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Notes on Collateral Constraints in a Simple Model of Housing

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Abstract

These notes provide the derivations of results stated without proof in Hornstein (2009). For a simple model of the demand for housing, it is shown that on a balanced growth path, the rate at which the relative price of housing changes over time is determined by the relative productivity growth rates of the housing sector and the rest of the economy. The model is then modified to include a collateral constrained consumer. We show that collateral constraints may affect the level of the housing price path, but they do not affect the growth rate of housing prices.

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Section 1 describes a simple model of the housing market where land is an essential input to the production of houses. This model is based on work by Davis and Heathcote (2005) and it attributes endogenous changes in the price of housing to changes in the relative scarcity of land. In order to understand long run trends in house prices, the model's balanced growth path is characterized. In section 2 the environment is modified to model the demand for mortgage financed housing. This section is based on the Campbell and Hercowitz (2006) representation of collateral constraints. It is shown that collateral constraints do not affect the growth rates of the balanced growth path, but they may affect the level of the balanced growth path.

1. A simple model of housing

We describe a simple general equilibrium model of the demand for housing where the price of housing is endogenous. A representative consumer has preferences over the consumption of nondurable goods and housing services. Housing services are proportional to the stock of housing. New housing is produced by combining new residential structures, structures for short, with land. New structures, together with nondurable consumption goods, are produced from aggregate output. The rate of transformation between nondurable consumption goods and structures is exogenous and determines the relative prices of structures. If aggregate output is increasing faster than the supply of land or if structures become relatively more expensive, then the relative price of housing will increase over time. The rate at which the relative price of housing increases depends on the share of land in the production of new housing. The bigger (smaller) is the production share of land, the closer will be the growth rate of relative house prices to the growth rate of aggregate output (the price appreciation rate of structures).

1.1. The environment

Time is continuous and the horizon is infinite. A representative agent derives utility from the consumption of a nondurable good, c, and the consumption of housing services, h. The agent's preferences are

$$\int_0^\infty e^{-\rho_0 t} \left\{\theta \ln h_0 (t) + (1-\theta) \ln c_0 (t)\right\} dt \tag{1.1}$$

with time preference rate $\rho_0 > 0$ and $0 < \theta < 1$. The consumption of housing services is proportional to the stock of housing units owned by the agent. In the following we will use the terms housing services and housing stock interchangeably.

The agent receives an exogenous endowment stream of a homogeneous good. The value of the endowment in terms of the nondurable consumption good is y_0. In the following we will express all prices in terms of the nondurable consumption good. The agent also receives l units of new land and the price of new land is p_l. The agent can use his income for consumption, the purchase of new housing units, x_h, at the relative price p_h, or he can save it at an interest rate r. The flow budget constraint of the household is

$$\dot{a}_0(t) + c_0(t) + p_h(t)\,x_{h0}(t) = y_0(t) + p_l(t)\,l(t) + r(t)\,a_0(t) \tag{1.2}$$

where a is the agent's net-financial wealth.[1] Housing depreciates at rate $\delta > 0$ and the stock of housing accumulates according to

$$\dot{h}_0(t) = x_{h0}(t) - \delta h_0(t). \tag{1.3}$$

Note that old land embodied in the housing stock disappears with the depreciated structures.

The homogenous good can be used to produce the nondurable consumption good or it can be used to produce structures, x_s. The rate of transformation between nondurable consumption goods and structures is exogenous and determines the relative price of structures, p_s. The relative price of structures is the inverse of the relative productivity of the structures sector. The aggregate resource constraint for nondurable consumption and structures is

$$c(t) + p_s(t)\,x_s(t) = y(t). \tag{1.4}$$

Structures are combined with new land to produce new housing units using a Cobb-Douglas technology

$$x_h(t) = x_s(t)^{\beta}\,l(t)^{1-\beta}, \tag{1.5}$$

with $0 \leq \beta \leq 1$. The production of all goods is competitive.

The representative agent owns all of the endowment of the homogeneous output good. Market clearing for the output good, the nondurable consumption good, new housing, structures, and the credit market implies (1.4), (1.5), and

$$y(t) = y_0(t),\ c(t) = c_0(t),\ x_h(t) = x_{h0}(t),\ \text{and } 0 = a_0(t). \tag{1.6}$$

We assume that the economy is growing over time. In particular, the endowments y and l and the relative price p_s all grow at constant rates, γ_y, γ_l, and γ_s,

$$y(t) = \bar{y}e^{\gamma_y t},\ l(t) = \bar{l}e^{\gamma_l t},\ \text{and } p_s(t) = \bar{p}_s e^{\gamma_s t}. \tag{1.7}$$

[1]The notation $\dot{z}(t) = \partial z(t)/\partial t$ denotes the time derivative of the variable z as a function of time t, and the growth rate of the variable z is denoted $\hat{z}(t) = \dot{z}(t)/z(t)$.

In the following we will drop the time index when not needed.

1.2. Optimal consumption and production

A convenient way to state the dynamic optimization problem in continuous time involves the Hamiltonian function, see e.g. Kamien and Schwartz (1981). The Hamiltonian function serves a similar purpose as the Lagrange function for static constrained optimization problems. The Hamiltonian for the representative agent is defined as

$$\begin{aligned} H_0 &= \theta \ln h_0 + (1-\theta) \ln c_0 \\ &\quad + \lambda_0 \left(r a_0 + p_l l + y_0 - c_0 - p_h x_{h0} \right) + \mu_0 \left(x_{h0} - \delta h_0 \right), \end{aligned} \tag{1.8}$$

where the constraint multipliers λ and μ denote the marginal utility value of wealth and housing.

The first order conditions for the optimal allocation are

$$c_0 \;:\; \lambda_0 = (1-\theta)/c_0 \tag{1.9}$$

$$x_{h0} \;:\; p_h \lambda_0 = \mu_0 \tag{1.10}$$

$$a_0 \;:\; \dot{\lambda}_0 = \rho_0 \lambda_0 - \frac{\partial H_0}{\partial a_0} = (\rho_0 - r)\lambda_0 \tag{1.11}$$

$$h_0 \;:\; \dot{\mu}_0 = \rho_0 \mu_0 - \frac{\partial H_0}{\partial h_0} = (\rho_0 + \delta)\mu_0 - \theta/h_0 \tag{1.12}$$

We can rewrite the first order conditions for wealth (1.11) and housing stock (1.12) as capital value equations for the marginal utility of wealth and housing stock in utility terms

$$\rho_0 \lambda_0 = r \lambda_0 + \dot{\lambda}_0 \tag{1.13}$$

$$\rho_0 \mu_0 = \theta/h_0 - \delta \mu_0 + \dot{\mu}_0 \tag{1.14}$$

Equation (1.13) states that the required return on a unit of wealth is equal to the interest paid plus the capital gain. Equation (1.14) states that the required return on a unit of the housing stock is equal to the marginal utility from housing less depreciation and the capital gain. Equations (1.9) and (1.10) then state that for the marginal unit of nondurable consumption or new housing stock the marginal costs and benefits are equalized. The marginal cost is the good's price times the marginal utility of wealth, and the marginal benefit is the marginal utility for nondurable consumption goods or the capital value of a marginal unit of housing stock.

Another way to look at the optimality conditions for optimal consumption of housing and nondurable consumption goods starts with equation (1.12), rewritten as

$$\theta / h_0 = (\rho_0 + \delta)\, \mu_0 - \dot{\mu}_0. \tag{1.15}$$

This expression equates the marginal utility from housing with the user cost of housing in utility terms. The user cost of housing is the implicit rental rate paid for the use of housing stock. Given the utility price of housing μ_0, the user cost is the required return on housing plus depreciation minus capital gains due to changes in the capital value of the housing stock. Combining equation (1.15) with equations (1.9) and (1.10), we get that

$$\frac{\theta / h_0}{(1-\theta)/c_0} = \frac{(\rho_0 + \delta - \hat{\mu}_0)\, \mu_0}{\lambda_0} = (\rho_0 + \delta - \hat{\mu}_0)\, p_h. \tag{1.16}$$

Equation (1.16) states that for the optimal consumption allocation the marginal rate of substitution between the consumption of housing services and nondurable consumption goods is equal to their relative price where the relative price of housing services is equal to the user cost of housing.

Competitive production of new housing implies that for the two inputs, structures and new land, the value of an input's marginal product is equal to its price

$$\beta p_h x_h / x_s = p_s, \tag{1.17}$$
$$(1-\beta)\, p_h x_h / l = p_l. \tag{1.18}$$

1.3. A balanced growth path

On a balanced growth (BGP) all variables grow at constant, but potentially different, rates. We first characterize the growth rates of the BGP and in the next subsection the level of the BGP.

The resource constraint for the output good (1.4) implies that on the BGP nondurable consumption and the value of structures grow at the same rate as does output

$$\hat{c} = \gamma_s + \hat{x}_s = \gamma_y. \tag{1.19}$$

The production function for new housing, equation (1.5), implies that investment in new housing grows at a rate that is a weighted average of the growth rates of new structures and land,

$$\hat{x}_h = \beta \hat{x}_s + (1-\beta)\, \gamma_l. \tag{1.20}$$

The market clearing conditions (1.6) imply that the representative household's choice variables grow at the same rates as the corresponding aggregate variables

$$\hat{c}_0 = \hat{y}_0 = \gamma_y, \text{ and } \hat{x}_{h0} = \hat{x}_h. \tag{1.21}$$

The accumulation equation for the housing stock, (1.3), implies that the stock of housing grows at the same rate as does investment in new housing,

$$\hat{h} = \hat{x}_h \text{ and } \frac{x_h}{h} = \hat{h} + \delta. \tag{1.22}$$

The growth rates for the price of new housing and land are determined by the first order conditions for optimal input use in the production of new housing, equations (1.17) and (1.18),

$$\hat{p}_h + \hat{x}_h = \hat{p}_l + \gamma_l = \gamma_s + \hat{x}_s = \gamma_y. \tag{1.23}$$

We can now express the growth rates for the housing stock and the relative price of housing on the BGP as functions of the exogenous growth rates of output, the relative price of structures, and the supply of new land. The impact of a higher output growth rate on the rate at which relative house prices increase is immediate. Combining expressions (1.20) and (1.22) yields the housing accumulation rate,

$$\hat{h} = \beta\left(\gamma_y - \gamma_s\right) + (1-\beta)\,\gamma_l, \tag{1.24}$$

and combining expressions (1.20) and (1.23) yields the rate at which the relative price of housing changes,

$$\hat{p}_h = \beta\gamma_s + (1-\beta)\left(\gamma_y - \gamma_l\right). \tag{1.25}$$

On the one hand, a higher growth rate of aggregate output increases both the rate of home price appreciation and the rate of housing stock accumulation. On the other hand, if the relative price of structures increases at a faster rate or the rate at which new land becomes available declines, the relative price of housing increases at a faster rate, but the housing stock is accumulated at a slower rate.

The impact of changes in the exogenous growth rates on the house price appreciation rate depends on the share of land in the production of homes. If land is not an input to the production of homes, that is, $\beta = 1$, then home production is proportional to the use of structures. Thus home price appreciation is determined by the rate at which the relative price of structures changes and is independent of output growth and the availability of new land. On the other hand, if new homes are in fixed supply, that is, $\beta = 0$, then home price appreciation depends on the difference between the output growth rate and the land supply growth rate.

Finally, we obtain the growth rates of the marginal value of wealth and housing from equations (1.9) and (1.10), together with equations (1.21) and (1.25),

$$\hat{\lambda}_0 = -\gamma_y, \tag{1.26}$$
$$\hat{\mu}_0 = \beta\left(\gamma_s - \gamma_y\right) - (1-\beta)\,\gamma_l. \tag{1.27}$$

1.4. The BGP level

We normalize all variables such that they remain constant on the BGP. If the variable z grows at the rate $\hat{z}$ on the BGP, we define its normalized value as

$$\tilde{z}(t) = z(t)\, e^{-\hat{z}t}. \tag{1.28}$$

Essentially, the normalized value of a variable represents the level of its growth path. By construction the normalized variables do not change on the BGP, that is, $\dot{\tilde{z}} = 0$. For the exogenous shift terms this means that

$$\tilde{y} = \bar{y},\ \tilde{p}_s = \bar{p}_s, \text{ and } \tilde{l} = \bar{l}. \tag{1.29}$$

We now characterize the competitive equilibrium allocation on the BGP in terms of the normalized variables.

From equations (1.13) and (1.26) the equilibrium interest rate on the BGP is determined by the household's time discount factor and output growth

$$r = \rho_0 + \gamma_y. \tag{1.30}$$

Rearranging equation (1.16) and substituting for the housing stock from equation (1.22) we get that the value of housing investment is proportional to nondurable consumption

$$\tilde{p}_h \tilde{x}_{h0} = \frac{\theta}{1-\theta}\frac{\delta+\hat{h}}{\rho_0+\delta+\hat{h}}\tilde{c}_0 = A_0\tilde{c}_0. \tag{1.31}$$

We can use this expression, together with the resource constraint (1.4), in the condition for the optimal use of structures as an input (1.17) and obtain that nondurable consumption and the value of housing investment are proportional to output

$$\tilde{c}_0 = \frac{1}{1+\beta A_0}\tilde{y} = C_0\bar{y}, \tag{1.32}$$
$$\tilde{p}_h \tilde{x}_{h0} = A_0 C_0 \bar{y}. \tag{1.33}$$

Since the agent is representative this is also the aggregate value of housing investment. To separate out real housing investment and the price of housing we use the expression for nondurable consumption, (1.32), together with the resource constraint for the output good, (1.4), and the production function for new housing, (1.5), and we get

$$\tilde{x}_h = \left[\frac{(1-C_0)\,\bar{y}}{\bar{p}_s}\right]^{\beta} \bar{l}^{1-\beta} \tag{1.34}$$

$$\tilde{p}_h = \bar{p}_s^{\beta} \left[\frac{\bar{y}\,(1-C_0)}{\bar{l}}\right]^{1-\beta} /\beta \tag{1.35}$$

On the BGP the normalized home price and home investment depend on the normalized output, price of structures, and new land supply. On the one hand, a permanently higher level of output implies a higher level of home prices and home investment. On the other hand, a permanently higher price of structures or a permanently lower supply of new land implies a higher home price level and a lower home investment level. Through the equilibrium share of nondurable consumption the normalized home price and investment also depend on the rate of housing stock accumulation. If the BGP is such that the household accumulates housing stock at a faster rate, e.g., output grows at a faster rate or the price of structures grows at a slower rate, then the output share of nondurable consumption declines and the normalized home price (investment) level decreases (increases).

A special case obtains when the supply of new homes is exogenous, that is, $\beta = 0$. For this case the resource constraints simplify to $\tilde{c} = \bar{y}$ and $\tilde{x}_h = \bar{l}$, and we get the home price as

$$\tilde{p}_h = A_0 C_0 \frac{\bar{y}}{\bar{l}}. \tag{1.36}$$

2. Collateral constraints for housing

We now introduce another representative consumer into the economy. This new consumer is more impatient than the agent studied above so that at the equilibrium interest rate the impatient agent will borrow from the patient agent. Henceforth we will therefore distinguish between the lender, type 0 agent, and the borrower, type 1 agent. The impatient agent is constrained in his borrowings by the collateral value of his housing stock.

2.1. The borrower

The representative borrower has preferences over the consumption of nondurable goods and housing

$$\int_0^{\infty} e^{-\rho_1 t} \left\{\theta \ln h_1(t) + (1-\theta) \ln c_1(t)\right\} dt \tag{2.1}$$

with time preference rate $\rho_1 > \rho_0$. The borrower earns income y_1 and he can borrow or save at the interest rate r. The agent borrows (saves) if $a < 0$ ($a > 0$). The flow budget constraint of the borrower is

$$\dot{a}_1(t) + c_1(t) + p_h(t)\, x_{h1}(t) = y_1(t) + r(t)\, a_1(t). \tag{2.2}$$

Housing depreciates at rate $\delta > 0$ and the stock of durable goods accumulates according to

$$\dot{h}_1(t) = x_{h1}(t) - \delta h_1(t). \tag{2.3}$$

The amount of credit that a borrower can obtain is limited by the collateral value of the housing stock he owns. We assume that the required equity share of a borrower for a vintage τ home is

$$\omega(\tau) = 1 - (1-\pi)\, e^{-(\phi-\delta)\tau}, \tag{2.4}$$

with $\phi \geq \delta$. The down payment requirement for the purchase of new housing is $\omega(0) = \pi \in [0,1]$. The required equity share remains constant if $\phi = \delta$, and increases with the age of the vintage to one if $\phi > \delta$. Given the stock of undepreciated durable goods, the total equity requirement for a borrower is

$$\begin{aligned} \bar{\omega}(t) &= p_h(t) \int_0^\infty \omega(\tau) \left[e^{-\delta\tau} x_{h1}(t-\tau)\right] d\tau \\ &= p_h(t) \left[\int_0^\infty e^{-\delta\tau} x_{h1}(t-\tau)\, d\tau - (1-\pi) \int_0^\infty e^{-\phi\tau} x_{h1}(t-\tau)\, d\tau\right]. \end{aligned}$$

The first term in the square brackets represents the undepreciated past purchases of housing, that is, the stock of housing, h_1. The second term in the square brackets represents the part of the housing stock against which the agent can borrow, q_1. The implied accumulation equation for the collateralized housing stock is

$$\dot{q}_1(t) = x_{h1}(t) - \phi q_1(t). \tag{2.5}$$

In the following we will refer to the collateralized housing stock as the 'collateral stock.' We can rewrite the expression for the total equity requirement as

$$\bar{\omega}(t) = p_h(t)\left[h_1(t) - (1-\pi)\, q_1(t)\right]. \tag{2.6}$$

The collateral constraint then requires that the total equity value is at least as high as the required equity value

$$p_h(t)\, h_1(t) + a_1(t) \geq \bar{\omega}(t)$$

which simplifies to

$$(1-\pi)\, p_h(t)\, q_1(t) \geq -a_1(t)\,. \tag{2.7}$$

We assume that the representative borrower interacts with the representative lender from section 1 in a competitive equilibrium. Production of nondurable consumption goods, structures, and new homes continues to be determined by equations (1.4) and (1.5). We assume that the lender receives a fraction α of the endowment of the output good and the remainder goes to the borrower,

$$y_0(t) = \alpha y(t) \text{ and } y_1(t) = (1-\alpha)\, y(t)\,. \tag{2.8}$$

We also continue to assume that the lender receives all of the endowment of new land. Market clearing for the nondurable consumption goods and new housing now implies that

$$c(t) = c_0(t) + c_1(t)\,,\ x_h(t) = x_{h0}(t) + x_{h1}(t)\,,\ 0 = a_0(t) + a_1(t)\,. \tag{2.9}$$

2.2. Optimal Consumption of the borrower

The borrower is assumed to maximize utility (2.1) subject to the budget constraint (2.2), the accumulation equations for the housing and collateral stock, (2.3) and (2.5), and the collateral constraint (2.7).

The Hamiltonian for the borrower's optimization problem is defined as

$$\begin{aligned} H_1 &= \theta \ln h_1 + (1-\theta)\ln c_1 \\ &\quad + \lambda_1 (r a_1 + y_1 - c_1 - p_h x_{h1}) + \mu_1 (x_{h1} - \delta h_1) \\ &\quad + \varphi_1 (x_{h1} - \phi q_1) + \kappa_1 [a_1 + (1-\pi)\, p_h q_1]\,, \end{aligned} \tag{2.10}$$

where the multiplier φ denotes the marginal utility value of the collateral stock, and κ denotes the marginal value of loosening the collateral constraint.

The first order conditions for optimal consumption of the borrower are analogous to the first order conditions of the lender with appropriate modifications for the collateral constraint,

$$c_1 \ :\ \lambda_1 = (1-\theta)/c_1 \tag{2.11}$$

$$x_{h1} \ :\ p_h \lambda_1 = \mu_1 + \varphi_1 \tag{2.12}$$

$$a_1 \ :\ \dot{\lambda}_1 = (\rho_1 - r)\,\lambda_1 - \kappa_1 \tag{2.13}$$

$$q_1 \ :\ \dot{\varphi}_1 = (\rho_1 + \phi)\,\varphi_1 - \kappa_1 (1-\pi)\, p_h \tag{2.14}$$

$$h_1 \ :\ \dot{\mu}_1 = (\rho_1 + \delta)\,\mu_1 - \theta/h_1. \tag{2.15}$$

For now we simply assume that the collateral constraint is binding, that is, equation (2.7) holds as an equality. Below we will show that this is the case on the BGP.

We can rewrite the first order conditions for wealth, collateral stock, and housing stock as capital value equations for the marginal utility of wealth, collateral, and housing,

$$\rho_1 \lambda_1 = r\lambda_1 + \kappa_1 + \dot{\lambda}_1 \tag{2.16}$$

$$\rho_1 \varphi_1 = \kappa_1 (1-\pi) p_h - \phi \varphi_1 + \dot{\varphi}_1 \tag{2.17}$$

$$\rho_1 \mu_1 = \theta / h_1 - \delta \mu_1 + \dot{\mu}_1. \tag{2.18}$$

Equation (2.16) states that the flow benefit of one more unit of wealth includes the value of a marginal relaxation of the collateral constraint, in addition to the usual capital gains and interest paid on wealth, see equation (1.13). Equation (2.17) defines the flow return on the capital value of collateral as the marginal relaxation of the collateral constraint from an additional unit of collateral, net of the depreciation and capital gains on collateral. Finally, the borrower's capital value equation for housing, (2.18), is the same as the corresponding equation (1.14) for the lender.

Equation (2.12) equates the marginal cost and marginal benefit (in utility terms) of an additional unit of housing stock. The marginal benefit now has two components. Besides the capital value of an additional unit of housing it also includes the capital value of an additional unit of collateral since having more housing allows the agent to borrow more. The borrower also equates the marginal rate of substitution between the consumption of housing services and nondurable consumption goods with the relative price of housing services,

$$\frac{\theta / h_1}{(1-\theta)/c_1} = (\rho_1 + \delta - \hat{\mu}_1)\left(p_h - \frac{\varphi_1}{\lambda_1}\right). \tag{2.19}$$

Relative to the analogous equation (1.16) for the lender, the user cost of housing is reduced because housing provides a collateral value for the borrower.

2.3. A BGP

The growth rates of aggregate variables on the BGP are determined as before by equations (1.19) and (1.20) since the aggregate resource constraints have not changed. From the definition of market clearing (2.9) it follows that on the BGP consumption of nondurable goods and housing, wealth, etc., for borrowers and lenders grow at the same rates

$$\hat{c}_i = \hat{a}_i = \gamma_y \text{ and } \hat{x}_{hi} = \hat{h}_i = \hat{q}_1 = \hat{h}, \text{ for } i = 0, 1, \tag{2.20}$$

and we normalize all variables as described by equation (1.28).

The interest rate on the BGP continues to be determined by the lender's time discount rate and the output growth rate, equation (1.30). One can show that on the BGP the collateral constraint is binding for the borrower since the borrower's marginal utility of wealth is positive and he is more impatient than the lender. From equation (2.11) it follows that on any BGP with positive consumption for the borrower, λ_1 is positive. Equation (2.13), together with the BGP value of the interest rate, equation (1.30), and the assumption that the borrower is more impatient than the lender then implies that the collateral constraint multiplier, κ_1, is positive, that is, the collateral constraint is binding.

On the BGP the borrower's value of housing investment is proportional to his nondurable consumption,

$$\tilde{p}_{h1}\tilde{x}_{h1} = \frac{\theta}{1-\theta}\frac{\delta+\hat{h}}{\delta+\hat{h}+\rho_1}\left[1-(1-\pi)\frac{\rho_1-\rho_0}{\rho_1+\phi+\hat{h}}\right]^{-1} = A_1\tilde{c}_1. \tag{2.21}$$

We obtain equation (2.21) by substituting for the BGP values of the capital values of wealth and collateral in (2.19). Combining the BGP expressions for the housing stock and the collateral stock yields the collateral stock proportional to the housing stock,

$$\tilde{q}_1 = \frac{\delta+\hat{h}}{\phi+\hat{h}}\tilde{h}_1. \tag{2.22}$$

Substituting this expression in the collateral constraint, (2.7), then yields total borrowing proportional to the borrower's home investment value

$$\tilde{a}_1 = -\frac{1-\pi}{\phi+\hat{h}}\tilde{p}_h\tilde{x}_{h1} = -B_1\tilde{p}_h\tilde{x}_{h1}. \tag{2.23}$$

Finally, substituting expressions (2.21) and (2.23) into the borrower's budget constraint, (2.2) we obtain the borrower's consumption of nondurable goods as a fraction of his flow income,

$$\tilde{c}_1 = \left[1+A_1\left(1+\rho_0 B_1\right)\right]^{-1}\tilde{y}_1 = C_1\bar{y}_1. \tag{2.24}$$

So far we have been able to characterize the optimal consumption allocation for the borrower, without any recourse to the decisions of the lender, except for the use of the equilibrium interest rate (1.30). In order to derive the BGP allocation for the lender and the equilibrium prices, we now use the lender's optimal decision rule for investment expenditures, (1.31), together with the lender's budget constraint and the market clearing conditions. After some more algebra we obtain the lender's consumption of nondurable goods as a share of aggregate output

$$\tilde{c}_0 = \frac{1-\alpha+\alpha A_1 C_1\left[1-\beta+\rho_0 B_1\right]}{1+\beta A_0}\bar{y} = C_0\bar{y}. \tag{2.25}$$

Aggregate expenditures on nondurable goods and new housing investment then are

$$\tilde{c} = \left(C_0+\alpha C_1\right)\bar{y} = C\bar{y} \tag{2.26}$$

$$\tilde{p}_h\tilde{x}_h = \left[A_0 C_0+\alpha A_1 C_1\right]\bar{y} = D\bar{y}. \tag{2.27}$$

We obtain investment and the price of new housing from the production function for new housing together with the resource constraint for the output good and the demand for nondurable consumption (2.26)

$$\tilde{x}_h = \left[\left(1-C\right)\bar{y}/\tilde{p}_s\right]^{\beta}\bar{l}^{1-\beta}, \tag{2.28}$$

$$\tilde{p}_h = \left(\frac{\bar{y}}{\bar{l}}\right)^{1-\beta}\left(\frac{\tilde{p}_s}{1-C}\right)^{\beta} D. \tag{2.29}$$

References

[1] Campbell, Jeffrey R., and Zvi Hercowitz (2006). The role of collateralized household debt in macroeconomic stabilization. Federal Reserve Bank of Chicago WP 2004-24.

[2] Davis, Morris A. and Jonathan Heathcote. 2005. Housing and the business cycle. *International Economic Review* 46, 751-784.

[3] Hornstein, Andreas. 2009. Problems for a Fundamental Theory of House Prices. Federal Reserve Bank of Richmond Economic Quarterly.

[4] Kamien, Morton I., and Nancy L. Schwartz. 1981. Dynamic Optimization: the calculus of variations and optimal control. Elsevier: New York-Amsterdam.