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Jeffrey M. Lacker
Federal Reserve Bank of Richmond

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ABSTRACT: In a simple risk-sharing environment with ex post private information, conditions are found under which a collateralized debt contract is the optimal allocation. The critical condition for optimality is that the borrower values the collateral good more highly than does the lender; otherwise the optimal contract does not resemble debt. Limited collateral can give rise to an endogenous borrowing constraint, driving a further wedge between the intertemporal marginal rates of substitution of the borrower and the lender. I argue that perhaps all debt contracts are implicitly collateralized.

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**Research Department, Federal Reserve Bank of Richmond, P.O. Box 27622, Richmond, VA 23261, 804-697-8279, e-mail: E1JML02@RICH.FRB.ORG

The most notable feature of the ubiquitous debt contract is that the repayment does not vary over a wide range of apparently relevant circumstances, an observation that is inconsistent with the full insurance predicted by complete-information, complete-markets models. This paper describes a new model of optimal debt contracts. The model builds directly on Kenneth J. Arrow's (1974) insight that contractual payments will not be contingent on ex post information that is private to one party: if the borrower can falsely pretend to have suffered an adverse shock, then a contingent contract is inconsistent with truthful reporting.

Observed debt contracts are often risky, however. The borrower makes a fixed payment but occasionally, as in a "default," something else happens. Contracts that are occasionally contingent pose a challenge for Arrow's approach to explaining debt. It seems hard to reconcile the observation that the borrower pays less in some circumstances with the ability of the borrower to report falsely; why wouldn't the borrower always plead distress?

The approach taken here is to focus on the subset of observed debt contracts that are collateralized; incomplete payment formally entitles the lender to possession of a specific good belonging to the borrower. Collateral thus serves as a useful tool for ensuring repayment (Robert J. Barro, 1976; Daniel K. Benjamin, 1978). If default requires surrendering collateral, the borrower in good circumstances considering a dishonest report will compare the value of the collateral to the value of the avoided repayment. As long as the former exceeds the latter, the borrower has no incentive to misreport.

I explore this approach in a model that combines Arrow's idea with the insights of Barro and Benjamin in a simple two-period risk-sharing framework with multiple goods. The borrower faces an ex post risk that can be costlessly falsified. Conditions are found under which the optimal arrangement is a "collateralized debt contract," in which the borrower makes a fixed payment of one good whenever there is enough. The collateral good is transferred only if the amount of the first good falls short of the full fixed payment. When the borrower has sufficient resources, paying the full obligation is preferred to paying less and surrendering the collateral.

The crucial condition for the optimality of a collateralized debt contract is that the borrower and lender place different relative valuations on the two goods; *the borrower values the collateral good more, relative to the payment good, than does the lender.* The difference in valuations implies that the optimal contract attempts to minimize, subject to resource and incentive constraints, the collateral good received by the lender. It would seem that something like this condition has to hold in any model of collateralized debt, for otherwise the contract would have the borrower hand over collateral quite frequently, and the contract would not resemble debt.[1]

Most of the goods that are usually thought of as suitable collateral seem to be valued more highly by borrowers than by lenders. In a literal interpretation of the model, a loan to be repaid out of a farmer's next harvest is collateralized by the farmer's chattels--durable, portable personal property. More generally, a plot of land or a home is usually more useful and valuable to the mortgagor than to the mortgagee, in part because sale to a third party can involve significant costs. An entrepreneur might value a capital good more highly than others, either because of a stock of knowledge acquired by using it, or because of a learning or setup cost involved in transferring it to the use of someone else. Unobserved efforts by the borrower might be required to maintain the collateral good, in which case the moral hazard problem will be mitigated by having the borrower retain the collateral as often as possible.[2]

Although collateralized debt is the tangible interpretation of the optimal contract here, the results provide some insight into debt contracts that are not explicitly collateralized. Indeed, it could be argued that perhaps all debt contracts are implicitly collateralized, in the sense that paying less than the specified amount at the specified date and place results in the borrower having to sacrifice something else, such as an obligation to make payment at a later date or to surrender nonexempt assets in a bankruptcy proceeding. In fact, under U.S. law an unsecured creditor can, in the event of a default, obtain a judgement against the debtor (allowing seizure of assets), have a judgement lien entered against the debtor's assets, have the debtor's wages garnished, or file an involuntary bankruptcy petition. The creditor's legal rights in these cases can be viewed as contingent claims, distinct from the promised repayment, that implicitly collateralize unsecured lending (Benjamin, 1978). If such claims are of less value to the creditor than to the debtor, perhaps due to deadweight costs of the legal process, then they could act as implicit collateral as in the model of the present paper.[3] Although such imperfections in contract enforcement deserve careful explicit treatment, the results presented here suggest that contracts are likely to have a similar structure in those settings.

Borrowing constraints emerge endogenously here that are similar in form to the constraints that are often imposed exogenously, and they provide a plausible theory of "credit rationing." Endogenous borrowing constraints can arise in this model because the amount of the collateral good imposes an upper bound on what the borrower can credibly promise to pay back, exactly as suggested by Irving Fisher (1930, 210-11).[4] As a result, no rational lender is willing to loan more than a certain limit determined by the amount of the borrower's collateral. This provides a generalization of "borrowing constraints" to situations in which some lending does take place and collateral is available but insufficient to support the desired level of borrowing. The model thus captures the notion of "credit rationing" in debt markets, in the sense that borrowers would like to borrow more at the existing contractual interest rate, but can not do so without more collateral.[5]

In the first model the collateral good is perfectly divisible, but in many debt contracts the collateral is an indivisible good, such as an automobile or a house. A second set of models relaxes this assumption. When the collateral good is indivisible, one must allow for randomized collateral transfers. When output is large enough the borrower makes a fixed payment of output and does not transfer the collateral -- in this respect the contract still resembles debt. But when output is insufficient to make the fixed payment the borrower hands over all of the output and transfers the collateral good with a probability that is a strictly decreasing function of output. Thus the payment is independent of whether or not collateral is transferred. Both the randomization feature and the independence of the payment from the collateral transfer suggest that these contracts do not resemble collateralized debt.

Strong precommitment abilities are crucial to such contracts. The parties must able to bind themselves to adopt prespecified mixed strategies following given announcements. With more limited abilities to commit, the contracts described above are no longer feasible. Under limited commitment the optimal contracts again resemble debt; when output is insufficient to make the fixed payment the borrower makes a smaller payment and transfers the collateral with probability one. The limited commitment constraint can be viewed as the unwillingness of an outside enforcement facility to implement punitive transfers that are random functions of the ex post state and transfers that have already been made. This constraint does not rule out randomization a priori, but it implies that mixed transfer strategies must be ex post rational for the borrower.

A restriction to renegotiation-proof contracts further constrains the set of optimal contracts. Undercollateralized contracts in which the lender is worse off in the default states are eliminated. In such contracts there are states -- small deficiencies in output -- under which the lender receives the collateral but would prefer to take most of the output instead of the collateral, leaving the borrower no worse off. If the borrower anticipates such renegotiation proposals from the lender, incentive constraints would be affected. The borrower would hide output in the nondefault states in order to induce renegotiation, and thus would never make the fixed notional payment. Renegotiation further restricts the total return that can be credibly promised to the lender, and so sharpens the ex ante borrowing constraint facing the borrower.

The renegotiation-proof contracts do not allow the lender to bear any losses in the event of default and transfer of the collateral. And yet there is empirical evidence that lenders do bear losses, on average, when they take collateral. The evidence, drawn from residential and commercial real estate lending, suggests that lender losses are related to ex post deterioration in collateral value.[6] In a model with uncertain but publicly observed ex post collateral value, optimal contracts can impose losses on a lender that takes depreciated collateral. In addition, contracts can have the feature that loans are "restructured" in response to declines in collateral value, even when the lender does not take the collateral.

The models presented here can viewed as complementing some other explanations of risky debt contracts have been proposed. Robert Townsend (1979), also building on Arrow's approach, explores an environment in which lenders could, at a cost, verify the true state of the borrower ex post (see also Douglas Gale and Martin Hellwig, 1985; and Williamson, 1987). Risk-sharing is possible across the states in which verification takes place, but because verification is costly it is used in only the most dire circumstances; in other states the payment is noncontingent. There are, however, some widely known problems with this account. First, as Townsend himself recognized, people would prefer randomized verification arrangements, similar to those employed in auditing situations by tax authorities and others.[7] Second, some have argued that ex post verification is not the predominant cost associated with lending (Cheng Wang and Stephen D. Williamson, 1996).

Douglas W. Diamond (1984) also builds on Arrow's approach, and argues that nonpecuniary penalties can be imposed on borrowers when the payment is insufficient. Since these are deadweight costs from society's point of view, the optimal arrangement attempts to minimize the expected value of such penalties. Thus a fixed payment is made whenever the borrower has sufficient resources, and penalties are assessed whenever the payment falls short. Diamond's model has a form very similar to the models presented here, since transfer of collateral is a deadweight cost that in effect punishes the borrower. In fact, his model is formally a special case of the first model below, since pecuniary penalties can be varied continuously (see note 8 below). Thus nonpecuniary penalties and collateralization share a common economic structure.

In a series of recent papers Oliver Hart and John Moore have pursued a variation on Arrow's approach (1989, 1994). They assume that the borrower's circumstances are observable to the lender, but are not verifiable by outside parties. As a result, enforceable contractual provisions cannot be contingent on the borrower's circumstances. Ex post renegotiation allows contingent outcomes where formally contingent contracts are unenforceable. The ex ante contract sets the threat points that determine the outcomes of their renegotiation game, but because contractual provisions are necessarily noncontingent, control over ex post outcomes is imperfect. Their setup is similar in some important ways to the models in this paper; collateral is worth more to the borrower than to the lender, and the possibility of renegotiation constrains attainable outcomes. Their information structure is quite different, however. The borrower's true output is observed by their lender but can not be observed by their court. My borrower can hide output, but both the lender and the court can observe the output the borrower actually displays.

Two other recent explanations of debt contracts depart from Arrow's approach and rely instead on adverse selection among borrowers. In the models of Robert D. Innes (1990), borrowers differ in the probability distribution governing their ex post return and lenders would like to distinguish good borrowers from bad. Good borrowers thus find it less costly, in terms of expected utility, to promise returns in the low output states, so the best separating contracts have the good borrowers pay as much as possible in the low output states. This, along with a monotonicity condition on repayment schedules, ensures that the equilibrium contract for the good borrowers is a risky debt contract. Wang and Williamson (1996) also rely on adverse selection and a monotonicity condition, but lenders can use screening probabilities as well as contract terms to separate borrower classes. Good borrowers are screened and receive a debt contract, again because repayment when output is low is a less costly commitment for a good borrower.

The basic economic environment is described in Section I. Section II considers the case of perfectly divisible collateral; the main result is a condition under which the optimal contract is a collateralized debt contract. That Section also discusses an extended model in which loan size is endogenous, and shows how the borrower's collateral can limit the amount that can be borrowed and drive a further wedge between the expected intertemporal marginal rates of substitution of the borrower and the lender. Section III covers the case of indivisible collateral. In the basic setup the optimal contract no longer resembles debt, even if the collateral is more valuable to the borrower than the lender. Under a limited commitment condition the optimal contract does resemble debt, but is not always renegotiation-proof. With commitment capabilities further restricted by renegotiation-proofness, contracts in which the lender bears some risk are no longer feasible. A final section concludes. An Appendix contains proofs.

I. The Environment

The model is a simple two-agent, two-period, two-good environment. In period 1 the lender makes a loan advance, which for simplicity is a fixed amount. In Section III I will briefly consider a model in which the amount of the loan is made endogenous, but until then notation for allocations in the first period is suppressed. In the second period the borrower has endowments of two goods. Good *1* will turn out to be the payment good and can be thought of as corn. Good *2* will end up serving as collateral and can be thought of as chattels. The borrower's corn endowment is uncertain ex ante, and for simplicity this is the sole source of uncertainty in the model. In the first set of models the collateral good is perfectly divisible; in Section III the collateral is indivisible.

Formally then, let θ denote the borrower's random endowment of good *1*. The borrower's endowment of good *2* is a known constant $k > 0$, and the lender's endowment of good *1* is a known constant $e > 0$. Attention will be restricted to the case in which the lender has no endowment of the collateral good.

The borrower (agent *a*) is to make transfers (possibly contingent) of y_1 of good *1* and y_2 of good *2* to the lender (agent *b*) after the endowments are received. After transfers take place the goods are consumed. Consumptions are given by

$$c_{a1} = \theta - y_1, \qquad c_{a2} = k - y_2,$$

$$c_{b1} = e + y_1, \qquad c_{b2} = y_2.$$

The borrower derives utility from consumption according to the function $u(c_{a1},c_{a2}) = u_1(c_{a1}) + u_2(c_{a2})$, where the functions u_i, i=1,2, are continuous, concave, twice continuously differentiable, strictly increasing, and have finite derivatives. The utility of the lender is given by $c_{b1} + \mu_b c_{b2}$, where the constant μ_b is the lender's valuation of the collateral good relative to the payment good, or, equivalently, the lender's marginal rate of substitution between the collateral good and the payment good.[8] Note that μ_b need not be positive, so that "acquisition" of the collateral good might leave agent *b* indifferent or even worse off. Thus having agent *a* surrender some of good 2 might be a pure punishment that provides no gain to agent *b*, or may in fact be costly.[9]

The borrower's corn endowment θ is governed by an absolutely continuous distribution with a density $f(\theta)$ that is strictly positive on $\Omega = [\theta,\theta]$, the support of θ, where $0\leq\theta_0<\theta_1<\infty$. The borrower's realized endowment of the random good is private information. To be specific, it is costless for agent *a* to hide the good and pretend that the realized amount is smaller than it actually is. In contrast, it is assumed to be prohibitively costly to falsify units of the good that do not exist, pretending that the realized amount is larger than it actually is.[10]

Finally, I assume that $-u_1''(c_{a1})/u_1'(c_{a1})$, the absolute risk aversion of agent *a* with respect to good 1, is nonincreasing. This assumption, together with the information technology, implies that optimal contracts need not contain extraneous randomization.

II. Divisible Collateral

In this section the collateral good is perfectly divisible. This is an important benchmark case, since it isolates the implications of the informational imperfection, apart from any frictions due to indivisibility. Examples of relatively divisible collateral abound, however: inventories, accounts receivable, security pools, and the personal belongings (up to the exclusion) that implicitly collateralize unsecured lending under U. S. bankruptcy law.

Agents meet in an initial period and, in exchange for a loan advance, agree to a repayment contract to be described in more detail below.[11] Then some time later the random endowment θ is realized. Agent *a* observes the realized value of θ and either allows the true value to be seen by agent *b*, or makes it seem that the realized value was some smaller value θ'. Given the amount displayed by agent *a*, transfers are to take place according to a schedule agreed to in the contract. For maximum generality, following Edward C. Prescott and Townsend (1984) and Townsend (1987), transfers are allowed to be random. For a given display, θ', the contract specifies a measure,

$\pi(dy_1,dy_2|\theta')$, a probability distribution over transfers (y_1,y_2).[12] A *contract* is a family of probability measures, $\pi(\ ,\ |\theta)$, one for each $\theta\in\Omega$, on the set of feasible transfers. Nonincreasing absolute risk aversion and the impossibility of faking goods, however, allow us to restrict attention to deterministic contracts, in which payments are nonrandom functions of the state.[13]

A *contract*, then, is a pair of functions of the state, $y_1(\theta)$ and $y_2(\theta)$. An application of the well-known Revelation Principle (Roger B. Myerson, 1979; Milton Harris and Townsend, 1981; Townsend, 1988), allows us to restrict attention to contracts that satisfy the following self-selection constraint.

$$u_1(\theta-y_1(\theta)) + u_2(k-y_2(\theta)) \geq u_1(\theta-y_1(\theta')) + u_2(k-y_2(\theta')) \qquad \text{(IC1)}$$
$$\forall(\theta,\theta')\in\Omega\times\Omega,\quad s.t.\ \theta'<\theta.$$

A contract is *incentive compatible* if it satisfies (IC1). For any given contract (not necessarily satisfying (IC1)), there exists a contract which satisfies (IC1) and which results in an identical allocation. Thus there is no loss in generality in restricting attention to contracts which satisfy the incentive compatibility condition (IC1). Under a contract that satisfies (IC1), agent *a* never has an incentive to falsify the state. A contract is *resource feasible* if it satisfies

$$-e \leq y_1(\theta) \leq \theta, \qquad \text{(RF1)}$$
$$0 \leq y_2(\theta) \leq k, \quad \forall\theta\in\Omega$$

An *optimal contract* is one that is resource feasible and incentive compatible, and for which there is no alternative resource feasible and incentive compatible contract that makes one agent better off (in the sense of ex ante expected utility) without making the other agent worse off. Optimal contracts can be found as solutions to a particular private information "Arrow-Debreu program," as in Townsend (1987). The program is to choose payment schedules $y_1(\theta)$ and $y_2(\theta)$ to

$$\text{(P1)} \qquad \text{MAX} \quad \int [u_1(\theta - y_1(\theta)) + u_2(k - y_2(\theta))] f(\theta) d\theta$$

s.t. (RF1), (IC1), and

$$\text{(VB1)} \qquad \int [e + y_1(\theta) + \mu_b y_2(\theta)] f(\theta) d\theta \quad \geq \quad \overline{v}_b$$

where $\overline{v}_b$ is the reservation utility of agent *b*. In (P1) the contract is chosen to maximize the expected utility of the borrower, subject to feasibility and incentive constraints and a minimum expected utility constraint for the lender.[14]

In the absence of incentive constraints the solution to (P1) would simply shift all of the output risk to the risk-neutral lender.[15] The effect of the incentive constraints is to force the borrower to bear much of the risk of the random output. Note that the incentive constraints imply the following local first-order condition.

$$\frac{\partial}{\partial\theta}\Big[u_1(\theta - y_1(\theta)) + u_2(k - y_2(\theta))\Big] \quad \geq \quad u_1^{\prime}(\theta - y_1(\theta)) \qquad \forall\theta \in (\theta_0, \theta_1)$$

The total derivative of the borrower's utility with respect to output must be greater than the marginal utility of output. In other words, the borrower must at least bear the direct risk of variations in output. The profile of the borrower's ex post utility across realizations of θ must be positively sloped, rather than constant as in the frictionless risk-sharing version of the model. The incentive constraints thus limit the transfer of risk to the lender.

A *collateralized debt contract*, or *debt contract* for short, is a contract, $(y_1^*(\theta, R), y_2^*(\theta, R))$, satisfying:

$$y_1^*(\theta,R) = \text{MIN}[\theta,R], \qquad \forall\theta\in[\theta_0,\theta_1],$$

$$y_2^*(\theta,R) = 0 \qquad \forall\theta\in[R,\theta_1]$$

$$y_2*(\theta,R) = k - \phi_2[u_2(k)-u_1'(0)(R-\theta)] \qquad \forall\theta\in[\theta_0,R)$$

where the function ϕ_2 is the inverse of u_2, and R is an arbitrary constant in (θ_0,θ_1). Proposition 1 below describes conditions under which a collateralized debt contract is optimal. R is the gross contractual repayment of good 1, and it indexes the family of collateralized debt contracts. The payment schedule for the collateral good is specified so that incentive constraints are satisfied locally with equality. An example of a debt contract is displayed in Figure 1. For realizations of θ that are large enough, agent *a* transfers a constant amount, R, of good *1*, and none of good *2*. If the realization of θ is less than R, then agent *a* transfers all of good *1*, and some of good *2*. The third line of the definition ensures incentive compatibility for θ below R; in this range the smaller the corn output the smaller the corn payment, but the collateral transfer is larger by enough to dissuade the borrower from hiding output. This condition makes $y_2^*(\theta,R)$ positive and strictly decreasing for θ between θ_0 and R. The collateral transfer schedule satisfies with equality the first-order condition implied by incentive compatibility.[16]

Collateralized debt contracts have the property, mentioned in the introduction, that they minimize the expected value of collateral transfers, subject to resource and incentive constraints. When no collateral transfers occur, incentive compatibility requires a fixed corn payment, R. Sometimes, however, the borrower's corn harvest is insufficient to make the payment R. In this case some collateral transfer is required in order to keep the borrower honest when output is high. In the collateralized debt contract, the collateral transfer schedule is for each level of output the smallest amount consistent with incentive compatibility.

The largest amount of the collateral good ever transferred under a given debt contract is $y_2^*(\theta_0,R)$, the collateral good transfer for the smallest realization of θ. This can be called the *collateral* associated with the contract R. The larger the nominal payment R, the larger the collateral requirement. The borrower's endowment of collateral may constrain the feasible values of R. There may be some value of R below θ_1, call it $\bar{R}$, such that $y_2^*(\theta_0,\bar{R}) = k$; the debt contract corresponding to $\bar{R}$ requires transfer of all of the available collateral good in the lowest state. $\bar{R}$ is implicitly defined by

$$u_1'(0)(\bar{R}-\theta_0) \quad = \quad u_2(k)$$

$\bar{R}$ is the value of R for which the collateral constraint $y_2(\theta_0) \leq k$ just binds. No collateralized debt contract with $R > \bar{R}$ is feasible, since it would require more collateral than agent a has.

For future reference let $c_{hi}^*(\theta,R)$ denote the consumption of agent h of good i in state θ under collateralized debt contract R. Some additional notation will be helpful as well. For a given debt contract R, define $v_a(\theta,R)$ and $v_b(\theta,R)$ as the ex post utilities of the two agents in state θ under the debt contract R:

$$v_a(\theta,R) \quad = \quad u_1(c_{a1}^*(\theta,R)) + u_2(c_{a2}^*(\theta,R))$$

$$v_b(\theta,R) \quad = \quad c_{b1}^*(\theta,R) + \mu_b c_{b2}^*(\theta,R)$$

Let $v_{hj}(\theta,R) \equiv \partial v_h(\theta,R)/\partial j$, for $h=a,b$, and $j=\theta,R$. From now on assume that R has been chosen to satisfy (VB1) with equality, so that $E[v_b(\theta,R)] = \bar{v}_b$, and that $\bar{v}_b$ is such that R exceeds θ_0.[17] For convenience, define the following functions.

$$\mu_a^*(\theta) \quad \equiv \quad \frac{u_2'(c_{a2}^*(\theta,R))}{u_1'(c_{a1}^*(\theta,R))}$$

$$\rho^*(\theta) \quad \equiv \quad -\frac{u_1''(c_{a1}^*(\theta,R))}{u_1'(c_{a1}^*(\theta,R))}$$

The function $\mu_a^*(\theta)$ is the borrower's marginal rate of substitution between the collateral good and the payment good. The function $\rho^*(\theta)$ is the coefficient of absolute risk aversion of the borrower with respect to good *1*.

The crucial condition for the optimality of a debt contract here is an inequality relating the marginal rates of substitution of the two agents.

PROPOSITION 1: *If a collateralized debt contract R satisfies* (VB1) *with equality and*

(1) $$\mu_b \quad < \quad \mu_a^*(\theta) \quad - \quad \rho^*(\theta)\mu_a^*(\theta)\phi(\theta), \qquad \forall\theta\in(\theta_0,\theta_1),$$

where

$$\phi(\theta) \quad \equiv \quad \frac{F(\theta)}{f(\theta)}\left[E[v_{aR}(\hat\theta,R)|\hat\theta\leq\theta]\frac{E[v_{bR}(\hat\theta,R)]}{E[v_{aR}(\hat\theta,R)]} \quad - \quad E[v_{bR}(\hat\theta,R)|\hat\theta\leq\theta]\right],$$

then R is the unique optimal contract.

The proof is in the Appendix. The term $\phi(\theta)$ is proportional to the costate variable from the optimization problem, and is always positive; it will be discussed below. Note that all of the terms in (1) can be calculated directly from knowledge of *R*, the candidate collateralized debt contract that satisfies (VB1) with equality.

It is easiest to understand Proposition 1 by first considering the special case in which the borrower is also risk neutral, so assume for a moment that $u_1(c) = c$ and $u_2(c) = \mu_a c$, where $\mu_a > 0$. In this case (1) simplifies to $\mu_b < \mu_a$, which simply says that the borrower values the collateral good more highly than does the lender. The optimal contract in this setting is the one that minimizes $E[(\mu_a-\mu_b)y_2(\theta)]$, the expected deadweight loss due to the transfer of collateral from the high-value user (the borrower) to the low-value user (the lender), subject to the resource and incentive constraints. Consider an arbitrary feasible and incentive compatible contract. The incentive constraints require that the value of the total payment from the borrower's point of view, $y_1(\theta) + \mu_a y_2(\theta)$, must be nonincreasing. For a given θ, therefore, changing $y_1(\theta)$ and $y_2(\theta)$ in a manner which leaves $y_1(\theta) + \mu_a y_2(\theta)$ unchanged leaves the incentive constraint unaffected as well. Since $\mu_a > \mu_b$, reducing the collateral payment $y_2(\theta)$ by δ and increasing the corn payment $y_1(\theta)$ by $\mu_a\delta$ makes the lender better off by $(\mu_a-\mu_b)\delta > 0$. The limit of such improvement is reached either when $y_2(\theta)$ reaches a minimum at $y_2(\theta) = 0$, or $y_1(\theta)$ reaches a maximum at θ. The former occurs for $\theta > R$, and the latter for $\theta < R$. Finally, further reductions in collateral transfer are available by making the value of the payment to the borrower, $y_1(\theta) + \mu_a y_2(\theta)$, constant rather than strictly decreasing.

The general case of a risk averse borrower is important because debt contracts are preeminently impediments to risk sharing. When the borrower is risk averse, the term subtracted on the right side of (1) is positive. In this case, optimality of the collateralized debt contract requires that the gap between the borrower and the lender's valuations of the collateral exceed $\rho^*(\theta)\mu_a^*(\theta)\phi(\theta)$. This expression measures the value of an improvement in risk-sharing that can be obtained by reducing the corn payment and increasing the collateral payment for the state θ. Consider a local perturbation around the collateralized debt contract that decreases $y_1(\theta)$ by δ and increases $y_2(\theta)$ by

ϵ to keep $u_1(\theta - y_1(\theta)) + u_2(k - y_2(\theta))$ unchanged. Locally, this requires $\delta = \mu_a^*(\theta)\epsilon$. This perturbation relaxes the incentive constraint $v_{a\theta}(\theta) \geq u_1'(\theta - y_1(\theta))$, however, because of the concavity of u_1. Therefore, the ex post utility profile of the borrower can be made flatter at θ, representing an improvement in risk-sharing. The last term in (1) is the value of such an improvement in risk-sharing: $\rho(\theta,R)\mu_a^*(\theta)$ measures the magnitude of the relaxation of the incentive constraints, and $\phi(\theta)$, proportional to the costate variable in the optimization problem, measures the value of resulting improvement in risk sharing. Proposition 1 states that the gain from relaxing incentive constraints in this manner must be less than the deadweight loss due to collateral transfer.

To see this another way, the constraint that the borrower not have an incentive to display $\theta'' < \theta$ when the true state is θ is unaffected by the perturbation described above, but consider the constraint that the borrower not have an incentive to display θ when the true state is $\theta' > \theta$:

$$u_1(\theta' - y_1(\theta')) + u_2(k - y_2(\theta'))$$
$$\geq u_1(\theta' - y_1(\theta)) + u_2(k - y_2(\theta)).$$

The perturbation keeps $u_1(\theta - y_1(\theta)) + u_2(k - y_2(\theta))$ constant, but since $u_1'(\theta' - y_1(\theta)) < u_1'(\theta - y_1(\theta))$, the right side of (IC1) has been reduced. The left side of (IC1) can then be reduced for $\theta' > \theta$, lowering the difference $v_a(\theta') - v_a(\theta)$. This effectively transfers more risk from the borrower to the lender.

Proposition 1 can be interpreted as a lower bound on $\mu_a^*(\theta) - \mu_b$, the gap between the marginal value of the collateral to the two agents. Note that if the coefficient of absolute risk aversion, $\rho^*(\theta)$, is small, the effect on incentive constraints is small and there is little to gain by giving the borrower less collateral and more corn. Thus (1) can also be interpreted as an upper bound on the absolute risk aversion of the borrower.

When condition (1) does not hold, there is no gap in valuation, meaning that for each state there is an allocation at which the marginal rates of substitution of the borrower and the lender are the same. In this case the optimal contract is like riskless debt. Transfers equate the borrower's marginal rate of substitution to the lender's. The incentive contraints still bind, however, and so the value of the payment is constant across output realizations. Note that the value of the payment is unambiguous here because both the borrower and the lender place the same value on the collateral good in equilibrium. If we imagine this model embedded in a market setting in which the relative price of the collateral good was equal to the lender's marginal rate of substitution, we could think of the borrower as selling enough corn and chattels to acquire a fixed amount of medium of exchange to pay to the lender. But this is a perfectly risk-free payment schedule and the chattels play no special role securing the loan.

Some interesting aspects of debt contracts in this setting are worth noting. In the collateralized debt contract the lender takes a "loss" of $R - \theta$ (in terms of good *1*) for $\theta < R$. The lender is compensated (if $\mu_b > 0$) by the transfer of collateral in these states, but not fully. The amount of the collateral transfered is set so that the borrower is indifferent, at the margin, between payment and collateral. Because the collateral is not as valuable to the lender as it is to the borrower, at the margin, the lender's ex post utility is always lower for $\theta < R$ than it is for $\theta \geq R$. From the lender's point of view the debt is undercollateralized, though the contract is fully collateralized when evaluated from the borrower's point of view. Note also that as long as the collateral good has a positive value to the lender ($\mu_b > 0$), it will be in the lender's interest to take possession of the collateral for nonpayment, as called for in the original contract. Thus the contract is fully time consistent in this case, unlike the costly verification setup.[18]

One might think that undercollateralization is a counterfactual feature of the optimal contract here. After all, it would be feasible in many cases for a contract to specify that the lender is at least "made whole" in the event of default; for example, if instead $y_2(\theta)=(R-\theta)/\mu_b$ for $\theta<R$, then the lender's ex post return is always R.[19] But the empirical evidence on this feature is mixed. Certainly in most explicitly collateralized debt contracts the nominal value of the collateral exceeds the repayment obligation. Yet it is still possible for lenders to realize less, on net, in the event of default due to legal fees and the internal costs associated with foreclosure and the disposition of collateral. The gap in valuations can be thought of as including such lender costs. Moreover, lenders often do report losses on collateralized loans. For example, for home mortgages in the U.S. loss rates of 20 to 30 percent, conditional on foreclosure, are common (Richard D. Evans, Brian A. Maris, and Robert I. Weinstein, 1985; Robert Van Order and Ann B. Schnare, 1994). For loans secured by commercial real estate average loss rates can be even higher (Timothy Curry, Joseph Blalock and Rebel Cole, 1991).

In one important respect optimal collateralized debt contracts in this setting differ significantly from the debt contracts in some other models. It is easy to establish that if $\mu_b > 0$, then the ex ante expected utility of the lender, $E[v_b(\theta,R)]$, is strictly increasing in R, since $v_b(\theta,R)$ is strictly increasing in R for each θ. As a result, the lender's expected utility has no interior maximum with respect to R: the lender always prefers a larger R. In this environment there will be nothing like the "credit rationing" in Williamson (1987) or Stiglitz and Weiss (1981), which depends on just such an interior maximum. However, as the next section explores, the borrower's endowment of collateral can constrain contracts in a way that can easily be interpreted as "credit rationing."

Endogenous Loan Size and Borrowing Constraints

We can learn more about the effects of collateral constraints by allowing the size of the loan advance in the first period to be endogenously determined. This allows us to derive an expression relating the expected intertemporal marginal rate of substitution (IMRS) of the two agents. The informational imperfection that makes a collateralized debt contract optimal also drives a wedge between the two agents' IMRS, even if the collateral constraint is not binding. A binding collateral constraint sharply limits the loan amount and further widens the wedge.

Now let c_{hti} and e_{hti} be the consumption and endowment, respectively, of agent h at date t of good i. At date $t=2$ the environment is exactly as described above, so c_{h2i} corresponds to c_{hi} in the previous notation, and so on. For simplicity, assume that the collateral good is durable and only provides utility at date *2*, so notation for the collateral good at date *1* can be suppressed. Utilities are now $u_1(c_{a11}) + \beta[u_1(c_{a21}) + u_2(c_{a22})]$ for the borrower and $c_{b11} + \beta[c_{b21} + \mu_b c_{b22}]$ for the lender, where $0 < \beta \leq 1$. At date *1* the lender, agent b, gives a loan advance of q units of good *1* to the borrower, agent a. The borrower's consumption at date *1* is $c_{a11} = e_{a11} + q$, and the lender's consumption at date 1 is $c_{b11} = e_{b11} - q$. The lender's date *2* consumption of goods *1* and 2 are $c_{b21}(\theta) = e_{b21} + y_1(\theta)$ and $y_2(\theta)$, respectively.

Resource feasibility at date *1* requires $-e_{a11} \leq q \leq e_{b11}$. Individual rationality on the part of the lender requires

$$q \leq \beta E[y_1(\theta) + \mu_b y_2(\theta)], \tag{2}$$

which replaces the (VB) constraint. All other constraints are the same as before, with appropriate modification of notation.

The following equation shows the relationship between the expected IMRS of the two agents.

$$(3) \qquad E\left[\frac{\beta u_1'(c^*_{a21}(\theta,R))}{u_1'(c^*_{a11})}\right] = \beta\left[1 - F(R) + F(R)E\left[\frac{\mu_b}{\mu^*_a(\theta)}\middle|\theta \le R\right]\right] - \eta$$

where η is the nonnegative multiplier on the constraint that the maximum collateral transfer cannot exceed the borrower's available collateral ($y_2(\theta_0) \le k$).[20] If the collateral constraint does *not* bind, then $\eta = 0$ and R and q are determined by (2) and (3). If no contract with $R < \bar{R}$ satisfies (2) and (3) with $\eta = 0$, then $R = \bar{R}$, (2) determines q, and (3) determines η.

The expected IMRS of the borrower is less than the IMRS of the lender (β) for two reasons. First, $\mu_b/\mu^*_a(\theta) < 1$, reflecting the gap between the borrower's and the lender's valuation of the collateral good. The loss to the lender in "default states" when collateral is transferred requires a premium $(R/q) - \beta^{-1}$ over the required rate of return in good states. This drives the borrower's expected IMRS below β even when the collateral constraint is not binding. Second, when the collateral constraint binds the associated multiplier, η, is positive, further reducing the borrower's IMRS. Insufficient collateral places an upper limit on the amount the borrower can credibly promise to repay, constraining the amount that can be borrowed. This limits the extent to which the borrower can shift consumption from the future to the present, driving the borrower's IMRS further below the lender's. This is precisely the effect Irving Fisher described.[21] The model thus provides a theory of endogenous borrowing constraints and "credit rationing" in a market for debt contracts.

Renegotiation

The optimal contracts of Proposition 1 are ex ante efficient, and thus they never provide an incentive to renegotiate. Nothing about the optimization problem (P1) required that contracts be renegotiation proof. Even though agents are able to commit not to reopen the contract, a renegotiation proof contract resulted nonetheless.

Imposing renegotiation proofness actually alters Proposition 1, however. Recall that the risk aversion term on the right side of (1) reflects the possibility of contracts that in some states give chattels to the lender and corn to the borrower. Such contracts would not be renegotiation proof, since someone could be made better off ex post if the borrower payed more in corn and less in chattels. Imposing renegotiation proofness means that the collateralized debt contract need not dominate such arrangements. The risk aversion term in (1) reflected the value of such arrangements. Thus imposing renegotiation proofness simplifies the optimality condition (1) by eliminating the last term on the right side. What remains is the simple condition that the borrower everywhere value the collateral good more highly than does the lender.

To formalize this result, let us suppose that renegotiation can take place after the borrower has displayed the output, and that the lender can make a take-it-or-leave-it offer which the borrower can then either accept or reject. In this setting, agents can achieve a given allocation in one of two ways: either they agree to a contract ex ante and then renegotiate the contract in some states of the world, or they agree to a contract that provides no incentive to renegotiate in any state. It is straightforward to demonstrate here that any allocation that can be achieved under an incentive compatible contract that is renegotiated in some states can also be achieved under an incentive compatible contract that provides no incentive to renegotiate.[22] Hence, we can safely restrict attention to contracts that are *renegotiation-proof* in the following sense.

DEFINITION: *A contract* (y_1,y_2) *is renegotiation-proof at* θ *if there does not exist a* (z_1,z_2), where $z_1\in[-e_{b1},\theta]$, *and* $z_2\in[0,k]$, *such that:*

(i) $e+z_1 + \mu_b z_2 > e+y_1(\theta) + \mu_b y_2(\theta)$*; and,*

(ii) $u(\theta - z_1, k - z_2(\theta)) \geq u(\theta - y_1(\theta), k - y_2(\theta))$.

A contract is renegotiation-proof (RP) *if it is renegotiation-proof for all* θ.

A contract is renegotiation-proof, according to this definition, if there is no other allocation that makes the lender better off without making the borrower worse off.[23] Renegotiation-proof, in this setting, is equivalent to ex post efficiency.[24]

The optimization problem is now to find a contract that maximizes the expected utility of the borrower, as in (P1), subject to (VB1), (RF1), (IC1), and now (RP): call this problem (P2). The requirements for optimality of the colalteralized debt contract are now less stringent.

PROPOSITION 2: *If a collateralized debt contract R satisfies* (VB1) *with equality and*

$$\mu_b < \mu_a^*(\theta) \quad \forall \theta \in (\theta_0, \theta_1), \tag{2}$$

then R is the unique optimal contract.

Condition (2) states that at every point along the collateralized debt contract the borrower values the collateral good more highly than does the lender. Since the collateralized debt contract lies along the boundary of the feasible set defined by (RF1), the concavity of utility implies that there are no feasible allocations that equate the marginal rates of substitution of the borrower and the lender. Any contract that lies in the interior of the feasible set for a given state will be renegotiated, since indifference curves cross there; the lender will offer to give up chattels in exchange for corn.[25]

To summarize then, when the agents can always renegotiate, the collateralized debt contract is optimal whenever the borrower universally value the collateral good more highly than does the lender. When agents can precommit to not renegotiate, the set of available contracts is larger, and a more stringent condition is required for optimality; the gap in valuations must be larger than a term proportional to the borrower's coefficient of absolute risk aversion.

III. Indivisible Collateral

In many debt contracts, the collateral good is a single, indivisible asset--for example, automobiles, houses, or buildings. In such settings it is impossible to hand over just a portion of the collateral, as the optimal contract of Proposition 1 requires. When the second good is indivisible, what is the optimal contract? Is it still recognizable as a collateralized debt arrangement?

Consider, then, the environment described in Section I but with good *2*, the collateral good, now indivisible. For simplicity, let us suppose that the borrower has just one unit of size k of good *2*. Resource feasibility now requires $y_2 \in \{0,k\}$ instead of just $0 \le y_2 \le k$. For maximum generality, we must allow for randomized transfers of good *2*, since the set of feasible transfers is not convex. A contract now consists of $\pi(\theta)$, the probability of transferring the collateral good when the revealed output is θ, along with two payment functions: $y_{10}(\theta)$ is paid if the collateral is not transferred and $y_{1k}(\theta)$ is paid if the collateral is transferred.[26] The incentive constraints are now

$$(1-\pi(\theta))u(\theta-y_{10}(\theta),k) \quad + \quad \pi(\theta)u(\theta-y_{1k}(\theta),0)$$

$$\text{(IC2)} \qquad \ge \quad (1-\pi(\theta'))u(\theta-y_{10}(\theta'),k) \quad + \quad \pi(\theta')u(\theta-y_{1k}(\theta'),0)$$

$$\forall(\theta,\theta')\in\Omega\times\Omega, \quad s.t.\ \theta'<\theta.$$

and the resource constraints are

$$\text{(RF2)} \qquad -e \ \le \ y_1(\theta) \ \le \ \theta, \quad y_2(\theta)\in\{0,k\}, \quad \forall\theta\in\Omega.$$

As before, an *optimal contract* can be found as the solution to a constrained maximization problem (returning to the full commitment case).

(P2) $$\text{MAX} \quad \int [(1-\pi(\theta))u(\theta-y_{10}(\theta),k) + \pi(\theta)u(\theta-y_{1k}(\theta),0)]\,f(\theta)d\theta$$

s.t. (RF2), (IC2), and

(VB2) $$\int [e + (1-\pi(\theta))y_{10}(\theta) + \pi(\theta)(y_{1k}(\theta)+\mu_b k)]\,f(\theta)d\theta \quad \geq \quad \bar{v}_b$$

If the borrower values the collateral good more highly than does the lender, then the optimal contract will again attempt to minimize the probability of transferring collateral. Thus the payment schedule will again be a fixed transfer *R*, whenever output is greater than *R* with no collateral transfer in this case. When output is less than *R*, there is a positive probability that the collateral is transferred to the lender. The probability is chosen to just satisfy the incentive constraint: the gain to the borrower from hiding output and making a smaller payment is offset by a larger probability of losing the collateral. A likely candidate contract therefore is

$$y^*_{10}(\theta,R) \quad = \quad y^*_{1k}(\theta,R) \quad = \quad \text{MIN}[\theta,R] \qquad \theta\in[\theta_0,\theta_1]$$

$$\pi^*(\theta,R) \quad = \quad \frac{(R-\theta)u_1'(0)}{(u_2(k)-u_2(0))} \qquad \theta\in[\theta_0,R)$$

$$\pi^*(\theta,R) \quad = \quad 0 \qquad \theta\in[R,\theta_1]$$

The probability of transferring collateral is a decreasing linear function of $\theta < R$. As in the divisible case, the crucial condition for the optimality of this contract is an inequality relating the marginal rates of substitution of the two agents, but also involving the borrower's risk aversion.

PROPOSITION 2: *If for a given R the contract* $(\pi^*,y^*_{10},y^*_{1k})$ *satisfies* (VB2) *with equality along with condition* (1), *where* $v_a(\theta,R)$ *and* $v_b(\theta,R)$ *are appropriately redefined and*

$$\mu_a^* \equiv \frac{(u_2(k)-u_2(0))}{u_1^{/}(c_{a1}^*(\theta,R))k},$$

then it is the unique optimal contract.

When the collateral good is indivisible, the borrower's marginal rate of substitution, μ_a^*, must be redefined as the per unit value of the collateral good in terms of the payment good. Otherwise Proposition 2 is virtually identical to Proposition 1; the gap between the borrower's and the lender's valuation of the collateral good must be greater than the value of improved risk-sharing that could be obtained by marginally reducing the corn payment and increasing the probability of collateral transfer.

Note that the payment of good *1* is MIN[θ,*R*], whether collateral is transferred or not. Randomized collateral payment when output is deficient is used as a punishment to discourage underreporting when output is high. The expected collateral transfer does not make the lender whole for the value "lost" due to insufficient output. This can be seen by writing the lender's ex post expected utility as

$$v_b(\theta,R) = e + R - (1 - \mu_b/\mu_a^*)\text{MAX}[R-\theta,0]$$

The term subtracted is the average loss suffered by the lender in the "default" states $\theta<R$, and is strictly positive because of the asymmetry in collateral valuation. Since the schedule of randomized collateral transfers is determined by the borrower's incentive constraint, the borrower is indifferent between a given transfer and the transfer for the next lowest output report. But since the lender does not value the collateral as highly as does the borrower, the lender strictly prefers the payment corresponding to the larger transfer.

Limited Commitment

Randomized transfer of collateral is, arguably, counterfactual. It is true that a lender's ability to obtain possession of a borrower's assets is often uncertain, but observed randomness does not appear to vary sensitively with payment deficiencies in the way predicted by the theory. Certainly explicit contractual provisions for ex post randomization are relatively rare. Moreover, the fact that the payment for a given output does not depend on whether or not the collateral is transferred is inconsistent with most debt contracts. Generally when the lender receives collateral the borrower owes less than if the collateral is kept.

One reason the randomization predicted by the theory is not more widely observed may be the difficulties of precommitment. In the contracts of Proposition 2, the borrower has a clear interest in the outcome of the random choice regarding collateral transfer. The lender's interest is equally clear, though opposite to the borrower's. If neither party can credibly precommit to randomize their future actions, attainable allocations will be constrained. The natural solution to this difficulty is to entrust randomization to a reliable third party.[27] For example, in the event of a payment shortfall, the arrangement might call for the lender to invoke the intervention of an outside agency that would inspect the ex ante contract and determine randomly, with pre-agreed (possibly contingent) probabilities, whether or not the collateral is to be transferred. Courts, the usual third party contract enforcement institutions, are not often seen to perform such randomization services, however; it would appear to conflict with the principle that an enforceable monetary obligation must be a "sum certain," calculable at the time of the suit. For whatever reason, courts appear unwilling to enforce uncertain obligations.

Suppose, then, that courts are limited in this way with regard to randomization. Specifically, let us assume that courts are only willing to impose remedies that are deterministic functions of the observable circumstances. Agent *a* observes the state θ, reveals the state θ', and then makes transfers (y_1,y_2). The lender can now bring suit. The contract instructs the court to make transfers (possibly negative) from *a* to *b*. The most general possible enforcement facility would allow the stipulated transfer schedule to be a random function of (θ',y_1,y_2). A facility that could impose prespecified random transfer schemes when invoked could be used to implement contracts like those in Proposition 2. The assumption here is that transfers must be deterministic functions of (θ',y_1,y_2).[28]

Transfer schedules can be designed here to support any incentive compatible payment schedule without the need to invoke the enforcement facility. For example, suppose the contract calls for given payments y_1 and y_2 when the displayed output is θ. If *a* pays too little of either good, *b* sues and the court takes everything away from *a* and gives it to *b*. Otherwise the court does nothing. With this threat the borrower pays exactly y_1 and y_2 when the displayed output is θ.

Note that randomized allocations have not been ruled out a priori; the contracting parties can themselves randomize payments. It must, however, be in their interests to do so. Suppose a contract specifies that for display θ the borrower with probability $\pi(\theta)$ keeps the collateral and pays $y_{1k}(\theta)$, and with probability $1-\pi(\theta)$ hands over the collateral and pays $y_{10}(\theta)$. The transfer schedule could specify taking all of the remaining good *1* if the borrower pays less than $y_{1k}(\theta)$ when the collateral is given to *b*, or if the borrower pays less than $y_{10}(\theta)$ when the collateral is not given to *b*; otherwise no transfer is required. Given θ, and given a collateral transfer $y_2\in\{0,k\}$, the borrower will then have an incentive to make the appropriate payment. But in order for the borrower to have an incentive to randomize over these two alternatives, transferring the collateral with probability $\pi(\theta)$, it must be the case that

(LC) $\pi(\theta) \in \mathrm{ARGMAX}_{\pi\in[0,1]}\{(1-\pi)u(\theta-y_{10}(\theta),k) + \pi u(\theta-y_{1k}(\theta),0)\}$

The constraint (LC) reflects the inability of the borrower to precommit to a randomized transfer schedule, and the absence of any third party that is able to impose randomized payments.[29]

The limited commitment constraint (LC) rules out the optimal contract of Proposition 2. Under that contract the borrower always declines when called upon to transfer collateral with positive probability. If randomized collateral transfers are to occur, the borrower must be indifferent between transferring collateral and not transferring collateral: $u(\theta-y_{10}(\theta),k) = u(\theta-y_{1k}(\theta),0)$. It seems plausible that under such a constraint the problematic feature of the contracts of Proposition 2 would disappear. Since the borrower would have to be indifferent when randomizing, the corn payment would have be correspondingly smaller when collateral is transferred. Surrendering collateral might then more closely resemble "making up" a payment deficiency.

The limited commitment constraint is strong enough to completely eliminate randomized contracts, as it turns out. Consider now the problem (P2) modified with the addition of constraint (LC).

(P3) $$\mathrm{MAX} \int[(1-\pi(\theta))u(\theta-y_{10}(\theta),k) + \pi(\theta)u(\theta-y_{1k}(\theta),0)]f(\theta)d\theta$$

$s.t.$ (RF2), (IC2), (VB2), and (LC)

We now have:

PROPOSITION 3: *If*

(4) $$\frac{u_2(k) - u_2(0)}{u_1'(0)} > \mu_b k$$

then the optimal contract takes the form of

$$y_{10}(\theta)=R \quad \pi(\theta)=0 \quad \text{for } \theta \geq R,$$

$$y_{1k}(\theta)=R_k \quad \pi(\theta)=1 \quad \text{for } \theta<R,$$

where $\theta_0 \geq R_k \geq R - c_k$*, and* c_k *is defined by* $u(c_k,0) = u(0,k)$*.* c_k *is strictly greater than* $\mu_b k$*.*

The contracts of Proposition 3 are recognizable as collateralized debt. A fixed payment R is made unless output is less than that, in which case the collateral is transferred to the lender and a smaller payment R_k is made. The parameter c_k is the amount of good 1 that would provide a borrower lacking collateral with the same utility as having the collateral and consuming none of the harvest. Because utility is concave, c_k is the smallest amount the borrower would accept as compensation for the surrender of collateral. The payment R_k must be large enough so that the borrower prefers to keep the collateral and pay R whenever possible; if $R_k < R - c_k$ the borrower would hide output when $\theta = R$ in order to pay R_k and transfer the collateral instead.

Condition (4) states that the value of the collateral to the borrower exceeds the value of the collateral to the lender, analogous to condition (2). This condition is weaker than (1) because there is no feasible opportunity for risk sharing via a contract that in any state gives the lender chattels and the borrower corn. With indivisible collateral, an interior contract would require randomization, and the limited commitment constraint imposes sharp conditions on mixed strategies. Any contract with $0 < \pi(\theta) < 1$ has $u(\theta-y_{10}(\theta),k) = u(\theta-y_{1k}(\theta),0)$, as noted above. This implies that $\hat{\pi}(\theta) < \pi(\theta)$ is feasible, provides the borrower with the same expected utility, and, because of (4), provides the lender with strictly greater expected utility, leaving incentive constraints unaffected. Thus any contract which has random collateral transfers in some state is strictly dominated by a contract with no collateral transfer in that state.

Proposition 3 allows a range of optimal contracts. At one extreme, the borrower pays $R_k = \theta_0$ when collateral is transferred. This is the maximal feasible transfer for the lowest possible output, and since the transfer is constant across states in which collateral is transferred this is the largest feasible value for R_k. This allows the lowest possible transfer R in the good states. At the other extreme $R_k = R - c_k$. This results in the smallest possible value of R_k and the largest possible value of R. At this end of the spectrum, the contract is *undercollateralized* in the sense that the lender's return in the default states, $\mu_b k + R_k = R - (c_k - \mu_b k)$, is smaller than the return in the nondefault states, R.[30]

The factors affecting the choice of contracts within the range allowed by Proposition 3 are easy to understand. One is the deadweight loss due to the transfer of collateral. Other things held constant, the contracting parties prefer to provide expected return to the lender by transferring output rather than collateral, since the former is less costly to the borrower. By itself, this favors contracts with the smallest possible probability of collateral transfer, and thus the smallest possible R. To compensate for a small output transfer in the good states, such a contract maximizes the corn payment in the default states: $R_k = \theta_0$. On the other hand, the borrower is risk averse, and may prefer to consume more in the low output default states and less in the good states. By itself, this tends to favor contracts with smaller R_k and larger R. Together, the balance of these two forces determines the optimal contract.

Finding a *unique* optimal contract requires a few mild conditions, which are detailed in the Appendix. To illustrate the factors affecting the optimal contract in this model, it helps to consider the two polar contracts.

PROPOSITION 4: (a) (*Minimally collateralized debt contract*) *Suppose the contract* (R, R_k) *satisfies* $R_k = R - c_k$ and (VB2) *with equality. If* (R, R_k) *is optimal for* (P3) *then*

(5) $$c_k - \mu_b k \leq \left[1 - \frac{E[u_1'(\theta - R) | \theta \geq R]}{E[u_1'(\theta - R_k) | \theta < R]}\right]\left(\frac{1-F(R)}{f(R)}\right).$$

(b) (*Overcollateralized debt contract*) *If the borrower is risk neutral, then the optimal contract has* $R_k = \theta_0$.

Condition (5) is necessary for the minimally-collateralized debt contract to be optimal. (Sufficient conditions appear in the Appendix.) The term in brackets reflects the risk aversion of the borrower, and is larger the greater is the difference between average marginal utility when collateral is transferred and average marginal utility when no collateral is transferred. The left side of (5) is the gap between the (consumption-equivalent) value of the collateral to the borrower and the value of the collateral to the lender. Optimality of the contract with $R_k = R - c_k$ requires that the bracketed risk aversion term dominate the deadweight loss of collateral transfer. Condition (5) can thus be viewed as an *upper* bound on the gap between valuations of the collateral good, or as a *lower* bound on risk aversion. Note that this is just the opposite of Proposition 1, where condition (1) is a lower bound on the valuation gap and an upper bound on risk aversion. But (1) was required for the optimality of debt contracts in that model; debt contracts are optimal here under the weaker condition (4) (a condition independent of risk aversion), while (5) just determines when the minimally-collateralized debt contract is optimal.

In the opposite case of a risk neutral borrower, the optimal contract in Proposition 4(b) minimizes the expected deadweight loss of collateral transfer, which requires the maximal value of R_k. The result is a debt contract that is overcollateralized, in the sense that the collateral transfer more than compensates the lender for the loss of payment: $\mu_b k > R - R_k$. Proposition 4(b) is just an extreme case of low risk aversion. For intermediate cases the optimal contract lies between the two extremes.

The value of the collateral held by the borrower again constrains the amount that can be credibly promised in payment, and will limit, if loan size is endogenous, the amount that can be borrowed. The loan contract that provides the greatest utility for the lender is determined by the upper bounds on R_k and R: $R_k=\theta_0$ and $R=R_k+c_k$. Thus the maximal loan contract R is defined by

$$u_1(\bar{R}-\theta_0)-u_1(0) \quad = \quad u_2(k)-u_2(0)$$

Note the similarity with the earlier expression for the divisible case. And just as in the divisible case, the amount k of collateral the borrower holds will constrain the amount that can be borrowed when the loan size is endogenous.

Renegotiation

The undercollateralized debt contracts of Proposition 4 have one problematic feature--they are not renegotiation-proof. When a contract is undercollateralized the return to the lender, $R_k + \mu_b k$, is strictly less than the usual payment, R. When output falls short of R by a small amount, it would be feasible to make both parties better off. Specifically, when θ is between $R_k + \mu_b k$ and R, the borrower could pay an amount $\theta-\epsilon$ of good 1 and keep the collateral. This provides the borrower with utility $u(\epsilon,k) > u(0,k)$ $= u(c_k,0) > u(\theta-R_k,0)$, strictly more than under the original arrangement. (Note that $c_k \geq R - R_k > \theta - R_k$ for θ in this range.) The lender is strictly better off as well, since $R_k + \mu_b k < \theta - \epsilon$ for small ϵ. Given the opportunity for an ex post Pareto-improvement, it is only natural to assume that the two agents would take advantage of it and renegotiate the original contract.

Anticipating renegotiation affects the contract they agree to ex ante. Suppose the borrower knows that whenever θ is between $R_k + \mu_b k$ and R, the required payment will be reduced and no collateral transfer will be required.

Then whenever θ is greater than R the borrower will want to hide output in order to induce renegotiation and a lower payment. But a lender anticipating such behavior will not view the promised payment R as credible.

As before, suppose that the lender makes a take-it-or-leave-it offer to the borrower. This takes place after the borrower's display of output but before any payments have taken place. Thus the renegotiated contract must still respect the limited commitment constraints (LC). This leads to the following definition.

DEFINITION: *A contract* (π, y_{10}, y_{1k}) *is renegotiation-proof at* θ *if there does not exist a* (p, z_{10}, z_{1k}), *where* $p \in [0,1]$, $z_{10} \in [-e_{b1}, \theta]$, *and* $z_{1k} \in [-e_{b1}, \theta]$, *such that:*

(i) (p, z_{10}, z_{1k}) *satisfies* (LC) *for* θ;

(ii) $$(1-p)(e+z_{10}) + p(e+z_{1k}+\mu_b k) > (1-\pi(\theta))(e+y_{10}(\theta)) + \pi(\theta)(e+y_{1k}(\theta)+\mu_b k);$$ *and,*

(iii) $$(1-p)u(\theta-z_{10},k) + pu(\theta-z_{1k},0) \geq (1-\pi(\theta))u(\theta-y_{10}(\theta),k) + \pi(\theta)u(\theta-y_{1k}(\theta),0).$$

A contract is renegotiation-proof (RP) *if it is renegotiation-proof for all* θ.

A contract is renegotiation-proof, according to this definition, if there is no other allocation that makes the lender better off without making the borrower worse off, subject to the limited commitment constraint. As noted above, this corresponds to an ex post renegotiation game in which the lender makes a take-it-or-leave-it offer and the borrower either accepts or rejects.[31] Renegotiation-proof, in this setting, is equivalent to ex post efficiency, subject to the limited commitment constraint.[32]

The optimization problem is now to find a contract that maximizes the expected utility of the borrower, as in (P3), subject to (VB2), (RF), (IC), (LC), and now (RP): call this problem (P4). The effect of adding the constraint (RP) is to rule out undercollateralized contracts. If the lender is worse off when collateral is transferred, then there exist states in which the contract can be improved upon by having the borrower keep the collateral and pay more of good 1 instead. Otherwise, the optimal contracts are identical in form to those of Proposition 3.

PROPOSITION 5: *If* (4) *holds, then the optimal contract in* (P4) *takes the form described in Proposition 3, except that* $\theta_0 \geq R_k \geq R - \mu_b k$.

Now contracts can be *just-collateralized*, in other words $R_k + \mu_b k = R$, or overcollateralized, as when $R_k + \mu_b k > R$. Undercollateralized contracts are ruled out.

The choice between overcollateralized versus just-collateralized contracts involves the same trade-offs as in the limited commitment case. The deadweight loss of transferring collateral to the lender favors a small R, infrequent collateral transfers, and a large default payment R_k. Such a contract provides the smallest possible default state consumption to the borrower, however. Risk aversion on the part of the borrower favors a lower default payment, a larger R, and more frequent collateral transfer. Paralleling Proposition 4, we have:

PROPOSITION 6: (a) (*Just-collateralized debt contract*) *Suppose the contract* (R, R_k) *satisfies* $R_k = R - \mu_b k$ and (VB2) *with equality. If* (R, R_k) *is optimal in* (P4) *then*

$$(6) \qquad \frac{u(0,k) - u(\mu_b k, 0)}{E[u_1'(\theta - R_k) \mid \theta < R]} \leq \left[1 - \frac{E[u_1'(\theta - R) \mid \theta \geq R]}{E[u_1'(\theta - R_k) \mid \theta < R]}\right] \left(\frac{1 - F(R)}{f(R)}\right).$$

(b) (*Overcollaterallized debt contract*) *If the borrower is risk neutral, then the optimal contract has* $R_k = \theta_0$.

As in Proposition 4, if a just-collateralized contract is optimal then an inequality must hold. The right side of (6) is identical to the right side of (5), and again captures the value to the borrower of the consumption smoothing provided by minimizing the payment when collateral is transferred. The left side of (6) differs from (5), but still measures the deadweight loss of collateral transfer. The left side of (5) measures the cost in units of the lender's (linear) utility, which is appropriate there because the lender bears the deadweight loss. In a just-collateralized contract the borrower bears the deadweight loss of collateral transfer, and thus the left side of (6) measures the deadweight loss in terms of the borrower's utility; $u(0,k) - u(\mu_b k,0)$ is the gap between the borrower's utility at $\theta=R$, and the borrower's utility for θ just below R.

Otherwise Proposition 6 matches Proposition 4 exactly. If the borrower is sufficiently risk averse the contract is just-collateralized. If the borrower is less risk averse the contract is overcollateralized to some degree. For the limiting case of a risk neutral borrower the contract is as overcollateralized as possible.

The amount of collateral the borrower has will again constrain the amount the borrower can credibly commit to repay, but the possibility of renegotiation tightens the constraint. The contract that maximizes the return to the lender now yields $R=\theta_0+\mu_b k$ in all states. In the limited commitment case the contract that maximizes the return to the lender provides $R=\theta_0+c_k$ when output is greater than θ_0+c_k, and $\theta_0+\mu_b k$ when output is less than θ_0+c_k. Since $c_k>\mu_b k$ (again because the collateral is worth less to the lender than to the borrower), the return that can be credibly promised to the lender is lower under the renegotiation-proof constraint.

Loan losses

As noted earlier, the empirical evidence suggests that it is quite common for lenders to suffer losses when they acquire collateral in lieu of payment. The renegotiation-proof contracts of Proposition 5 do not allow the lender to bear any risk. Can the indivisible collateral model with renegotiation be reconciled with occasional lender losses? The settings from which the evidence is drawn -- lending secured by residential homes or commercial real estate -- suggests an answer. In most such loans the value of the collateral is initially expected to exceed the loan balance, but occasionally a negative shock drives the collateral value down. Perhaps the large losses associated with default and foreclosure stem from ex post deterioration in the value of collateral.

To explore this possibility, suppose now that the value of the collateral to both borrower and lender is uncertain ex ante but publicly observed ex post. For concreteness, let us suppose that collateral k is k_H, unless a shock to the collateral value occurs, in which case it takes on a lower value k_L. (Expected utilities are as above with the realized value substituted for k.) Since the collateral value is public ex post, payments can now be indexed by the collateral value. The optimal contract will specify two distinct collateralized debt contracts -- one for each collateral value. Thus the optimal contract will have the form:

$$R(\theta,T) = \begin{cases} R_{Lk}, & \theta<R_L,\ T=L \\ R_L, & \theta\geq R_L,\ T=L \\ R_{Hk}, & \theta<R_H,\ T=H \\ R_H, & \theta\geq R_H,\ T=H \end{cases}$$

where (R_{Lk},R_L,R_{Hk},R_H) are four scalars. When the collateral value is high, collateral is transferred for $\theta<R_H$. When collateral value is low, collateral is transferred for $\theta>R_L$.

The most realistic case is one in which the collateral constraint binds when the collateral value is low; that is,

$$\theta_0 = R_{Lk} = R_L - \mu_b k_L.$$

To match up with the observation of lender losses when collateral is transferred, $R_{Lk}+\mu_b k_L$ would have to fall short of the lender's required rate of return. In this case the remaining contract parameters R_{Hk} and R_H must be set to provide the lender with more than the required rate of return, in order to make up for the loss suffered when there is a collateral shock. As before, the relative magnitude of R_{Hk} and R_H is determined by the considerations described in Proposition 6; the contract is just collateralized (for the high collateral value) if the deadweight loss is not too large or if the borrower is sufficiently risk averse.

Contracts of this type have four possible outcomes. As long as the collateral value is high and output is large enough, the borrower makes the nominal payment R_H and keeps the collateral. If output is insufficient to make the payment R_H, but the collateral value is high, the borrower hands over the collateral and makes a smaller payment, R_{Hk}. If there is a shock to the value of collateral, the borrower threatens to hand over the collateral. If the borrower has enough cash (output is greater than $\mu_b k_L$, the value of the collateral to the lender) the lender accepts a cash payment equal to $\mu_b k_L$ in lieu of the collateral. If the borrower does not have enough cash (output is less than $\mu_b k_L$) then the lender takes the collateral and a small supplemental payment, Θ_0.

This model can capture the empirical observation that lenders suffer losses, on average, when collateral is transferred. If, as one would expect, the ex post value of the collateral is positively correlated with the borrower's return, then the lender's expected return will be lower in states in which collateral is transferred. The model also displays the feature that loans are "restructured" in response to declines in collateral value, even when the lender does not take the collateral. In essence, the ability to hide output lets the borrower threaten to plead poverty and hand over the collateral. In response, the lender must accept a lower cash payment and "forgive" some of the nominal debt R_H.

IV. Concluding Remarks

This paper presents a model of the striking feature of risky debt contracts that they are only occasionally contingent on the borrower's ex post circumstances. The approach adopted here is to formalize the notion that collateral requirements act as repayment incentives -- something else the borrower surrenders in the event that payment cannot be made exactly as promised. In a simple risk-sharing model with two goods -- one risky, one certain -- I look for conditions under which the optimal contract resembles collateralized debt. When the second good is perfectly divisible, and commitment powers can prevent renegotiation, the conditions require that the second good is valued more highly by the borrower than the lender, and that the difference in valuations outweighs the value of a certain indirect opportunity for risk-sharing (a term proportional to the risk aversion of the borrower). This condition can be seen as a lower bound on the valuation gap or as an upper bound on the borrower' risk aversion. When renegotiation can not be prevented, the optimality of collateralized debt merely requires that the borrower value the collateral good more highly than does the lender.

When the second good is indivisible, the optimal contract does not resemble debt, even with a positive valuation gap. Under a limited commitment constraint, however, the optimal contract does resemble debt, and all that is required is that the valuation gap be positive. An additional limit on commitment abilities is to require that contracts be renegotiation-proof. Such a constraint further reduces the set of possible optimal contracts by eliminating undercollateralized contracts in which the lender bears a loss in the event that collateral is transferred. If the ex post value of collateral is uncertain, however, contracts involve lender losses conditional on collateral transfer.

Clearly, providing an incentive to repay is not the only role that collateral plays in financial contracting. For instance, formal collateral provisions are critical in determining the priority of claims in situations in which more than one lender is involved. This suggests that the distinction between explicit and implicit collateralization might be most relevant to this priority function, and less relevant to the role in repayment incentives modeled here. If so, the broad range of remedies available to unsecured creditors can be seen as playing the role of implicit collateral for debt contracts. Collateral, more broadly defined, might then be an appealing explanation of the ubiquitous debt contract.

APPENDIX

Proposition 1: In this appendix I show that if the collateralized debt contract satisfies (VB1) with equality and (1), then it satisfies the necessary and sufficient conditions for the maximization problem (P1). I first reformulate (P1). Define $x_1(\theta) = u_1(\theta - y_1(\theta))$ and $x_2(\theta) = u_2(e_{a2} - y_2(\theta))$, and let $x(\theta) = (x_1(\theta), x_2(\theta))'$. A contract is now a function $\boldsymbol{x}:\Omega \to \mathbb{R}^2$. Restrict attention to the set **A** of absolutely continuous functions from Ω to $\mathbb{R}^2$. In the working paper version (1991), I show that collateralized debt is uniquely optimal within the larger set of contracts that are functions of bounded variation. Define ϕ_i as the inverse of u_i for $i=1,2$. Given $\boldsymbol{x}$, we can recover $y_1(\theta) = \theta - \phi_1(x_1(\theta))$, and $y_1(\theta) = \theta - \phi_1(x_1(\theta))$.

The program (P1) can be simplified by replacing (IC1) with a weaker set of "local" incentive constraints:

$$\text{(A.1)} \qquad -y_1'(\theta)u_1'(\theta - y_1(\theta)) - y_2'(\theta)u_2'(e_{a2} - y_2(\theta)) \geq 0 \qquad \forall\theta\in(\theta_0,\theta_1)$$

which must hold wherever the derivatives exist. Using $x_1'(\theta) =$

$(1 - y_1'(\theta))u_1'(\theta - y_1(\theta))$ and $x_2'(\theta) = -y_2'(\theta)u_2'(e_{a2} - y_2(\theta))$, we can rewrite (A.1) as

$$\text{(IC}'\text{)} \qquad x_1'(\theta) + x_2'(\theta) \geq u_1'(\phi_1(x_1(\theta))) \equiv g(x_1(\theta)), \qquad \forall\theta\in(\theta_0,\theta_1)$$

which again must hold wherever the derivatives exist. Among absolutely continuous contacts, (IC′) is necessary and sufficient for (IC1) due to concavity and nonincreasing absolute risk aversion, but for discontinuous contracts sufficiency fails. Since collateralized debt contracts satisfy (IC1) by construction, it suffices to prove that the debt contract is optimal under the weaker condition (IC′). Let $\boldsymbol{X}(\theta) = [\underline{x}_1, \overline{x}_1(\theta)]\times[\underline{x}_2, \overline{x}_2]$, $\underline{x}_1 = u_1(0)$,

$\overline{x}_1(\theta) = u_1(\theta+e_{b1})$, $\underline{x}_2 = u_2(0)$, $\overline{x}_2 = u_2(e_{a1}+e_{b2})$, and $\boldsymbol{Z}(\theta,x) = \{z\in\mathbb{R}^2 | z_1+z_2 \geq g(x_1)\}$.

For the collateralized debt contract, $x_1^*(\theta) = \underline{x}_1$ for $\theta\in[\theta_0,R]$, and

$x_1^*(\theta)\in(\underline{x}_1,\overline{x}_1(\theta))$ for $\theta\in(R,\theta_1]$, while $x_2^*(\theta)\in(\underline{x}_2,\overline{x}_2)$ for $\theta\in(\theta_0,R)$, and $x_2^*(\theta) = \overline{x}_2$

for $\theta\in[R,\theta_1]$. If $x_2^*(\theta_0) = \underline{x}_2$, then the contract is "collateral constrained,"

and $R = \overline{R}$. Let λ be the nonnegative multiplier on the constraint (VB1).

Define the Lagrangian function as

$$L(\theta,x,z) \quad = \quad -(x_1+x_2)f(\theta) \; - \; \lambda[e_{b1}+\theta-\phi_1(x_1)+\mu_b(e_{b2}+e_{a2}-\phi_2(x_2))]f(\theta)$$

if $x\in\boldsymbol{X}(\theta)$ and $x'\in\boldsymbol{Z}(\theta,x)$, and $+\infty$ otherwise. Define the functional

$$J_L(\boldsymbol{x}) \quad = \quad \int_{\theta_0}^{\theta_1} L(\theta,x(\theta),x'(\theta))d\theta.$$

Our problem is now: (P′) Choose $\boldsymbol{x}\in\boldsymbol{A}$ to minimize $J_L(\boldsymbol{x})$. This problem falls in a class of problems studied by Rockafellar (1972).

I will now show that if the collateralized debt contract satisfies (VB1) with equality and the inequality (1), then it satisfies the Hamiltonian and endpoint conditions for (P′). The costate function is a mapping from Ω to $\mathbb{R}^2$, but for a nontrivial solution it must lie in the normal cone of $\boldsymbol{Z}(\theta,x(\theta))$ for each θ, which requires that the two functions be everywhere equal and nonpositive. Thus the Hamiltonian conditions can be written in terms of a single costate function $\boldsymbol{p}:\Omega\rightarrow\mathbb{R}$. The Hamiltonian conditions evaluated at the collateralized debt contract are as follows.

$$\text{(A.2)} \qquad \begin{aligned} p'(\theta) \; - \; \rho^*(\theta)p(\theta) \; &\in \; L_1^*(\theta) \\ p'(\theta) \; &\in \; L_2^*(\theta) \end{aligned}$$

where

$$\rho^*(\theta) \equiv -\frac{u_1''(\phi_1(x_1^*(\theta)))}{u_1'(\phi_1(x_1^*(\theta)))}$$

$$L_1^*(\theta) = \begin{cases} (-\infty, \nu_1(\theta)] & \text{if } x_1^*(\theta) = \underline{x}_1, \\ \{\nu_1(\theta)\} & \text{if } x_1^*(\theta) \in (\underline{x}_1, \overline{x}_1(\theta)), \\ [\nu_1(\theta), +\infty) & \text{if } x_1^*(\theta) = \overline{x}_1(\theta), \end{cases}$$

$$L_2^*(\theta) = \begin{cases} (-\infty, \nu_2(\theta)] & \text{if } x_2^*(\theta) = \underline{x}_2, \\ \{\nu_2(\theta)\} & \text{if } x_2^*(\theta) \in (\underline{x}_2, \overline{x}_2), \\ [\nu_2(\theta), +\infty) & \text{if } x_2^*(\theta) = \overline{x}_2, \end{cases}$$

$$\nu_1(\theta) = \left(-1 + \frac{\lambda}{u_1'(\phi_1(x_1^*(\theta)))}\right) f(\theta),$$

$$\nu_2(\theta) = \left(-1 + \frac{\lambda \mu_b}{u_2'(\phi_2(x_2^*(\theta)))}\right) f(\theta),$$

and ***p*** is absolutely continuous. The endpoint conditions are

(A.3)

$$\text{if } R < \overline{R} \text{ then } p(\theta_0) = 0, \qquad \text{if } p(\theta_0) < 0 \text{ then } R = \overline{R}$$

$$\text{if } R > -e_{b1} \text{ then } p(\theta_1) = 0, \qquad \text{if } p(\theta_1) > 0 \text{ then } R = -e_{b1}.$$

The collateralized debt contract is optimal if there exists an absolutely continuous and nonpositive ***p*** and a nonnegative λ that satisfy (A.2) and (A.3). ***p*** satisfies (A.2) if and only if

$$\text{(A.4)} \qquad p'(\theta) = \begin{cases} \nu_2(\theta) & \text{for } \theta \in (\theta_0, R) \\ \rho^*(\theta)p(\theta) + \nu_1(\theta) & \text{for } \theta \in (R, \theta_1), \end{cases}$$

and

$$\text{(A.5)} \qquad \rho^*(\theta)p(\theta) + \nu_1(\theta) \geq \nu_2(\theta).$$

Solving (A.4) with endpoint $p(\theta_1) = 0$ (since $R > -e_{b1}$) yields

$$\text{(A.6)} \qquad \begin{aligned} -p(\theta_0)u_1'(0) &= \int_{\theta_0}^{\theta_1} -u_1'(\theta - y_1^*(\theta, R))f(\theta)d\theta \\ &\quad + \lambda\int_{\theta_0}^{R} \frac{\mu_b u_1'(\theta - y_1^*(\theta, R))}{u_2'(e_{b2} - y_2^*(\theta, R))} f(\theta)d\theta \\ &\quad + \lambda\int_{R}^{\theta_1} f(\theta)d\theta \\ &= E[v_{aR}(\theta, R)] + \lambda E[v_{bR}(\theta, R)]. \end{aligned}$$

Solving for $p(\theta)$ yields

$$\text{(A.7)} \qquad \begin{aligned} p(\theta)u_1'(\theta - y_1^*(\theta, R)) &= p(\theta_0)u_1'(0) \\ &\quad + E[v_{aR}(\hat\theta, R) | \hat\theta \leq \theta]F(\theta) + \lambda E[v_{bR}(\hat\theta, R) | \hat\theta \leq \theta]F(\theta). \end{aligned}$$

Setting $p(\theta_0) = 0$ and using (A.6) to substitute for λ yields

$$- \frac{p(\theta)u_1'(\theta - y_1^*(\theta,R))}{\lambda f(\theta)}$$

$$= \frac{F(\theta)}{f(\theta)}\left[E[v_{aR}(\hat{\theta},R)|\hat{\theta}\le\theta]\frac{E[v_{bR}(\hat{\theta},R)]}{E[v_{aR}(\hat{\theta},R)]} - E[v_{bR}(\hat{\theta},R)|\hat{\theta}\le\theta]\right]$$

$$\equiv \phi(\theta).$$

It is straightforward to verify that $\lambda > 0$ and $p(\theta) < 0$ for all $\theta\in(\theta_0,\theta_1)$.

Rearranging (A.5) and substituting yields

$$\mu_b \le \frac{u_2'(e_{b2}-y_2^*(\theta,R))}{u_1'(\theta-y_1^*(\theta,R))}\left[1 + \rho^*(\theta)\frac{p(\theta)u_1'(\theta-y_1^*(\theta,R))}{\lambda f(\theta)}\right]$$

$$= \mu_a^*(\theta) - \rho^*(\theta)\mu_a^*(\theta)\phi(\theta),$$

which is implied by (1). This proves that if the collateralized debt contract satisfies (1), then it satisfies the Hamiltonian and endpoint conditions for (P$'$).

To establish uniqueness, suppose that $\boldsymbol{x}^*$ is an optimal collateralized debt contract and $\hat{\boldsymbol{x}}$ is some optimal contract that differs from $\boldsymbol{x}^*$ on a set of positive measure. Since $\hat{\boldsymbol{x}}$ is feasible, $\hat{x}'(\theta)\in\boldsymbol{Z}(\theta,\hat{x}(\theta))$ for all θ. From the definition of subgradients

$$w_1(\hat{x}_1(\theta) - x_1^*(\theta)) + w_2(\hat{x}_2(\theta) - x_2^*(\theta)) \le L(\theta,\hat{x}(\theta),\hat{x}'(\theta)) - L(\theta,x^*(\theta),x^{*\prime}(\theta))$$

$$\forall\ w_1\in L_1^*(\theta),\ w_2\in L_2^*(\theta).$$

For $\theta < R$, $x_1^*(\theta) = \underline{x}_1$, $\hat{x}_1(\theta) - x_1^*(\theta) \ge 0$, $p'(\theta) - \rho^*(\theta)p(\theta) < \nu_1(\theta)$, and

$p'(\theta) = \nu_2(\theta)$. For $\theta > R$, $x_2^*(\theta)$, $\hat{x}_2(\theta) - x_2^*(\theta) \le 0$, $p'(\theta) - \rho^*(\theta)p(\theta) = \nu_1(\theta)$, and $p'(\theta) > \nu_2(\theta)$. Therefore

$$(p'(\theta) - \rho^*(\theta)p(\theta))(\hat{x}_1(\theta) - x_1^*(\theta)) + p'(\theta)(\hat{x}_2(\theta) - x_2^*(\theta))$$

$$\text{(A.8)} \qquad \leq \nu_1(\theta)(\hat{x}_1(\theta) - x_1^*(\theta)) + \nu_2(\theta)(\hat{x}_2(\theta) - x_2^*(\theta))$$

$$\leq L(\theta,\hat{x}(\theta),\hat{x}'(\theta)) - L(\theta,x^*(\theta),x^{*\prime}(\theta)).$$

Also note that

$$\hat{x}_1'(\theta) + \hat{x}_2'(\theta) \geq u_1'(\phi_1(\hat{x}_1(\theta)))$$

$$\text{(A.9)} \qquad \geq u_1'(\phi_1(x_1^*(\theta))) - \rho^*(\theta)(\hat{x}_1(\theta) - x_1^*(\theta))$$

$$= x_1^{*\prime}(\theta) + x_2^{*\prime}(\theta) - \rho^*(\theta)(\hat{x}_1(\theta) - x_1^*(\theta)).$$

Therefore,

$$J_L(\hat{\boldsymbol{x}}) - J_L(\boldsymbol{x}^*)$$

$$= \int_{\theta_0}^{\theta_1} [L(\theta,\hat{x}(\theta),\hat{x}'(\theta)) - L(\theta,x^*(\theta),x^{*\prime}(\theta))]d\theta$$

$$\text{(A.10)} \qquad \geq \int_{\theta_0}^{\theta_1}[p'(\theta)-\rho^*(\theta)p(\theta)](\hat{x}_1(\theta)-x_1^*(\theta)) + p'(\theta)(\hat{x}_2(\theta)-x_2^*(\theta))]d\theta$$

$$= -\int_{\theta_0}^{\theta_1} p(\theta)[\hat{x}_1'(\theta) + \hat{x}_2'(\theta) + \rho^*(\theta)(\hat{x}_1(\theta) - x_1^*(\theta)) - x_1^{*\prime}(\theta) - x_2^{*\prime}(\theta)]d\theta$$

$$\geq 0.$$

If $\hat{x}_1(\theta) \neq x_1^*(\theta)$ for a set of positive measure in $[\theta_0,R)$, then (A.8) is strict for such θ, and the first inequality in (A.10) is strict. If $\hat{x}_2(\theta) \neq x_2^*(\theta)$ for set of positive measure in $(R,\theta_1]$, then (A.8) is strict for such θ and the first inequality in (A.10) is strict. If neither of these conditions holds, then incentive compatibility implies that either $\hat{x}_2'(\theta) > x_2^{*\prime}(\theta)$ for a set of positive measure in $[\theta_0,R)$, or $\hat{x}_1'(\theta) > x_1^{*\prime}(\theta)$ for a set of positive measure in $(R,\theta_1]$. In either case (A.9) is strict for such θ and the second inequality in (A.10) is strict. Thus $J_L(\hat{\boldsymbol{x}}) > J_L(\boldsymbol{x}^*)$ and $\boldsymbol{x}^*$ is the unique optimal contract.

Endogenous Loan Size: Equation (3) can be obtained by substituting the new first order condition, $\lambda = u_1'(c_{a11})$, for λ in (A.6) and rearranging. The multiplier η is given by

$$\eta \quad = \quad -\frac{p(\theta_0)u_1'(0)}{u_1'(c_{a11})}.$$

Note that $p(\theta_0) > 0$ now, and is given by (A.6) with $\lambda = u_1'(c_{a11})$.

Proposition 2: The proof parallels that of Proposition 1; this is a streamlined presentation. Reformulate the problem by defining $x_1(\theta) = (1-\pi(\theta))u(\theta-y_{10}(\theta),e_{a2})$, $x_2(\theta) = \pi(\theta)u(\theta-y_{1k}(\theta),0)$, and $x_3(\theta) = \pi(\theta)$. Let $x(\theta) = (x_1(\theta),x_2(\theta),x_3(\theta))'$. A contract is now a function $\boldsymbol{x}(\theta):\Omega\rightarrow\mathbb{R}^3$.

Resource feasibility requires

$$0 \leq x_3 \leq 1$$

$$-e_{b1} \leq \theta - \phi_0(x_1/(1-x_3)) \leq \theta$$

$$-e_{b1} \leq \theta - \phi_k(x_2/x_3) \leq \theta$$

for each θ, where the functions ϕ_0 and ϕ_k are defined by $u(\phi_K(v),k-K) = v$ for $K=0,k$. Define $\boldsymbol{X}(\theta)$ as the set of $x\in\mathbb{R}^3$ that satisfy the above resource constraints; $\boldsymbol{X}(\theta)$ is a tetrahedron. The "local" incentive constraints are

$$x_1'(\theta) + x_2'(\theta) \geq g(x(\theta)), \qquad \forall\theta\in(\theta_0,\theta_1)$$

where

$$g(x) = x_3u_1'(\phi_k(x_2/x_3)) + (1-x_3)u_1'(\phi_0(x_1/(1-x_3)))$$

Let $\boldsymbol{Z}(\theta,x) = \{z\in\mathbb{R}^3 | z_1+z_2\geq g(x)\}$. Suppose x^* corresponds to the candidate contract that satisfies (VB2) with equality. With the Lagrangian suitably modified, the problem again falls into the class studied by Rockafellar (1972). The costate function is now a mapping p from Ω to $\mathbb{R}^3$, but as before it must lie in the normal cone of $\boldsymbol{Z}$, which requires here that $p_1 = p_2 \leq 0$, and $p_3 = 0$. Thus the Hamiltonian conditions can be written in terms of a single costate function p. The Hamiltonian conditions are satisfied if and only if (A.4) and (A.5) hold with $\nu_1(\theta)$ defined as before but with $\nu_2(\theta)$ now given by

$$\nu_2(\theta) \quad = \quad \left[\frac{\lambda\mu_b e_{a2}}{u_2(e_{a2})-u_2(0)} \quad - \quad 1\right] f(\theta)$$

The remainder of the proof is identical.

<u>**Proposition 3:**</u> Suppose an arbitrary contract (π, y_{10}, y_{1k}) satisfies (RF2), (IC2), (LC), and (VB2) with equality. Suppose there exists a measurable subset A of Ω on which $\pi(\theta) > 0$ and $v_a(\theta) \geq u(0,k)$. Then there exists an alternative contract with $\pi(\theta) = 0$ for all θ in A, and everything else unchanged. Since $u(\theta - y_{10}(\theta),k) = u(\theta - y_{1k}(\theta),0)$ (if $\pi(\theta) = 1$ then this choice of $y_{10}(\theta)$ is without loss of generality), agent a is indifferent between the two contracts in every state. Under condition (4), it is easy to show that agent b is strictly better off. Therefore, an optimal contract must have $\pi(\theta) = 0$ whenever $v_a(\theta) \geq u(0,k)$. If $v_a(\theta) < u(0,k)$ then it is not feasible to find a $y_{10}(\theta)$ such that $u(\theta - y_{10}(\theta),k) = v_a(\theta)$ as would be required by (LC) if $\pi(\theta) < 1$. Therefore $\pi(\theta) = 1$ whenever $v_a(\theta) < u(0,k)$.

Since $v_a(\theta)$ is strictly increasing, we can now write a contract in terms of a cutoff value θ': $\pi(\theta) = 1$ for $\theta < \theta'$, and $\pi(\theta) = 0$ for $\theta \geq \theta'$. Incentive compatibility requires that y_{1k} and y_{10} are nonincreasing on $[\theta_0, \theta')$ and $[\theta', \theta_1]$ respectively. In addition, we require that $\theta' - y_{1k}(\theta') \leq c_k$, where $y_{1k}(\theta')$ is the left-hand limit of y_{1k} at θ': otherwise θ' could be reduced with no change in $v_a(\theta)$ but an increase in $v_b(\theta)$ for some θ. Consider an arbitrary contract

$(\theta', y_{10}, y_{1k})$ satisfying these conditions and resource feasibility. Replace y_{1k} and y_{10} with their certainty equivalents; the resulting contract is feasible and Pareto-improving so the original could not have been optimal. The Problem (P3) has now been reduced to choosing three numbers, θ', $y_{1k}(\theta) = R_k\ \forall\theta<\theta'$, and $y_{10}(\theta) = R\ \forall\theta\geq\theta'$. These numbers must satisfy the following constraints.

$$\text{(A.11)}\qquad \begin{aligned} \theta_0 \;\geq\; R_k \;\geq\; \theta' - c_k \\ \theta' \;\geq\; R \;\geq\; -e_{b1} \end{aligned}$$

I claim that $\theta' = R$ in an optimal contract. Take an arbitrary contract (θ', R, R_k) that satisfies (VB2) with equality and (A.11). For $\delta \geq 0$, define $\theta'(\delta) = \theta' - \delta$, and define $R(\delta)$ by

$$e_{b1} + R(\delta)(1-F(\theta'(\delta))) + (R_k+\mu_b k)F(\theta'(\delta)) = \bar{v}_b$$

Define $R_k(\delta) = R_k$. With $v_a = \mathrm{E}[v_a(\theta)]$, we have

$$\frac{\partial v_a}{\partial\delta} = \Big[u(\theta'(\delta)-R_k,0) - u(\theta'(\delta)-R(\delta),k) + (R_k+\mu_b k-R(\delta))\mathrm{E}[u_1'(\theta-R(\delta))|\theta\geq\theta']\Big]\, f(\theta')$$

Condition (4) implies that the expression in the brackets is strictly positive for feasible values of δ. Therefore, if $\theta' > R$ the contract can always be improved by increasing δ. Only when $\theta' = R$ is $\delta > 0$ infeasible. The fact that $c_k > \mu_b k$ follows from (4) and the concavity of u_1. This proves Proposition 3.

Proposition 4: In view of Proposition 3, (P3) can be reduced to choosing two numbers, R and R_k, to maximize the borrower's expected utility subject to (VB2) and

$$\theta_0 - R_k \geq 0 \qquad (\lambda_0)$$

$$R_k - R + c_k \geq 0 \qquad (\lambda_k)$$

Let λ_θ and λ_k be the nonnegative multipliers associated with their respective constraints. Let $R_k(R)$ be the implicit function defined by (VB2) holding with equality. The first order necessary conditions can then be written as

$$\mu_b k + R_k(R) - R - \frac{u(0,k) - u(R-R_k(R),0)}{E[u_1'(\theta-R_k(R))|\theta\le R]}$$

$$+ \left[1 - \frac{E[u_1'(\theta-R)|\theta\ge R]}{E[u_1'(\theta-R_k(R))|\theta\le R]}\right]\left(\frac{1-F(R)}{f(R)}\right) \tag{A.12}$$

$$= \frac{(1+(R_k(R)+\mu_b k-R)f(R))\lambda_k}{f(R)F(R)E[u_1'(\theta-R_k(R))|\theta\le R]} + \frac{(1-F(R)+(R_k(R)+\mu_b k-R)f(R))\lambda_0}{f(R)F(R)E[u_1'(\theta-R_k(R))|\theta\le R]}$$

$$\lambda_k \ge 0, \qquad \lambda_k(R_k(R)-R+c_k) = 0 \tag{A.13}$$

$$\lambda_0 \ge 0, \qquad \lambda_0(\theta_0-R_k(R)) = 0 \tag{A.14}$$

PROPOSITION 7: *If*

(i) $\dfrac{(c_k-\mu_b k)f(R)}{1-F(R)} \le 1,$

(ii) $E[u_1'(\theta-R)|\theta\ge R]$ is nondecreasing in R, and

(iii) $\dfrac{f(R)}{1-F(R)}$ is nondecreasing,

then there is a unique optimal contract (R,R_k) *determined by* (A.12)-(A.14).

Proof, which is omitted, hinges on showing that under these assumptions $R_k(R)$ is decreasing, the left side of (A.12) is strictly decreasing in R, and the coefficients on λ_k and λ_0 are both positive. Either the left side of (A.12) is positive at $R_k(R) = \theta_0$, or the left side of (A.12) equals zero (determining R), or the left side of (A.12) is negative at $R_k(R) = R - c_k$.

Proposition 5: Fix θ and suppose that $\pi(\theta)\in(0,1)$. Then (LC) implies that $u(\theta - y_{10}(\theta), k) = u(\theta - y_{1k}(\theta), 0)$. Together with (4) this implies that $y_{10}(\theta) > y_{1k}(\theta) + \mu_b k$. The borrower is indifferent to setting $\pi(\theta) = 0$ instead, while the lender is strictly better off. Thus the original contract was not renegotiation-proof at θ and we must have $\pi(\theta) \in \{0,1\}$.

Suppose $\pi(\theta) = 1$ and $u(\theta - y_{1k}(\theta), 0) \geq u(0, k)$. Then there exists a feasible $y_{10}(\theta)$ such that $u(\theta - y_{10}(\theta), k) = u(\theta - y_{1k}(\theta), 0)$. Setting $\pi(\theta) = 0$ now dominates the original contract, which thus could not have been renegotiation-proof. Therefore, if $\pi(\theta) = 1$ then $u(\theta - y_{1k}(\theta), 0) < u(0, k)$. As before, there exists a cutoff θ' such that $\pi(\theta) = 1$ for $\theta < \theta'$, and $\pi(\theta) = 0$ for $\theta \geq \theta'$. At θ' we must have $\theta' - y_{1k}(\theta') \leq c_k$, where $y_{1k}(\theta')$ is the left hand limit of $y_{1k}(\theta)$ at θ'.

Suppose $y_{1k}(\theta'') + \mu_b k < \theta''$ for some $\theta'' < \theta'$: then this contract is not renegotiation-proof at θ''. For $y_{10} \in (y_{1k}(\theta'') + \mu_b k, \theta'')$, changing $\pi(\theta'')$ to 0 with $y_{10}(\theta'') = y_{10}$, makes the lender strictly better off. An implication of (4) is $u(c, k) > u(c + \mu_b k, 0)$ for all $c \geq 0$, and thus $u(\theta'' - y_{1k}(\theta'') - \mu_b k, k) > u(\theta'' - y_{1k}(\theta''), 0)$. Since y_{10} can be made arbitrarily close to $y_{1k}(\theta'') + \mu_b k$, there exists a y_{10} such that $u(\theta'' - y_{10}, k) \geq u(\theta'' - y_{1k}(\theta''), 0)$, as is required for the contract to be renegotiated. Therefore, (RP) implies that $y_{1k}(\theta'') + \mu_b k \geq \theta''$ for all $\theta'' < \theta'$.

Given $\pi(\theta) \in \{0,1\}$, it is straightforward to show that (IC2) again implies that $y_{10}(\theta)$ is nonincreasing on $\{\theta | \pi(\theta)=0\}$, and $y_{1k}(\theta)$ is nonincreasing on $\{\theta | \pi(\theta)=1\}$. In addition, feasibility implies that $y_{1k}(\theta_0) \leq \theta_0$ and $y_{10}(\theta_1) \geq -e_{b1}$. Collecting results, I have shown that (RF2), (IC2), (LC) and (RP) together imply that $\pi(\theta) = 1$ for $\theta < \theta'$, $\pi(\theta) = 0$ for $\theta \geq \theta'$, $y_{10}(\theta)$ and $y_{1k}(\theta)$ are nonincreasing, $y_{1k}(\theta_0) \leq \theta_0$, $y_{10}(\theta_1) \geq -e_{b1}$, and $y_{1k}(\theta) + \mu_b k \geq \theta$ for all $\theta < \theta'$. The converse is easily proved as well. The rest of the proof follows that of Proposition 3, except that in constraint (A.12), $R_k \geq \theta' - c_k$ is replaced by $R_k \geq \theta' - \mu_b k$.

Proposition 6: Parallels the proof of Proposition 4, with (A.13) replaced by

(A.15) $$\lambda_k \geq 0, \qquad \lambda_k(R_k(R) - R + \mu_b k) = 0$$

Equation (6) follows immediately from (A.12).

NOTES

1. The contract might then resemble ownership of the collateral by the "lender," but such arrangements lie outside the scope of the present paper.

2. Mark Bagnoli and Kenneth Snowden (1992) describe a model in which ex post moral hazard in the care and maintenance of the collateral gives rise to the asymmetry in valuations that makes collateralized debt optimal below. It is straightforward to extend the model presented here so that the asymmetric valuation arises endogenously from a moral hazard problem.

3. It is worth noting in this regard that the most important legal distinction between a secured and an unsecured loan is how the claim stands vis-a-vis third parties, suggesting that explicit collateralization is consequential mainly within multilateral financial arrangements (Thomas J. Jackson and Anthony T. Kronman, 1979). For example, under the current U.S. law governing secured transactions the difference between secured and unsecured parties is slight when there is only one creditor. See Article 9 of the Uniform Commercial Code. An exception is when the collateral is an "exempt asset" under bankruptcy law, and is thus out of reach of unsecured creditors. Also, a secured lender can often recover collateral without judicial process.

4. Benjamin (1978) also advances this idea. Martin Hellwig (1977) presents a model of endogenous borrowing constraints in the absence of collateral. Collateral also constrains borrowing in the models of Hart and Moore (1989, 1994), but their information structure is quite different from mine: see below.

5. The credit rationing that emerges here is very different from that of Joseph E. Stiglitz and Andrew Weiss (1981) or Stephen D. Williamson (1987). In their models lenders reach a point at which raising the nominal interest rate on the loan, even though it is feasible, fails to increase the expected return; in Stiglitz and Weiss the mix of borrower types deteriorates rapidly enough at higher interest rates, while in Williamson the increase in expected monitoring costs swamps the increase in gross returns. Here lenders reach a point at which no increase in the nominal interest rate is feasible and incentive compatible.

6. Richard D. Evans, Brian A. Maris, and Robert I. Weinstein (1985), Robert Van Order and Ann B. Schnare (1994), Timothy Curry, Joseph Blalock and Rebel Cole (1991).

7. See Dilip Mookherjee and Ivan Png (1989) and Townsend (1988), although Boyd and Smith (1994) argue that in empirically relevant cases the extent of randomization is small. Note that randomization survives when the lender can not precommit to mixed verification strategies: Jeffrey M. Lacker (1990), Robert Moore (1989).

8. The ever popular assumption of risk neutrality on the part of the lender is without loss of generality here; the results easily carry over to a setting with a risk averse lender. See Lacker (1991b).

9. Diamond's (1984) setup can be obtained by setting $e_{b1} = e_{b2} = 0$, $\theta_0 = 0$, $\mu_b \equiv 0$, $u_1(c_{a1}) = c_{a1}$, $u_2(c_{a2}) = c_{a2}$, and then relaxing Assumption 1(d) to allow

costless faking of goods that do not exist. The latter is inconsequential, so his environment is effectively a special case of Assumption 1. Diamond's "nonpecuniary penalties" are $u_2(e_{a2}) - u_2(e_{a2}-y_2)$. An environment that is formally identical to mine is described in Chapter 5 of Townsend (1993). See Lacker (1991a) for discussion of a discrete-state-space version.

10. The constraint that one cannot produce evidence of goods that do not exist does not bind for the equilibrium collateralized debt contract, but plays a role in ruling out randomized contracts; see Lacker (1991b). See Lacker and John A. Weinberg (1989) for a discussion of costly falsification.

11. The loan advance will be treated as fixed, so that we may focus on the characteristics of the contract governing repayment. Later in this section the amount of the loan is made endogenous.

12. This presumes that agents have access to an enforcement facility that can punish agent *a* for withholding stipulated transfers, but can not overcoming the informational imperfection of costless falsification.

13. For proof see the working paper version, Lacker (1991b).

14. Attention is restricted to contracts that are functions of bounded variation.

15. Under the assumption that $\mu_b < u_2'(c_{a1})/u_2'(c_{a2})$ for all feasible c_{a1} and c_{a2} (see Proposition 1), the optimal contract without (IC1) is $y_1(\theta) = \text{MAX}[\theta-\bar{c}_{a1},-e]$ for some $\bar{c}_{a1}$ between 0 and θ_1+e, and $y_2(\theta) = 0$.

16. Nonincreasing absolute risk aversion ensures global incentive compatibility.

17. It is easy to show that $E[v_b(\theta,R)]$ is monotonic in R, so a unique value satisfies (VB) with equality.

18. See Lacker (1990) and Moore (1989) for a discussion of time consistency in the costly verification environment.

19. This is only feasible if $R \le \theta_0+\mu_b k$, a condition more stringent than $R \le \bar{R}$.

20. Equation (3) presumes $-e_{a11} < q < e_{b11}$.

21. Fisher (1930, pp. 210-211) noted that a borrower's collateral will limit the amount he can borrow, and "(i)n consequence of this limitation upon his borrowing power, the borrower may not succeed in modifying his income stream sufficiently to bring his rate of preference for present over future income down to agreement with the rate or rates of interest ruling in the market." (p. 211)

22. In a series of recent papers Hart and Moore (1989, 1994) examine very similar environments in which information is observed by the contracting parties but not by the enforcement facility. As a result, enforcable contractual terms can not be contingent on the relevant ex post information. In their model some allocations require renegotiation because contractual terms can not be state contingent in the same way as renegotiation outcomes. In my model the displayed output can be observed by the enforcement facility -- it is *verifiable* in the language of Hart and Moore. Moreover, there is no

way to extract further information about the borrower's output beyond the amount displayed. Thus contractual terms can replicate renegotiation outcomes.

23. I assume that if the borrower is indifferent the offer is accepted.

24. In this environment, if there is a feasible (z_1,z_2) that makes the borrower better off without making the lender worse off, then there exists a (z_1,z_2) that makes the lender better off without making the borrower worse off.

25. The constraint that contracts be renegotiation proof does not, by itself, imply that the collateralized debt contract is optimal, since it does not imply that incentive constraints bind. For example, a contract like the collateralized debt contract but with y_1 strictly decreasing for $\theta>R$ is also renegotiation proof.

26. As before, we could allow for more randomization. A more general contract would be a measure $\pi(y_1,y_2|\theta)$ specifying the probability of transfer (y_1,y_2) given display θ. With nonincreasing absolute risk aversion, however, it is easy to show that conditional on the transfer of collateral no further randomization is ever desirable: $\pi(y_1|y_2,\theta)$ is deterministic.

27. Something of this sort must underlie any model with lotteries if agents can't precommit.

28. More formally, a contract now consists of: a strategy $\pi(dy_1,dy_2,d\theta'|\theta)$ for the borrower, a probability measure over displays θ' and voluntary transfers (y_1,y_2); a strategy $\pi(s|y_1,y_2,\theta')$ for the lender, a probability measure over $s\in${"don't sue","sue"}; and a transfer schedule $\pi(d\tau_1,d\tau_2|y_1,y_2,\theta')$, a probability measure over feasible transfers for the court to impose. The court, if invoked by the lender's suit, follows $\pi(d\tau_1,d\tau_2|y_1,y_2,\theta')$ mechanically. Given the transfer schedule, the lender's strategy must be a best response for each (y_1,y_2,θ). Given the transfer schedule and the lender's strategy, the borrower's strategy must be a best response for each θ. The restrictive assumption here is that transfer schedules must be deterministic, so that $\pi(d\tau_1,d\tau_2|y_1,y_2,\theta')$ reduces to a pair of functions, $\tau_1(y_1,y_2,\theta')$ and $\tau_2(y_1,y_2,\theta')$. Strategies can, in principle, be mixed, but they must be in each agent's interest, ex post. We are also assuming, quite naturally, that the court cannot observe the probabilities with which agents choose random actions, if that is what their agreement required of them.

29. In principle a similar limited commitment condition applies to the choices of agent b. There is a property here analogous to the Revelation Principle, however: any feasible and incentive compatible allocation also satisfying limited commitment conditions for both agents can be achieved by a feasible and incentive compatible contract satisfying (LC) in which the enforcement facility is never invoked. Thus we can safely neglect the lender's strategy choice.

30. Monotonicity of R_k with respect to R across contracts that satisfy (VB2) with equality requires that $(c_k-\mu_b k)f(R)\leq(1-F(R))$. See the Appendix for details.

31. I implicitly assume that if the borrower is indifferent the offer is accepted.

32. In this environment, if there is a feasible (p, z_{10}, z_{1k}) that makes the borrower better off without making the lender worse off, and satisfies (LC), then there exists a (p, z_{10}, z_{1k}) that makes the lender better off without making the borrower worse off. Note that the contracts in Proposition 2 are renegotiation proof, and thus renegotiation-proofness alone does not deliver collateralized debt contracts here.

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