no discontinuity pertaining to the impulse response functions around the critical value, according to the MOD solution. But with the alternative ordering (yielding a solution analogous to that with ϕ_{23} given by (21) in Section 1) there is a drastic change (relative to the IRFs for the only stable solution with $\mu_1 < 105$) resulting from a very small change in one parameter value (μ_1) (see Figure 3). This contrast seems to be strongly suggestive of the idea that the MOD solution is, in this case, much more plausible than the alternative solution, dynamic stability of the latter notwithstanding.

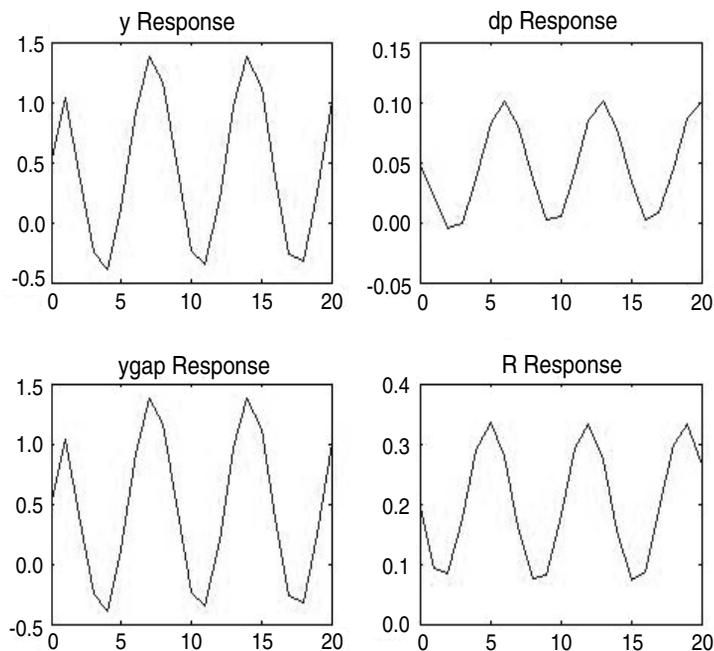
²⁷The different arrangements of the complex eigenvalues yield identical impulse response functions.

Figure 2 Responses to Unit Shock to IS, $\mu_1 = 106$ 

4. SUNSPOT EQUILIBRIA

Thus far, we have seen that nonfundamental equilibria of the form considered in (5')–(6') are not learnable, thereby lending support to the position that strenuous inflation forecast targeting is not dangerous. We have not, however, considered all possible forms of indeterminacy. In particular, in his critical review of the argument pertaining to inflation-forecast targeting in McCallum (2003), Woodford (2003b; 1,181–2) has shown that in cases with indeterminacy, solutions of a certain “sunspot” type—one that depends on an extraneous variable that evolves exogenously according to a finite-state Markov chain—can be learnable. This result accords with the analysis of Honkapohja and Mitra (2004; 1,753–4), who conduct a detailed analysis of Markov SSE (stationary sunspot equilibria) cases as well as non-Markov SSE cases. For some relevant discussion, see Appendix B.

It is nevertheless my belief that the indeterminacy under discussion pertains to RE solutions that are not plausible. This belief is based not on any refutation of the formal learnability analysis, but instead on a judgment

Figure 3 Responses to Unit Shock to IS, $\mu_1 = 106$ (alt. soln.)

concerning the way in which the formal analysis is used to gain insight into actual behavior in real-world economies. Specifically, the learning process described above in previous sections postulates individual agents who in effect base their expectations on forecasting rules implied by correctly specified (but unconstrained) vector-autoregression (VAR) models constructed using data from previous periods. To me this seems to be going as far as common sense allows in attributing sophistication to the expectation-formation processes of individuals in actual economies. But the forecasting rules needed for achievement of the Markov SSE solutions are not implied by VARs of this type; they require state-dependent intercept terms. That is, estimation of basic VARs over indefinitely long spans of time would not lead to forecast rules with the type of state-dependent parameters needed to support sunspot SSEs. Thus, I contend that such equilibria are simply implausible and should not be

considered when discussing possible RE equilibria relevant for the design of monetary policy.²⁸

Now, I can well imagine that some readers might be inclined to respond to this argument with the objection that rational expectations is itself implausible, so that there is an inconsistency in this argument. But of course, taken literally, RE itself *is* implausible—as early critics emphasized. Nevertheless, RE is rightly regarded by mainstream researchers as the appropriate assumption for the purpose of economic analysis, especially in the context of macroeconomic policy analysis. That is the case because RE is fundamentally the assumption that agents optimize with respect to their expectational behavior—just as they do (according to basic neoclassical economic analysis) with respect to other regular economic activities such as selection of consumption bundles, selection of quantities produced and inputs utilized, etc.—for a necessary condition for optimization is that individuals eliminate any systematically erroneous component of their expectational behavior. Moreover, RE is doubly attractive (to researchers) from a policy perspective, for it assures that a researcher does not propose policy rules of a type that is designed to exploit allegedly consistent patterns of suboptimal expectational behavior by individuals.

Accordingly, I contend that there is no inconsistency in using RE as one's expectational hypothesis while placing some limit on the scope of learning processes that can lead to RE equilibria. This is, as mentioned above, fundamentally a specification regarding information availability. In standard RE analysis it is assumed that agents have knowledge of the values of endogenous (and some exogenous) variables only in the present and past, not the future. Furthermore, in some influential papers, such as Lucas (1972), only partial information concerning current endogenous variables is available. There seems to be no difference in principle from our preceding argument in assuming that agents may not have observations on the current state of the system needed for the learning analysis of the Markov SSE variety.

There is also an alternative class of sunspot equilibria that will be dynamically stable when the determinacy condition of Section 1 is not satisfied. Specifically, in terms of our model (2) and (4), an arbitrary sunspot variable ξ_t with the property $E_{t-1}\xi_t = 0$ can be added to expressions like (5') and (6'), leading to stable solutions of that form. Honkapohja and Mitra (2004; 1,756) find, however, that “non-fundamental equilibria of the form (3) and (5) [their equation numbers] are never E-stable.” Thus, this alternative class of

²⁸ An alternative way of presenting this position would be to *define* an RE equilibrium in a dynamic model for monetary policy analysis so as to require LS learnability (on the basis of *basic* VAR forecasting rules) as a necessary condition—one that represents *informational feasibility*.

sunspots does not, according to the learnability criterion promoted here, pose a problem for actual monetary policy.²⁹

Some expository material pertaining to the two classes of sunspot solutions, and an apparent (but not actual) inconsistency in the results of Honkapohja and Mitra (2004) and E&H (2003), is provided in Appendix B.

5. CONCLUSIONS

In the contemporary mainstream literature on monetary policy analysis, it is typically contended that inflation forecast targeting—use of an interest rate policy rule that responds to currently expected inflation for some future period(s)—will, if applied too strongly, generate indeterminacy in the sense of a multiplicity of dynamically stable RE solutions. It is also concluded in this literature that this outcome represents a practical difficulty for monetary policymakers. By contrast, the present article argues that these findings of indeterminacy do not pose any actual problem for monetary policymakers. The reason is that in these analyses only one of the RE solutions possesses the property of least-squares learnability, a concept that is necessary for the plausibility of any rational expectations solution. Accordingly, other RE solutions could not plausibly prevail because there is no way for individual agents in the model (designed to depict reality) to obtain enough quantitative information about the economy’s dynamics to form expectations in a way that would support the solution in question. Typically, however, this objection does not pertain to a single RE solution, which is the “natural” fundamentals solution. Thus, this article contends that indeterminacy of the type in question represents a pseudo-problem, not an actual problem for actual policymakers.

APPENDIX A

Here we provide additional development of results summarized in Section 2. Continuing from three lines below equation (27), the following analysis centers around (26). Since we do not assume that A is invertible, we write

²⁹That finding is highly agreeable from the perspective of my argument. But even if it did not hold, I would continue to suggest that these arbitrary sunspot solutions are highly implausible. The crux of the matter is that the learning analysis treats the unspecified sunspot variables as observable by individual agents. But sunspot variables are, by definition, ones that represent no component of tastes, technology, government behavior, or institutional constraints—they represent merely the arbitrary beliefs of (individual and independent) market participants.

$$\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Omega^2 \\ \Omega \end{bmatrix} = \begin{bmatrix} I & -C \\ I & 0 \end{bmatrix} \begin{bmatrix} \Omega \\ I \end{bmatrix}, \quad (\text{A1})$$

in which the first row reproduces the matrix quadratic (26). Let the $2m \times 2m$ matrices on the left and right sides of (A1) be denoted \bar{A} and \bar{C} , respectively. Then instead of focusing on the eigenvalues of $\bar{A}^{-1}\bar{C}$, a matrix that does not exist when A is singular, we instead solve for the (generalized) eigenvalues of \bar{C} with respect to \bar{A} . Thus, instead of diagonalizing $\bar{A}^{-1}\bar{C}$, as in Blanchard and Khan (1980), we use the Schur generalized decomposition, which establishes that there exist unitary matrices Q and Z such that $Q\bar{C}Z = T$ and $Q\bar{A}Z = S$ with T and S triangular.³⁰ Then eigenvalues of \bar{C} with respect to \bar{A} are defined as t_{ii}/s_{ii} . Some of these are “infinite,” in the sense that some s_{ii} may equal zero. This will be the case, indeed, whenever A and therefore \bar{A} are of less than full rank since then S is also singular. All of the foregoing is true for any ordering of the eigenvalues and associated columns of Z (and rows of Q). For the present, let us focus on the arrangement that places the t_{ii}/s_{ii} in order of decreasing modulus.³¹

To begin the analysis, premultiply equation (A1) by Q . Since $Q\bar{A} = SH$ and $Q\bar{C} = TH$, where $H \equiv Z^{-1}$, the resulting equation can be written as

$$\begin{bmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \Omega^2 \\ \Omega \end{bmatrix} = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \Omega \\ I \end{bmatrix}. \quad (\text{A2})$$

The first row of (A2) reduces to

$$S_{11} (H_{11}\Omega + H_{12}) \Omega = T_{11} (H_{11}\Omega + H_{12}). \quad (\text{A3})$$

Then, if H_{11} is invertible, the latter can be used to solve for Ω as

$$\Omega = -H_{11}^{-1}H_{12} = -H_{11}^{-1}(-H_{11}Z_{12}Z_{22}^{-1}) = Z_{12}Z_{22}^{-1}, \quad (\text{A4})$$

where the second equality comes from the upper right-hand submatrix of the identity $HZ = I$, provided that H_{11} is invertible, which we assume without significant loss of generality.^{32,33}

³⁰ Provided only that there exists some λ for which $\det[\bar{C} - \lambda\bar{A}] \neq 0$. See Golub and Van Loan (1996, 377) or Klein (2000). Note that in McCallum (2007) the matrices \bar{A} and A are denoted A and A_{11} , respectively.

³¹ The discussion proceeds as if none of the t_{ii}/s_{ii} equals 1.0 exactly. If one does, the model can be adjusted by multiplying some relevant coefficient by (e.g.) 0.9999.

³² This invertibility condition, also required by King and Watson (1998) and Klein (2000), obtains except for degenerate special cases of (1) that can be solved by simpler methods than considered here. Note that the invertibility of H_{11} implies the invertibility of Z_{22} , given that Z and H are unitary.

³³ Note that it is not being claimed that all solutions are of the form (9).

As mentioned above, there are many solutions Ω to (26). These correspond to different arrangements of the eigenvalues, which result in different groupings of the columns of Z and therefore different compositions of Z_{12} and Z_{22} . Here, with the eigenvalues t_{ii}/s_{ii} arranged in order of decreasing modulus, the diagonal elements of S_{22} will all be nonzero provided that S has at least m nonzero eigenvalues, which we assume to be the case.³⁴ Clearly, for any solution under consideration to be dynamically stable, the eigenvalues of Ω must be smaller than 1.0 in modulus. In McCallum (2007) it is shown that

$$\Omega = Z_{22}S_{22}^{-1}T_{22}Z_{22}^{-1}, \quad (\text{A5})$$

so Ω has the same eigenvalues as $S_{22}^{-1}T_{22}$. The latter is triangular, moreover, so the relevant eigenvalues are the m smallest of the $2m$ ratios t_{ii}/s_{ii} (given the decreasing-modulus ordering). For dynamic stability, the modulus of each of these ratios must then be less than 1. (In many cases, some of the m smallest moduli will equal zero.)

Let us refer to the solution under the decreasing-modulus ordering as the MOD solution. Now suppose that the MOD solution is stable. For it to be the only stable solution, there must be no other arrangement of the t_{ii}/s_{ii} that would result in a Ω matrix with all eigenvalues smaller in modulus than 1.0. Thus, each of the t_{ii}/s_{ii} for $i = 1, \dots, m$ must have modulus greater than 1.0, some perhaps infinite. Is there some $m \times m$ matrix whose eigenvalues relate cleanly to these ratios? Yes, it is $F \equiv (I - A\Omega)^{-1}A$, which appears frequently in Binder and Pesaran (1995).³⁵ Regarding F , it is shown that, for any ordering such that H_{11} is invertible, including the MOD ordering, we have the equality

$$H_{11} F H_{11}^{-1} = T_{11}^{-1}S_{11}, \quad (\text{A6})$$

which implies that F has the same eigenvalues as $T_{11}^{-1}S_{11}$. In other words, the eigenvalues of F are the same, for any given arrangement, as the *inverses* of the values of t_{ii}/s_{ii} for $i = 1, \dots, m$. Under the MOD ordering these are the inverses of the first (largest) m of the eigenvalues of the system's matrix pencil. Accordingly, for solution (A4) to be the only stable solution, all the eigenvalues of the corresponding F must be smaller than 1.0 in modulus. This

³⁴ From its structure it is obvious that \bar{A} has at least m nonzero eigenvalues so, since Q and Z are nonsingular, S must have rank of at least m . This necessary condition is not sufficient for S to have at least m nonzero eigenvalues, however; hence, the assumption.

³⁵ There is no general proof of invertibility of $[I - A\Omega]$, but if $A\Omega$ were by chance to have some eigenvalue exactly equal to 1.0, that condition could be eliminated by making some small adjustment to elements of A or C .

is a generalization of a result of Blanchard and Khan (1980) for a model with nonsingular A .

Thus, we have established notation and reported results showing that the existence of a unique stable solution requires that all eigenvalues of the defined Ω matrix and the corresponding F must be less than 1.0 in modulus. It will be convenient to express that condition as follows: all $|\lambda_\Omega| < 1$ and all $|\lambda_F| < 1$.

APPENDIX B

To illustrate some concepts pertaining to sunspot equilibria, consider the simple univariate model

$$x_t = aE_t x_{t+1} + u_t \quad a \neq 0, a \neq 1, \quad (\text{B1})$$

where u_t is generated by an AR(1) process with AR parameter ρ , assuming $|\rho| < 1$.³⁶ Then the fundamental RE solution will be of the form $x_t = \phi u_t$ so $E_t x_{t+1} = \rho \phi u_t$ and the undetermined-coefficient procedure relationship $\phi u_t = a\rho\phi u_t + u_t$ implies that $\phi = 1/(1 - a\rho)$. Thus, the fundamental solution is

$$x_t = \frac{1}{1 - a\rho} u_t. \quad (\text{B2})$$

To introduce sunspot phenomena, consider solutions of the form

$$x_t = \phi_1 x_{t-1} + \phi_2 u_t + \phi_3 \xi_t, \quad (\text{B3})$$

where ξ_t is a “sunspot” variable—i.e., an extraneous variable generated by any stochastic process such that $E_{t-1} \xi_t = 0$. Then we have

$$E_t x_{t+1} = \phi_1 (\phi_1 x_{t-1} + \phi_2 u_t + \phi_3 \xi_t) + \phi_2 \rho u_t + 0, \quad (\text{B4})$$

and substitution of (B3) and (B4) into (B1) yields

$$\phi_1 x_{t-1} + \phi_2 u_t + \phi_3 \xi_t = a [\phi_1 (\phi_1 x_{t-1} + \phi_2 u_t + \phi_3 \xi_t) + \phi_2 \rho u_t] + u_t. \quad (\text{B5})$$

³⁶ Symbols are used here with meanings potentially different from those in the body of the article and in Appendix A.

For the latter to hold for all values of x_{t-1} , u_t , and ξ_t —i.e., to be a solution—it is necessary that

$$\phi_1 = a\phi_1^2, \tag{B6a}$$

$$\phi_2 = a\phi_1\phi_2 + \phi_2a\rho + 1, \text{ and} \tag{B6b}$$

$$\phi_3 = a\phi_1\phi_3. \tag{B6c}$$

The first of these equations has two solutions, $\phi_1 = 1/a$ and $\phi_1 = 0$. The latter gives the fundamentals solution, but the former gives other solutions. Thus, with $\phi_1 = 1/a$, (B6b) becomes $\phi_2 = -1/a\rho$, while (B6c) reduces to $\phi_3 = \phi_3$ (i.e., is satisfied by any finite value). Accordingly, there are sunspot solutions satisfying

$$x_t = (1/a)x_{t-1} - (1/a\rho)u_t + \phi_3\xi_t, \tag{B7}$$

where ϕ_3 can be any real number. If $|a| < 1$, these solutions are explosive, but we have an infinity of stable sunspot solutions if $|a| > 1$.

The learnability of solutions (B2) and (B7) has been studied by E&H (2003), who show that the fundamentals solution is E-stable if $a < 1$ and is not E-stable if $a > 1$, whereas the sunspot solutions (B7) are not E-stable (or learnable) for any value of a .

E&H (2003) also consider, however, a special class of K-state Markov sunspots such that $x_t = \bar{x}(i)$ when the state variable s_t equals the constant $s(i)$, for $i = 1, 2, \dots, K$, and evolves in accordance with a K-state Markov process with fixed transition probabilities. In this case, it transpires that non-fundamental solutions can be E-stable, even though such solutions can be written as special cases of the form (B3). That seemingly contradictory result is actually compatible with the result of the previous paragraph because adoption of the K-state Markov specification places additional structure on the system that is not implied by (B3), and consequently uses, in effect, a different “perceived law of motion” for the learnability analysis. E&H remark that “. . . one way to view our results is that the learning processes are attempting to learn different things for the different representations: for the AR(1) form of SSEs the learning rule corresponds to least squares estimation of the coefficients, while for the two-state Markov representation of 2-SSEs the learning scheme in effect estimates the support of the distribution” (2003, 179). My reasons for rejecting these solutions are developed in Section 5.

REFERENCES

- Bean, Charles, and Nigel Jenkinson. 2001. "The Formulation of Monetary Policy at the Bank of England." *Bank of England Quarterly Bulletin* 41 (Winter): 434–41.
- Bernanke, Ben S., and Michael Woodford. 1997. "Inflation Forecasts and Monetary Policy." *Journal of Money, Credit, and Banking* 24 (November): 653–84.
- Binder, Michael, and M. H. Pesaran. 1995. "Multivariate Rational Expectations Models and Macroeconometric Modelling: A Review and Some New Results." In *Handbook of Applied Econometrics*, edited by M. H. Pesaran and M. Wickens. Oxford, U.K.: Basil Blackwell Publishers.
- Blanchard, Olivier, and Charles M. Kahn. 1980. "The Solution of Linear Difference Models Under Rational Expectations." *Econometrica* 48 (July): 1,305–11.
- Bullard, James. 2006. "The Learnability Criterion and Monetary Policy." *Federal Reserve Bank of St. Louis Review* 88 (May): 203–17.
- Bullard, James, and Kaushik Mitra. 2002. "Learning About Monetary Policy Rules." *Journal of Monetary Economics* 49 (September): 1,105–29.
- Clarida, Richard, Jordi Galí, and Mark Gertler. 2000. "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory." *Quarterly Journal of Economics* 115 (February): 147–80.
- Evans, George W. 1986. "Selection Criteria for Models with Non-Uniqueness." *Journal of Monetary Economics* 18 (September): 147–57.
- Evans, George W., and Seppo Honkapohja. 2001. *Learning and Expectations in Macroeconomics*. Princeton, N.J.: Princeton University Press.
- Evans, George W., and Seppo Honkapohja. 2003. "Expectational Stability of Stationary Sunspot Equilibria in a Forward-Looking Linear Model." *Journal of Economic Dynamics and Control* 28 (October): 171–81.
- Galí, Jordi. 2008. *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework*. Princeton, N.J.: Princeton University Press.
- Goodfriend, Marvin. 2005. "Inflation Targeting in the United States?" In *The Inflation Targeting Debate*, edited by B. S. Bernanke and M. Woodford. Chicago: University of Chicago Press.

- Goodfriend, Marvin. 2007. "How the World Achieved Consensus on Monetary Policy." *Journal of Economic Perspectives* 21 (Fall): 47–68.
- Honkapohja, Seppo, and Kaushik Mitra, 2004. "Are Non-Fundamental Equilibria Learnable in Models of Monetary Policy?" *Journal of Monetary Economics* 51 (November): 1,743–70.
- King, Robert G. 2000. "The New IS-LM Model: Language, Logic, and Limits." Federal Reserve Bank of Richmond *Economic Quarterly* 86 (Summer): 45–103.
- King, Robert G., and Mark W. Watson. 1998. "The Solution of Singular Linear Difference Systems Under Rational Expectations." *International Economic Review* 39 (November): 1,015–26.
- Klein, Paul, 2000. "Using the Generalized Schur Form to Solve a Multivariate Linear Rational Expectations Model." *Journal of Economic Dynamics and Control* 24 (September): 1,405–23.
- Lubik, Thomas A., and Frank Schorfheide. 2004. "Testing for Indeterminacy: An Application to U.S. Monetary Policy." *American Economic Review* 94 (March): 190–217.
- Lucas, Robert E., Jr. 1972. "Expectations and the Neutrality of Money." *Journal of Economic Theory* 4 (April): 103–24.
- McCallum, Bennett T. 1983. "On Non-Uniqueness in Rational Expectations Models: An Attempt at Perspective." *Journal of Monetary Economics* 11: 139–68.
- McCallum, Bennett T. 2001. "Monetary Policy Analysis in Models Without Money." Federal Reserve Bank of St. Louis *Review* 83 (July): 145–60.
- McCallum, Bennett T. 2003. "Multiple-Solution Indeterminacies in Monetary Policy Analysis." *Journal of Monetary Economics* 50 (July): 1,153–75.
- McCallum, Bennett T. 2007. "E-Stability vis-a-vis Determinacy Results for a Broad Class of Linear Rational Expectations Models." *Journal of Economic Dynamics and Control* 31 (April): 1,376–91.
- McCallum, Bennett T. 2008. "How Important is Money in the Conduct of Monetary Policy? A Comment." *Journal of Money, Credit and Banking* 40 (December): 1,783–90.
- Svensson, Lars E. O. 1997. "Inflation Forecast Targeting: Implementing and Monitoring Inflation Targets." *European Economic Review* 41 (June): 1,111–46.
- U.K. Treasury. 2003. *Policy Frameworks in the U.K. and EMU*. London: HM Treasury.

Walsh, Carl E. 2003. *Monetary Theory and Policy*, 2nd ed. Cambridge, Mass.: MIT Press.

Woodford, Michael. 1994. "Nonstandard Indicators for Monetary Policy: Can Their Usefulness Be Judged From Forecasting Regressions?" In *Monetary Policy*, edited by N. G. Mankiw. Chicago: University of Chicago Press.

Woodford, Michael. 2003a. *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton, N.J.: Princeton University Press.

Woodford, Michael, 2003b. "Comment on: Multiple-Solution Indeterminacies in Monetary Policy Analysis." *Journal of Monetary Economics* 50 (July): 1,177–88.

Woodford, Michael. 2008. "How Important is Money in the Conduct of Monetary Policy?" *Journal of Money, Credit and Banking* 40 (December): 1,561–98.