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How Large Are Returns to Scale in the U.S.? A View Across the Boundary

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In this article, I investigate the size of the returns to scale in aggregate U.S. production. I do so by estimating the aggregate returns to scale within a theory-consistent general equilibrium framework using Bayesian methods. This approach distinguishes this article from much of the empirical literature in this area, which is largely based on production-function regressions and limited-information methods. The production structure within a general equilibrium setting, on the other hand, is subject to cross-equation restrictions that can aid and sharpen inference. My investigation proceeds against the background that increasing returns are at the core of business cycle theories that rely on equilibrium indeterminacy and sunspot shocks as the sources of economic fluctuations (e.g., Benhabib and Farmer 1994; Guo and Lansing 1998; Weder 2000).

Specifically, the theoretical literature has shown that multiple equilibria can arise when the degree of returns to scale is large enough. At the same time, the consensus of a large empirical literature is that aggregate production exhibits constant returns. However, equilibrium indeterminacy is a characteristic of a system of equations and can therefore not be assessed adequately with production function regressions. Instead, empirical researchers should apply full-information, likelihood-based methods to conduct inference along these lines. as not

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allowing for indeterminacy leaves the empirical model misspecified. I therefore estimate the returns to scale in a theory-consistent manner using econometric methods that allow for indeterminate equilibria. I apply the methodology developed by Lubik and Schorfheide (2004) to bridge the boundary between determinacy and indeterminacy and estimate a theoretical model over the entire parameter space, including those parameter combinations that imply indeterminacy. This view across the boundary allows me to detect the possibility that data were generated under indeterminacy and provides the correct framework for estimating the returns to scale.

I proceed in three steps. First, I estimate a standard stochastic growth model with increasing returns to scale in production. In this benchmark specification, I estimate the model only on that region of the parameter space that implies a unique, determinate equilibrium to get an assessment of what a standard approach without taking into account indeterminacy would result in. The estimated model is based on the seminal paper of Benhabib and Farmer (1994). The mechanism that leads to increasing returns is externalities in the production process: individual firms have production functions with constant returns, but these are subject to movements in an endogenous productivity component that depends on the production decisions by all other firms in the economy. The key assumption is that individual firms take this productivity component as given and thereby do not take into account that increases in individual factor inputs also raise this productivity component. In the aggregate, the feedback effect from this mechanism can lead to increasing returns in the economy-wide production function. Benhabib and Farmer (1994) show analytically that if the strength of this feedback effect, tied to an externality parameter, is large enough, the resulting equilibria can be indeterminate in the sense that there are multiple adjustment paths to the steady state.

In this benchmark model with externalities, I find estimates that are tightly concentrated around the case of constant returns. Moreover, I also find that aggregate labor supply is fairly inelastic. This finding presents a problem for the existence of indeterminate equilibria due to increasing returns. It can be shown algebraically that the threshold required for an indeterminate equilibrium to arise depends on how elastic the labor supply is. Even with only mildly increasing returns, crossing the boundary into indeterminacy requires a perfectly elastic labor supply, both of which factors I can rule out from my estimation. Based on this baseline model with externalities, it would therefore seem unlikely that equilibrium indeterminacy would arise since the parameter estimates are far away from their threshold values.

In the second step, I therefore estimate a modified version of the benchmark model that allows for variable capacity utilization based on the influential paper by Wen (1998). He shows that the indeterminacy threshold is considerably closer to the constant-returns case when production is subject to variable capacity utilization, that is, when firms can vary the intensity with which the capital stock is used. Given typical parameter values from the literature, the required degree of increasing returns for an indeterminate equilibrium is within the range of plausible empirical estimates. When I estimate the model with variable capacity utilization, I find mildly increasing returns, but the statistical confidence region includes the constant-returns case. As in the benchmark model, I find an inelastic labor supply. In Wen's model, the threshold value of the returns-to-scale parameter is a function of the labor supply elasticity. The threshold attains a minimum for a perfectly elastic labor supply but rises sharply when labor becomes less elastic. Even with mildly increasing returns, these results indicate that indeterminacy will likely not arise in the framework with variable capacity utilization on account of the labor supply parameter.

A caveat to this conclusion is that the results are obtained by restricting the estimation to the determinate region of the parameter space. If the data are generated under parameters that imply indeterminacy, the thus-estimated model would be misspecified and the estimates biased. This potential misspecification would manifest itself as a piling up of parameter estimates near or at the boundary between determinacy and indeterminacy (Canova 2009; Morris 2016) or it might not be detected at all if there is a local mode of the likelihood function in the determinacy region.

In a third step, I therefore apply the methodology developed by Lubik and Schorfheide (2004) that takes the possibility of indeterminacy into account and allows a researcher to look across the boundary.[1] Reestimating the two models over the entire parameter space leave the original results virtually unchanged. Using measures of fit, I find that it is highly unlikely that U.S. data are generated from an indeterminate equilibrium and are driven by nonfundamental or sunspot shocks. The combination of at best mildly increasing returns and inelastic labor supply rule out indeterminacy even after correcting for potential biases in the estimation algorithm.[2]

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[1] This notion is discussed in further detail in Lubik and Schorfheide (2004) and An and Schorfheide (2007).

[2] Conceptually, this article is closest to Farmer and Ohanian (1999). They estimate a model with variable capacity utilization and preferences that are nonseparable in consumption and leisure. This specification requires only a small degree of increasing returns to generate indeterminacy. Their empirical estimates indicate that returns

The article is structured as follows. In the next section, I specify the benchmark model, namely a standard stochastic growth model with externalities in production, and I discuss how this can imply increasing returns to scale and equilibrium indeterminacy. Section 2 describes my empirical approach and discusses the data used in the estimation. In the third section, I present and discuss results from the estimation of the benchmark model, while I extend the standard model in Section 4 to allow for variable capacity utilization. I address the issue of an indeterminate equilibrium as the source of business cycle fluctuations within this context in Section 5. The final section concludes and discusses limitations and extensions of the work contained in this article.

1. A FIRST PASS: THE STANDARD RBC MODEL WITH EXTERNALITIES

The benchmark model for studying returns to scale is the standard stochastic growth model with an externality in production. I use this model as a data-generating process from which I derive benchmark estimates for the returns to scale from aggregate data. Moreover, this model has been used by Benhabib and Farmer (1994) and Farmer and Guo (1994) to study the implications of indeterminacy and sunspot-driven business cycles. It will therefore also serve as a useful benchmark for capturing the degrees to scale when the data are allowed to cross the boundary between determinacy and indeterminacy.

In the model economy, a representative agent is assumed to maximize the intertemporal utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\log c_t - \chi_t \frac{n_t^{1+\gamma}}{1+\gamma} \right], \tag{1}$$

subject to sequences of the budget constraint:

$$c_t + k_{t+1} = A_t \bar{e}_t k_t^{\alpha} n_t^{1-\alpha} + (1-\delta) k_t, \tag{2}$$

by choosing sequences of consumption $\{c_t\}_{t=0}^{\infty}$, labor input $\{n_t\}_{t=0}^{\infty}$, and the capital stock $\{k_{t+1}\}_{t=0}^{\infty}$. The structural parameters satisfy the restrictions: $0 < \beta < 1$, $\gamma \geq 0$, $0 < \alpha < 1$, $0 < \delta < 1$, whereby β is the discount factor, γ the inverse of the Frisch labor supply elasticity, α the capital share, and δ the depreciation rate.

to scale are, in fact, increasing, but that U.S. data are nevertheless better described by the standard RBC model without sunspot shocks. This paper differs from theirs in that they estimate the model equation by equation without imposing cross-equation restrictions. Secondly, they do not formally test whether U.S. time series are better represented by a specification that allows for sunspot shocks. In this article, I conduct a formal test that can distinguish between the two variants.

The externality in the production process, $\overline{e}_t$, is taken parametrically by the agent. Conceptually, this means that when computing first-order conditions for the agent's problem, $\overline{e}_t$ is taken as fixed. It is only when equilibrium conditions are imposed ex post that the functional dependence of $\overline{e}_t$ on other endogenous variables is realized.[3] I assume that $\overline{e}_t$ depends on the average capital stock $\overline{k}_t$ and labor input $\overline{n}_t$:

$$\overline{e}_t = \left[\overline{k}_t^{\alpha} \overline{n}_t^{1-\alpha}\right]^{\eta-1}, \tag{3}$$

where the externality parameter $\eta \geq 0$ captures the returns to scale. When $\eta = 1$, production exhibits constant returns, while for $\eta > 1$ increasing returns are obtained. In equilibrium, $\overline{k}_t = k_t$ and $\overline{n}_t = n_t$. The social production function is thus given by:

$$y_t = A_t k_t^{\alpha\eta} n_t^{(1-\alpha)\eta}. \tag{4}$$

The model economy is driven by two exogenous shocks, technology A_t and preference χ_t, which captures variations in the disutility of working. I assume that A_t is a stationary first-order autoregressive process. Specifically, the level of technology is assumed to evolve according to:

$$A_t = (A_{t-1})^{\rho_A} e^{\varepsilon_t^A},\ \varepsilon_t^A \sim \mathcal{N}\left(0, \sigma_A^2\right), \tag{5}$$

where $0 \leq \rho_A < 1$ and mean technology is normalized to one. The shock ε_t^A is a zero-mean Gaussian innovation with variance σ_A^2. The preference process χ_t is also assumed to follow a stationary AR(1) process:

$$\chi_t = \left(\chi_{t-1}\right)^{\rho_\chi} e^{\varepsilon_t^\chi},\ \varepsilon_t^\chi \sim \mathcal{N}\left(0, \sigma_\chi^2\right), \tag{6}$$

where $0 \leq \rho_\chi < 1$. The preference shock alters the marginal rate of substitution between consumption and leisure.

The first-order conditions for this model form a system of equations that needs to be solved in order to provide a reduced form representation that serves as an input into the estimation procedure. This can be accomplished by approximating the equilibrium conditions in the neighborhood of the steady state using log-linearization. The resulting linear rational expectations model can then be solved using standard methods. I list the linearized equations that are used to estimate the model in the Appendix.

[3] A social planner would recognize this dependence and impose it ex ante, that is, before taking first-order conditions. It is this asymmetry that leads to lower social welfare in the benchmark case and creates a channel for welfare-improving tax policy, for instance. In addition, it creates the underpinning for equilibrium indeterminacy as Benhabib and Farmer (1994) show.

In a seminal paper, Benhabib and Farmer (1994) demonstrate that if the degrees of scale in production are large enough, then the model exhibits equilibrium indeterminacy. This has two implications for the behavior of the model. First, there are multiple adjustment paths to the unique steady state. Second, equilibrium dynamics can change markedly when compared to the determinate case in that nonfundamental shocks, "sunspots," can affect equilibrium outcomes and generate additional volatility. Benhabib and Farmer (1994) derive a simple analytical threshold condition for indeterminacy to arise in a continuous-time framework. The corresponding conditions for the discrete-time case, which are relevant for the model that I take to the data, are considerably more complicated, lengthy, and in parts not very intuitive. I list and discuss them in the Appendix. In order to develop intuition, I therefore derive insights based on the well-known Benhabib-Farmer condition first.

A necessary condition for indeterminacy to arise in Benhabib and Farmer (1994) is that the returns-to-scale parameter η is above a certain threshold given by the following:

$$\eta > \frac{1+\gamma}{1-\alpha}. \tag{7}$$

It has to be larger than the ratio between the exponent on the disutility of labor $1+\gamma$ and the labor share in production. Since the latter is a value between zero and one and typically found to be around two-thirds, indeterminacy in this model requires quite high increasing returns. This high level of a threshold is further exacerbated if the labor supply is less than perfectly elastic, that is, if $\gamma > 0$.

The intuition behind the condition is that if the returns to scale are large enough, the aggregate labor demand schedule is upward-sloping. In the standard case, workers are employed until their marginal product equals their wage. Hiring an additional worker reduces firm profits since the competitive wage would be higher than what the worker could produce at the margin. With production externalities as in (3), however, an additional feedback effect arises. At the margin, additional labor input raises economy-wide total factor productivity through its effect on $\overline{e}_t$, which feeds back on the competitive wage and counters the declining marginal product of labor. When this effect is large enough, labor demand starts sloping upward since the externality factor becomes dominant. In this scenario, the economy becomes susceptible to the influence of sunspot shocks that are unrelated to fundamentals such as productivity disturbances. When firms believe employment is higher than it should be given the fundamentals, this belief is self-validating in an indeterminate equilibrium: higher labor input leads to a stronger externality, which raises production and, in turn, requires more labor input.

Since I am interested in taking this model to the data, I employ a discrete-time model. I list the corresponding analytical determinacy conditions in the Appendix. Generally speaking, the intuition from the continuous-time condition (7) carries over to discrete time, specifically the fact that the labor-demand schedule needs to be upward-sloping. I now turn to the first empirical exercise, where I estimate the standard RBC model with externalities to determine the returns to scale in the aggregate production function for the U.S. economy. I will do so against the background of the possibility of an indeterminate equilibrium in the data in case the indeterminacy conditions apply. Whether they do so is naturally an empirical question.

2. EMPIRICAL APPROACH

Bayesian Estimation

My empirical approach to the questions raised in this article is Bayesian DSGE estimation. This methodology is discussed in detail in An and Schorfheide (2007). The main object of investigation is the parameter vector θ, on which inference is conducted by extracting information from the observed data $Y^T = \{y_t\}_{t=1}^T$, with a sample size of T. The data are interpreted through the lens of a structural model, which provides restrictions necessary for parameter identification. A log-linear DSGE model can be written in terms of a state-space representation for y_t:

$$y_t = \Xi(\theta) s_t, \quad \Gamma_0(\theta) s_t = \Gamma_1(\theta) s_{t-1} + \Psi(\theta) \epsilon_t + \Pi(\theta) \eta_t, \qquad (8)$$

where the vector s_t collects the state variables of the theoretical model and where the coefficient matrices are shown as generally dependent on the structural parameters θ. ϵ_t is a vector of fundamental shocks, and the vector η_t collects the endogenous forecast errors of the rational expectations formation process in the parlance of Sims (2002). The model can be solved under determinacy and indeterminacy by the method described in Lubik and Schorfheide (2003).

Empirical evaluation in this Bayesian framework starts by specifying a probability distribution of the structural shocks $\{\epsilon_t\}_{t=1}^T$, from which a likelihood function $L\left(\theta|Y^T\right)$ can be obtained by means of the Kalman filter. The next step is to specify a prior distribution $p(\theta)$ over the structural parameters. The data Y^T are then used to update the prior through the likelihood function. The main concept in Bayesian inference is the posterior distribution $p(\theta|Y^T)$, which is the distribution of the parameters conditional on having seen the data. Moments of the posterior can then be used to characterize the parameter estimates.

The posterior distribution is computed according to Bayes' Theorem:

$$p(\theta|Y^T) = \frac{L(\theta|Y^T)p(\theta)}{\int L(\theta|Y^T)p(\theta)d\theta}, \tag{9}$$

whereby the denominator is the marginal data density, which can serve as a measure of overall model fit. Finally, the prior and posterior can be used to directly compare two different models or specifications, H_0 and H_1, as to which explains a given data set better. This is done by conducting a posterior odds test, which is similar to computing likelihood ratios. I apply this test later on to assess whether U.S. data are more likely to have been generated under determinacy or indeterminacy.

Data and Priors

I estimate all models in this article on quarterly U.S. data from 1954:3 to 2007:4.[4] I estimate the benchmark specifications on two data series, namely output and employment. Aggregate output y_t is measured as (the natural logarithm of) real per capita GDP. Since I assume that the model is driven by stationary shock processes, I need to remove any trends. I do so by passing the output series through an HP filter with smoothing parameter $\lambda = 1600$, which is standard for quarterly data. Employment n_t is measured as average weekly hours times employment from the Household Survey divided by population. I assume that the employment series is stationary, so that no further transformation is necessary. In the extended model discussed in Section 4, I also include a measure of capacity utilization in the data set. This is measured by the series available from the Board of Governors and reported as a percentage of industrial production. No further transformation is applied to these data series.

In a Bayesian DSGE estimation approach, a prior distribution needs to be specified for the model parameters. I largely choose prior means to be consistent with values established previously in the literature. The prior distributions are reported in Table 1. The specific form of the density is predicated by the type of parameter. A parameter restricted to lie on the unit interval is assumed to have a beta-distribution, while parameters on the real line are typically chosen to have gamma distributions, whereas variances are described by inverse gamma densities. I choose tight priors on the capital share and depreciation, but looser priors on the labor-supply elasticity and the returns-to-scale parameter.

[4] I choose to end my sample period at the onset of the Great Recession. The sharp decline in GDP would be difficult to capture even with HP-filtered data. Moreover, the

Table 1 Prior Distribution

Name	Range	Density	Mean	Std. Deviation
α	$[0, 1)$	Beta	0.34	0.020
β	$[0, 1)$	Beta	0.99	0.002
γ	$\mathbb{R}^+$	Gamma	2.00	0.500
δ	$[0, 1)$	Beta	0.025	0.005
η	$\mathbb{R}^+$	Gamma	1.00	0.500
ρ_A	$[0, 1)$	Beta	0.20	0.100
ρ_χ	$[0, 1)$	Beta	0.95	0.050
σ_A	$\mathbb{R}^+$	InvGamma	N.A.	N.A.
σ_χ	$\mathbb{R}^+$	InvGamma	N.A.	N.A.

Note: The inverse gamma priors are of the form $p(\sigma|\nu, s) \propto \sigma^{-\nu-1}e^{-\nu s^2/2\sigma^2}$, where $\nu = 1$ and s equals 0.015. The prior is truncated at the boundary of the determinacy region.

Specifically, I set a very tight prior for the discount factor β with a mean of 0.99. The prior on the capital share α has a mean of 0.34 with a standard deviation of 0.02, while the depreciation rate δ has a mean of 0.025 with a standard deviation of 0.005. These are standard values in the calibration literature, but I allow for some flexibility in these parameters to somewhat adjust to the model environment at hand. I impose a more agnostic prior on the labor-supply elasticity parameter γ, where I impose some curvature on the disutility of labor with a mean of 2.0 and a standard deviation of 0.5. This value is somewhat distant from the case of a perfectly elastic labor supply with $\gamma = 0$. I choose the higher value since there is considerable evidence, both microeconomic and macroeconomic, that labor supply is not perfectly elastic and can be quite inelastic. I allow for some variation in this parameter because of the uncertainty surrounding this value.

The key parameter in this article is the degree of returns to scale, η. I center this value at the constant-returns case of 1 but assume a large standard deviation of 0.5. My underlying motivation is that I want the data to clearly dominate the posterior estimate. Finally, the parameters governing the two exogenous shock processes, technology A_t and preferences χ_t, are based on prior experience. The autocorrelation parameter for the technology process ρ_A has a mean of 0.95, while the corresponding value for ρ_χ is a slightly less persistent 0.9.

apparent shift in the level path of GDP that is visible in the data from 2008 on might affect parameter estimates.

3. SOME BASELINE ESTIMATION RESULTS

As my benchmark, I estimate the RBC model with externalities by letting all parameters vary freely over the admissible range as discussed in the model section above. Some of the parameter combinations would imply indeterminacy given the condition (7). As discussed before, the RBC model with externalities requires both very high labor supply elasticity (a small γ) and increasing returns for indeterminacy (a high enough $\eta > 1$). In particular, it would require values that are beyond those usually found in the literature. Studies using production function data such as Basu and Fernald (1997) typically find at best only mildly increasing returns at the aggregate level.

In the benchmark specification, I adopt a naive approach to the potential presence of indeterminate equilibria. I let myself be guided by prior studies that use limited information or single-equation methods that have nothing to say about indeterminacy since it is a property of a dynamic general equilibrium system (see the discussion in Lubik and Schorfheide [2004]). Prior inspection shows that it is highly unlikely that the returns to scale are large enough to meet the indeterminacy threshold. For instance, even with perfectly elastic labor supply, the indeterminacy condition in the continuous case would require a returns-to-scale parameter of $\eta > 1.5$ for $\alpha = 1/3$. It therefore seems a priori unlikely that the benchmark model would produce indeterminate outcomes based on typical parameter values found in the literature.

I thus proceed by estimating the model only over the determinate region over the parameter space. This procedure establishes a baseline as to what the parameter estimates that define the threshold between determinacy and indeterminacy would be if the model were restricted to a subset of the full admissible parameter space. I implement this numerically by penalizing the region of the parameter space that would imply indeterminacy for all possible draws from the joint prior distribution. I do so by throwing out all parameter combinations for which the solution algorithm of Sims (2002) returns an indeterminate equilibrium. More precisely, the solution algorithm rejects all draws that fall outside the determinacy bounds established by the analytical conditions given in the Appendix. This procedure implies that the prior distribution is restricted to the determinacy region only so that the search algorithm for the maximum of the likelihood function cannot venture into the indeterminacy region.

Table 2 reports the estimation results from the RBC model. The column labeled "Baseline Model" contains those from the baseline specification described above. The estimates of the capital share α, the discount factor β, and the depreciation rate δ are consistent with those commonly used in the calibration literature, respectively, 0.33, 0.99,

Table 2 Parameter Estimation Results, RBC Model

	Baseline Model		Restricted Model: $\eta = 1$		Restricted Model: $\gamma = 0$	
	Mean	**90% Interval**	**Mean**	**90% Interval**	**Mean**	**90% Interval**
α	0.331	[0.301, 0.349]	0.335	[0.310, 0.368]	0.329	[0.285, 0.371]
β	0.986	[0.979, 0.995]	0.990	[0.982, 0.994]	0.988	[0.979, 0.995]
γ	2.061	[1.573, 2.671]	2.332	[1.871, 2.904]	0.000	–
δ	0.022	[0.014, 0.030]	0.025	[0.017, 0.031]	0.026	[0.021, 0.030]
η	0.982	[0.894, 1.060]	1.000	–	0.912	[0.796, 0.923]
ρ_A	0.980	[0.968, 0.998]	0.981	[0.971, 0.998]	0.987	[0.971, 0.999]
ρ_χ	0.945	[0.901, 0.987]	0.974	[0.921, 0.995]	0.979	[0.960, 0.985]
σ_A	0.005	[0.003, 0.008]	0.018	[0.009, 0.029]	0.018	[0.014, 0.022]
σ_χ	0.019	[0.010, 0.025]	0.042	[0.030, 0.051]	0.030	[0.021, 0.040]

Note: The table reports posterior means and 90 percent coverage regions (in brackets). The posterior summary statistics are calculated from the output of the posterior simulator.

and 0.02, although the latter is contained in a fairly wide 90 percent probability interval. Posterior estimates of the autoregressive parameters tend to be high, which is a common observation in small-scale Bayesian DSGE models. The posterior mean of the scale parameter η is 0.98 with a 90 percent coverage range of $[0.89, 1.06]$. The posterior for this parameter is thus firmly centered on a small region around the constant-returns-to-scale case, which would rule out any possibility of indeterminacy. In addition, the labor supply parameter γ has a posterior mean of 2.06. For this value, the minimum required degree of returns to scale to result in indeterminacy would have to be 4.57.

To be fair, the joint prior distribution over the parameter space put virtually no probability mass on the indeterminacy region even before restricting the solution to determinate equilibria, and it was centered on constant returns. To gauge the sensitivity of the estimation, I experimented with various alternative starting values and priors. The results proved to be robust to different starting values, as the iterations of the algorithm quickly approached the benchmark posterior mode, even for high values of η. There was also no evidence that the algorithm would pile up at the boundary of the parameter space, that is, the threshold between determinacy and indeterminacy, which Morris (2016) suggests is evidence of misspecification. I obtained similar results when varying the prior distribution, specifically in the direction of a higher mean of η and a tighter distribution. Posterior mode estimates quickly converged to the benchmark case. This suggests the conclusion that restricting the model to the determinacy regions does not bias the findings since the indeterminacy regions are far away from plausible parameterizations consistent with the data.

As a second exercise, I estimate the model under the restriction $\eta = 1$. The results are reported in Table 2 in the column labeled "Restricted Model: $\eta = 1$." This is the case of the standard RBC model as in King, Plosser, and Rebelo (1988). It is well-known that the standard RBC model does not admit indeterminate equilibria, so that I do not have to restrict the parameter space over which the model is estimated. The parameter estimates were virtually unchanged with the exception of the labor parameter γ, which increased to 2.33. As the benchmark results indicated before, the estimation algorithm settles quickly and closely on the constant-returns-to-scale case. Therefore, conditioning on this value, $\eta = 1$, should not affect the other parameter estimates much.

Alternatively, I fix $\gamma = 0$ (see Table 2, last column). This specification corresponds to the benchmark case of Benhabib and Farmer (1994) with perfectly elastic labor supply. Under this specification, the required returns to scale for equilibrium indeterminacy are considerably lower, namely at 1.5 given the standard parameterization of $\alpha = 1/3$. This restriction results in a posterior mean of $\eta = 0.91$. The algorithm thus pushed the returns-to-scale parameter in an opposite direction of what would be needed to cross the indeterminacy threshold. I explored this specification a bit further by imposing a tight prior on η with a mean of 1.60 and a standard deviation of 0.05. Even in this case, the resulting posterior mean is 0.99, as in the unrestricted benchmark case. It seems clear that the information in the data strongly prefers mildly decreasing returns to scale in production.[5]

I can formally compare the different specifications by computing their marginal data densities (MDD). These can be thought of as comparable to maximum likelihood values in that they capture the value of the posterior with all parameters integrated out. They also form the basis of posterior odds tests, which allow econometricians to discriminate between two alternative models in terms of overall fit. Given even prior probabilities on the two competing models, the model with the higher MDD can be considered as the better descriptor of the data. I report the MDDs in Table 3. Clearly, the information in the data draws the posterior strongly toward decreasing returns. The restricted model with $\eta = 1$ dominates all others, as can be seen in the first row.

[5] Arguably, the standard RBC model is misspecified in that it assumes constant returns to scale. However, the degree of uncertainty around this value is such that it encompasses constant returns.

Table 3 Marginal Data Densities and Posterior Odds Tests

Marginal Data Densities				
	Baseline	$\eta = 1$	$\gamma = 0$	Sunspot
RBC Model	128.75	129.81	88.43	–
Cap. Util.	138.92	130.01	–	120.34

Note: Marginal data densities are approximated by Geweke's (1999) harmonic mean estimator.

Moreover, comparison of the MDDs allows us to reject the specification with a perfectly elastic labor supply by a wide margin.[6]

In order to get a sense of the driving forces behind the data as interpreted through this specific model, I also compute variance decompositions. The results are broadly similar across different model specifications. Therefore, I only report those for the baseline model in Table 4. Technology shocks determine about 80 percent of fluctuations in output, the remainder are made up by shocks to preferences, namely the disutility of working. In contrast, these labor supply shocks are the main determinants of labor input in the amount of roughly two-thirds of the overall variability.

I can draw some preliminary conclusions at this point. Overall, I do not find any evidence of increasing returns in aggregate U.S. data under the assumption that the standard RBC model with production externalities is the data-generating process. The results show that the estimates are tightly clustered around the constant-returns case with more probability mass on decreasing returns. Even if we are willing to allow for increasing returns in contrast to what the data say, estimates for η are not at the level required for indeterminacy in an environment with perfectly elastic labor supply. In addition, the estimated aggregate labor supply elasticity is far too low to generate indeterminacy at any remotely plausible level of increasing returns.[7,8]

[6] The difference between the two values of the MDDs is almost 50 on a log scale. With even prior odds, that is equal prior probability on each model being the data-generating process, this amounts to a probability one acceptance of the constant-returns-to-scale model with inelastic labor supply.

[7] To the best of my knowledge, no empirical study has found increasing returns of that magnitude. Baxter and King (1991) come closest with $\eta = 1.6$.

[8] The main caveat for this conclusion is that the model is estimated under the restriction that the equilibrium is determinate. By doing so, I rule out any possibility of finding considerable returns to scale a priori. In effect, the model is misspecified along this dimension. The robustness checks that I performed show, however, that this is not the case. In that sense, the additional restriction to the space of determinate equilibria is not much of a restriction at all. This may be different for other models.

Table 4 Variance Decompositions

	Technology		Preference		Sunspot/ Measurement	
	Mean	90% Interval	Mean	90% Interval	Mean	90% Interval
	Standard RBC					
Output	0.81	[0.74, 0.90]	0.19	[0.08, 0.29]		
Labor	0.36	[0.30, 0.51]	0.64	[0.58, 0.71]		
	Variable Capacity Utilization					
Output	0.94	[0.89, 0.98]	0.06	[0.02, 0.11]		
Labor	0.13	[0.09, 0.19]	0.87	[0.81, 0.89]		
	Variable Capacity Utilization with Sunspots					
Output	0.81	[0.74, 0.86]	0.04	[0.01, 0.06]	0.15	[0.12, 0.23]
Labor	0.08	[0.04, 0.12]	0.21	[0.15, 0.29]	0.71	[0.60, 0.82]
	Variable Capacity Utilization with Utilization Data					
Output	0.92	[0.88, 0.95]	0.02	[0.00, 0.04]	0.06	[0.01, 0.12]
Labor	0.01	[0.00, 0.02]	0.67	[0.58, 0.76]	0.32	[0.22, 0.42]
Utilization	0.32	[0.24, 0.39]	0.06	[0.02, 0.09]	0.62	[0.52, 0.74]

4. VARIABLE CAPACITY UTILIZATION AND INCREASING RETURNS

The main conclusion from my empirical analysis of the Benhabib and Farmer (1994) model is that the degree of returns to scale necessary for indeterminacy to arise is implausibly large. This has been noted in the literature, which evolved toward developing frameworks that lead to a lower threshold value. A key paper following up on this issue is Wen (1998), who introduces variable capacity utilization into an otherwise standard Benhabib-Farmer model.[9] He is able to show that the degree of increasing returns required for indeterminacy is considerably less than in the standard model. I now use his framework to reassess the conclusion drawn in the previous section. I proceed as before in that I first estimate the model by restricting the parameter space to the determinacy region. This establishes a baseline to assess whether disregarding the possibility of indeterminacy has an effect on parameter estimates. This issue is addressed in the subsequent section.

[9] It has long been recognized that variable capacity utilization is an important component of business cycle analysis. In a key paper, Burnside and Eichenbaum (1996) demonstrate that variable capital utilization can significantly reduce the volatility of technology shocks required to replicate observed business cycles in otherwise standard models. Moreover, Basu and Fernald (1997) point out that production function regressions need to allow for variable capacity utilization in order to be able to remove endogenous components from total factor productivity and to get unbiased estimates of the returns to scale.

I assume that a representative agent maximizes the intertemporal utility function (1) as before. The budget constraint is modified by introducing variable capacity utilization u_t:

$$c_t + k_{t+1} = A_t \overline{e}_t \left(u_t k_t\right)^\alpha \left(n_t\right)^{1-\alpha} + (1-\delta_t)k_t. \tag{10}$$

$u_t \in (0,1)$ is the rate of capacity utilization. Given the capital stock k_t, which is predetermined in the current period, changes in utilization affect production and present an additional margin of adjustment. This captures the idea that the capital stock is sometimes left idle and that in general the utilization rate of machinery varies over time, depending on demand conditions, shift work, the work week, and other factors. Varying productive capacity gives firms a margin along which profits can be optimized by preemptively hoarding capital in anticipation of future demand conditions. However, changes in utilization come at a cost since capacity variation affects the depreciation rate. The more intensely the capital stock is utilized, the faster it depreciates. As in Wen (1998), I assume for simplicity a monotonic relationship between u_t and the depreciation rate δ_t:

$$\delta_t = \frac{1}{\theta} u_t^\theta, \tag{11}$$

where θ is a parameter. I can find the first-order conditions by maximizing the utility function (1) subject to the budget constraint and the definition of the depreciation rate by choosing sequences of consumption $\{c_t\}_{t=0}^\infty$, labor input $\{n_t\}_{t=0}^\infty$, capacity utilization $\{u_t\}_{t=0}^\infty$, and capital stock $\{k_{t+1}\}_{t=0}^\infty$.

As in the standard RBC model, I assume that $\overline{e}_t$ captures the externality in the production process and is taken parametrically by the agent. Under this specification, $\overline{e}_t$ depends on the average capital stock $\overline{k}_t$, labor input $\overline{n}_t$, and capacity utilization $\overline{u}_t$:

$$\overline{e}_t = \left[\left(\overline{u}_t \overline{k}_t\right)^\alpha \left(\overline{n}_t\right)^{1-\alpha}\right]^{\eta-1}, \tag{12}$$

where the externality parameter $\eta \geq 0$ captures the returns to scale. As before, production exhibits constant returns when $\eta = 1$ and returns to scale are increasing for $\eta > 1$. The determinacy conditions for this model are listed in the Appendix.

I estimate the model using Bayesian methods as discussed above. For comparison purposes, I estimate the model on the same two data series, output and labor input, and for the same two shocks, technology and labor disutility. In a robustness check, I further utilize data on capacity utilization and allow for the presence of sunspot shocks and measurement error. A convenient feature of the choice of the depreciation cost function is that it implies the same number of independent parameters to be estimated. The existence of a steady state imposes a parametric restriction between θ and the depreciation rate: $\theta = \frac{1-\beta(1-\delta)}{\beta\delta}$. That is, the depreciation cost elasticity is not an independent parameter, but is determined by the steady-state depreciation rate and vice versa. I can therefore choose to treat steady-state depreciation parametrically. Consequently, I impose the same prior on δ and on the other parameters in the model. This implies a prior mean of $\theta = 1.40$. The empirical difference between the benchmark and the extended model only lies in the different dynamics via the introduction of capacity utilization and endogenous depreciation but not in different priors.

The estimation results for the extended model are reported in Table 5. The first set of results is contained in the left column, labeled "Baseline Model," where I allow all parameters to vary freely over the determinacy regions. That is, I throw out all parameter draws that would imply an indeterminate equilibrium just as I did in the benchmark case for the standard model. The parameter estimate that stands out is a high $\gamma = 8.46$, which implies a very inelastic labor supply and thereby likely rules out the possibility of indeterminate equilibria on account of increasing returns. The baseline estimates also show a lower capital elasticity of $\alpha = 0.27$ and a higher depreciation rate of $\delta = 0.05$ than in the standard RBC model. These estimates are consistent with those found in the literature on variable capacity utilization and reflect the impact of the latter on adjusting input margins in production as suggested by Burnside, Eichenbaum, and Rebelo (1995). Moreover, the implied estimate at the posterior means of the depreciation cost parameter is $\theta = 1.20$.

As to the question of increasing returns, I estimate the externality parameter $\eta = 1.09$ with a 90 percent coverage region of $[0.98, 1.17]$. This is higher than in the standard RBC model, although the constant-returns case is included in this coverage region. Incidentally, this value is right at the preferred estimate of Laitner and Stolyarov (2004), who estimate a full set of structural equations derived from a business cycle model using a methods of moments approach that is independent of whether the data are generated from a determinate or indeterminate equilibrium. What is intriguing about this result is that the returns to scale are at the threshold for indeterminacy in the baseline calibration in Wen (1998). Yet, as I argued in the previous section, the other critical parameter is the labor supply elasticity. In his benchmark calibration, Wen (1998) assumes perfectly elastic supply with $\gamma = 0$, whereas the posterior mean in my estimation is considerably higher. While I cannot rule out mild increasing returns empirically, the other parameter estimates imply that the equilibrium is not indeterminate.

Table 5 Parameter Estimation Results, Variable Capacity Utilization

	Baseline Model		Restricted Model: $\eta = 1$		Restricted Model: $\gamma = 0$	
	Mean	90% Interval	Mean	90% Interval	Mean	90% Interval
α	0.274	[0.261, 0.286]	0.254	[0.241, 0.275]	0.201	[0.182, 0.259]
β	0.991	[0.989, 0.994]	0.993	[0.987, 0.996]	0.994	[0.989, 0.999]
γ	8.459	[6.987, 9.801]	12.90	[10.45, 14.86]	0.903	[0.420, 1.681]
δ	0.049	[0.043, 0.055]	0.058	[0.053, 0.064]	0.089	[0.082, 0.099]
η	1.087	[0.975, 1.174]	1.000	–	1.384	[1.121, 1.605]
ρ_A	0.982	[0.969, 0.996]	0.067	[0.059, 0.072]	0.966	[0.958, 0.980]
ρ_χ	0.958	[0.949, 0.968]	0.965	[0.944, 0.991]	0.850	[0.791, 0.921]
σ_A	0.036	[0.030, 0.041]	0.041	[0.036, 0.044]	0.048	[0.042, 0.056]
σ_χ	0.094	[0.085, 0.099]	0.086	[0.070, 0.110]	0.079	[0.054, 0.097]
σ_ζ					0.163	[0.081, 0.303]

Note: The table reports posterior means and 90 percent probability intervals (in brackets). The posterior summary statistics are calculated from the output of the posterior simulator.

As a first robustness check, I estimate a restricted version of the model where I fix $\eta = 1$, which shuts down the externality feedback. The effects on the parameter estimates are somewhat larger than in the corresponding exercise for the RBC model. The posterior mean of α declines to 0.25, while depreciation rate δ increases to 0.06. The labor supply parameter is now estimated at 12.90. However, as the MDDs in the second row of Table 3 show, the unrestricted version is much preferred in terms of overall fit. Interestingly enough, the model with variable capacity utilization also dominates the standard RBC model in explaining labor input and GDP. Finally, I also compute the variance decompositions, which are reported in Table 4. The relative importance of the two shocks, technology and preference, in explaining the two data series is unchanged compared to the first model specification.

As a second robustness check, I also estimate the model using the Federal Reserve's data on capital utilization.[10] Adding a third observable variable to the model requires an additional source of uncertainty in order to avoid a singular likelihood function. I choose to add a measurement error to the observation equation that links the data series to its counterpart in the model instead of introducing an additional

[10] Available at: https://www.federalreserve.gov/releases/g17/

shock. I find that the parameter estimates do not change in any significant manner. The likely reason is that the utilization series mirrors the output series very closely and thus does not contain enough information to improve the empirical model.[11]

Bayesian estimates of a standard RBC model with variable capacity utilization that allows for increasing returns to scale via production externalities show that the U.S. economy is characterized by mildly increasing returns. This stands in contrast with the results derived from the model without capacity utilization, which found constant returns. This leaves open the possibility that the equilibrium in the U.S. economy may be indeterminate given the mechanism described in this article. What goes against this argument is that indeterminacy also requires a low labor supply elasticity. Estimates from both models show that labor is, in fact, fairly inelastically supplied. However, since I restricted the estimation to the determinacy region of the parameter space, I cannot be confident of the soundness of this conclusion. In the next section, I therefore look across the boundary of the determinacy region and estimate the model under indeterminacy.

5. ARE U.S. BUSINESS CYCLES DRIVEN BY SUNSPOT FLUCTUATIONS?

I now follow the implications of the theoretical model to their logical end and assess whether the observed U.S. data are generated under indeterminacy. As the discussion above shows, equilibrium indeterminacy requires a high degree of increasing returns (a large enough estimate of η) and a high labor supply elasticity (a low enough estimate of γ). In all estimated specifications, the labor supply elasticity turned out to be too low for the equilibrium to be indeterminate even if the externalities parameter was within a range that would otherwise have put the economy across the boundary, namely in the model with variable capacity utilization. However, these estimates should be understood against the background that I ruled out indeterminate equilibria a priori by restricting the prior to that region of the parameter space where there is a unique equilibrium.[12]

[11] At the same time, the measurement error explains about one-third of the fluctuations in the utilization series (see Table 4), which does suggest the model is not well-specified to capture movements in utilization that are independent of the output series.

[12] I do not find any indication across the various specifications that the posterior estimates are clustering near the indeterminacy threshold. As discussed in Canova (2009) and Morris (2016), this pile-up of probability mass near the boundary could be seen as evidence that the model is misspecified since indeterminacy is not explicitly accounted for. Nevertheless, it cannot be ruled out that a posterior mode is well within the indeter-

I therefore reestimate the two model specifications over the full parameter space using the methodology developed by Lubik and Schorfheide (2003, 2004), who show how to write the full set of indeterminate equilibria in a reduced form. The estimation algorithm can be used to reveal which of the many indeterminate equilibria the data reflect. At the same time, the indeterminate solution allows for the influence of an additional exogenous disturbance, namely nonfundamental sunspot shocks, in addition to the two fundamental shocks from before. I use the same data series for the estimation as in the benchmark case to ensure comparability across the result. I should note, however, that allowing for indeterminacy and sunspot shocks gives the estimation algorithm additional degrees of freedom to fit the data.

In estimating the models under indeterminacy, the first issue I face is that my chosen benchmark prior puts only small probability mass on the indeterminacy region. This is particularly problematic in the standard RBC model where even in the case of $\gamma = 0$, the required threshold value for η equals 1.5. Allowing for a wider dispersion in these two key parameters does not seem to make much of a difference. I therefore experimented with shifting the prior means. I found that a prior mean of $\eta = 2.6$ with a standard deviation of 0.1 and almost perfectly elastic labor supply would be needed to support a posterior estimate in the indeterminacy region. Since these values are far outside what can be considered a plausible range, it seems safe to rule out estimates based on this prior. Consequently, I argue that the Benhabib and Farmer (1994) model cannot be used to support the notion of sunspot-driven business cycles since it is simply inconsistent with the data.

I face a similar issue in the case of the Wen (1998) model. Under the benchmark prior, there is not much mass in the indeterminacy region. The limiting factor is again the labor supply elasticity parameter γ, which needs to be close to zero to be able to support an indeterminate equilibrium. Experimenting with the prior, I find, however, that a prior mean for η of 1.7 with a standard deviation of 0.2 puts enough mass beyond the boundary. Using this prior, I reestimate the specification with variable capacity utilization. The results are reported in the last column of Table 5. The posterior mean of the elasticity parameter $\eta = 1.38$, which is higher than in the benchmark case. At the same time, the estimate of $\gamma = 0.90$, which guarantees that the equilibrium is indeterminate. As Table 3 shows, however, the MDDs indicate that the indeterminacy specification is rejected relative to the

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minacy region and can therefore not be detected when the parameter space is restricted to determinacy.

benchmark specification even when taking into account the higher degrees of freedom afforded by the model solution under indeterminacy. Table 4 reports the variance decompositions for the indeterminacy specification. Although I can conclude that the data are unlikely to have been generated under indeterminacy, it is interesting to determine how much of an effect sunspot shocks may have on economic fluctuations. The contribution to output fluctuations is small, around 15 percent, whereas sunspots drive a substantial fraction of labor input.

Are U.S. business cycles driven by sunspot fluctuations? Not if one believes that the source of these sunspot fluctuations lies in increasing returns to scale. Based on the results in this section, I can rule out the standard RBC model with production externalities as in Benhabib and Farmer (1994) as the data-generating process for a possible sunspot equilibrium. The extension of Wen (1998) to include variable capacity utilization is more promising, but the statistical support for indeterminacy is quite weak. As the results from the preceding sections show, aggregate U.S. production likely exhibits constant returns to scale, which rules out equilibrium indeterminacy a priori.

6. CONCLUSION

This article studies the returns to scale in aggregate U.S. data by estimating various specifications of the standard RBC model. In order to allow for the possibility of increasing returns in production, so as not to impose constant returns a priori, I introduce aggregate production externalities as in the framework of Benhabib and Farmer (1994). The degree of returns to scale can then be tied to a single parameter that measures the strength of the externality effect. In a second model specification, I also introduce variable capacity utilization as in Wen (1998), who generally reduces the required degree of increasing returns needed to support indeterminacy. All model specifications present in this paper admit the possibility of equilibrium indeterminacy to the effect that business cycles could be driven by extraneous, nonfundamental shocks.

I estimate the various specifications using Bayesian DSGE methods. I find strong evidence for constant returns to scale in aggregate U.S. data. Specifications that impose increasing returns are rejected based on standard model selection criteria. I show in a simple robustness exercise that a substantial degree of increasing returns can only be supported by imposing implausible priors. Equilibrium indeterminacy in the modeling frameworks used in this article requires a high enough degree of increasing returns and a low enough labor supply elasticity. My estimates show that even if increasing returns were present, we can rule out indeterminacy on account of an inelastic labor supply.

Therefore, a theory of sunspot-driven business cycles should not rely on increasing returns to scale in production.

The empirical results are to some extent model-dependent. My conclusion as to the possibility of indeterminate equilibria appears robust as the framework based on production externalities requires implausible labor supply elasticities. Nevertheless, alternative model setups may imply different, less stringent requirements for indeterminacy. Prime candidates are models with alternative utility functions, such as nonseparability in consumption and leisure (see Bennett and Farmer 2000) or models with multiple sectors (see Benhabib, Meng, and Nishimura 2000). Finally, if researchers are interested in sunspot shocks as potential driving forces for business cycles, exploring other avenues than production externalities seems a more promising option. For instance, Lubik and Schorfheide (2004) show that the Great Inflation of the 1970s was caused by sunspot shocks since the Federal Reserve pursued monetary policy that was not aggressive enough in fighting inflation. More recently, Golosov and Menzio (2015) have proposed a novel theoretical framework that generates sunspot-driven business cycles through idiosyncratic and firm-specific uncertainty over the quality of their workers.

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APPENDIX: INDETERMINACY CONDITIONS

Benhabib and Farmer (1994) derive analytical conditions that are necessary for indeterminacy in a continuous-time version of the RBC model with externalities. As it turns out, the corresponding conditions for the discrete-time version are considerably more complex. Meng and Xue (2009) derive these conditions for general forms of utility and production with externalities. Under the restriction $\eta \geq 1$ and logarithmic utility, the necessary and sufficient conditions for indeterminacy are (see Meng and Xue [2009], Proposition 4, case (i)):[13]

$$\Gamma_2 \eta < \frac{1+\gamma}{1-\alpha} < \Gamma_1 \eta,$$

where $\Gamma_1 = \frac{(2-\delta)(1+\beta(1-\delta)) - \frac{\beta}{\alpha}\left(\frac{1-\beta}{\beta}+\delta\right)^2}{2\left(2+\eta\frac{1-\beta}{\beta}+\delta(\eta-1)\right)}$ and $\Gamma_2 = \frac{(1-\beta)(1-\delta)}{\eta\frac{1-\beta}{\beta}+\delta(\eta-1)}$. The condition has the familiar form that links a minimum value of the externalities parameter η to the labor supply elasticity and the capital share but is harder to interpret than the corresponding continuous-time restriction. If we just look at the necessary condition, then we have:

$$\eta > \frac{1+\gamma}{1-\alpha}\frac{1}{\beta(1-\delta)}.$$

Wen (1998) derives necessary and sufficient conditions for equilibrium indeterminacy in his model with capacity utilization. The general analytical conditions are more cumbersome than those for the standard RBC model with externalities. Wen (1998) therefore restricts his analysis to the case such that $\alpha\eta < \theta$, whereby $\theta = \frac{1-\beta(1-\delta)}{\beta\delta}$, based on the steady-state restriction linking the endogenous depreciation rate δ and the parameter θ. Under this restriction, necessary and sufficient conditions for indeterminacy are:

$$\begin{aligned}
\eta &< \frac{1}{\alpha}, \\
\eta &> 1 + \frac{\theta\left(1+\gamma-\beta\left(1-\alpha\right)\right) - \left(1+\gamma\right)\alpha}{\beta\left(1-\alpha\right)\theta + \left(1+\gamma\right)\alpha - \frac{1-\beta}{1+\beta}\left(1+\gamma\right)\theta}, \\
\eta &> 1 + \frac{\theta\left(1+\gamma-\beta\left(1-\alpha\right)\right) - \left(1+\gamma\right)\alpha + \frac{1-\beta}{1+\beta}\left(1+\gamma\right)\beta\delta\left(\theta-\alpha\right)\frac{1-\alpha\theta}{2\alpha}}{\beta\left(1-\alpha\right)\theta + \left(1+\gamma\right)\alpha - \frac{1-\beta}{1+\beta}\left(1+\gamma\right)\left(\theta - \frac{1}{2}\left(\theta-\alpha\right)\left(1-\beta\right)\right)}.
\end{aligned}$$

[13] Meng and Xue (2009) consider two additional cases where indeterminacy arises when $\eta < 1$, that is, when there are decreasing returns to scale. Although I allowed for these cases in the benchmark specification based on a wide prior centered on $\eta = 1$, I did not encounter indeterminate equilibria in this region when estimating the model.

The third condition differs from the second by additional terms in the numerator and the denominator. As Wen (1998) demonstrates, they are virtually identical for β closest to one. It is fairly straightforward to show that the threshold value for η, beyond which an indeterminate equilibrium arises is increasing in γ. That is, the less elastic labor supply is, the less likely is an indeterminate equilibrium.