Supplementary Material to Working Paper 08-07

Sectoral vs. Aggregate Shocks: A Structural Factor Analysis of Industrial Production

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1 The Model

$$\max E_t \sum_{t=0}^{\infty} \beta^t \sum_{j=1}^{N} \left(\frac{C_{jt}^{1-\sigma} - 1}{1-\sigma} - \psi L_{jt} \right)$$

subject to

$$Y_{jt} = C_{jt} + \sum_{i=1}^{N} M_{jit} + K_{jt+1} - (1-\delta) K_{jt}$$

and

$$Y_{jt} = A_{jt} K_{jt}^{\alpha_j} \prod_{i=1}^{N} M_{ijt}^{\gamma_{ij}} L_{jt}^{1-\alpha_j - \sum_{i=1}^{N} \gamma_{ij}}.$$

The first-order necessary conditions are:

$$C_{jt}: \ C_{jt}^{-\sigma} = \lambda_{jt},$$

$$L_{jt}: \ \psi = \lambda_{jt} \frac{Y_{jt}}{L_{jt}} \left(1 - \alpha_j - \sum_{i=1}^{N} \gamma_{ij} \right).$$

Combining these two equations gives

$$\psi = C_{jt}^{-\sigma} \frac{Y_{jt}}{L_{jt}} \left(1 - \alpha_j - \sum_{i=1}^{N} \gamma_{ij} \right).$$

$$M_{ijt}: \ \lambda_{it} = \lambda_{jt} \gamma_{ij} \frac{Y_{jt}}{M_{ijt}},$$

or

$$C_{it}^{-\sigma} = C_{jt}^{-\sigma} \gamma_{ij} \frac{Y_{jt}}{M_{ijt}}.$$

$$K_{jt+1}: \ C_{jt}^{-\sigma} = \beta E_t \left[C_{jt+1}^{-\sigma} \left(\alpha_j \frac{Y_{jt+1}}{K_{jt+1}} + 1 - \delta \right) \right]$$

2 Dynamics of the System

The dynamics are described by a set of $4N + N^2$ equations in $4N + N^2$ unknowns. When $N = 117$, this amounts to 14157 equations, but preliminary algebraic manipulations help keep the system tractable.

$$\psi L_{jt} = C_{jt}^{-\sigma} \left(1 - \alpha_j - \sum_{i=1}^{N} \gamma_{ij} \right) Y_{jt}$$

$$C_{it}^{-\sigma} = C_{jt}^{-\sigma} \gamma_{ij} \frac{Y_{jt}}{M_{ijt}}$$

$$C_{jt}^{-\sigma} = \beta E_t \left[C_{jt+1}^{-\sigma} \left(\alpha_j \frac{Y_{jt+1}}{K_{jt+1}} + 1 - \delta \right) \right]$$

$$Y_{jt} = C_{jt} + \sum_{i=1}^{N} M_{jit} + K_{jt+1} - (1-\delta) K_{jt}$$

and

$$Y_{jt} = A_{jt} K_{jt}^{\alpha_j} \prod_{i=1}^{N} M_{ijt}^{\gamma_{ij}} L_{jt}^{1-\alpha_j - \sum_{i=1}^{N} \gamma_{ij}}.$$

3 Log-linearized Equations

The "hat" notation stands for percent deviation from steady state.

$$\widehat{L}_{jt} = -\sigma \widehat{C}_{jt} + \widehat{Y}_{jt}$$

$$-\sigma \widehat{C}_{it} = -\sigma \widehat{C}_{jt} + \widehat{Y}_{jt} - \widehat{M}_{ijt}$$

$$-\sigma \widehat{C}_{jt} = -\sigma E_t \widehat{C}_{jt+1} + \widetilde{\beta} \widehat{Y}_{jt+1} - \widetilde{\beta} \widehat{K}_{jt+1}$$

where $\widetilde{\beta} = 1 - \beta + \beta\delta$.

$$\widehat{Y}_{jt} = S_{C_j} \widehat{C}_{jt} + S_{K_j} \widehat{K}_{jt+1} - (1-\delta) S_{K_j} \widehat{K}_{jt} + \sum_{i=1}^{N} S_{M_{ji}} \widehat{M}_{jit}$$

$$\widehat{Y}_{jt} = \widehat{A}_{jt} + \alpha_j \widehat{K}_{jt} + \sum_{i=1}^{N} \gamma_{ij} \widehat{M}_{ijt} + \left(1 - \alpha_j - \sum_{i=1}^{N} \gamma_{ij} \right) \widehat{L}_{jt}.$$

Let $c_t = [\widehat{C}_{1t}, ..., \widehat{C}_{Nt}]$, etc... and $m_t = [\widehat{M}_{11t}, ..., \widehat{M}_{1Nt}, \widehat{M}_{21t}, ..., \widehat{M}_{NNt}]$. The log-linearized equations can be written in matrix form as follows:

$$l_t = -\sigma c_t + y_t, \tag{1}$$

$$m_t = M_y y_t + M_c c_t \tag{2}$$

where

$$M_y = 1_{N \times 1} \otimes I \text{ and } M_c = \sigma (I \otimes 1_{N \times 1}) - \sigma (1_{N \times 1} \otimes I),$$

$$-\sigma c_t = -\sigma E_t c_{t+1} + \widetilde{\beta} E_t y_{t+1} - \widetilde{\beta} k_{t+1} \tag{3}$$

$$y_t = S_c c_t + S_m m_t + S_k k_{t+1} - S_k (1-\delta) k_t, \tag{4}$$

where

$$S_c = \begin{bmatrix} S_{C_1} & & \\ & ... & \\ & & S_{C_N} \end{bmatrix}, \text{ etc... and } S_m = \begin{bmatrix} S_{M_{11}} & S_{M_{12}} & ... & 0 & 0 \\ & & & & \\ 0 & 0 & ... & S_{M_{N,N-1}} & S_{M_{NN}} \end{bmatrix},$$

$$y_t = a_t + \alpha_d k_t + \widetilde{\Gamma} m_t + \Phi l_t, \tag{5}$$

where

$$\widetilde{\Gamma}_{N\times N^2} = \begin{bmatrix} \gamma_{11} & 0 & \ldots & \gamma_{21} & 0 & \ldots & \gamma_{N1} & 0 & \ldots & & & \\ 0 & \gamma_{12} & \ldots & 0 & \gamma_{22} & \ldots & & \gamma_{N2} & & & & \\ & & \gamma_{13} & & & \gamma_{23} & & & \gamma_{N3} & & & \\ & & & \ldots & & & \ldots & & & \ldots & & \\ & & & & \gamma_{1N} & 0 & \ldots & \gamma_{2N} & & & \gamma_{NN} \end{bmatrix}$$

and

$$\Phi = I - \alpha_d - \begin{bmatrix} \sum_i \gamma_{i1} & & & \\ & \sum_i \gamma_{i2} & & \\ & & \ldots & \\ & & & \sum_i \gamma_{iN} \end{bmatrix}$$

4 System Reduction

Use equation (1), (2) and (5) to obtain

$$\underbrace{[I - \widetilde{\Gamma} M_y - \Phi]}_{\alpha_d} y_t = a_t + \alpha_d k_t + \underbrace{[\widetilde{\Gamma} M_c - \Phi\sigma]}_{\Omega_{yc}} c_t$$

or, equivalently

$$y_t = \alpha_d^{-1} a_t + k_t + \alpha_d^{-1} \Omega_{yc} c_t.$$

Note that $\Omega_{yc} = \alpha_d(I - (I - \Gamma')\alpha_d^{-1})$ when $\sigma = 1$. Substituting this equation in equation (3) gives

$$-\sigma c_t = -\sigma c_{t+1} + \widetilde{\beta}[\alpha_d^{-1} a_{t+1} + k_{t+1} + \alpha_d^{-1} \Omega_{yc} c_{t+1}] - \widetilde{\beta} k_{t+1}$$

or

$$-\sigma c_t = [-\sigma I + \widetilde{\beta} \alpha_d^{-1} \Omega_{yc}] c_{t+1} + \widetilde{\beta} \alpha_d^{-1} a_{t+1}. \tag{7}$$

Use the resource constraint (4) to obtain

$$y_t = S_c c_t + S_m [M_y y_t + M_c c_t] + S_k k_{t+1} - S_k (1 - \delta) k_t$$

or

$$(I - S_m M_y) y_t = (S_c + S_m M_c) c_t + S_k k_{t+1} - S_k (1 - \delta) k_t$$

which gives

$$(I - S_m M_y)[\alpha_d^{-1} a_t + k_t + \alpha_d^{-1} \Omega_{yc} c_t] = (S_c + S_m M_c) c_t + S_k k_{t+1} - S_k (1 - \delta) k_t$$

or finally

$$
\begin{aligned}
S_k k_{t+1} \;=\; & [(I - S_m M_y)\alpha_d^{-1}\Omega_{yc} - (S_c + S_m M_c)]c_t + [S_k(1-\delta) + (I - S_m M_y)]k_t \qquad (8) \\
& +(I - S_m M_y)\alpha_d^{-1} a_t
\end{aligned}
$$

We can write equations (7) and (8) as:

$$
\begin{aligned}
& \begin{bmatrix} -\sigma I + \widetilde{\beta}\alpha_d^{-1}\Omega_{yc} & 0 \\ 0 & S_k \end{bmatrix} E_t \begin{bmatrix} c_{t+1} \\ k_{t+1} \end{bmatrix} \\
= & \begin{bmatrix} -\sigma I & 0 \\ (I - S_m M_y)\alpha_d^{-1}\Omega_{yc} - (S_c + S_m M_c) & S_k(1-\delta) + (I - S_m M_y) \end{bmatrix} \begin{bmatrix} c_t \\ k_t \end{bmatrix} + \\
& \begin{bmatrix} 0 \\ (I - S_m M_y)\alpha_d^{-1} \end{bmatrix} a_t + \begin{bmatrix} -\widetilde{\beta}\alpha_d^{-1} \\ 0 \end{bmatrix} E_t(a_{t+1}) \qquad (9)
\end{aligned}
$$

At this stage, the dynamics of the system can be solved using standard linear rational expectations toolkits such as Blanchard and Kahn (1980), King, Plosser, Rebelo (1988), and Klein (2000). The results presented in the text are based on King and Watson (2002). To use these methods, however, one must first obtain the steady state of the system. In this model, this can be achieved analytically.

5 Finding the Steady State Analytically

The steady state solution only requires inverting $N \times N$ matrices. The steady state equations for labor, materials, and capital are respectively

$$
\psi L_j = \lambda_j (1 - \alpha_j - \sum_{i=1}^{N} \gamma_{ij}) Y_j
$$

$$
M_{1j} = \frac{\lambda_j}{\lambda_i} \gamma_{ij} Y_j
$$

$$
K_j = \underbrace{\alpha_j [\frac{1}{\beta} - 1 + \delta]^{-1}}_{\phi_{K_j}} Y_j .
$$

Now take the logs of these equations to obtain (small letters denote logs)

$$
l_j = -\ln\psi + \ln\lambda_j + \ln(1 - \alpha_j - \sum_{i=1}^{N} \gamma_{ij}) + y_j ,
$$

$$
m_{ij} = \ln\lambda_j - \ln\lambda_i + \ln\gamma_{ij} + y_j
$$

$$k_j = \ln \phi_{K_j} + y_j. \tag{12}$$

The log steady state equations can be written in matrix form to summarize the entire system. Let $l = [l_1, ..., l_N]$, etc... and $m = [m_{11}, m_{12}, ...m_{1N}, m_{21}, ...m_{NN}]$. Then, we have that

$$l = -\ln \psi + \ln \widetilde{\Phi} + \ln \lambda + y,$$

where

$$\ln \widetilde{\Phi} = \begin{bmatrix} \ln(1 - \alpha_1 - \sum_i \gamma_{i1}) \\ ... \\ \ln(1 - \alpha_N - \sum_i \gamma_{iN}) \end{bmatrix}.$$

Similarly, the equation for steady state materials can be expressed as,

$$m = M_\lambda \ln \lambda + M_y y + vec(\ln \Gamma'),$$

where

$$M_\lambda = 1_{N\times 1} \otimes I - I \otimes 1_{N\times 1}, \text{ and } M_y = 1_{N\times 1} \otimes I.$$

Finally, we have that

$$k = \ln \phi_K + y,$$

where

$$\ln \phi_K = \begin{bmatrix} \ln \phi_{K_1} \\ ... \\ \ln \phi_{K_N} \end{bmatrix}$$

The log of production in all sectors can be expressed as

$$y = a + \alpha_d k + \widetilde{\Gamma} m + \Phi l,$$

where $\widetilde{\Gamma}$ is defined as above. Making the appropriate substitutions yields

$$y = a + \alpha_d(\ln \phi_K + y) + \widetilde{\Gamma}[M_\lambda \ln \lambda + M_y y + vec(\ln \Gamma')] + \Phi(-\ln \psi + \ln \widetilde{\Phi} + \ln \lambda + y)$$

or, equivalently,

$$\underbrace{[I - \alpha_d - \widetilde{\Gamma} M_y - \Phi]}_{=0} y = a + \alpha_d \ln \phi_K + \widetilde{\Gamma} vec(\ln \Gamma') + \Phi \ln \widetilde{\Phi} - \Phi \ln \psi + (\widetilde{\Gamma} M_\lambda + \Phi) \ln \lambda.$$

It follows that we can solve for (shadow) prices in the steady state in closed form,

$$\ln \lambda = -(\widetilde{\Gamma} M_\lambda + \Phi)^{-1}[a + \alpha_d \ln \phi_K + \widetilde{\Gamma} vec(\ln \Gamma') + \Phi \ln \widetilde{\Phi} - \Phi \ln \psi],$$

and $\lambda = e^{\ln \lambda}$.

To solve for the vector Y, write the resource constraints as

$$\lambda^{-\frac{1}{\sigma}} + \delta\phi_K^d Y + M_r Y = Y$$

where

$$\phi_K^d = \begin{bmatrix} \phi_{K_1} & & & \\ & \cdots & & \\ & & \phi_{K_{N-1}} & \\ & & & \Phi_{K_N} \end{bmatrix}, \text{ and } M_r = \begin{bmatrix} \gamma_{11} & \gamma_{12}\frac{\lambda_2}{\lambda_1} & \cdots & \gamma_{1N}\frac{\lambda_N}{\lambda_1} \\ \gamma_{21}\frac{\lambda_1}{\lambda_2} & \gamma_{22} & \cdots & \gamma_{2N}\frac{\lambda_N}{\lambda_2} \\ \cdots & & \cdots & \\ \gamma_{N1}\frac{\lambda_1}{\lambda N} & \gamma_{N2}\frac{\lambda_2}{\lambda_N} & \cdots & \gamma_{NN} \end{bmatrix},$$

and ϕ_K^d is a diagonal matrix with ϕ_K on its diagonal. The solution for Y is then given by

$$Y = [I - \delta\phi_K^d - M_r]^{-1}\lambda^{-\frac{1}{\sigma}}.$$

Solving for the remaining variables in the steady state is then straightforward.

6 Output from King and Watson (2002) programs

The policy functions take the form (with 2 sectors as an example):

$$\begin{bmatrix} c_{1t} \\ c_{2t} \\ k_{1t} \\ k_{2t} \end{bmatrix} = \begin{bmatrix} \cdots & & & \\ & & & \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_{1t} \\ k_{2t} \\ \delta_{1t} \\ \delta_{2t} \end{bmatrix},$$

$$\underbrace{\begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix}}_{x_t} = \begin{bmatrix} \cdots & & 1 & 0 \\ & & 0 & 1 \end{bmatrix} \begin{bmatrix} k_{1t} \\ k_{2t} \\ \delta_{1t} \\ \delta_{2t} \end{bmatrix},$$

and

$$\begin{bmatrix} k_{1t+1} \\ k_{2t+1} \\ \delta_{1t+1} \\ \delta_{2t+1} \end{bmatrix} = \begin{bmatrix} M_k & M_a \\ 0 & I \end{bmatrix} \begin{bmatrix} k_{1t} \\ k_{2t} \\ \delta_{1t} \\ \delta_{2t} \end{bmatrix} + H\varepsilon_t.$$

More generally, we can write these equations as

$$\begin{bmatrix} c_t \\ k_t \end{bmatrix} = \begin{bmatrix} \Pi_{ck} & \Pi_{ca} \\ I & 0 \end{bmatrix} \underbrace{\begin{bmatrix} k_t \\ \delta_t \end{bmatrix}}_{s_t},$$

$$x_t = \begin{bmatrix} & Q \end{bmatrix} s_t$$

and,

$$\begin{bmatrix} k_{t+1} \\ \delta_{t+1} \end{bmatrix} = \begin{bmatrix} M_k & M_a \\ 0 & I \end{bmatrix} \begin{bmatrix} k_t \\ \delta_t \end{bmatrix} + H\varepsilon_t.$$

7 Obtaining the Filtering Matrices

Since we assume that the logarithm of sectoral productivity follows a random walk, $Q = I$ in the procedure governing the driving process (i.e. drp.gss) of King and Watson (2002). Then, we have that

$$k_{t+1} = M_k k_t + M_a a_t$$

while

$$c_t = \Pi_{ck} k_t + \Pi_{ca} a_t.$$

Recall that

$$y_t = \alpha_d^{-1} a_t + k_t + \alpha_d^{-1} \Omega_{yc} c_t.$$

Therefore,

$$\begin{aligned} y_t &= \alpha_d^{-1} a_t + k_t + \alpha_d^{-1} \Omega_{yc} [\Pi_{ck} k_t + \Pi_{ca} a_t] \\ &= \underbrace{\alpha_d^{-1} [I + \Omega_{yc} \Pi_{ca}]}_{\Pi_a} a_t + \underbrace{[I + \alpha_d^{-1} \Omega_{yc} \Pi_{ck}]}_{\Pi_k} k_t \end{aligned}$$

so that

$$k_t = \Pi_k^{-1} y_t - \Pi_k^{-1} \Pi_a a_t.$$

Using these equations, we have that

$$\begin{aligned} y_{t+1} &= \Pi_k k_{t+1} + \Pi_a a_{t+1} \\ &= \Pi_k (M_k k_t + M_a a_t) + \Pi_a a_{t+1} \\ &= \Pi_k M_k (\Pi_k^{-1} y_t - \Pi_k^{-1} \Pi_a a_t) + \Pi_k M_a a_t + \Pi_a a_{t+1} \end{aligned}$$

or

$$y_{t+1} = \underbrace{\Pi_k M_k \Pi_k^{-1}}_{\varrho} y_t + \underbrace{\Pi_k (M_a - M_k \Pi_k^{-1} \Pi_a)}_{\Xi} a_t + \Pi_a a_{t+1}.$$

Under the assumptions made in the paper regarding the process for a_t, it follows that

$$\Delta y_{t+1} = \varrho \Delta y_t + \Xi \varepsilon_t + \Pi_a \varepsilon_{t+1},$$

so that the filtering is carried out according to

$$\varepsilon_{t+1} = \Pi_a^{-1}\Delta y_{t+1} - \Pi_a^{-1}\varrho\Delta y_t - \Pi_a^{-1}\Xi\varepsilon_t.$$

where ε_0 is set to zero.[1]

Let

$$\eta_{t+1} = \Xi\varepsilon_t + \Pi_a\varepsilon_{t+1},$$

Then, if $var(\varepsilon_t) = I$,

$$\Sigma_{\eta\eta} = \Xi\Xi' + \Pi_a\Pi_a'.$$

References

[1] Blanchard, O. and C. Kahn (1980), "The solution of linear difference models under rational expectations, Econometrica, 48:1305-1311.

[2] King, R. G., Plosser, C. I, and S. T. Rebelo (1988), "Production, growth, and business cycles, technical appendix," manuscript, University of Rochester.

[3] King, R. G., and M. W. Watson (2002), "System reduction and solution algorithms for singular linear difference systems under rational expectations," 20:57-86.

[4] Klein, P. (2000), "Using the generalized Schur form to solve a multivariate linear rational expectations model," Journal of Economic Dynamics and Control, 24:1405-1423.

[1] For the various calibrations presented in the text, the eigenvalues of $\Pi_a^{-1}\Xi$ have modulus less than one.