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How Can Consumption-Based Asset-Pricing Models Explain Low Interest Rates?

Felipe Schwartzman

The Great Recession gave way to a period of very low short-term nominal and real interest rates. As the recovery proceeds and the Federal Reserve starts to decide the rhythm with which it intends to raise policy rates, one fundamental question is whether the low interest rates are just a symptom of a recessionary period (even if prolonged) in which the Federal Reserve chose to take a deliberately expansionary stance, or if they reflect longer-run fundamental forces that may not dissipate easily. In the latter case, optimal policy may warrant a slow increase of the policy interest rate, so that it remains low by historical standards even when inflation and the labor market are close to their long-run levels. Currently, Federal Open Market Committee members appear to forecast such a slow increase, as documented in the Summary of Economic Projections.

The purpose of this article is to use consumption-based asset-pricing models to gain some insight into the determinants of the "natural interest rate," that is, the interest rate that would prevail in the absence of nominal rigidities. Since this natural rate is not itself a function of central bank decisions, it can be used as a yardstick for the stance of monetary policy. In particular, in terms of modern monetary theory (Woodford 2003), one can say that the policy stance is expansionary

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if the interest rate is below the "natural rate of interest" and contractionary otherwise.[1] The question about the optimal pace of interest rate liftoff can thus be recast in terms of the speed with which the natural rate of interest is likely to increase.

Consumption-based asset-pricing models are a natural starting point for the discussion of the fundamental determinants of interest rates for macroeconomists since they share conventional assumptions of most workhorse macroeconomic models: rational expectations, frictionless asset markets, and a representative household. This contrasts with behavioral economics models, which emphasize departures from rational expectations, and with segmented markets models, in which asset prices are determined by only a subset of households.[2] While these alternatives are certainly worthy of further discussion, the purpose of this article is to provide a first look at the progress that one can make with this more familiar baseline.[3] I will review three main strands within the consumption-based asset-pricing literature: habit formation, long-term risk, and disaster risk. Rather than provide a comprehensive review of the literature within each of those strands, I will discuss some of the main ideas based on a small number of influential articles.[4] At the end of each section I include a short discussion of how the model could be used to explain low interest rates. Those discussions are meant to be illustrative rather than conclusive, in that they delimit promising areas for further research rather than provide a complete answer to how well consumption-based asset-pricing models can explain currently low interest rates.

As we will see in the models reviewed, interest rates can be low either because market participants expect consumption growth to be low, because they perceive consumption risk to be high, or because

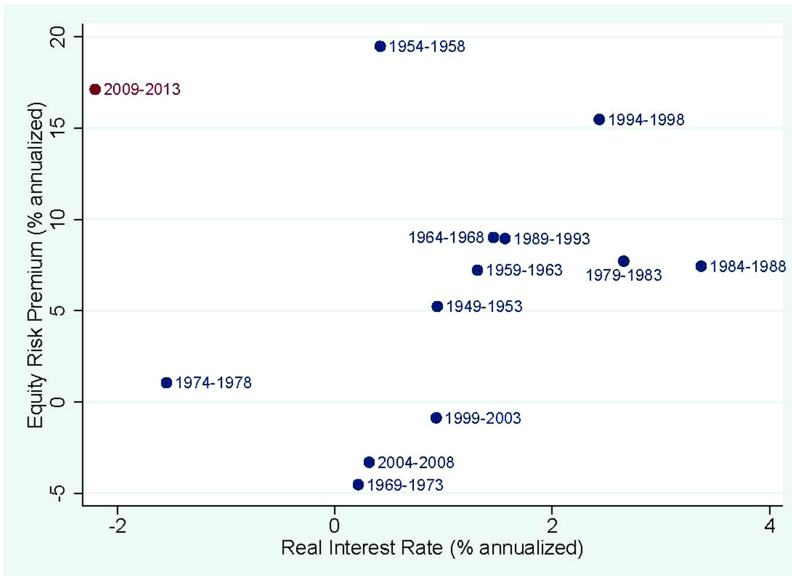
[1] Naturally, the central bank chooses the nominal interest rate, with the real interest rate being determined endogenously, whereas the "natural" rate of interest is typically understood to be a real rate. For more on the link between real and nominal interest rates from a consumption-based asset-pricing perspective, see Sarte (1998) and Wolman (2006).

[2] In particular, Mehra and Prescott (2008) question the assumption about whether the highly liquid Treasury bill rate is an appropriate measure of the interest rate that households use to save for retirement and smooth consumption.

[3] For examples of articles relying on segmented markets to account for the reduction in interest rates post-2008, see Del Negro et al. (2010), Guerrieri and Lorenzoni (2011), and Eggertson and Krugman (2012), and more generally, Vissing-Jørgensen (2002) and Vissing-Jørgensen and Attanasio (2003) for a discussion of how market segmentation affects interest rates. Seminal articles in the behavioral finance literature are Barberis, Shleifer, and Vishny (1998); Daniel, Hirshleifer, and Subrahmanyam (1998); and Hong and Stein (1999). See also Shleifer (2000) and Barberis and Thaler (2003) for reviews.

[4] In fact, to a large extent the material in this article is a reorganization of material in more detailed reviews by Campbell (2003), Barro and Ursúa (2011), and Cochrane (2011). While this article is written so as to be largely self-contained, the reader is referred to those texts for many of the details (including some of the derivations).

Figure 1 The Equity Premium and the Risk-Free Rate



they have low risk tolerance. In contrast, equity risk premia do not depend on expected consumption growth. Hence, one can gain some insight into the driving force behind low interest rates by examining the behavior of the risk premium. The evolution over time in the two variables can be seen in Figure 1. It depicts the postwar values of the real interest rate, measured by the 30-day Treasury bill rate deflated by the consumer price index, and of the equity risk premium, both of which averaged over various five-year periods.[5] The five years since the onset of the Great Recession stand out not only because of the exceptionally low real rate of interest, but also because of a historically high equity risk premium. Given the models reviewed, the high risk premium suggests that low interest rates in the recent period are likely to be either a consequence of a perception that consumption risk is particularly high, or of very low risk tolerance.

The article is structured as follows: In the following section, I lay out the notation used in the article as well as common conventions, simplifications, and approximations. Each subsequent section discusses

[5] To calculate the equity risk premium, I use the value weighted equity returns index from the Center for Research in Security Prices.

one variety of consumption-based asset-pricing models: the Mehra and Prescott (1985) benchmark, the recursive utility and long-run risk extensions of Weil (1989) and Bansal and Yaron (2004), the disaster-risk model of Rietz (1988) and Barro (2006), and Campbell and Cochrane's (1999) habit-formation models. The final section concludes.

1. NOTATION, CONVENTIONS, SIMPLIFICATIONS, AND APPROXIMATIONS

Assets are claims on streams of dividends. In particular, purchasing some asset, i, provides an economic agent with a stochastic stream of dividends $\left\{D_{t+s}^{i}\right\}_{s=0}^{\infty}$ for as long as the agent holds it. In consumption-based asset-pricing models there are no liquidity constraints or other transaction costs, so agents can trade assets freely at each period. If the price of asset i is given by P_t^i, then we can define its return between periods t and $t+1$ as

$$R_{i,t+1} \equiv \frac{P_{i,t+1} + D_{i,t+1}}{P_{i,t}}. \tag{1}$$

Asset pricing concerns itself either with determining the price-dividend ratio for an asset, $\frac{P_t^i}{D_t^i}$, or its expected returns, $E_t\left[R_{i,t+1}\right]$. Typically, higher returns are associated with lower price-dividend ratios.

While the literature discusses the pricing of many kinds of assets, the three main ones are the risk-free asset, a market portfolio of equities, and total wealth. The risk-free asset (denoted by $i = f$) is exactly what the name implies: an asset that pays the same dividend in all states of nature. As an empirical matter, the asset-pricing literature identifies the risk-free asset with short-term Treasury bills. Thus, the predictions of the models under review for the risk-free rate are going to be the most relevant ones for the purpose of monetary policy analysis.

The market portfolio of equities ($i = e$) refers to a well-diversified portfolio of shares issued by firms and traded in stock markets with prices summarized by indices such as the S&P 500. This is, in turn, different from total wealth ($i = w$), which is a fictitious asset (in the sense that there are no formal markets for it) that pays out aggregate consumption as dividends. It includes equity, bonds, housing, and human capital. Oftentimes studies of equity pricing at first identify equity with the wealth portfolio and then in refinements treat the two as distinct. The distinction between equity and the wealth portfolio normally focuses on the fact that firms are leveraged, both because they issue bonds and because salaries are normally insulated from high-frequency fluctuations in output. Therefore, for any change increase in aggregate endowment, dividends should change by a greater amount. The simplest way of modeling this leverage is to assume that aggregate dividends on equity are a deterministic function of consumption, with $D_t^e = (D_t^w)^\lambda = C_t^\lambda$, for some $\lambda > 1$.

One simplification used by the asset-pricing literature to obtain analytical results is to rely on log normality assumptions. If the log of asset returns is normally distributed, one can use the fact that for any normally distributed x, $E\left[e^x\right] = e^{E[x]+\frac{1}{2}Var[x]}$. Thus, if returns $R_{i,t+1}$ are log-normally distributed,

$$\ln\left(E\left[R_{i,t+1}\right]\right) = E\left[r_{i,t+1}\right] + \frac{1}{2}Var\left[r_{i,t+1}\right],$$

where we use small letters to denote the natural logarithm.

A further simplification, used in disaster models, is the use of a continuous time formulation to study disaster risk. Denote by dt the length of a period of time. Let $e^{r_{i,t+1}dt}$ be the gross return per period of time of that asset. Suppose the return on some asset i is either $e^{\bar{r}dt}$ with probability e^{-pdt} or $(1-b)\,e^{\bar{r}dt}$ with probability $1-e^{-pdt}$. Then

$$E\left[e^{r_{i,t+dt}dt}\right] = \left[e^{-pdt} + \left(1-e^{-pdt}\right)(1-b)\right]e^{\bar{r}}.$$

Taking logs and dividing by dt yields

$$\frac{\ln E\left[e^{r_{i,t+dt}dt}\right]}{dt} = \bar{r} + \frac{\ln\left[e^{-pdt} + \left(1-e^{-pdt}\right)(1-b)\right]}{dt}.$$

Taking the limit as $dt \to 0$ and applying l'Hopital's rule,

$$E\left[r_{i,t+dt}\right] = \bar{r} - pb.$$

The continuous time approximation yields an intuitive expression for expected log returns. Those are equal to $\bar{r}$, except that with probability p they fall by b.

Finally, a common approximation used in the analytical literature is to log-linearize equation (1) to obtain

$$r_{i,t+1} = \rho p_{i,t+1} + (1-\rho)\,d_{i,t+1} - p_{i,t},$$

where ρ is the average $\frac{P}{P+D}$ ratio and is typically calibrated to some value close to 1. Rearranging and iterating forward up to some time $t+T$ with $T>0$ yields

$$p_{i,t} - d_{i,t} = \sum_{s=0}^{T} \rho^s \Delta d_{t+1+s} - \sum_{s=0}^{T} \rho^s r_{i,t+1+s} + \rho^{T+1} p_{i,T+1}.$$

The expression is useful in that it breaks down three different determinants of the price-dividend ratio. The first term on the right-hand side is a discounted sum of future dividends growth. The faster dividends are expected to grow, the more a portfolio that pays off the consumption good as dividends is worth. The second term is a discounted sum of returns. All else constant, if prices are low in spite of high dividend growth, then the returns will be high as prices catch up with dividends. The third term is a "bubble" term. In most asset-pricing applications, one assumes that the bubble term goes to zero almost as surely as T increases. Given the no-bubble condition,

$$p_{i,t} - d_{i,t} = \sum_{s=0}^{\infty} \rho^s \Delta d_{t+1+s} - \sum_{s=0}^{\infty} \rho^s r_{i,t+1+s}.$$

The equation highlights that a high price-dividend ratio can forecast either a high growth in dividend payments or low future rates of returns.

Taking expectations and rearranging,

$$\begin{aligned} &(E_{t+1} - E_t)\, r_{i,t+1} \\ &= (E_{t+1} - E_t) \sum_{s=0}^{\infty} \rho^s \Delta d_{t+1+s} - (E_{t+1} - E_t) \sum_{s=1}^{\infty} \rho^s r_{i,t+1+s}, \end{aligned} \tag{2}$$

where $(E_{t+1} - E_t)\, r_{i,t+1} \equiv r_{i,t+1} - E_t r_{i,t+1}$ denotes the surprise in returns. The latter equation is useful to assess the sources of volatility in an asset return. It emphasizes that the volatility in returns for any asset can be a function of either the volatility of news concerning its future dividend flows or news concerning its future returns.

2. THE MEHRA AND PRESCOTT BENCHMARK

We start by examining a simplified version of the power utility benchmark case examined by Mehra and Prescott (1985). This corresponds to the common setup in macroeconomic models in which households are endowed with a separable power utility of consumption. As commonly done in the finance literature, Mehra and Prescott follow Lucas (1978) and focus on the case of an endowment economy in which households consume and trade claims on immediately perishable fruits that fall from an infinitely lived tree.[6]

Individual households determine how much to consume in each period of time and how much to invest in a portfolio of assets that it has available. We assume that there are N different assets, indexed $i \in \{1, ..., N\}$, and that those assets completely span the shocks that the households are subject to so that markets are complete. The problem of the household is

$$\max_{\{x_t^i\}_{(t,i)=(0,1)}^{(\infty,N)}} E_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma} - 1}{1-\gamma} \right]$$
$$s.t. : C_t + \sum_{i=1}^{N} P_{i,t} x_{i,t} = \sum_{i=1}^{N} x_{i,t-1} \left(P_{i,t} + D_{i,t} \right),$$

where x_t^i is the amount of shares of asset i held by the household at time t and, as before, P_t^i is the realized price and D_t^i is its realized dividend. The parameter γ is the coefficient of relative risk aversion and governs the tolerance that households have for risk. It is also the inverse of the intertemporal elasticity of substitution, governing the household's desire to smooth consumption over time. The optimality condition for the household is

$$P_{i,t} C_t^{-\gamma} = \beta E_t \left[C_{t+1}^{-\gamma} \left(P_{i,t+1} + D_{i,t+1} \right) \right].$$

Let $R_{i,t+1} \equiv \frac{P_{i,t+1} + D_{i,t+1}}{P_{i,t}}$ be the return on asset i. Returns, like prices, are equilibrium objects determined endogenously. Given expected future prices and dividends, higher returns are tied to lower prices at t. Given the definition of returns and the optimality condition, we have that

$$1 = \beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{i,t+1} \right]. \tag{3}$$

The ratio of marginal utilities $\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$ is the *pricing kernel* in this economy. In order to hold a positive and finite amount of an

[6] The analysis of asset-pricing models to environments with production ("Production Based Asset Pricing") is itself an active area of research that we will leave undiscussed. For important contributions in that literature, see Cochrane (1991); Jermann (1998); Boldrin, Christiano, and Fisher (2001); and Gomes, Kogan, and Zhang (2002), among many others.

asset, a risk-neutral household ($\gamma = 0$) requires that the return of the asset i be, on average, equal to β^{-1} irrespective of its variance. If $\gamma > 0$, the household instead requires β^{-1} to be equal to a weighted average of returns, giving more weight to states of the world where its consumption growth is lowest. The implication of this weighting is easiest to see if one rewrites equation (3) as

$$\beta^{-1} = E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right] E_t\left[R_t^i\right] + cov\left(\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}, R_t^i\right).$$

Suppose there is a risk-free asset, denoted by $i = f$, so that $var\left(R_t^f\right) = 0$. Then

$$R_t^f = \frac{\beta^{-1}}{E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]}, \tag{4}$$

and

$$\frac{E_t\left[R_t^i\right] - R_t^f}{R_t^f} = -\beta cov\left(\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}, R_t^i\right), \tag{5}$$

so that households request a higher premium over the risk-free rate for assets in which the covariance between the pricing kernel and the rate of returns is negative. It is possible to express equations (4) and (5) in log-linear form if one is willing to assume that the logs of consumption growth and asset returns are normally distributed. Then,

$$E_t\left[r_{i,t+1}\right] - r_{f,t+1} + \frac{\sigma_i^2}{2} = \gamma\sigma_{ic}, \tag{6}$$

with

$$r_{f,t+1} = -\log\beta + \gamma E_t\Delta c_{t+1} - \frac{\gamma^2\sigma_c^2}{2}, \tag{7}$$

where $r_{i,t+1}$ are the log returns on asset i, $r_{f,t+1}$ are the log returns on the risk-free asset, σ_i^2 is the variance of the logarithm of the returns on asset i, σ_c^2 is the variance on the logarithm of consumption growth, and σ_{ic} is the covariance between log returns and log consumption growth.

The first two terms on the left-hand side of equation (6) are just the differences between the expected return on some asset i and the risk-free asset. The third term is a Jensen's inequality adjustment term, accounting for the fact that, since logarithm is a concave function, the logarithm of an expected variable is always larger than the expectation of the logarithm.[7] The term on the right-hand side has two components. The second, σ_{ic}, is the covariance between the asset return and consumption growth and can be interpreted as the "quantity of risk" in the asset. The first, γ, is the coefficient of relative risk aversion and it can be interpreted as the "price" of risk. Under power utility, the price of risk is constant, and asset prices only depend on the risk one period ahead.

As famously demonstrated by Mehra and Prescott (1985), the model performs poorly in quantitative terms. In their baseline exercise, they equate equity with the wealth portfolio, i.e., an asset that pays out aggregate consumption as dividends.[8] Given that consumption growth does not vary much, the quantity of risk σ_{ic} is very low. Because of that, Mehra and Prescott find that for reasonable values of γ (10 and under), the equity risk premium implied by the right-hand side of equation (6) is an order of magnitude smaller than the one found in the data. This observation has spurred a very large literature and is a cornerstone of modern asset-pricing research.

For a large enough γ, the model is of course able to match the equity premium. However, setting γ to a very large number also has implications for the risk-free rate that do not fit the data. In an average quarter, consumption growth $E_t\left[\Delta c_{t+1}\right]$ is close to 2 percent in yearly terms and the standard deviation has a similar magnitude. If we take the coefficient of risk aversion to be $\gamma = 10$, close to Mehra and Prescott's upper bound, then matching the risk-free rate of 1 percent in yearly terms would require a discount rate of close to -19 percent per year. In a period of time where expected consumption is 1 percent instead of 2 percent, the interest rate would fall from 1 percent to -9 percent.

Intuitively, the reason for the tradeoff between matching the high risk premium and the low interest rate is that γ captures how unwilling households are to let consumption vary, be it over time or between states of nature. The higher γ, the more households dislike variation in consumption along either dimension. Hence, if a household with a high γ foresees that its consumption will grow slower, it will be very willing to borrow in order to keep consumption smoothed out over time. In equilibrium, this leads to a sharp reduction in the interest rate.

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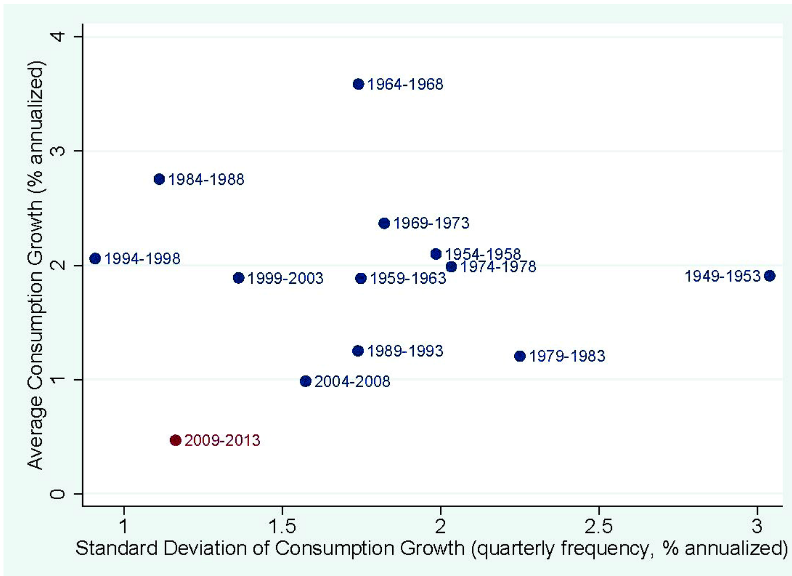
[7] Formally, Jensen's inequality states that if g is a concave (convex) function, then $g\left(E\left[x\right]\right) > (<)\ E\left[g\left(x\right)\right]$. In that specific case, the left-hand side is $\log\left(E\left[\frac{R_{i,t+1}}{R_{f,t+1}}\right]\right) > E\left[\log\left[\frac{R_{i,t+1}}{R_{f,t+1}}\right]\right] = E_t\left[r_{i,t+1}\right] - r_{f,t+1}$.

[8] As a robustness, they also consider the case where leverage increases the volatility of equities.

Figure 2 Consumption Growth and Risk



Implications for the Interest Rate in the Recent Period

While, in quantitative terms, the Mehra and Prescott benchmark fails as an explanation of asset pricing, it is still a useful benchmark in that it highlights which factors are likely to matter for interest rates in consumption-based asset-pricing models. In what follows, I use this benchmark as a qualitative guide to the factors driving the risk-free interest rate and show how they have evolved in the current recession. For convenience, I restate equation (6) for the risk-free rate below:

$$r_{f,t+1} = -\log \beta + \gamma E_t \Delta c_{t+1} - \frac{\gamma^2 \sigma_c^2}{2}. \tag{8}$$

As equation (8) makes clear, interest rates can either be low because market participants expect consumption growth to be low or because they perceive consumption risk to be high.

Figure 2 shows the average and standard deviations of quarterly consumption growths, both expressed in annualized terms and averaged over various five-year periods.[9] While 2009–13 does feature exceptionally low consumption growth for historical standards, it also features exceptionally low consumption variance. Hence, in qualitative terms, the model would have to account for the low interest rates through low expected consumption growth.

It is worth highlighting that, given the Mehra and Prescott (1985) benchmark, there is a tension between Figures 1 and 2, since equation (6) implies that, if consumption is correlated with dividends, a high variance of consumption growth ought to be associated with a high equity premium.[10] In contrast, we observe a low variance of consumption growth and a high equity premium. As we will see, alternative consumption-based asset-pricing models can provide potential resolutions to this inconsistency, as they allow either for the possibility that the "price" of risk may be changing (as in habit formation models) or that the kind of short-term consumption risk depicted in Figure 2 may not be the best measure of the kind of risk that asset holders are mostly concerned with when making their portfolio decisions.

3. RECURSIVE UTILITY AND LONG-RUN RISK

As discussed above, a major challenge facing common power-utility models is the difficulty in matching both households' willingness to let their consumption change over time (captured by a low interest rate) and their unwillingness to let it vary across states of nature (captured by the high equity risk premium). One possible solution to this tension is to allow for the possibility that the desire for intertemporal smoothing is governed by a different parameter than the desire for insurance. This is provided by the recursive utility function proposed by Epstein and Zin (1989) and Weil (1989, 1990), based on prior work by Kreps and Porteus (1978). In particular, the recursive utility function

[9] The consumption series is taken from Martin Lettau's website and is defined in Lettau and Ludvigson (2001). In particular, it excludes durable goods, shoes, and clothing.

[10] Consumption variance is an important factor in explaining the equity risk premium under the assumption that that consumption growth is i.i.d. and that growth in stock dividends is perfectly correlated with consumption growth, so that $\Delta d_{e,t} = \lambda \Delta c_{e,t}$. Then, if we guess that equity returns are also i.i.d., from equation (2) we have that

$$(E_{t+1} - E_t)\, r_{e,t+1} = (E_{t+1} - E_t)\, \Delta d_{t+1}.$$

Since dividend growth is i.i.d., the guess that equity returns are i.i.d. is verified. In this case, the covariance between consumption and equity returns σ_{ec} is simply $\lambda var\,(\Delta c_{e,t})$. Hence, from equation (6), higher consumption variance is associated with a higher equity risk premium.

provides for the representation of preferences over lotteries in which agents rank them in terms of the time in which uncertainty is resolved. For example, an agent may face two different lotteries that pay the same amounts at some distant date depending on the flip of a coin, but in one lottery the coin flip takes place immediately, whereas in the other it only takes place much later. Under this kind of preference, agents may prefer the first lottery to the second even though the distribution of outcomes is identical.[11] The Epstein-Zin-Weil (EZW) utility function can be written as

$$U_t = \left\{ (1-\beta) \left[C_t \right]^{1-\frac{1}{\psi}} + \beta \left[\left(E_t U_{t+1}^{1-\gamma} \right)^{\frac{1}{1-\gamma}} \right]^{1-\frac{1}{\psi}} \right\}^{\frac{1}{1-\frac{1}{\psi}}}, \tag{9}$$

where U_t is the utility at time t. Preferences for early resolution of uncertainty emerge if $\frac{1}{\psi} < \gamma$.

The parameter ψ can be interpreted as the intertemporal elasticity of substitution. This interpretation becomes most clear in the deterministic case. Without uncertainty, the exponents in $1-\gamma$ around U_{t+1} cancel out and, with a slight rearrangement, equation (9) collapses to the usual Bellman equation format, with period utility of consumption given by $C_t^{1-\frac{1}{\psi}}$.

The parameter γ can be interpreted as a risk-aversion parameter. Heuristically, this can be seen in a version of the problem where the household only consumes in $t = 2$ so that there are no intertemporal choices to be made. Then, $U_t = 0$ for $t > 2$, $U_2 = C_2$, and $U_1 = \beta^{\frac{1}{1-\frac{1}{\psi}}} E_t \left[C_2^{1-\gamma} \right]^{\frac{1}{1-\gamma}}$, so that the problem of the household is equivalent to maximizing expected utility $E_t \left[C_2^{1-\gamma} \right]$.[12]

Finally, it is also straightforward to check that, if $\frac{1}{\psi} = \gamma$, equation (9) collapses back to a recursive version of the benchmark power-utility case, in which the coefficient of relative risk aversion is equal to the inverse of the intertemporal elasticity of substitution.

Given this utility function, one can derive the following Euler equation for portfolio decisions:[13]

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[11] This is a violation of the independence axiom for preferences so that with Epstein-Zin-Weil preferences, utility will not necessarily be separable across states of nature.

[12] Stictly speaking, for this example we would need $\psi < 1$, so that the utility function is still well defined for $C_t = 0$.

[13] See the Appendix for a derivation.

$$1 = E_t \left[\left\{ \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \right\}^{\theta} \left(\frac{1}{R_{w,t+1}} \right)^{1-\theta} R_{i,t+1} \right], \qquad (10)$$

where $R_{w,t+1}$ is the return on total household wealth and $\theta \equiv \frac{1-\gamma}{1-\frac{1}{\psi}}$, so that in the benchmark power-utility case, $\theta = 1$. The pricing kernel is $\left\{ \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \right\}^{\theta} \left(\frac{1}{R_{w,t+1}} \right)^{1-\theta}$ and is a weighted average of the pricing kernel obtained in the benchmark separable utility case and the reciprocal of the return on wealth, $R_{w,t+1}$.The return on wealth in the pricing kernel captures the impact of news about future consumption on agent's marginal utility. To see this, recall that, from equation (2), surprises in the returns to the wealth portfolio satisfy

$$\begin{aligned}&(E_{t+1} - E_t)\, r_{w,t+1} \\ &= (E_{t+1} - E_t) \sum_{s=0}^{\infty} \rho^s \Delta c_{t+1+s} - (E_{t+1} - E_t) \sum_{s=1}^{\infty} \rho^s r_{w,t+1+s}, \qquad (11)\end{aligned}$$

where we use the fact that, by definition, the dividends on the wealth portfolio are equal to aggregate consumption. Thus, surprises to the returns on wealth reflect surprises in future consumption growth, discounted by surprises to the future returns on wealth itself.

The reason why returns on wealth are factors in the pricing kernel under EZW preferences is because of the nonseparability between utility for current and future consumption. With power utility, preferences are separable. Given that agents are able to completely change their portfolio each period, they need not concern themselves with consumption flows in the far future when evaluating which portfolio to hold between two adjacent periods. This is no longer true with EZW preferences.

If the logs of consumption growth and returns are normally distributed, we can write the following expression for the risk premium associated with any given asset i:

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \theta \frac{\sigma_{ic}}{\psi} + (1-\theta)\,\sigma_{iw}, \qquad (12)$$

and for the risk-free rate,

$$r_{f,t+1} = -\log\beta + \frac{1}{\psi} E_t \left[\Delta c_{t+1} \right] + \frac{\theta - 1}{2} \sigma_w^2 - \frac{\theta}{2\psi^2} \sigma_c^2, \qquad (13)$$

where now σ_{iw} is the covariance between the returns on asset i and the return on total household wealth, and σ_w^2 is the variance of the total returns on wealth.

Recursive preferences allow one to account for the equity premium puzzle in two ways. First, as highlighted by Weil (1989), there is no longer a tradeoff between matching the equity risk premium and the risk-free rate, as there is an additional parameter to be calibrated. Furthermore, as explored in detail by Bansal and Yaron (2004), with $\theta \neq 1$, the covariance of the asset return with the return on total wealth σ_{iw} becomes an additional factor in determining the equity premium. Thus, if, for example, the variation in total return on wealth is similar to the variation in equity returns, then returns on total wealth are clearly much more volatile than consumption, so that σ_{iw} is potentially much larger than σ_{ic}.

One problem with evaluating equations (12) and (13) is that the variance of total wealth is hard to measure since total wealth includes human capital. One can make some progress by imposing structure on the process for consumption. In particular, suppose that the consumption growth Δc_{t+1} is the sum of a predictable component z_t and an unpredictable one $\epsilon_{c,t+1}$ as in

$$\begin{aligned} \Delta c_{t+1} &= z_t + \sigma_c \epsilon_{c,t+1} \\ z_{t+1} &= (1-\phi)\, g + \phi z_t + \sigma_z \epsilon_{z,t+1}, \end{aligned}$$

with $\epsilon_{c,t+1}$ and $\epsilon_{z,t+1}$ i.i.d. standard normal variables.[14] With ϕ close to 1 and high σ_c, this structure allows for consumption growth to have a predictable, stochastic, long-term component, even if at high frequencies overall consumption growth is hard to predict.[15] For the wealth portfolio, the dividends are equal to aggregate consumption, so that $R_{w,t} = \frac{P_{w,t+1}+C_{t+1}}{P_{w,t}}$. From equations (12) and (13) we have that, if risk doesn't vary over time (so that it is homoscedastic), then

$$E_t r_{w,t+1} = \mu + \frac{1}{\psi} E_t \left[\Delta c_{t+1}\right],$$

where μ is a constant that depends on the variances. This allows us to substitute out the returns from the right-hand side of equation (2) to obtain

[14] It is straightforward but tedious to allow for correlation between $\epsilon_{c,t+1}$ and $\epsilon_{z,t+1}$, so we will assume that they are uncorrelated.

[15] The question of whether or not consumption growth rate has a persistent component is hard to settle, since ϕ is hard to estimate in small samples. Bansal and Yaron (2004) show that a large ϕ is not inconsistent with observed autocovariance of consumption growth and observed variances of consumption growth at different horizons.

$$
\begin{aligned}
& (E_{t+1} - E_t)\, r^w_{t+1} \\
= \; & (E_{t+1} - E_t)\, \Delta c_{t+1+s} \\
& + \left(1 - \frac{1}{\psi}\right) (E_{t+1} - E_t) \sum_{s=1}^{\infty} \rho^s \Delta c_{t+1+s}. \qquad (14)
\end{aligned}
$$

We can now use the expression just derived to describe the sources of one-step-ahead variation in returns to the wealth portfolio. The first component on the right-hand side is the innovation in consumption growth, with variance σ_c^2. The second component is a discounted sum of future consumption growth. It changes as news about future consumption growth arrives, in the form of innovations to z_{t+1}. This second component incorporating news about future consumption is what allows returns on the wealth portfolio, and hence the pricing kernel, to be significantly more volatile than consumption growth. If, instead, consumption growth were i.i.d. so that this component would be equal to zero, the variance of returns on wealth would be as small as the variance of consumption growth. The higher variance of the pricing kernel associated with persistence in consumption growth is what allows models with EZW preferences to imply substantially larger risk premia than models with power utility for a given value of the risk-aversion parameter γ, as one can see from equations (12) and (13) determining the risk premium and the risk-free rate.

Bansal and Yaron (2004) emphasize that a reasonable parameterization of the model requires both $\gamma > 1$ and $\psi > 1$. They choose $\gamma = 10$, at the upper bound of Mehra and Prescott's (1985) exercise, and $\psi = 1.5$. The choice of ψ is subject to debate, as many empirical studies of consumption behavior over time point to very low values for ψ. Bansal and Yaron (2004) counter that stochastic variance in consumption introduces a downward bias in estimates of ψ and that, furthermore, studies with more disaggregated consumption data support lower ψ. Importantly, they also point out that one can discipline the value of ψ through the correlation between asset prices and news about consumption growth and consumption volatility. This can be seen in equation (14), where, with $\psi > 1$, news about future consumption growth leads to an increase in the returns on wealth, but with $\psi < 1$ such news leads to a reduction.

As emphasized by Bansal and Yaron (2004), recursive preferences imply that risk premia vary not only with news about future consumption growth, but also with news about its variance. A higher variance of innovations to future consumption growth increases the variance of returns on the wealth portfolio and, hence, of the pricing kernel, leading to a higher equity premium and lower risk-free rates. Therefore, time variation in the variance of long-run growth ("long-run risk") can be an important factor explaining the variance in risk premia observed in the data.

Implications for the Interest Rate in the Recent Period

For convenience, I restate the equation describing the determinants of the risk-free rate:

$$r_{f,t+1} = -\log \beta + \frac{1}{\psi} E_t \left[\Delta c_{t+1} \right] + \frac{\theta - 1}{2} \sigma_w^2 - \frac{\theta}{2\psi^2} \sigma_c^2.$$

Note that under the calibration adopted by Bansal and Yaron (2004), $\frac{1}{\psi} = \frac{2}{3}$, $\frac{\theta-1}{2} = -14$, and $\frac{\theta}{2\psi^2} = -6$, so that the weight placed on the two risk factors is comparatively large. This equation holds for the case of homoscedastic risk. Bansal and Yaron (2004) also provide a derivation of the risk-free rate when risk is time varying so that σ_w^2 and σ_c^2 are functions of time. In that case, the coefficients change but the essential factors determining the risk-free rate remain the same.[16]

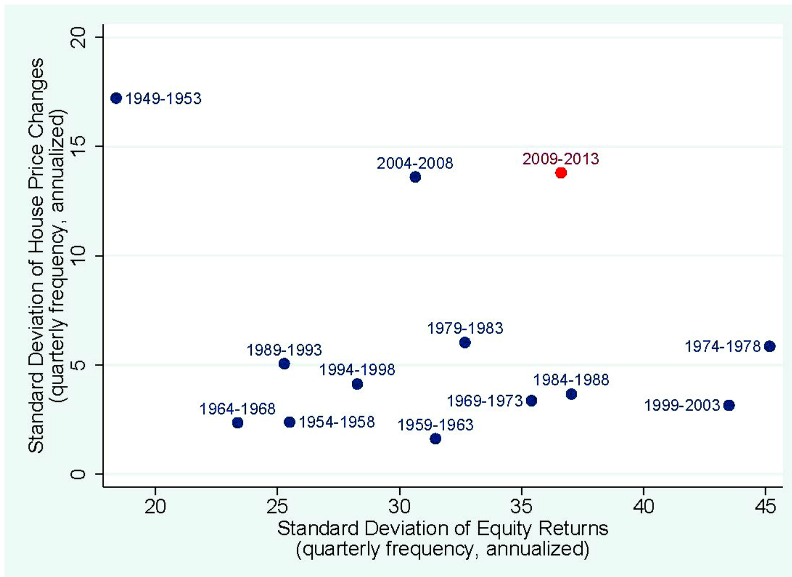
The recursive preferences model implies that the risk-free rate changes not only with the expected growth rate of consumption or with the variance of that growth rate, but also with changes in the mean and variance of returns on wealth, σ_w^2. As previously discussed, these are, in turn, functions of the variance of the long-term component of consumption growth. Given the calibration advocated by Bansal and Yaron (2004), a reduction in the interest rate could thus stem not only from the same factors that explain the reduction in interest rates in the benchmark time-separable model, but also from an increase in the variance of the long-run component of consumption growth.

Total wealth in the economy includes not only equity in firms, but also housing and human capital. Figure 3 depicts the volatilities of equity returns and house price increases over five-year periods.[17] Both volatilities were high by historical standards in the 2009–13 period, most notably the volatility of housing returns. Thus, long-run risk could, in principle, help explain the low interest rates while accounting for the disconnect between high risk premia and the low volatility of consumption growth in that period. More generally, however, the

[16] Bansal and Yaron consider a case in which there is only one stochastic risk factor so that σ_w^2 and σ_c^2 co-move perfectly.

[17] The housing price data is from Shiller (2015). House price increases are a good approximation for housing returns so long as rents are stable.

Figure 3 Wealth Risk



correlation between these volatilities and the equity premium is questionable. For example, the 2004–08 period exhibits very high house price volatility even as the equity risk premium is very low (see Figure 1). Likewise, the 1999–2003 period exhibits very low equity risk premia together with a very volatile equity premium. Naturally, these are only rough correlations based on period averages using arbitrary cutoffs, so this should not be seen as grounds for rejecting the long-run risk model. Also, we have ignored the hard to measure contribution of volatility in returns to human capital.

4. DISASTER RISK

One early reaction to Mehra and Prescott's (1985) equity premium puzzle is that the distribution of asset returns and consumption growth is prone to rare but large disasters. If those disasters are likely to have a larger impact on the dividends paid out by equities than on the return on sovereign bonds, they can generate a large premium between stocks and bonds as private agents seek to insure themselves against those rare occurrences.

The argument was first put forward by Rietz (1988). Barro (2006) makes a case for the argument by using international data to calculate the probabilities and magnitudes of large disasters, putting a 1.7 percent probability of a collapse in consumption of, on average, 30 percent.[18] He also calculates the probability of sovereign default in the event of a disaster and the recovery rate that investors can expect in those events. He finds that with a coefficient of relative risk aversion as small as four and a discount rate of 3 percent per year, it is possible to obtain equity premia and risk-free rates that are closer to the data.

Barro (2006, 2009) considers an environment where the aggregate endowment follows a random walk with drift g and variance σ_c^2 most of the time, but with probability p it collapses permanently to a fraction $1-b$ of its value, where b is itself a random variable drawn from the empirical distribution of disasters that he documents. Taking a continuous-time limit, Barro (2009) arrives at expressions that, after a substitution, yield the following expressions for the risk-free rate:[19]

$$r^f = -\log\beta + \gamma g - \frac{1}{2}\gamma^2\sigma^2 - p\left[E\left(1-b\right)^{-\gamma} - 1\right],$$

and, if one takes, as he does, equity to incorporate all of the wealth portfolio, for the risk premium:

$$r^e - r^f = \gamma\sigma^2 + p\left[E\left(1-b\right)^{-\gamma} - E\left(1-b\right)^{1-\gamma} - E\left[b\right]\right].$$

Thus, an increase in the probability of disasters leads to a reduction in the riskless rate and an increase in the equity risk premium. One important result is that asset returns are nonlinear functions of the size of disasters b. This enhances the ability of disasters generating large risk premia and low interest rates since, as b approaches 1, the marginal utility of consumption in the disaster state approaches infinity. Furthermore, as emphasized by Barro, the model can accommodate "bonanzas," which are as large as the disasters and still generate large risk premia, since households will be much more concerned with the disaster states (in which they have high marginal utility) than with the bonanza states (in which their marginal utility is low).

Barro and Ursúa (2011) provide a comprehensive review of the small literature that has emerged around the notion of disaster risk being a key driver of asset-pricing data. This literature has expanded the model

[18] In particular, Barro defines a disaster as an event in which gross domestic product (GDP) drops by 15 percent or more, and equate the change in consumption with the observed change in GDP.

[19] The substitution in question is from the expected consumption growth $E_t\left[\frac{C_{t+1}-C_t}{C_t}\right]$ (denoted g^* in Barro [2009]) for its determinants, $g + \frac{1}{2}\sigma^2 - p \times E\left[b\right]$. The substitution singles out g since it is likely to be closer to observed average log consumption growth then g^*.

to allow for time-varying disaster risk, thus allowing it to explain time-varying risk premia (Gourio 2010), and disasters that are correlated across countries and happen slowly rather than quickly (Nakamura et al. 2010), as well as to evaluate implications of the model for additional asset pricing facts (Gabaix 2008).

Implications for the Interest Rate in the Recent Period

For convenience, I restate the equation describing the determinants of the risk-free rate:

$$r^f = -\log\beta + \gamma g - \frac{1}{2}\gamma^2\sigma^2 - p\left[E\left(1-b\right)^{-\gamma} - 1\right].$$

In addition to the determinants of interest rates in the other models (expected growth and one-step-ahead volatility of consumption), models with economic disasters imply that interest rates ought to change in response to changes in the probability of disaster or to changes in the expected size of disasters.

It is plausible that, in the aftermath of the Great Recession, economic agents have updated upward their subjective probabilities of such an episode occurring again. This could go some way in explaining the smaller interest rate observed in the recent period. In particular, consumption dropped 2.7 percent between Q2:2007 and Q4:2009. Relative to a 2 percent per year trend, the reduction was 7.9 percent. Suppose that, given that observation, agents assign a probability of 5 percent to a drop in consumption of 5 percent in any given period, so that such a disaster occurs on average every 20 years.[20] Then, if they have a risk aversion of four, they will request a risk-free rate that is $0.05\left(0.95^{-4}-1\right) = 1.14$ percent in yearly terms smaller than before. This revision is unlikely to dissipate very quickly since, given the small probabilities of a disaster occurring, the fact that another one hasn't come to fruition should weigh little on the probability assessment. Importantly, apart from helping explain the lower interest rate, the disaster risk could allow one in principle to reconcile the low volatility of

[20] A 5 percent probability would be high compared to the 1.7 percent calculated by Barro (2006), but, in contrast, the reduction in consumption of 5 percent is less extreme than the average 30 percent reduction found in that study. One obvious caveat is that the drop in consumption occurred smoothly, over two and a half years, whereas the model assumes that the whole change occurs instantaneously. Nakamura et al. (2010) show how the rare disaster model can accomodate slow disasters if agents have EZW preferences.

consumption with the high equity premium in the post-Great Recession era.

5. HABITS

In both the discussion of rare disasters and long-term risk, the time variation in expected risk premia is understood primarily as stemming from time variation in the quantity of risk that households face. Under habit formation, this same time variation is explained as stemming from variation in the risk tolerance of households, which determines the price of risk.

In habit-formation models, the marginal utility of consumption depends on a time-varying state variable that evolves as a function of past consumption decisions. The key idea is that as households become habituated to certain consumption levels, their marginal utility of consumption becomes higher for a given level of consumption. Habit-formation models differ along several dimensions, including whether habits are "internal" (where habit depends on individual household consumption) or "external" (where habit depends on aggregate consumption), whether habits enter in the utility function multiplicatively or additively, and whether habits change more or less quickly with consumption.[21] In what follows, we discuss the model by Campbell and Cochrane (1999).

Campbell and Cochrane (1999) point out that habit models are successful in generating volatile time-varying risk premia because they increase the volatility of the marginal utility of consumption. They assume habits enter additively and are "external" so that

$$u\left(C_t, X_t\right) = \frac{\left(C_t - X_t\right)^{1-\gamma} - 1}{1-\gamma},$$

where X_t is the stock of habits. Then the curvature of the utility function with respect to C_t is given by

$$\frac{u_{cc}\left(C_t, X_t\right) C_t}{u_c\left(C_t, X_t\right)} = -\frac{\gamma}{S_t},$$

where $S_t \equiv \frac{C_t - X_t}{C_t}$ is "surplus consumption," the gap between consumption and the habit. It follows that the curvature is higher in absolute

[21] See Campbell (2003) for a more detailed discussion. Models of habits in finance include Sundaresan (1989), Abel (1990, 1999), Constantinides (1990), and Campbell and Cochrane (1999), among others.

terms when consumption is closest to its habit level X_t. This time-varying curvature implies that the pricing kernel $\frac{u_c(C_{t+1},X_{t+1})}{u_c(C_t,X_t)}$ is also likely to vary with S_t.

To complete the specification of preferences, Campbell and Cochrane (1999) need to specify how habits evolve over time. Rather than using a more conventional specification in which the habit stock X_t evolves as a log-linear function of C_t, they recur to a nonlinear specification in which S_t is a log-linear function of changes in log C_t. One advantage of this specification is that it ensures that surplus consumption S_t is always positive, which is necessary for the utility to be well defined. They define the evolution of surplus consumption to be given by

$$s_{t+1} = \phi\bar{s} + (1-\phi)\, s_{t-1} + \lambda\,(s_t)\,(c_{t+1} - c_t - E\,[c_t])\,,$$

where $\bar{S}$ is the steady-state level of the habit (and $\bar{s}$ its log), and $\lambda\,(s_t)$ is a nonlinear function of s_t. The nonlinear term $\lambda\,(s_t)$ helps them deal with one important difficulty with habit-formation models. This is that, while a time-varying pricing kernel helps generate volatile expected risk premia, it can also give rise to counterfactually volatile interest rates. In Campbell and Cochrane's specification, the risk-free rate is given by

$$r_t^f = -\ln(\beta) + \gamma E_t\,[\Delta c_{t+1}] - \gamma\,(1-\phi)\,(s_t - \bar{s}) - \frac{\gamma^2\sigma^2}{2}\,[1 + \lambda\,(s_t)]^2\,.$$

The first two terms are the ones obtained in a model without habits. The following two include the effect of habits. The third term summarizes the effect of habits on intertemporal substitution. Surplus consumption is expected to mean revert at the rate $1-\phi$. If it is above its steady-state levels, then households expect it to become smaller over time, which is to say that they expect their marginal utility to become smaller. Thus, they become more patient, leading to a smaller equilibrium risk-free rate. The last term on the right-hand side captures the effect of consumption risk on the risk-free rate. Now, apart from the usual reason through which consumption risk generates precautionary savings, households also seek to keep their consumption risk low because it is correlated with their habit formation. In periods in which realizations of consumption are high, surplus consumption also increases.

Campbell and Cochrane (1999) discipline their choice of $\lambda\,(s_t)$ by adding three requirements. Two of them are technical. They impose that X_t is pre-determined in steady state and that it is always increasing in shocks to c_t. These conditions ensure that, close to steady state, their process for habits resembles more common specifications. The third requirement is that risk-free rates do not vary with habits. Thus, by construction, their model delivers a low volatility for the risk-free rates, as in the data. This allows them to focus more sharply on the variation in risk premia. Given Campbell and Cochrane's (1999) calibration, the interest rate is

$$r_{t+1}^{f}=-\log\left(\beta\right)+\gamma E_{t}\left[\Delta c_{t+1}\right]-\left(\frac{\gamma}{\bar{S}}\right)^{2}\frac{\sigma_{c}^{2}}{2}.$$

Note that $\frac{\gamma}{\bar{S}}$ is the curvature of the utility function with respect to consumption in steady state and is thus a measure of the risk tolerance of households. If $\bar{S}<1$, it is possible for the model to have a large steady-state curvature with respect to consumption ($\frac{\gamma}{\bar{S}}$), leading to high risk premia, even if it has a relatively low γ. This, in turn, allows it to admit more moderate interest rates. Specifically, Campbell and Cochrane calibrate $\gamma = 2.372$ and $\bar{S} = 0.049$, so that the curvature of the utility function close to steady state is approximately equal to 48.

Campbell and Cochrane (1995) also consider an extension of the model in which they choose $\lambda\left(s_t\right)$ to ensure that risk-free rates are a linear function of log habits, decreasing when surplus consumption is high. They pick the intercept to correspond to a 1 percent real interest rate and the slope so that the lower bound for the real interest rate is zero.

Implications for the Interest Rate in the Recent Period

For convenience, I restate the equation describing the determinants of the risk-free rate:

$$r_{t+1}^{f}=-\log\left(\beta\right)+\gamma E_{t}\left[\Delta c_{t+1}\right]-\left(\frac{\gamma}{\bar{S}}\right)^{2}\frac{\sigma_{c}^{2}}{2}.$$

As calibrated by Campbell and Cochrane, the factors determining the real interest rate in the model with habits are the same as in the Mehra and Prescott (1985) benchmark, the only difference being that the model with habits assigns a greater weight to consumption volatility.

The models with long-run or disaster risk are able to explain the reduced interest rate with the introduction of risk factors that cannot be easily discerned by measured consumption volatility. The model with habits stands in contrast to that. Thus, like the Mehra and Prescott (1985) benchmark, it needs to rely on the historically low consumption growth rate to account for the low interest rates. However, for any choice of γ, the habit model also puts a greater weight on the variance of consumption growth σ_c (since $\bar{S} < 1$), which was also low by historical standards in the post-2009 period. Therefore, for any choice of γ, the habit model would imply that the risk-free rate should have fallen by less than what is implied by the Mehra and Prescott (1985) benchmark.

One significant advantage of the habit formation model over the Mehra and Prescott model is that it can also accommodate the historically high equity risk premium, since the reduction in consumption in the aftermath of the Great Recession would have meant that "surplus consumption" S_t would be particularly low, leading to increased risk aversion.

6. SUMMARY AND CONCLUSION

The large drop in interest rates following the 2008 recession has given rise to discussions about whether the reduction was mainly due to policy or whether policy was following as best it could the "natural" rate and, in the latter case, what the determinants of that reduction could be. While explanations focusing on market segmentation have gained prominence, asset-pricing models in frictionless environments might also be able to provide sensible explanations for that drop.

In the text above, I discussed, on top of the benchmark power utility of Mehra and Prescott (1985), three leading varieties of consumption-based asset-pricing models with special focus on the determinants of the risk-free rate: long-run risk, disaster risk, and habit formation. All variants suggest that interest rates ought to be a function of expected consumption growth. This implication is consistent with the fact that consumption growth was low by historical standards in the 2009–13 period. At the same time, within this period there was a reduction in the volatility of consumption growth, which could enhance the effect of the reduced growth rate.

The challenge for the benchmark Mehra and Prescott (1985) framework is that this period also exhibits an equity premium that is high by historical standards, but consumption volatility is small. The three variants discussed are able to resolve that tension in different ways. Under long-run risk and disaster-risk models, agents' risk perception would increase because of, respectively, higher variance in the long-run component of consumption growth or a perceived increase in the probability of a large consumption decrease. The former is consistent with historically high equity market volatility, and the latter with an upward revision of the probability of disaster following the Great Recession. Under the habit-formation model, the tension can potentially be resolved by the observation that the reduction in consumption following the Great Recession led to increased risk aversion as households found themselves closer to their "subsistence" level of consumption. The explanations based on increased risk diverge from the habit formation in that the same increase in perceived risk that leads to an increased equity risk premium can also be an added factor explaining the reduced interest rate. In contrast, in the benchmark calibration adopted by Campbell and Cochrane (1995) for the habit-formation model, the presence of habits have no direct impact on how interest rates change over time but could reinforce the dampening effects of reduced one-step-ahead consumption volatility.

A priori, there is no reason why the different models cannot be combined. In particular, Nakamura et al. (2010) investigate asset-pricing implications of disasters that take multiple quarters to unfold when households have EZW preferences. Such disasters can be viewed as an intermediate case between the one-off disaster risk in Barro (2006) and the consumption growth rate uncertainty in Bansal and Yaron (2004). It is unclear whether extending a habit-formation model to allow for disaster risk would yield any additional insight. Combining habit formation with long-run risk would present a challenge since it would involve combining two forms of nonseparability in preferences.

APPENDIX: EULER EQUATION UNDER EZW PREFERENCES

The Euler equation under EZW preferences is obtained from the first-order conditions of the household subject to the budget constraint:

$$C_t + \sum_{i=1}^{N} P_{i,t} x_{i,t} = \sum_{i=1}^{N} x_{i,t-1} \left(P_{i,t} + D_{i,t} \right).$$

To derive the Euler equation under EZW preferences, we define household wealth as

$$W_{t+1} \equiv \sum_{i=1}^{N} x_{i,t-1} \left(P_{i,t} + D_{i,t} \right) = \sum_{i=1}^{N} x_{i,t-1} P_{i,t-1} R_{i,t}.$$

Given that definition, we can rewrite the budget constraint as

$$C_t + \sum_{i=1}^{N} P_{i,t} x_{i,t} = W_t.$$

Given that restated budget constraint, start with the "guess" that we can express the utility function as a linear function of wealth:

$$U_t = A_t W_t,$$

for some A_t to be determined. Note that A_t is time-varying, reflecting the fact that, if returns are not i.i.d., the utility of the household will vary as a function of the state of the economy. This is a reasonable guess since realized wealth is the only state variable in the household's problem and the utility function is homogeneous of degree 1 in U_t and C_t. Given that redefinition and that "guess," the household's problem becomes

$$\begin{aligned}
&A_t W_t \\
&\quad = \max_{W_{t+1}, C_t, \{x_{i,t}\}_{i=1}^{I}} \left\{ \begin{array}{c} (1-\beta)\left[C_t\right]^{1-\frac{1}{\psi}} \\ +\beta \left[\left(E_t \left[(A_{t+1} W_{t+1})^{(1-\gamma)} \right] \right)^{\frac{1}{1-\gamma}} \right]^{1-\frac{1}{\psi}} \end{array} \right\}^{\frac{1}{1-\frac{1}{\psi}}}, \\
&s.t. : C_t + \sum P_{i,t} x_{i,t} = W_t \\
&\quad W_{t+1} = \sum_{i=1}^{N} P_{i,t} x_{i,t} R_{i,t+1}.
\end{aligned}$$

The first-order conditions are

$$\begin{aligned}
\text{for } W_{t+1} : \omega_{t+1} &= \beta U_t^{\frac{1}{\psi}} \left(E_t \left[(A_{t+1} W_{t+1})^{(1-\gamma)} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma} - 1} A_{t+1}^{1-\gamma} \left(W_{t+1}\right)^{-\gamma} \\
\text{for } C_t : \lambda_t &= (1-\beta) U_t^{\frac{1}{\psi}} C_t^{-\frac{1}{\psi}} \\
\text{for } x_{i,t} : \lambda_t &= E_t \left[\omega_{t+1} R_{i,t+1} \right].
\end{aligned}$$

Note that there are in fact multiple first-order conditions for W_{t+1} since W_{t+1} will vary as a function of the ex-post realized state. There are accordingly multiple ω_{t+1}. The pricing kernel is given by

$$\frac{\omega_{t+1}}{\lambda_t} = \frac{\beta \left(E_t \left[(A_{t+1} W_{t+1})^{(1-\gamma)} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma} - 1} A_{t+1}^{1-\gamma} \left(W_{t+1}\right)^{-\gamma}}{(1-\beta) C_t^{-\frac{1}{\psi}}},$$

which can be rearranged as

$$\frac{\omega_{t+1}}{\lambda_t} = \left(\frac{E_t \left[\left(A_{t+1} W_{t+1} \right)^{(1-\gamma)} \right]}{\left(A_{t+1} W_{t+1} \right)^{1-\gamma}} \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma} - 1} \frac{\beta A_{t+1}^{1-\frac{1}{\psi}} \left(W_{t+1} \right)^{-\frac{1}{\psi}}}{\left(1-\beta \right) C_t^{-\frac{1}{\psi}}}.$$

Given the guess for the functional form of U_t, the envelope condition is

$$A_t = \lambda_t = \left(1-\beta \right) U_t^{\frac{1}{\psi}} C_t^{-\frac{1}{\psi}}.$$

Substituting into the first-order condition for C_t and using the guess that $U_t = A_t W_t$, we can write the envelope condition as

$$A_t = \left(1-\beta \right) A_t^{\frac{1}{\psi}} W_t^{\frac{1}{\psi}} C_t^{-\frac{1}{\psi}}.$$

So that, rearranging

$$A_t^{1-\frac{1}{\psi}} = \left(1-\beta \right) \left(\frac{W_t}{C_t} \right)^{\frac{1}{\psi}}.$$

Lead this expression one period and use substitute out A_{t+1} from the second term in the pricing kernel:

$$\frac{\omega_{t+1}}{\lambda_t} = \left(\frac{E_t \left[\left(A_{t+1} W_{t+1} \right)^{(1-\gamma)} \right]}{\left(A_{t+1} W_{t+1} \right)^{1-\gamma}} \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma} - 1} \frac{\beta \left(1-\beta \right) \left(\frac{W_{t+1}}{C_{t+1}} \right)^{\frac{1}{\psi}} \left(W_{t+1} \right)^{-\frac{1}{\psi}}}{\left(1-\beta \right) C_t^{-\frac{1}{\psi}}}.$$

The expression then simplifies to

$$\frac{\omega_{t+1}}{\lambda_t} = \beta \left(\frac{E_t \left[\left(A_{t+1} W_{t+1} \right)^{(1-\gamma)} \right]}{\left(A_{t+1} W_{t+1} \right)^{1-\gamma}} \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma} - 1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}}.$$

We can obtain the policy function for consumption by rearranging the envelope condition to obtain

$$C_t = \left(1-\beta \right)^{\psi} A_t^{1-\psi} W_t \equiv \mu_t W_t,$$

so that consumption is linear in wealth.

To obtain an expression for next-period wealth as a function of current wealth, we can write the second constraint alternatively as

$$W_{t+1} = R_{w,t+1} \times \sum_{i=1}^{N} P_{i,t} x_{i,t},$$

where $R_{w,t+1} \equiv \sum_{i=1}^{N} \frac{P_{i,t} x_{i,t}}{\sum_{i=1}^{N} P_{i,t} x_{i,t}} R_{i,t}$ is the return on total wealth. Since, in equilibrium, $x_{i,t}$ equals the supply of different assets i, $R_{w,t+1}$ can be taken as exogenous to the household's problem. With this change in notation, we can combine the two constraints on the household's problem to obtain

$$W_{t+1} = R_{w,t+1} \left(W_t - C_t \right) \equiv R_{w,t+1} \left(1 - \mu_t \right) W_t.$$

Finally, one can use the envelope condition to write A_t as a function of μ_t:

$$A_t^{1-\frac{1}{\psi}} = (1-\beta)\, \mu_t^{-\frac{1}{\psi}}.$$

With these two expressions, we can verify the "guess" that utility is linear in wealth. Substitute them into the utility function to obtain

$$\begin{aligned} & A_t W_t \\ = & \left\{ \begin{array}{c} (1-\beta) \left[\mu_t \right]^{1-\frac{1}{\psi}} \\ +\beta \left[\left(E_t \left[\left(A_{t+1} R_{w,t+1} \left(1-\mu_t \right) \right)^{(1-\gamma)} \right] \right)^{\frac{1}{1-\gamma}} \right]^{1-\frac{1}{\psi}} \end{array} \right\}^{\frac{1}{1-\frac{1}{\psi}}} W_t. \end{aligned}$$

We can then cancel out W_t from both sides, to obtain an expression relating A_t and μ_t:

$$A_t = \left\{ \begin{array}{c} (1-\beta) \left[\mu_t \right]^{1-\frac{1}{\psi}} \\ +\beta \left[\left(E_t \left[\left(A_{t+1} R_{w,t+1} \left(1-\mu_t \right) \right)^{(1-\gamma)} \right] \right)^{\frac{1}{1-\gamma}} \right]^{1-\frac{1}{\psi}} \end{array} \right\}^{\frac{1}{1-\frac{1}{\psi}}}.$$

Rearranging,

$$\begin{aligned} A_t^{1-\frac{1}{\psi}} &= (1-\beta) \left[\mu_t \right]^{1-\frac{1}{\psi}} \\ &\quad +\beta \left[\left(E_t \left[\left(A_{t+1} R_{w,t+1} \right)^{(1-\gamma)} \right] \right)^{\frac{1}{1-\gamma}} \right]^{1-\frac{1}{\psi}} \left(1-\mu_t \right)^{1-\frac{1}{\psi}}. \end{aligned}$$

Substituting in $A_t^{1-\frac{1}{\psi}} = (1-\beta)\, \mu_t^{-\frac{1}{\psi}}$,

$$\begin{aligned}(1-\beta)\,\mu_t^{-\frac{1}{\psi}} &= (1-\beta)\,[\mu_t]^{1-\frac{1}{\psi}} \\ &+\beta\left[\left(E_t\left[(A_{t+1}R_{w,t+1})^{(1-\gamma)}\right]\right)^{\frac{1}{1-\gamma}}\right]^{1-\frac{1}{\psi}}(1-\mu_t)^{1-\frac{1}{\psi}}.\end{aligned}$$

Now substitute in the expression for growth in wealth as a function of returns, $W_{t+1} = R_{w,t+1}\,(1-\mu_t)\,W_t$,

$$\begin{aligned}&(1-\beta)\,[\mu_t]^{-\frac{1}{\psi}} \\ =\ &(1-\beta)\,[\mu_t]^{1-\frac{1}{\psi}} \\ &+\beta\left[\left(E_t\left[\left(\frac{A_{t+1}W_{t+1}}{(1-\mu_t)\,W_t}\right)^{(1-\gamma)}\right]\right)^{\frac{1}{1-\gamma}}\right]^{1-\frac{1}{\psi}}(1-\mu_t)^{1-\frac{1}{\psi}},\end{aligned}$$

so that, rearranging

$$\frac{E_t\left[(A_{t+1}W_{t+1})^{(1-\gamma)}\right]}{(A_{t+1}W_{t+1})^{(1-\gamma)}} = \left[(1-\beta)\,\beta^{-1}\left(\frac{1-\mu_t}{\mu_t}\right)^{\frac{1}{\psi}}\right]^{\frac{1-\gamma}{1-\frac{1}{\psi}}}\left(\frac{(1-\mu_t)\,W_t}{A_{t+1}W_{t+1}}\right)^{1-\gamma}.$$

Note that

$$W_{t+1} = R_{w,t+1}\,(1-\mu_t)\,W_t,$$

so that

$$\frac{E_t\left[(A_{t+1}W_{t+1})^{(1-\gamma)}\right]}{(A_{t+1}W_{t+1})^{(1-\gamma)}} = \left[(1-\beta)\,\beta^{-1}\left(\frac{1-\mu_t}{\mu_t}\right)^{\frac{1}{\psi}}\right]^{\frac{1-\gamma}{1-\frac{1}{\psi}}}\left(\frac{1}{A_{t+1}R_{w,t+1}}\right)^{1-\gamma}.$$

Also,

$$\frac{C_{t+1}}{\mu_{t+1}} = R_{w,t+1}\frac{1-\mu_t}{\mu_t}C_t,$$

so that

$$\begin{aligned}&\frac{E_t\left[(A_{t+1}W_{t+1})^{(1-\gamma)}\right]}{(A_{t+1}W_{t+1})^{(1-\gamma)}} \\ =\ &\left[(1-\beta)\,\beta^{-1}\left(\frac{1}{\mu_{t+1}R_{w,t+1}}\frac{C_{t+1}}{C_t}\right)^{\frac{1}{\psi}}\right]^{\frac{1-\gamma}{1-\frac{1}{\psi}}}\left(\frac{1}{A_{t+1}R_{w,t+1}}\right)^{1-\gamma}.\end{aligned}$$

Finally, since $A_{t+1}^{1-\frac{1}{\psi}} = (1-\beta)\,\mu_{t+1}^{-\frac{1}{\psi}}$,

$$\frac{E_t\left[(A_{t+1}W_{t+1})^{(1-\gamma)}\right]}{(A_{t+1}W_{t+1})^{(1-\gamma)}}$$
$$= \quad \left[(1-\beta)\,\beta^{-1}\left(\frac{1}{\mu_{t+1}R_{w,t+1}}\frac{C_{t+1}}{C_t}\right)^{\frac{1}{\psi}}\right]^{\frac{1-\gamma}{1-\frac{1}{\psi}}}\left(\frac{1}{(1-\beta)^{\frac{1}{1-\frac{1}{\psi}}}\,\mu_{t+1}^{-\frac{1}{\psi}\frac{1}{1-\frac{1}{\psi}}}\,R_{w,t+1}}\right)^{1-\gamma}.$$

Note that μ_{t+1} cancels out. Collecting terms,

$$\frac{E_t\left[(A_{t+1}W_{t+1})^{(1-\gamma)}\right]}{(A_{t+1}W_{t+1})^{(1-\gamma)}} = \left[\beta^{-1}\left(\frac{C_{t+1}}{C_t}\right)^{\frac{1}{\psi}}R_{w,t+1}^{-1}\right]^{\frac{1-\gamma}{1-\frac{1}{\psi}}}.$$

Substitute back in the expression for the pricing kernel to obtain

$$\frac{\omega_{t+1}}{\lambda_t} = \beta\left(\left[\beta^{-1}\left(\frac{C_{t+1}}{C_t}\right)^{\frac{1}{\psi}}R_{w,t+1}^{-1}\right]^{\frac{1-\gamma}{1-\frac{1}{\psi}}}\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}-1}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}.$$

Rearranging,

$$\frac{\omega_{t+1}}{\lambda_t} = \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\right]^{\frac{1-\gamma}{1-\frac{1}{\psi}}} R_{w,t+1}^{\frac{1-\gamma}{1-\frac{1}{\psi}}-1},$$

which, given $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$, corresponds to the pricing kernel in equation (10).

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