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WP 17-12

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Abstract

We study global and local dynamics of a simple search and matching model of the labor market. We show that the model can be locally indeterminate or have no equilibrium at all, but only for parameterizations that are empirically implausible. In contrast to the local results, we show that the model exhibits chaotic and periodic dynamics for reasonable parameter values both in backward and forward time. In contrast to earlier work, we establish these results analytically without placing numerical restrictions on the parameters.

JEL Classification: C62, C65, E24, J64

Key Words: Indeterminacy, Bifurcation, Chaos, Backward Map, Forward Map

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1 Introduction

The search and matching model of the labor market has proved to be a convenient framework for studying the joint behavior of unemployment and job vacancies. Much of the qualitative and quantitative analysis in this framework relies on linear approximations and local solutions of fundamentally nonlinear environments, as does most of the dynamic literature in macroeconomics. Yet, researchers have increasingly come to realize that the global dynamics of such frameworks can have quite different implications than those derived from local counterparts. In particular, nonlinear dynamics can be periodic or chaotic, which a linear approach cannot capture. Moreover, a purely linear approach may rule out steady state equilibria as unstable concluding explosive dynamics, and miss on cyclical equilibria or stable dynamics elsewhere in the economic domain. Without a full characterization of the nature of the processes that generate economic data, any conclusions drawn based on a local approach might be misleading (see, for example, Wolman and Couper, 2003).

In this paper, we therefore study the global and local dynamics of the simple search and matching model. Specifically, we add to the literature by showing analytically that the model exhibits periodic and chaotic dynamics for a wide range of plausible parameterizations that have been used in the quantitative literature. We employ analytic proofs that are derived without placing numerical restrictions on the parameters. We are aided in this effort by the specific structure of the search-and-matching framework, which can be reduced to a recursive two-variable system. The model is thereby amenable to analytical characterization of its local and global properties. The key dynamics arise from the model's job-creation condition, whereby we show that this is the case both in backward and forward time. For the backward dynamics, we derive a mapping that can be easily analyzed after introducing a variable change. This mapping ensures that the evolution of labor market dynamics is both economically meaningful in the sense that trajectories of the model's variables are well defined, and that it is consistent with the model's job-creation condition in terms of uniqueness of the steady state. This differentiates our work from prior analysis of this model, which uses a map that can have more than one steady state for certain parameter values and may not always be defined on its domain.

Our paper contributes to the literature along two dimensions. First, the phenomenon of chaotic dynamics in economic models is interesting on its own. However, most work has focused on the Real Business Cycle (RBC) model or variants of the New Keynesian model, whereby study of the global dynamics of the search-and-matching model is scant. Second, our paper emphasizes the importance of considering global dynamics more broadly, especially since there is a growing awareness that reliance on local dynamics can be misleading. For example, as is the case with our model, it can be shown that under certain conditions the steady state may be a repeller, and it may therefore be tempting to conclude the model exhibits explosive dynamics. We show that this is not always the case - the loss of stability in the steady state coincides with emergence of cyclical behavior.

Our work builds on papers by Medio and Raines (2007) and Mendes and Mendes (2008).[1] The former authors study backward dynamics in general economic models and provide the general template for our analysis. Specifically, they develop general conditions under which periodic and chaotic dynamics can arise in a wide class of economic models that can be conveniently characterized by classes of mappings. Mendes and Mendes (2008) apply some of these insights to a labor market framework that is similar to ours. They show that the backward dynamics in the search and matching model can undergo a period-doubling bifurcation that leads to chaos. However, this result is established under strict restrictions on parameter values. Moreover, they show period doubling and existence of periodic points of period 3 and 5 only numerically. Our work improves upon theirs by establishing existence of periodic and chaotic solutions in the model analytically, which allows us to extend the range of acceptable parameter values under which cycles and chaos can occur. From a technical perspective, Mendes and Mendes (2008) use symbolic dynamics and inverse limit theory to establish cycles and chaos going forward in time, when backward dynamics exhibits similar behavior. Using the result established by Kennedy and Stockman (2008), we establish chaotic and periodic solutions in forward time more generally, without imposing numerical restrictions on parameter values.[2]

This paper is also close in spirit to recent contributions that analyze global dynamics in Real Business Cycle models, such as Coury and Wen (2009), Growiec et al. (2015), and Sorger (2016). In a RBC model with production externalities Coury and Wen (2009) show that the unique steady state is surrounded by stable deterministic cycles, which implies global indeterminacy that is not apparent from a local analysis. Their paper is similar to

[1] There is also earlier work by Bhattacharya and Bunzel (2003a,b), who study global dynamics in a search and matching framework but impose parametric restrictions and only consider the social-planner solution of the model. They establish the potential for n-period cycles in the model, but the modeling restrictions have been criticized by Shimer (2004). Our paper can be seen as contribution that unifies and clarifies these previous results.

[2] Another paper that studies global dynamics in a search-and-matching setting in labor and capital markets is Ernst and Semmler (2010). Their model has multiple steady states, one of which is a local attractor while another is saddle-path stable. Their analysis is fully numerical based on value-function iteration, whereas we solve the nonlinear equilibrium conditions that emerge from the first-order conditions.

ours in that we also work in a two-equation environment that is amenable to straightforward, and intuitive, analytical and numerical analysis. As in our paper, they find that indeterminacy is more pervasive than previously believed. Sorger (2016) extends this analysis to show that under standard monotonicity and convexity assumptions on technology and preferences the basic RBC model can have periodic solutions of any period as well as chaotic solutions. However, this does not arise for typical parameterizations that are employed in the literature. In contrast, we show that in the search-and-matching framework chaotic dynamics arise even under standard parameterization. Finally, Growiec at al. (2015) show that in an extended version of the RBC model limit cycles can explain the empirical evidence on substantial medium-to-long run, pro-cyclical swings in the labor share in the U.S.

The paper is structured as follows. In the next section, we describe the simple search and matching model of the labor market and derive the equilibrium conditions used in the local and global analysis. We also discuss our calibration approach as it pertains to the interpretation of our results. Section 3 presents the local determinacy properties of the model. In Section 4, we turn to global dynamics. This section is the central part of the paper, where we study the various global equilibria that arise in different regions of the parameter space and contrast our findings with those from the local analysis. The final section concludes. Relevant mathematical concepts and proofs are presented in the Appendix.

2 A Simple Search and Matching Model of the Labor Market

We develop a simple version of the search and matching model of the labor market. The model has become the workhorse framework for studying unemployment and vacancy dynamics and employment flows more generally, especially since Shimer's (2005) seminal contribution. The exposition below follows closely Krause and Lubik (2010), to which we refer for further details.

2.1 The Model

We assume time is discrete and the model period is one quarter. A continuum of identical firms employs workers who inelastically supply one unit of labor.[3] Output y of a typical

[3]For expositional convenience, we present the problem of a representative firm only. We abstract from indexing the individual variables.

firm is linear in employment n:

$$y_t = A_t n_t, \tag{1}$$

where A_t is an exogenous aggregate productivity process to be specified later.

Matching between workers and firms is captured by the function $m(u_t, v_t) = m u_t^{\xi} v_t^{1-\xi}$, with unemployment u, vacancies v, and parameters $m > 0$ and $0 < \xi < 1$. m is the match efficiency and measures the effectiveness of the matching process, while ξ is the match elasticity. The matching function describes the number of newly formed employment relationships that arise from the contacts between unemployed workers and firms seeking to fill open positions. Unemployment is defined as:

$$u_t = 1 - n_t, \tag{2}$$

which is the measure of all potential workers in the economy who are not employed at the beginning of the period and are thus available for job search activities.

We can write the law of motion for employment as follows:

$$n_t = (1-\rho)[n_{t-1} + m(u_{t-1}, v_{t-1})], \tag{3}$$

where new hires add to the existing stock of workers. The end-of-the-period workforce is subject to separation at the rate $0 < \rho < 1$.[4] We define $q(\theta_t)$ as the probability of filling a vacancy, or the firm-matching rate, where $\theta_t = v_t/u_t$ is labor market tightness. In terms of the matching function, we can write this as $q(\theta_t) = m(u_t, v_t)/v_t = m\theta_t^{-\xi}$. Similarly, the job-finding rate is $p(\theta_t) = m(u_t, v_t)/u_t = m\theta_t^{1-\xi}$. An individual firm is atomistic in the sense that it takes the aggregate matching rate $q(\theta_t)$ as given. The employment constraint on the firm's decision problem is therefore linear in vacancy postings:

$$n_t = (1-\rho)[n_{t-1} + v_{t-1} q(\theta_{t-1})]. \tag{4}$$

Firms maximize profits, using the discount factor $\beta^t \frac{\lambda_t}{\lambda_0}$ (to be determined below):

$$\max_{\{v_t, n_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t \frac{\lambda_t}{\lambda_0} \left[A_t n_t - w_t n_t - \kappa v_t\right] + \tag{5}$$
$$+ \sum_{t=0}^{\infty} \beta^t \frac{\lambda_t}{\lambda_0} \mu_t \left[(1-\rho)[n_{t-1} + v_{t-1} q(\theta_{t-1})] - n_t\right].$$

Wages paid to the workers are w, while $\kappa > 0$ is a firm's cost of posting a vacancy. μ is the Lagrange-multiplier on the firm's employment constraint. It can be interpreted as the

[4]Note that newly matched workers who are separated from their job within the period reenter the matching pool immediately.

marginal value of a filled position. Firms decide how many vacancies to post and how many workers to hire. The first-order conditions are:

$$n_t \quad : \qquad \mu_t = A_t - w_t + \beta(1-\rho)\frac{\lambda_{t+1}}{\lambda_t}\mu_{t+1}, \tag{6}$$

$$v_t \quad : \qquad \kappa = \beta(1-\rho)\frac{\lambda_{t+1}}{\lambda_t}\mu_{t+1}q(\theta_t), \tag{7}$$

which imply the job-creation condition (JCC):

$$\frac{\kappa}{q(\theta_t)} = (1-\rho)\beta\left(\frac{\lambda_{t+1}}{\lambda_t}\right)\left[A_t - w_{t+1} + \frac{\kappa}{q(\theta_{t+1})}\right]. \tag{8}$$

This optimality condition trades off expected hiring cost $\kappa/q(\theta_t)$ against the benefits of a productive match. This consists of the output accruing to the firm net of wage payments and the future savings on hiring costs when the current match is successful.

As is common in the literature, we assume the economy is populated by a representative household. The household is composed of workers, who are either unemployed or employed. If they are unemployed, they are compelled to search for a job, but they can draw unemployment benefits b. Employed members of the household receive pay w, but share this with the unemployed. They do not suffer disutility from working and supply a fixed number of hours.[5] Since the household's only choice variable is consumption, and since there is no mechanism to transfer resources intertemporally, the utility maximization problem is trivial. Assuming constant relative risk aversion, this determines the marginal utility of wealth, $\lambda_t = C_t^{-\sigma}$, where C is consumption and σ^{-1} is the intertemporal elasticity of substitution. In equilibrium, total income accruing to the household equals net output in the economy, which is composed of production less real resources lost in the search process, $Y_t = y_t - \kappa v_t$. Since $C_t = Y_t$, we can now derive the stochastic discount factor $\beta^t \frac{\lambda_t}{\lambda_0} = \beta^t \frac{Y_t^{-\sigma}}{Y_0^{-\sigma}}$.

Finally, we need to derive how wages are determined. We assume that wages are set according to the Nash bargaining solution.[6] As this is a lengthy, but standard, derivation, we refer to Krause and Lubik (2010) for further exposition. The Nash-bargained wage is thus:

$$w_t = \eta\left(A_t + \kappa\theta_t\right) + (1-\eta)b. \tag{9}$$

This equation can be substituted into the JCC to derive:

$$\frac{\kappa}{m}\theta_t^{\xi} = \beta(1-\rho)\frac{Y_t^{\sigma}}{Y_{t+1}^{\sigma}}\left[(1-\eta)\left(A_t - b\right) - \eta\kappa\theta_{t+1} + \frac{\kappa}{m}\theta_{t+1}^{\xi}\right]. \tag{10}$$

This completes the description of the model.

[5]We thus assume income pooling between employed and unemployed households and abstract from potential incentive problems concerning labor market search. This allows us to treat the labor market separate from the consumption choice. See Merz (1995) and Andolfatto (1996) for a discussion of these issues.

[6]This is a standard assumption in the literature. Shimer (2005) provides further discussion.

2.2 Steady State

We first establish that the model has a unique steady state. Steady state θ_{SS} solves the following nonlinear equation:

$$\theta_{SS}^{\xi} - \beta(1-\rho)\theta_{SS}^{\xi} = \beta(1-\rho)(1-\eta)m\frac{A-b}{\kappa} - \beta(1-\rho)\eta m\theta_{SS}, \tag{11}$$

which is derived from the JCC (10) after rearranging terms. We now prove the following Lemma.

Lemma 1 *The job-creation condition has a unique steady state θ_{SS}.*

Proof. Consider the lefthand side and the righthand side of the above equation separately. The lefthand side $f_1 = [1-\beta(1-\rho)]\,\theta_{SS}^{\xi}$ has an intercept at the origin and is strictly increasing in θ_{SS} since $1-\beta(1-\rho) > 0$. The righthand side $f_2 = \beta(1-\rho)(1-\eta)m\frac{A-b}{\kappa} - \beta(1-\rho)\eta m\theta_{SS}$ is linear in θ_{SS} and strictly decreasing with a positive intercept. It therefore follows that the two functions intersect once and that there is a unique steady state θ_{SS}. ■

We thus find that the simple search and matching model does not suffer from the multiple steady-state problem identified, for example, by Benhabib et al. (2001) in a monetary model with an interest rate feedback rule for monetary policy. They show that the interaction of the Fisher-equation, that is, the relationship between nominal and real interest rates and expected inflation, with an ad-hoc policy rule results in the existence of two steady states, one stable and one that is unstable globally. The key finding is that the globally unstable steady state is locally saddle-path stable and is actually the one that is imposed in linearized analyses. Benhabib et al. (2001) therefore argue that policy recommendations based on local analysis can be perilous in the global context (see also Wolman and Couper, 2003, for further discussion). This is not an issue in our model. Instead, our focus of investigation is whether the unique steady state is locally and globally stable or unstable and whether there are chaotic endogenous dynamics.

The remaining steady-state values can be computed in a straightforward manner. The steady-state unemployment rate u_{SS} can be computed from the law of motion for employment; that is, $\frac{\rho}{1-\rho}\frac{1-u_{SS}}{u_{SS}} = m\theta_{SS}^{1-\xi}$. The rest of the variables then follow immediately. Given the specific type of nonlinearity of the steady-state JCC, there is no closed-form solution for θ_{SS}. Instead, we have to compute the value of θ_{SS} numerically for a given parameterization. This makes it more burdensome to study the dynamic properties of the model since they have to be evaluated for each new set of parameter values. An alternative to computing the steady-state values directly is to target specific steady-state values and thereby impute the implied values of fixed parameters. If done judiciously, this would help avoid issues with nonlinearity when solving for the steady state. We describe such an approach in the next section when we discuss calibration.

2.3 Calibration

We now describe our choice for parameter values that we use in the numerical analysis of the local and global properties. In our analysis, we assume, as in Shimer (2005), that households are risk-neutral, that is, $\sigma = 0$. This simplifies derivation of analytical results considerably and in fact makes it possible to obtain analytical results for global dynamics (see also Bhattacharya and Bunzel, 2003a). We pursue two different strategies to assign numerical values to the structural model parameters. One strategy sets the parameter values directly. The advantage is that we can directly determine the impact of any parameter changes on the behavior of the model. A drawback is that there is not much independent information available for some of the parameters, and certain parameter choices can lead to a priori implausible steady-state values. We can address this issue by imposing plausible bounds on parameter values. Nevertheless, the steady state of the model is given by a unique mapping from the structural parameters to the endogenous variables, as Lemma 1 shows. However, the mapping from the parameters to objects of interests may not admit an analytical solution, of which one example is the computation of the steady state. Our alternative strategy treats endogenous variables as parameters to be calibrated. In order to obtain a specific target value, a parameter thus needs to be adjusted endogenously. The advantage of this approach is that a judicious choice of setting steady-state values can allow for analytical solutions.

We set the discount factor $\beta = 0.99$ and normalize the productivity level $A = 1$. The separation rate $\rho \in (0, 1)$. A typical value for quarterly data is $\rho = 0.1$, which is consistent with the evidence reported in Shimer (2005). The bargaining parameter $\eta \in (0, 1)$. The vast majority of the literature assumes $\eta = 0.5$, as independent observations on its value are not obvious to obtain. The match elasticity $\xi \in (0, 1)$. In a well-known study, Petrongolo and Pissarides (2001) advocate for values between 0.5 and 0.7. The plausibility of this range is supported by the evidence in Lubik (2013). However, values outside this range can be considered as well. The level parameter in the matching function $m > 0$ can be used to scale, for instance, the unemployment rate, but it is otherwise left unrestricted in the literature. However, we restrict this parameter to obey $m \in (0, 1)$ based on the following argument.

The job matching and job finding rates are defined as, respectively, $q(\theta) = m\theta^{-\xi}$ and $p(\theta) = m\theta^{1-\xi}$. These should properly be interpreted as the probabilities of a firm filling a vacancy and a worker finding a job. It is a quirk of the discrete-time matching model that mathematically these variables can take on values above one. Intuitively, at a low enough frequency, everyone in the pool of searchers will transition out of unemployment at least once, which translates into a job finding rate of above one. While this is conceptually valid - the rate counts the number of new matches per searchers over a long enough period - it violates the spirit of the search and matching model in that successful matching is probabilistic. We note that this is not a problem for the continuous time version of the search and matching model since $q(\theta)$ and $p(\theta)$ are instantaneous transition rates and thus are true probabilities. In what follows, we therefore restrict these rates to lie on the unit interval (see also Bhattacharya and Bunzel, 2003b, and Shimer, 2004). The following Lemma establishes the necessary parametric restriction.

Lemma 2 *The transition rates $q(\theta)$ and $p(\theta)$ are less than one if $m < 1$.*

Proof. Define $q(\theta) = m\theta^{-\xi}$ and $p(\theta) = m\theta^{1-\xi}$. $q(\theta) < 1$ implies $\theta > \left(\frac{1}{m}\right)^{-1/\zeta}$; $p(\theta) < 1$ implies $\theta < \left(\frac{1}{m}\right)^{1/(1-\xi)}$. For both transition rates to be less than one, this requires: $\left(\frac{1}{m}\right)^{-1/\zeta} < \theta < \left(\frac{1}{m}\right)^{1/(1-\xi)}$. This is a nonempty interval for θ if $m < 1$. ■

As for the remaining parameters, benefits $b \in (0, A)$ since they cannot exceed the marginal product of the firm, in which case a firm could not offer any wage that would induce an unemployed person to work. Given our normalization $A = 1$, this restricts b to the unit interval. Typical values in the literature range from $b = 0.4$ (Shimer, 2005) to $b = 0.9$ (Hagedorn and Manovskii, 2008). Vacancy posting cost $\kappa > 0$. It is a scale variable that can be measured in terms of resource loss as a percentage of GDP. Typical values are in the low percentage points when measured relative to output.

We also consider calibrating the steady-state unemployment rate u_{SS} directly. Using the law of motion for employment (3), we can then compute $\theta_{SS} = \left(\frac{1}{m}\frac{\rho}{1-\rho}\frac{1-u_{SS}}{u_{SS}}\right)^{1/(1-\xi)}$ without having to solve the nonlinear equation (11). Similarly, we can fix the job finding rate $p_{SS} = p(\theta_{SS})$, which implies $\theta_{SS} = (p_{SS}/m)^{1/(1-\xi)}$. The JCC then delivers the following restriction on the imputed parameter: $\frac{A-b}{\kappa} = \frac{\eta}{1-\eta}\theta_{SS} + \frac{1}{1-\eta}\frac{1-\beta(1-\rho)}{\beta(1-\rho)}\frac{\theta_{SS}^{\xi}}{m}$, from which we can obtain either b, κ, or even η. We note that in this expression b and κ are not separately identifiable. However, the term $\frac{A-b}{\kappa}$ scales various expressions, and we will discuss its importance for global analysis further below. In terms of numerical values assigned to the steady-state values, u_{SS} can be chosen to correspond to observed sample means, which typically is around 5%. An alternative approach is to target the observed employment rates, which would imply an unemployment rate that is much higher, for instance, 25%. Both approaches have been used in the literature, with different implications for the dynamic behavior of the calibrated model.[7]

3 Local Dynamics

The local dynamics of the simple search and matching model have been studied by Krause and Lubik (2010). We replicate their analysis here as a reference point for our study of global dynamics. As discussed above we consider the case $\sigma = 0$ so that we can obtain an analytical characterization. The job creation condition then becomes:

$$\frac{\kappa}{m}\theta_t^{\xi} = \beta(1-\rho)\left[(1-\eta)(A-b) - \eta\kappa\theta_{t+1} + \frac{\kappa}{m}\theta_{t+1}^{\xi}\right]. \tag{12}$$

We linearize this equation around the steady state, which results in:

$$\widehat{\theta}_t = \beta(1-\rho)\left(1 - \frac{\eta}{\xi}m\theta_{SS}^{1-\xi}\right)\widehat{\theta}_{t+1}, \tag{13}$$

where $\widehat{\theta}_t = \theta_t - \theta_{SS}$ is the deviation from the steady state θ_{SS}.[8] This is an autonomous first-order linear difference equation in θ, the dynamic properties of which depend on the coefficient $\beta(1-\rho)\left[1 - \frac{\eta}{\xi}m\theta_{SS}^{1-\xi}\right]$. Since this is a forward-looking equation, a unique and determinate equilibrium requires that the eigenvalue lies within the unit circle (see Blanchard and Kahn, 1980). More formally, we establish the following Theorem.

Theorem 3 *The equilibrium dynamics of the job-creation condition are locally unique if:*

$$0 < p(\theta_{SS}) < \frac{1+\beta(1-\rho)}{\beta(1-\rho)}\frac{\xi}{\eta}.$$

The equilibrium dynamics are locally indeterminate if:

$$1 > p(\theta_{SS}) > \frac{1+\beta(1-\rho)}{\beta(1-\rho)}\frac{\xi}{\eta}$$

Proof. The equilibrium is locally unique if $\left|\beta(1-\rho)\left(1 - \frac{\eta}{\xi}m\theta_{SS}^{1-\xi}\right)\right| < 1$. Consider the boundaries in turn. Denote $p(\theta_{SS}) = m\theta_{SS}^{1-\xi}$. $\beta(1-\rho)\left(1 - \frac{\eta}{\xi}p(\theta_{SS})\right) < 1$ implies $p(\theta_{SS}) >$

[7] The idea is to capture both measured unemployment in terms of recipients of unemployment benefits and potential job searchers that are only marginally attached to the labor force but are open to job search. Since we do not model labor force participation decisions, this is a shortcut to capturing effective labor market search. This approach has been taken by Cooley and Quadrini (1999) and Trigari (2009).

[8] *Log*-linearizing this equation around the steady state would result in the same dynamic properties.

$0 > -\frac{\xi}{\eta}\frac{1-\beta(1-\rho)}{\beta(1-\rho)}$, which is always true. Second, $-1 < \beta(1-\rho)\left(1-\frac{\eta}{\xi}p(\theta_{SS})\right)$ implies $p(\theta_{SS}) < \frac{\xi}{\eta}\frac{1+\beta(1-\rho)}{\beta(1-\rho)}$. Since $0 < p(\theta_{SS})$, this proves the first part of the theorem. The equilibrium is indeterminate if $\left|\beta(1-\rho)\left(1-\frac{\eta}{\xi}m\theta_{SS}^{1-\xi}\right)\right| > 1$ ■

The theorem is stated in terms of boundary conditions for the job finding rate $p(\theta_{SS})$. We find this useful for developing intuition as to the determinants of local and global dynamics in the model. Moreover, as we pointed out above, the search and matching model is in practice often calibrated in terms of target values for job matching and job finding rates. Theorem 3 establishes that for a wide range of parameter values the dynamic equilibrium is locally unique. Since the forward dynamics, obtained by iterating the linearized JCC forward, are stable, the unique equilibrium is $\widehat{\theta}_t = 0$, which puts labor market tightness at its steady-state value as the only possible equilibrium. The steady state θ_{SS} is locally unstable in this case. As time goes forward, the path for tightness will become unbounded unless the economy is placed on the initial condition $\theta_0 = \theta_{SS}$. Starting values in a small neighborhood of the steady state will lead to explosive paths.

An alternative way to see this is by inverting the linearized JCC. This implies the backward-looking representation $\widehat{\theta}_{t+1} = \left[\beta(1-\rho)\left(1-\frac{\eta}{\xi}m\theta_{SS}^{1-\xi}\right)\right]^{-1}\widehat{\theta}_t$. The root of this representation is the inverse of the root of the forward equation. Forward stability therefore implies backward instability, and vice versa. Given the parametric restrictions established in Theorem 3, the JCC would have explosive dynamics if expressed backward. Consequently, the only solution to be consistent with local stability is $\widehat{\theta}_t = 0$. One important insight is that in the linear case the roots of the forward and the backward representation of the difference equation in question are the inverse of each other. Local analysis can therefore rely on either representation. This is, in general, not the case for global dynamics.

The equilibrium is locally indeterminate if the job-finding rate is high enough. In this case, the JCC is a stable difference equation, one solution of which is, in fact:

$$\widehat{\theta}_{t+1} = \left[\beta(1-\rho)\left(1-\frac{\eta}{\xi}m\theta_{SS}^{1-\xi}\right)\right]^{-1}\widehat{\theta}_t. \tag{14}$$

The steady state is therefore an attractor. All paths with starting values in a small neighborhood around θ_{SS} converge to it. The different adjustment paths are indexed by their starting values θ_0^i, which correspond to the various dynamic equilibria. The threshold for switching from a unique to an indeterminate local equilibrium is given by the term $\frac{1+\beta(1-\rho)}{\beta(1-\rho)}\frac{\xi}{\eta}$. High enough job-finding rates result in indeterminacy. Job-finding rates can be high when the labor market is tight, that is, when there is a relatively high number of vacancies compared to the pool of unemployed. In this case, equilibria can be self-fulfilling because firms post additional vacancies even without supporting underlying fundamentals, such as high productivity, because the high job-finding rates stimulate job search by the unemployed and thereby validate the original vacancy posting.

Figure 1 depicts indeterminacy regions for various parameter combinations.[9] The panels with various parameter combinations show that for wide ranges of the parameter space, the steady-state equilibrium is locally unique. Indeterminacy generally arises when the match elasticity ξ is small. A special case is when $\xi = \eta$. This parameterization implements the so-called Hosios-condition, under which the decentralized allocation is identical to the social planner solution. In this case, $p(\theta_{SS}) < 1 < \frac{1+\beta(1-\rho)}{\beta(1-\rho)}$, and local indeterminacy can never arise, which is the main finding by Bhattacharya and Bunzel (2003a,b). In the lower righthand panel, the Hosios-condition can be represented by the 45-degree line where $\eta = \xi$. The indeterminacy region lies in the upper lefthand corner where the bargaining parameter η is high and ξ small, which is consistent with Theorem 3.

However, the Hosios-condition is empirically violated as the literature has amply demonstrated, and we do not regard this as a likely parameterization.[10] The threshold is tightened as the term $\frac{\xi}{\eta}$ becomes smaller. Low values of the match elasticity and high values for the bargaining share are therefore more likely to imply indeterminacy. For instance, for $\beta = 0.99$, $\rho = 0.1$, $\xi = 0.4$, and $\eta = 0.9$, the threshold coefficient is 0.94. den Haan et al. (2000) report an estimate for $p(\theta)$ of 0.45. Although this is far away from the threshold, we would nevertheless regard the possibility of local indeterminacy as more than a curiosity.

4 Global Dynamics

We now turn to an analysis of the global dynamics. We first provide some general insights into the properties of the nonlinear search and matching model and set up the map that we use to study global equilibria. We start with the stability analysis of the steady state and focus on the bifurcation that occurs when the dynamics switch from stable to unstable in backward time. We show that this switch corresponds to cyclical behavior in the model and establish the presence of chaotic dynamics in backward and forward time. Finally, we explicitly link the bifurcation to the structural parameters of the economic model.

[9] The figure also shows regions where the equilibrium does not exist. But for the purposes of this paper we rule these out on account of Lemma 2.

[10] See Lubik (2013) and the references cited therein.

4.1 Preliminaries

We analyze the benchmark case of an economy with risk-neutral agents as described above. The dynamics are governed by the job-creation condition, which we replicate here for convenience:

$$\frac{\kappa}{m}\theta_t^{\xi} = \beta(1-\rho)\left[(1-\eta)(A-b) - \eta\kappa\theta_{t+1} + \frac{\kappa}{m}\theta_{t+1}^{\xi}\right]. \tag{15}$$

This is an autonomous first-order nonlinear difference equation in θ. It describes the evolution of labor market tightness θ_t and can be solved independently from the rest of the model. As in the analysis of the local dynamics, this allows us to study the evolution of θ_t in isolation.[11] We rewrite the JCC by isolating terms in θ_{t+1} on the lefthand side:

$$\beta(1-\rho)\theta_{t+1}^{\xi} - \beta(1-\rho)m\eta\theta_{t+1} = \theta_t^{\xi} - \beta(1-\rho)(1-\eta)m\frac{A-b}{\kappa}. \tag{16}$$

In coefficient form, we have:

$$a\theta_{t+1}^{\xi} - c\theta_{t+1} = \theta_t^{\xi} - d, \tag{17}$$

where:

$$\begin{aligned} 0 &< a = \beta(1-\rho) < 1, \\ 0 &< c = \beta(1-\rho)m\eta < 1, \\ 0 &< d = \beta(1-\rho)(1-\eta)m\frac{A-b}{\kappa}. \end{aligned}$$

Equation (17) is in implicit form. The forward map is not invertible; that is, we cannot explicitly relate θ_{t+1} to θ_t. In other words, the forward dynamics are captured by a correspondence. In this case, it is therefore much more convenient to study the global properties using the backward dynamics.[12] We then use the results in Kennedy and Stockman (2008) to relate the backward dynamics to those in forward time.

In the previous literature, for example Mendes and Mendes (2008) and Bhattacharya and Bunzel (2003 a,b), the backward dynamics are defined via the map $g(\theta)$ by rearranging (17) to isolate θ_t:

$$\theta_t = \left(a\theta_{t+1}^{\xi} - c\theta_{t+1} + d\right)^{1/\xi} = g(\theta_{t+1}). \tag{18}$$

[11] Under risk aversion, the dynamics depend on the time path of output y_t. Output is a function of employment n_t, which evolves based on the law of motion (3). Since this feeds back onto the JCC via the definition of $\theta_t = v_t/u_t$, it results in an interconnected two-equation system that cannot be solved analytically.

[12] The relationship between the backward and forward dynamics of nonlinear systems is an area of active research (see, for example, Kennedy and Stockman (2008) and references thereof). This distinction is immaterial for the study of linear systems since they are always invertible in this sense. That is, the properties of the forward dynamics are the 'inverse' of the properties of the backward map. If, on the other hand, one of the dynamic maps is a correspondence, this equivalence fails.

However, the choice of the map g is problematic, since it can produce results that are inconsistent with the logic of the JCC. We show in Appendix A.2 that, for plausible values of the parameter ξ, the map g can have multiple positive steady states, whereas the positive steady state θ_{SS} in JCC is unique (see Lemma 1). To avoid this inconsistency, we therefore introduce a change of variables such that for $\theta_t \geq 0$:

$$z_t = \theta_t^{\xi}. \tag{19}$$

This allows us to rewrite the equation in (17) as the following backward recurrence relation:

$$z_t = az_{t+1} - cz_{t+1}^{1/\xi} + d = f(z_{t+1}), \tag{20}$$

where the coefficients are defined as above. Backward solutions of (17) and (20) are well-defined for any $\xi \in (0,1)$, as long as z_t and θ_t are non-negative. The solutions of (17) in terms of the original variable can be obtained by using $\theta_t = z_t^{1/\xi}$. The model in the form of (20) thus provides us with a more convenient and consistent means for studying the backward dynamics of the model.

4.2 Stability Properties

We now study the dynamics of the backward map $z_t = f(z_{t+1})$. We first establish the properties of the function f. We then study the stability properties of the steady state, where we distinguish between two broad areas of dynamics in the backward map, namely stable and unstable. The backward dynamics are governed by the properties of the map:

$$f(z) = az - cz^{1/\xi} + d, \tag{21}$$

where $f(0) = d > 0$ defines the intercept. The first derivative of f is given by:

$$f'(z) = a - \left(\frac{c}{\xi} z\right)^{\frac{1-\xi}{\xi}}, \tag{22}$$

and the map f has a global maximum at:

$$\left(\frac{a\xi}{c}\right)^{\frac{\xi}{1-\xi}} := z_{max}. \tag{23}$$

Given that f is increasing on $[0, z_{max})$ and decreasing on (z_{max}, ∞), the map f is a Type-B map as defined in the Appendix A.3. The point $z_0 > 0$ is the unique positive intersection point such that $f(z_0) = az_0 - cz_0^{1/\xi} + d = 0$.[13] Note that the coefficients are independent of

[13] It is straightforward to show that z_0 is unique over $[0, \infty)$. Since $f(0) = d > 0$ and f is increasing on $[0, z_{max})$, then $f(z_{max}) > 0$. Given that f is decreasing on $[z_{max}, \infty)$, the intersection point z_0 such that $f(z_0) = 0$ is unique. Note that the point z_0 corresponds to the point q in Appendix A.3.

the match elasticity ξ, which therefore only determines the shape of the mapping but not its location in (θ_{t+1}, θ_t)-space. Furthermore, the term $\frac{A-b}{\kappa}$ scales the intercept d but does not affect other coefficients.

We can express the maximum of f in terms of the structural parameters of the model: $z_{\max} = \left(\frac{a\xi}{c}\right)^{\frac{\xi}{1-\xi}} = \left(\frac{\xi}{m\eta}\right)^{\frac{\xi}{1-\xi}}$. The maximum only depends on three parameters, which reduce to two under the Hosios-condition $\xi = \eta$. In this special case, $z_{\max} > 1$ since $m < 1$. In the general case, $z_{\max}$ can be less than one if $m > \frac{\xi}{\eta}$. Notably, the location of the maximal point does not depend on other parameters, chiefly the scale term $\frac{A-b}{\kappa}$. In the next step, we establish that the equation in (20) has a unique positive steady state. It is straightforward to see that $\theta_{SS} = (z_{SS})^{1/\xi}$ is the same as established in Lemma 1; that is, $\theta_{SS} = (f(z_{SS}))^{1/\xi}$.

Lemma 4 *Equation (20) has a unique positive steady state* z_{SS}.

Proof. The steady state(s) of (20) must be the fixed point(s) of the map f in (21). The fixed point(s) $\bar{z}$ of f must satisfy the equation

$$a\bar{z} - c\bar{z}^{1/\xi} + d = \bar{z}.$$

To show that such a point $\bar{z}$ exists, we define a function $\phi(z) = (1-a)z + cz^{1/\xi} - d$. Since $\phi(0) = -d < 0$ and $\phi\left(\frac{d}{1-a}\right) = c\left(\frac{d}{1-a}\right)^{1/\xi} > 0$, there exists a point $\bar{z} \in \left(0, \frac{d}{1-a}\right)$ such that $\phi(\bar{z}) = 0$. To show that $\bar{z}$ is unique, we note that the derivative $\phi'(z) = (1-a) + \frac{c}{\xi}z^{\frac{1-\xi}{\xi}} > 0$ for $z > 0$. Therefore, ϕ is increasing on $(0, \infty)$, and hence its zero is unique. It follows that $\bar{z} = z_{SS}$ is the unique positive fixed point of the map f and the equation in (20) has a unique positive steady state. ■

We now analyze the stability properties of the steady state. These properties are determined around the thresholds ± 1 and are given by the first derivative $f'(z_{SS})$. If $|f'(z_{SS})| < 1$, the steady state z_{SS} is stable in its backward dynamics and unstable otherwise. At $|f'(z_{SS})| = 1$ a bifurcation occurs, and the dynamics switch from stable to unstable. Clearly, the stability properties of z_{SS} translate directly to that of θ_{SS}. $|f'(z_{SS})| < 1$ if and only if:

$$-1 < a - \left(\frac{c}{\xi}z_{SS}\right)^{\frac{1-\xi}{\xi}} < 1, \tag{24}$$

or, alternatively:

$$a - 1 < \left(\frac{c}{\xi}z_{SS}\right)^{\frac{1-\xi}{\xi}} < 1 + a. \tag{25}$$

Since the value of the parameter a is less than one, the first inequality above holds trivially, given Lemma 4 and the restrictions on the parameters. This also means that whenever $f'(z_{SS})$ is positive, it is less than one. In other words, if $z_{SS} \leq z_{\max}$, then z_{SS} is stable. Since a bifurcation cannot occur in this region of the map, we focus our attention on the other case, namely $f'(z_{SS}) > -1$. In the case when $z_{SS} > z_{\max}$, $f'(z_{SS})$ is negative, and the steady state z_{SS} may or may not be stable. Since $f'(z_{SS}) > -1$ implies that:

$$\frac{c}{\xi}(z_{SS})^{\frac{1-\xi}{\xi}} < 1 + a, \tag{26}$$

then z_{SS} is stable if (26) holds; it is unstable otherwise. For reference, we collect these results in Table 1.

We can also express these conditions in terms of the underlying structural parameters of the model. The stability condition (26) can be written in terms of θ_{SS} as follows:

$$\theta_{SS} < \frac{1+\beta(1-\rho)}{\beta(1-\rho)}\frac{\xi}{m\eta}. \tag{27}$$

This is the counterpart to the case identified for local dynamics under indeterminacy, which required a high enough job-finding rate $p(\theta_{SS})$ or a high steady-state value θ_{SS}. The loss of stability of θ_{SS} translates into stable adjustment dynamics for the forward map. The solution to the model is such that starting from an initial value in a close neighborhood of the steady state, the economy will converge nonmonotonically (since $f'(z_{SS}) < 0$) to the steady state, which is an attractor in the forward dynamics. The law of motion is given by the correspondence (17).

We note that the conditions in (26) and, equivalently, in (27) are in implicit form since z_{SS} and θ_{SS} are functions of the composite parameters a, c, d, and ξ, and ultimately the structural parameters; at the same time, the analytical expression for z_{SS} cannot always be obtained explicitly. We find it therefore convenient to express the threshold conditions in terms of endogenous variables for ease of economic interpretation. In that sense, condition (27) is an outcome of a stable equilibrium, in which the steady-state labor market tightness is less than the given threshold. For the benchmark case $\xi = 0.5$, we can solve directly for z_{SS} and express the equation (29) and (27) in terms of model parameters only. We will present results and discuss the benchmark case in the next subsection. We now turn to the main section of the paper and an analysis of its key result.

4.3 Periodic and Chaotic Solutions

We establish in this section that the transition from stable and unstable dynamics at $f'(z_{SS}) = -1$ in backwards time corresponds with another type of a bifurcation, namely the emergence of a sequence of cycles of doubling periods. That is, as the steady-state z_{SS} loses its stability and goes through this bifurcation, a new attractor emerges with double the period of the steady state (i.e. a 2-cycle). The model's solution oscillates between these values but in a nonexplosive manner. As the value of the bifurcation parameter changes, new attractors continue to appear of double the period of the previous ones. This eventually leads to bounded aperiodic and chaotic fluctuations in the dynamics of the model. We first establish this result analytically in terms of the composite coefficients of the map defined in the section on preliminaries above. We choose d as the bifurcation parameter, since it scales in $\frac{A-b}{\kappa}$. Whereas the parameters in the coefficients a and c are comparatively tightly restricted, the parameters in this scale coefficient are less economically restricted as discussed in the calibration section.

The change in stability of z_{SS} occurs at $f'(z_{SS}) = -1$, or:

$$\frac{c}{\xi}(z_{SS})^{\frac{1-\xi}{\xi}} = 1 + a. \tag{28}$$

We can rewrite (28) in terms of d as follows. At the steady state, we have $f(z_{SS}) = az_{SS} - cz_{SS}^{1/\xi} + d = z_{SS}$, or $a - c(z_{SS})^{\frac{1-\xi}{\xi}} + \frac{d}{z_{SS}} = 1$. Hence, $c(z_{SS})^{\frac{1-\xi}{\xi}} = a - 1 + \frac{d}{z_{SS}}$. We can plug this expression this into (28) and find an alternative and equivalent condition to (28):

$$\frac{d}{z_{SS}} = \xi(1+a) + (1-a), \quad \text{or:} \quad d = \mu z_{SS}, \tag{29}$$

where $\mu = \xi(1+a) + 1 - a$. When $d < \mu z_{SS}$, $-1 < f'(z_{SS}) < 0$; when $d > \mu z_{SS}$, $f'(z_{SS}) < -1$. In addition, note that if $\xi \geq 0.5$, $\mu \geq 1 + 0.5(1-a) > 1$, since $a < 1$. We can now establish the period doubling.[14]

Theorem 5 *The point $d^* = \mu z_{SS}^*$ is a bifurcation point for period doubling. Passing of d through the bifurcation threshold corresponds with emergence of cycles of period 2, 4, 8, etc.*

Proof. It is relatively straightforward to check the conditions required for period doubling bifurcations given in Elaydi (2007) and summarized in Appendix A.1. For fixed values of

[14] The condition in (29) is expressed in implicit form since z_{SS} depends on d, as well as a, c, and ξ. We show explicit analytical results in terms of model parameters only for the benchmark case $\xi = 0.5$ in the next subsection.

a, c, ξ, the parameterized map is:

$$f_d(z, d) = az - cz^{1/\xi} + d.$$

It is easy to verify that the parameterized map $f_d(z_{SS}) = z_{SS}$ for $d > 0$, i.e., f_d has a unique positive fixed point z_{SS} for all positive values of d. At the bifurcation point $d^* = \mu z^*_{SS}$, $\frac{\partial f_d}{\partial z_{SS}}(z_{SS}) = -1$. Finally, $\frac{\partial^2 f^2}{\partial d \partial z_{SS}}(d^*, z^*_{SS})$ cannot be zero, where f_d^2 is the composition of the map f with itself, i.e., $f_d^2(d, z_{SS}) = f_d(f_d(d, z_{SS}))$. We have:

$$f^2(d, z) = a(az - cz^{1/\xi} + d) - c(az - cz^{1/\xi} + d)^{1/\xi} + d,$$

and:

$$\frac{\partial^2 f_d}{\partial z \partial d} = -\frac{c}{\xi}\frac{1-\xi}{\xi}(az - cz^{1/\xi} + d)^{\frac{1-2\xi}{\xi}}(a - \frac{c}{\xi}z^{\frac{1-\xi}{\xi}}) \neq 0,$$

for $z_{\max} < z_{SS} < z_0$, which is our domain of analysis. ■

We illustrate this result by means of a simple example. Figure 2 demonstrates the period doubling numerically. For fixed $\xi = 0.5$, $a = 0.891$ and $c = 0.4$ and a range of parameter values of $d > 0$, we generate 500 iterates of the map f and plot the last 200 against d. For low values of d, the iterates converge to a single steady state. Around $d = 2.5$, we see emergence of a 2-cycle. Four cycles appear as d passes through 3.7, and one can also discern the emergence of an 8-cycle past 4. For higher values of d, the shaded region of the diagram shows aperiodic oscillations and is referred to as a chaotic region. A little past $d = 5$, we can also discern a stable 3-cycle. This parameterization is only one example of emergence of periodic doubling and chaos. These values are, however, economically plausible, as $a = 0.891$ is obtained by setting the discount rate $\beta = 0.99$ and the job separation rate $\rho = 0.1$. Under the Hosios-condition $\eta = \xi = 0.5$, the value of $c = 0.4$ implies that m is around 0.9. For values of d between 5 and 5.5, the implied steady-state unemployment rate u_{SS} ranges between 0.167 and 0.174, which is well within economically plausible bounds.

Appendix A.3 presents additional sufficient and general conditions for existence of periodic and chaotic solutions in Type-B maps. The results of Theorems 20, 21, and Corollary 22 therefore apply since our map f is of this type. We refer to the Appendix for a detailed discussion and further proofs. In what follows, we interpret some of the conditions stated in these theorems. First, Theorems 20 and 21 require that $f(z_{max}) = z_0$ where $z_0 > z_{max}$ is the preimage of 0, i.e., $f(z_0) = 0$. However, in most cases, we cannot solve for z_0 analytically. An alternative way of formulating this requirement is the following: for $f(z_{max}) = z_0$, it is necessary that the second iterate of $z_{\max}$ maps to zero, i.e., $f^2(z_{max}) = 0$. Since

$f(z_{\max}) = a\left(\frac{a\xi}{c}\right)^{\frac{\xi}{1-\xi}} - c\left(\frac{a\xi}{c}\right)^{\frac{1}{1-\xi}} + d$, the condition $f^2(z_{\max}) = 0$ implies:

$$f^2(z_{\max}) = a^2\left(\frac{a\xi}{c}\right)^{\frac{\xi}{1-\xi}} - ac\left(\frac{a\xi}{c}\right)^{\frac{1}{1-\xi}} + ad - c\left[a\left(\frac{a\xi}{c}\right)^{\frac{\xi}{1-\xi}} - c\left(\frac{a\xi}{c}\right)^{\frac{1}{1-\xi}} + d\right]^{\frac{1}{\xi}} + d = 0 \tag{30}$$

The equation in (30) is in implicit form and defines a surface in a four-dimensional coefficient space for a, d, c, ξ. For any fixed values of two of these parameters, one can plot implicit curves in two dimensions to show parameter ranges that satisfy this condition. Figure 3 shows the plot of a family of such curves that also satisfy $d < z_{max}$ (see Theorem 21) for ranges of parameters when a is set at 0.891 and $0 < c < 1$.

Denoting z_+ as the right preimage of $z_{\max}$, i.e., $z_+ > z_{\max}$ is a point such that $f(z_+) = z_{\max}$, then the case $d > z_{max}$ and $f(d) \geq z_+$ in Theorem 21 (b) implies that $f^2(d) \leq z_{\max}$ (since, again, we cannot always solve for z_+ analytically). This corresponds with the inequality:

$$a(ad - cd^{1/\xi} + d) - c\left(ad - cd^{1/\xi} + d\right)^{1/\xi} + d \leq \left(\frac{a\xi}{c}\right)^{\frac{\xi}{1-\xi}}. \tag{31}$$

The family of curves that satisfy (30) and the above inequality are presented graphically in Figure 4 for ranges of parameters $a = 0.891$ and $0 < c < 1$. Finally, the condition $f(d) = z_0$ exactly pins down the three cycle $\{\ldots, 0, d, z_0, 0, d, z_0, \ldots\}$, as $0 \to d \to f(d) = z_0 \to 0$. This condition can be rewritten as $f^2(d) = 0$, or:

$$a(ad - cd^{1/\xi} + d) - c\left(ad - cd^{1/\xi} + d\right)^{1/\xi} + d = 0, \tag{32}$$

as depicted in Figure 5.

In summary, Figures 3 - 5 give an idea for which parameterizations periodic and chaotic dynamics can arise in the nonlinear model. Previously, Mendes and Mendes (2008) have established chaos for $\xi = 0.2$. This is consistent with our finding that conditions in (30) and $d < z_{max}$, as well as (32), imply low values of the match elasticity ξ, e.g. as in Figure 5. However, empirical estimates of the parameter ξ in the literature are considerably higher. Petrongolo and Pissarides (2001) find a value ξ of 0.7, while Hall and Schulhofer-Wohl (2015) report estimates that range between 0.28 and 0.7 from a wide variety of studies, data, and empirical approaches. In addition, and in contrast to the earlier literature on nonlinear dynamics in the search and matching model, we find that the conditions in (30) and (31) yield values of ξ that are consistent with these empirical estimates. We can summarize these results in the following theorem. They are also collected in Table 1.

Theorem 6 *For certain parametrization, the map f is chaotic, i.e. the equation in (20) has chaotic as well as coexisting periodic equilibria of multiple periods. In particular,*

(i) If (30) holds and $z_{\max} > d$, then the equation in (20) has periodic equilibria of every period $p \geq 3$, as well as aperiodic and chaotic solutions.

(ii) If (30) and (31) hold and $z_{\max} < d$, then the equation in (20) has periodic equilibria of every period $p \geq 5$, as well as aperiodic and chaotic solutions.

(iii) If (32) holds, then the equation in (20) has a 3-cycle.

As a final step, we relate the equilibria in backward time to those of their forward representation. Translating cycles from backward to forward time (and vice-versa) is straightforward. Kennedy and Stockman (2008) show that equilibria in forward dynamics are chaotic if and only if they are chaotic in backward time. This gives us our last theorem, which comes as a consequence of Corollary 22 and Theorem 18 of the Appendix.

Theorem 7 *Under certain parameterization, the equations in (20) and (17) have periodic as well as chaotic equilibria going both forward and backward in time.*

4.3.1 A Simple Illustration for a Benchmark Parameterization

We can derive analytic conditions for the existence of periodic and chaotic equilibria for the special case when the match elasticity $\xi = 0.5$. This parameterization, which is empirically plausible as discussed above, implies that the map f becomes the quadratic:

$$f(z) = az - cz^2 + d. \tag{33}$$

We can therefore explicitly solve for z_{SS}, z_0, and z_+ as $z_{SS} = \frac{1}{2c}\left((1-a) + \sqrt{(1-a)^2 + 4cd}\right)$, $z_0 = \frac{1}{2c}\left(a + \sqrt{a^2 + 4cd}\right)$ and $z_+ = \frac{1}{2c}\left(a + \sqrt{a^2 + 4cd - 2a}\right)$. The critical value for period doubling bifurcation $d^* = \mu z_{SS}$ then corresponds to:

$$d^* = \frac{1}{2c}\left(1 + \frac{1-a}{2}\right)\left((1-a) + \sqrt{(1-a)^2 + 4cd^*}\right),$$

where the coefficients a and c are defined above. These curves are plotted in Figure 6 for the same parameterization as discussed before. Over the admissible region $0 < c < 1$, there is a wide range of parameter combinations where chaotic behavior can occur. We will dig deeper into the parameter regions that can imply periodic behavior in the next section.

We also want to highlight on additional case under this benchmark parameterization. Whereas in Section 4.3 we provide sufficient conditions, these are not the only instances where chaotic equilibria occur. For the quadratic case $\xi = 0.5$, it is also straightforward to pin down the values of parameters a, c, d that establishe qualitative (or topological) equivalence of the dynamic behavior of the iterates of the map f to those of the logistic map $r(\mu) = \mu r(1 - r)$. The logistic map is a canonical example for demonstrating chaos in one-dimensional maps, as, for example, in Elaydi (2007). Using the benchmark case, we can solve for period-3 points of the map f numerically. Specifically, we solve for the fixed points of the composite map f^3 that are different from z_{SS}. The parameter values where such points are found are given in Figure 7 for the same ranges as before.

4.4 Chaos Regions for Structural Parameters

In this final section, we provide further insight into the economic determinants of the range of equilibria by expressing them in terms of the structural parameters of the underlying model. This is not quite straightforward since the search-and-matching model is richly parameterized in the sense that the equilibrium and the dynamic behavior of one endogenous variable, namely labor market tightness θ, is determined by eight parameters in the JCC. In order to identify the relevant regions where chaos can occur, we therefore have to judiciously condition on specific parameters values. Recall that the location of $z_{\max} = \left(\frac{1}{m}\frac{\xi}{\eta}\right)^{\frac{\xi}{1-\xi}}$ is determined by three parameters only. Furthermore, we keep the separation rate ρ and the discount factor β fixed for the purposes of this analysis. We therefore find it convenient to analyze the equilibria in terms of the other model parameters that affect the type of equilibria and the shape of the map. This leaves the scale term $\frac{A-b}{\kappa}$ as the crucial coefficient to analyze, whereby we normalize $A = 1$ without loss of generality. It can be ascertained immediately that $\frac{A-b}{\kappa}$ shifts the map $f(z)$ vertically, thereby changing the location of the steady state z_{SS} and the intercept z_0 with the zero line. The shape of the map, however, is unaffected.

Similar to the analysis of the local dynamics, we find it convenient to describe the analytical properties of the map in terms of the steady-state values of endogenous variables. Specifically, we consider the steady-state unemployment rate u_{SS} as a calibration target. Since $\theta_{SS} = z_{SS}^{1/\xi}$, we can then back out the implied labor market tightness $\theta_{SS} = \left(\frac{1}{m}\frac{\rho}{1-\rho}\frac{1-u_{SS}}{u_{SS}}\right)^{1/(1-\xi)}$ from the law of motion for employment. However, we note that the steady state θ_{SS} and therefore u_{SS} are themselves functions of all of the structural parameters of the model. Given the JCC, this strategy then restricts the parameterization of either b or κ based on the relationship $\frac{A-b}{\kappa} = \frac{\eta}{1-\eta}\theta_{SS} + \frac{1}{1-\eta}\frac{1-\beta(1-\rho)}{\beta(1-\rho)}\frac{\theta_{SS}^{\xi}}{m}$. It thus leaves one remaining parameter for which we can do comparative static analysis. Intuitively, this approach targets a specific unemployment rate by setting, for instance, benefits b at a specific level. Changing of b (or κ) necessarily changes the steady state u_{SS}. Instead of discussing the effects of changes in this parameter on the implied u_{SS}, this reparameterization allows us a more direct and economically intuitive consideration.

We now establish the following Lemma, which describes the regions of the structural parameter space for which the first derivative of the map f is negative. We concentrate on this case since it admits a bifurcation.

Lemma 8 *For $z_{\max} < z_{SS} < z_0$, $f'(z_{SS}) < 0$. We distinguish three different regions of the parameter space:*

(i) $0 > f'(z_{SS}) > -1$*:*

$$\frac{1}{1 + \frac{\beta^{-1}+(1-\rho)}{\rho}\frac{\xi}{\eta}} < u_{SS} < \frac{1}{1 + \frac{1-\rho}{\rho}\frac{\xi}{\eta}}$$

(ii) $f'(z_{SS}) = -1$*:*

$$u_{SS} = \frac{1}{1 + \frac{\beta^{-1}+(1-\rho)}{\rho}\frac{\xi}{\eta}}$$

(iii) $-1 > f'(z_{SS})$*:*

$$u_{SS} < \frac{1}{1 + \frac{\beta^{-1}+(1-\rho)}{\rho}\frac{\xi}{\eta}}$$

If the parameterization is such that condition (ii) holds, that is, if the endogenous unemployment rate is equal to the threshold $\left[1 + \frac{\beta^{-1}+(1-\rho)}{\rho}\frac{\xi}{\eta}\right]^{-1}$, then the equilibrium undergoes a bifurcation. We can depict this scenario in terms of the structural parameters of the model. This is presented in Figure 8. These graphs are created for ranges of composite parameter values where chaos is observed. For a given point in the shaded region (for instance m and u), there exist parameter values a, c, d, and $\xi = 0.5$, which result in chaotic behavior in the model.[15] We observe that chaos is prevalent in the nonlinear model for economically plausible parameter values. Generally, we observe chaos for values of the match efficiency parameter m above 0.5, which is consistent with empirical estimates. At

[15]Except for the case shown in m-ρ space of Figure 8, a is fixed at 0.891, which is obtained by setting $\beta = 0.99$ and $\rho = 0.1$. We also impose the Hosios-condition that sets $\xi = \eta$.

the same time, chaotic behavior requires a high bargaining parameter η, a comparatively low job matching rate q, and is consistent with a moderately high steady-state unemployment rate u. On the other hand, combinations of the separation rate ρ and m that imply chaos are in the plausible range for the former but not the latter.

It is instructive to contrast these findings to the results on local dynamics as described in Section 3 and depicted in Figure 1. While we cannot talk about chaotic and periodic behavior locally, one conclusion we can draw from analysis of the local dynamics is that for a wide region of the parameter space the search-and-matching model exhibits locally unique dynamics. It is only in extreme regions of the parameter space that we observe explosive or excessively stable behavior, or in the language of local analysis nonexistence or indeterminacy, respectively. In contrast, the possibility of periodic and chaotic dynamics is pervasive in the sense that we can obtain such equilibria for an economically plausible and reasonably large region of the parameter space.

5 Conclusion

This paper demonstrates that periodic and chaotic dynamics are an economically plausible feature of the simple search-and-matching model of the labor market. This stands in contrast with findings from the literature on local dynamics that the model exhibits locally unique and stable equilibria over the wide range of the parameter space. In contrast to previous literature on global dynamics in this class of models, we are able to characterize analytically a larger set of regions implying chaotic dynamics. Specifically, we show existence of chaos for calibrations that have been used in the literature. Moreover, we do so by utilizing some recent results from the literature on chaotic equilibria.

Our paper is purely theoretical in nature, albeit with a focus on economically plausible outcomes. What remains to be seen, however, is the extent to which the fully nonlinear model with parameters that fall into the chaotic region is an actual data-generating process. That is, do actual labor market time series, such as the unemployment rate and vacancies, exhibit behavior of the type that are consistent with, say, periodic equilibria? For instance, Figure 9 depicts time paths of labor market tightness with periodic and chaotic behavior generated under different values of the match elasticity ξ. At first pass, this seems unlikely since we do not typically observe such oscillating behavior in the data. This would, of course, have to be established more formally. A second issue is the degree to which local theoretical models or linear empirical methods fail in describing such global dynamics. Research in this area is still sparse. However, the analytic results in this paper can serve as a background against which to conduct such an analysis.

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A Mathematical Appendix

In this appendix, we list definitions and results necessary for the establishment of periodic and chaotic solutions in the search and matching model, as discussed in the paper.

A.1 Preliminaries

Let $f : \mathbb{R} \to \mathbb{R}$ be a map and consider the first-order difference equation given by:

$$x_{t+1} = f(x_t). \tag{34}$$

Definition 9 *(Invariance)* The interval $I \subset \mathbb{R}$ is *invariant* under f if $f(I) \subseteq I$. For the first-order equation in (34), the above definition implies that if the initial value $x_0 \in I$, then $x_t \in I$ for $t > 0$.

Definition 10 *(Periodic points)* Let p be a nonnegative integer and let $f^p = f \circ f \circ \cdots \circ f$ be the composition of the map f with itself p times. The point $s \in \mathbb{R}$ is a p-periodic point of the map f if $f^p(s) = s$. The first-order equation in (34) has a periodic solution of period p if the map f has a p-periodic point. In this case, we say that the equation in (34) has a periodic solution of period p (or a p-cycle), i.e. $x_{t+p} = x_t$ for all $t \geq 0$.

The following result in Block and Coppel (1986) establishes sufficient conditions for existence of periodic points of odd periods.

Lemma 11 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. If for some odd integer $p > 1$ there exists a point x such that:*

$$f^p(x) \leq x < f(x) \quad \text{or} \quad f(x) < x \leq f^p(x),$$

then f has a periodic point of period p.

We next list Sharkovski's ordering of positive integers defined as follows (see Elaydi, 2007, for more):

$$
\begin{aligned}
&3 \lhd 5 \lhd 7 \lhd \dots \\
&2 \cdot 3 \lhd 2 \cdot 5 \lhd 2 \cdot 7 \lhd \dots \\
&2^2 \cdot 3 \lhd 2^2 \cdot 5 \lhd 2^2 \cdot 7 \lhd \dots \\
&\dots \\
&2^n \cdot 3 \lhd 2^n \cdot 5 \lhd 2^n \cdot 7 \lhd \dots \\
&\dots \\
&2^n \lhd 2^{n-1} \lhd \dots \lhd 2^2 \lhd 2 \lhd 1
\end{aligned}
$$

Now the theorem.

Theorem 12 *(Sharkovski, 1964)* Let $f : I \to I$ be a continuous map on the interval I, where I may be finite, infinite, or the whole real line. If f has a periodic point of period k, then it has a periodic point of period r for all r with $k \lhd r$.

Given Sharkovski's ordering, the above theorem states that if a function f has a periodic point of period 3, then it has periodic points of all periods, which is stated as a theorem below.

Theorem 13 *(Li and Yorke, 1975)* Let $f : I \to I$ be a continuous map on an interval $I \subseteq \mathbb{R}$. If f has a periodic point in I of period 3, then f has a periodic point of every integer period $k \geq 1$.

There are several, not necessarily equivalent, definitions of chaos in mathematical literature. The more commonly used ones are those in the sense of Li and Yorke, Devaney, and Block and Coppel (see Aulbach and Kieninger (2001) for more details). For the purpose of this paper, below we list the definition of chaos in the sense of Block and Coppel and refer to the result in Aulbach and Kieninger (2001) that establishes equivalence between chaos in the sense of Block and Coppel to that of Devaney.

Definition 14 A map $f : I \to I$ is called *turbulent* if there exist compact subintervals J, K of I with at most one common point such that

$$J \cup K \subseteq f(J) \cap f(K).$$

If J and K are disjoint, then f is said to be *strictly turbulent*.

Theorem 15 *(Chaos in the sense of Block and Coppel)* A continuous map $f : I \to I$ on a nontrivial compact interval I is chaotic in the sense of Block and Coppel if and only if one of the following equivalent conditions is satisfied:

(i) f^m is turbulent for some $m \in \mathbb{N}$.

(ii) f^m is strictly turbulent for some $m \in \mathbb{N}$.

(iii) f has a periodic point whose period is not a power of 2.

Theorem 16 *(Aulbach and Kieninger, 2001)* A continuous map $f : I \to I$ on an interval I is chaotic in Devaney sense if and and only if it is chaotic in the Block and Coppel sense.

Our next theorem is from Elaydi (2007) and lists conditions under which period doubling bifurcations occur.

Theorem 17 *(Period Doubling Bifurcation)* Let a one-parameter family $F_\mu(x)$ be written as a map of two variables, i.e. $H(\mu, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and let x^* be the fixed point of $F_m u$. Suppose that

(i) $H_\mu(x^*) = x^*$ for all μ in an interval around a threshold point μ^*.

(ii) $H'_{\mu^*}(x^*) = -1$

(iii) $\frac{\partial^2 H^2}{\partial \mu \partial x^*}(\mu^*, x^*) \neq 0$

where $H^2(\mu, x) = H(H(\mu, x))$. Then there exists an interval I about x^* and a function $p : I \to \mathbb{R}$ such that $H_{p(x)}(x) \neq x$, but $H^2_{p(x)}(x) = x$.

Finally, we state the result of Kennedy and Stockman (2008) that relates the solutions of a map iterated backward in time to those of forward representation. For establishment of periodic solutions in the forward map, existence of periodic solutions in the backward map is sufficient. Using the same notation as in Kennedy and Stockman (2008), the map f^{-1} is defined for the map f on a metric space X with $f : X \to X$, regardless whether f is multi-valued or not. Their main result states:

Theorem 18 Let $f : X \to X$ be continuous on a metric space X. Then f is chaotic on X in the sense of Devaney if and only if f^{-1} is chaotic on X.

The above theorem is an important result showing that models with backward dynamics are chaotic going forward in time if and only if they are chaotic going backward in time. Hence, establishment of chaotic solutions in backward dynamics is sufficient for existence of chaotic forward dynamics.

A.2 Fixed points of the g-map

Theorem 19 The map $g(x) = (ax^{\xi} - cx + d)^{\frac{1}{\xi}}$ can have two positive fixed points.

Proof. The fixed points of the map g must satisfy the expression:

$$x = (ax^{\xi} - cx + d)^{\frac{1}{\xi}},$$

or for $x \neq 0$

$$h(x) := \frac{(ax^{\xi} - cx + d)^{\frac{1}{\xi}}}{x} = 1$$

The derivative of $h(x)$ is given by:

$$h'(x) = \frac{1}{\xi x^2}\left[(ax^{\xi} - cx + d)^{\frac{1}{\xi}-1}(\xi a x^{\xi-1})x - (ax^{\xi} - cx + d)^{\frac{1}{\xi}}\right],$$

which can be rewritten as:

$$h'(x) = \frac{1}{\xi x^2}[(ax^{\xi} - cx + d)^{\frac{1-\xi}{\xi}}][(\xi - 1)ax^{\xi} - d].$$

Next, we determine the behavior of $h(x)$ via the sign of its derivative $h'(x)$. First, note that $\lim_{x\to 0^+} h(x) = \infty$. Since $0 < \xi < 1$, then $(\xi - 1)ax^{\xi} - d < 0$ for all $x \geq 0$. Also, we let:

$$\phi(x) = ax^{\xi} - cx + d \text{ with } \phi(0) = d > 0, \;\; \phi(d/a)^{\frac{1}{\xi}}) = -c(d/a)^{\frac{1}{\xi}} < 0,$$

hence there exists a point $x^* \in \left(0, \left(\frac{d}{a}\right)^{\frac{1}{\xi}}\right)$ such that $\phi(x^*) = 0$. Moreover, $\phi(x) > 0$ for $x \in (0, x^*)$, $\phi(x) < 0$ for $x > x^*$, and $h(x^*) = g(x^*) = 0$.

Now, if $\frac{1}{\xi} = 2k$ for some positive integer $k \geq 1$, then $\frac{1}{\xi} - 1$ is odd, which means that $(\phi(x))^{1/\xi - 1}$ is positive on $(0, x^*)$ and negative on (x^*, ∞). This means $h(x)$ is decreasing on $(0, x^*)$ and increasing on (x^*, ∞) and is exactly 0 at x^*. Therefore, there exist precisely, two points x' and x'', at which $h(x') = h(x'') = 1$, hence x' and x'' are the two fixed points of $g(x)$, which proves the above claim.

If, on the other hand, $\frac{1}{\xi} = 2k + 1$, then $(\phi(x))^{1/\xi - 1} > 0$ for $x > 0$, hence h is decreasing on $(0, \infty)$ and is equal to one at precisely one point, and in this case, the positive fixed point of the map g is unique. ■

See an example of the map g at $a = 0.8, c = 0.7$, and $d = 2$ with $\xi = 0.5$ in Figure 10. The two steady states are clearly discernible at the intersection points of the map g with the identity line.

A.3 Chaos in Type-B maps

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function with a critical point $m > 0$ such that f is increasing on $[0, m)$ and decreasing on (m, ∞) and $f(0) = d > 0$. Under appropriate scaling, this type of a map has been characterized by Medio and Raines (2007) as a Type-B map. We establish sufficient conditions for existence of periodic and chaotic solutions for a general class of such maps.

Given that f is decreasing on (m, ∞) and $f(m) > 0$, there exists a real number $q > m$, such that $f(q) = 0$ (i.e. q is the preimage of 0). This gives us the following result.

Theorem 20 *If $f(m) \leq q$, then the interval $[0, q]$ is invariant under f.*

Proof. Let $x \in [0, q]$. Then $f(x) \leq f(m) \leq q$ for all $x \geq 0$. Further, if $0 \leq x \leq m$, then $f(x) \geq f(0) = d > 0$ since f is increasing on $[0, m)$, and if $m \leq x \leq q$, then $f(x) \geq f(q) = 0$ since f is decreasing on $[0, q]$. ■

Now, for the Type-B map defined above, for any point $y \in [0, q]$, there exists a pair of real numbers y_- and y_+ such that $f(y_-) = f(y_+) = y$, i.e. y_- and y_+ are preimages of y. Moreover, if $z < y$, then:

$$z_- < y_- < m < y_+ < z_+.$$

We use this to establish sufficient conditions for existence of odd periodic points in Type-B maps.

Theorem 21 Let f be a Type-B map defined above.

(i) If $m > d$ and $f(m) = q$, then f has a periodic point of period 3 in $[0, q]$.

(ii) If $f(d) \geq m_+$ and $f(m) = q$, then f has a periodic point of period 5 in $[0, q]$.

(iii) If $f(d) = q$, then f has a periodic point of period 3 in $[0, q]$.

Proof. (i) If $m > d$, then $f(m) = q \geq m$, $f^2(m) = f(q) = 0$ and $f^3(m) = f(0) = d < m$, hence:

$$d = f^3(m) < m \leq f(m) = q,$$

and the result follows by Lemma 11.

(ii) If $f(d) > m_+$, then:

$$m_+ \to m \to q \to 0 \to d \to f(d),$$

i.e. $f(d) = f^5(m_+)$ and:

$$m = f(m_+) < m_+ \leq f^5(m_+),$$

and the result follows again by Lemma 11.

(iii) Setting $f(d) = q$ pins down exactly the cycle $\{\ldots 0, d, q, 0, d, q, \ldots\}$ as $f(0) = d$, $f(d) = q$, $f(q) = 0$. ■

As a corollary, we also have the following result.

Corollary 22 If any of the hypotheses (i), (ii), or (iii) in Theorem 21 hold, then f has periodic points of every periods in $[0, q]$ (except for 3 in case of (ii)) and is chaotic in the sense of Block and Coppel, and Devaney.

Table 1: Summary of Results

Conditions	Outcomes	Comments
(26)	Existence of a unique steady state z_{SS}	The steady state is stable in backward map.
(29)	Bifurcation threshold	Cycles of doubling periods emerge as d passes through the threshold.
(30) and $d < z_{\max}$	Existence of a 3-cycle Cycles of all periods $k \geq 3$ and chaos	 Occurs for low values of ξ
(30), $d > z_{\max}$ and (31)	Existence of a 5-cycle Cycles of all periods $k \geq 5$ and chaos	Occurs for empirically plausible values of ξ.
(32)	Sets the 3-cycles $\{0, d, z_0\}$	Occurs for low values of ξ. Solutions may become negative.

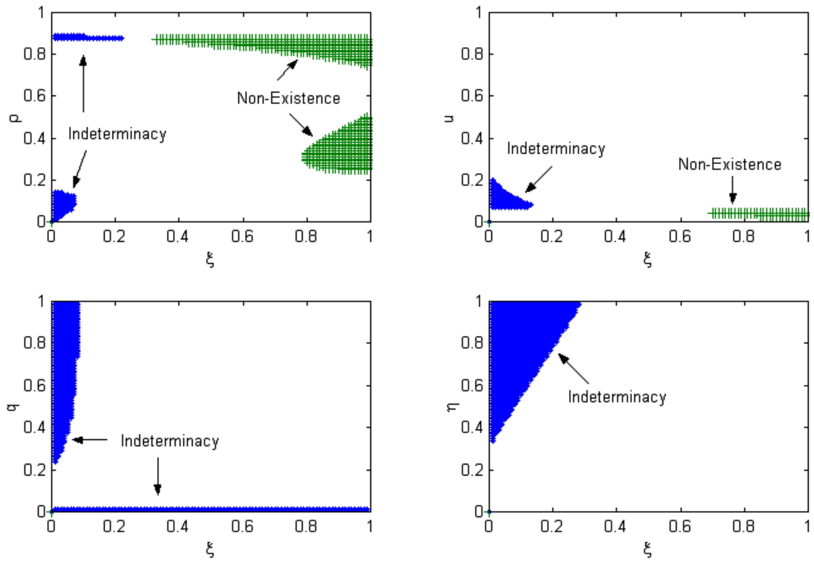

Figure 1: Local Dynamics: Determinacy Regions

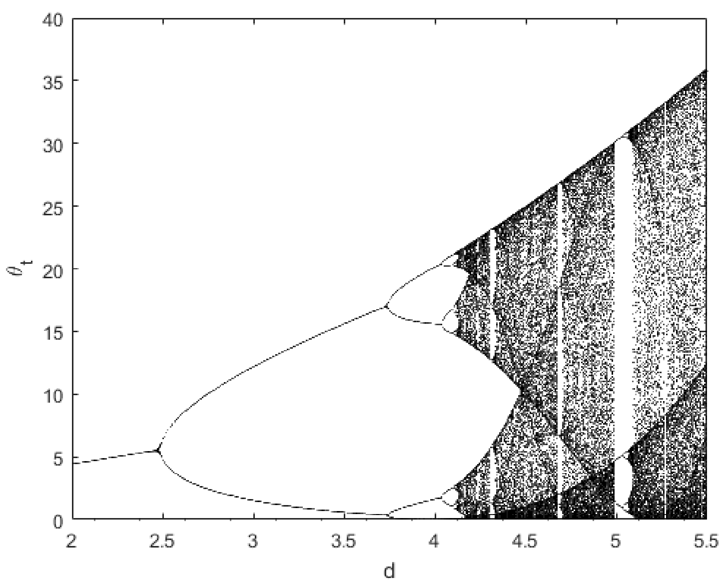

Figure 2: Bifurcation Diagram for $a = 0.891, c = 0.4, \xi = 0.5$

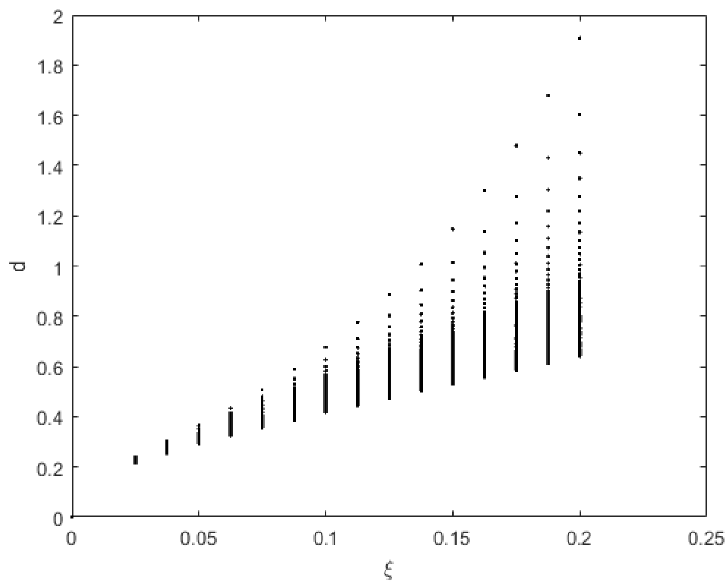

Figure 3: Implicit Curves for $f(z_{max}) = z_0, d < z_{\max}$, $a = 0.891, 0 < c < 1$

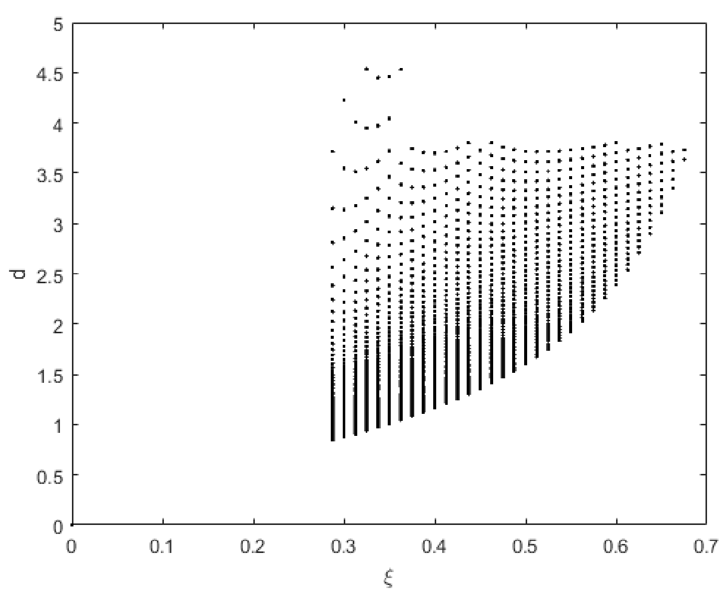

Figure 4: Implicit Curves for $f(z_{\max}) = z_0$, $d > z_{\max}$ and $f(d) > z_+$, $a = 0.891, 0 < c < 1$

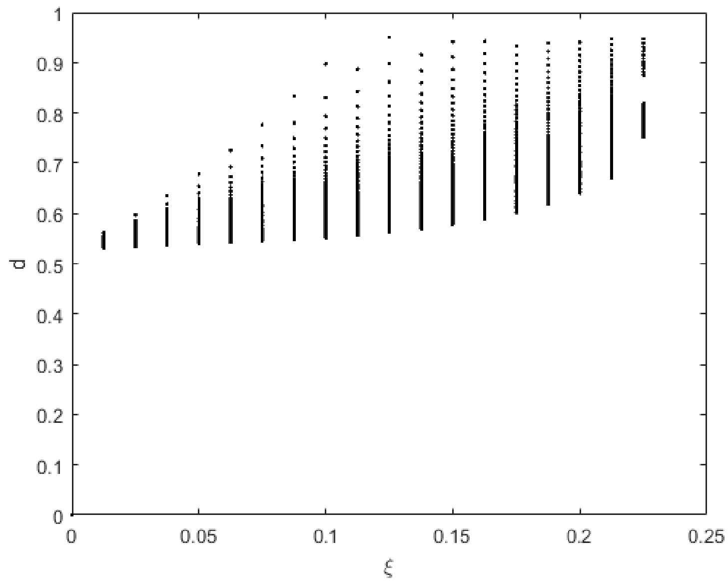

Figure 5: Implicit Curves for $f(d) = z_0$, $a = 0.891, 0 < c < 1$

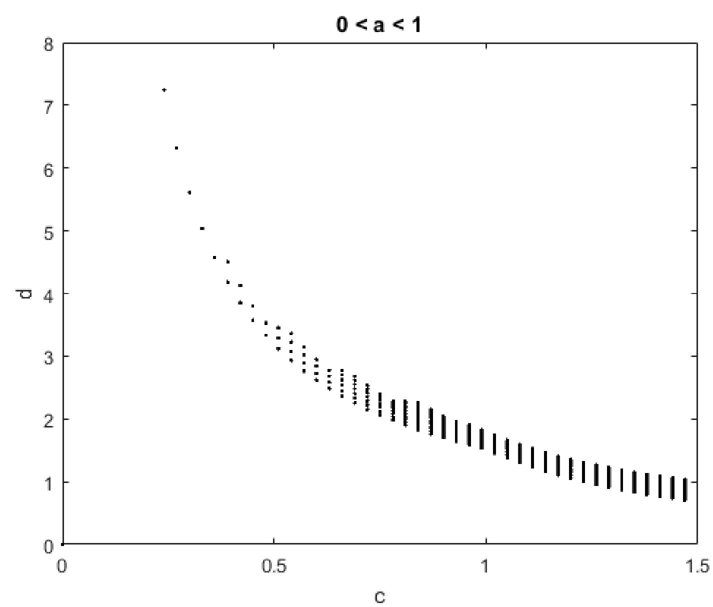

Figure 6: Values of d and c at Bifurcation Threshold, $0 < a < 1, \xi = 0.5$

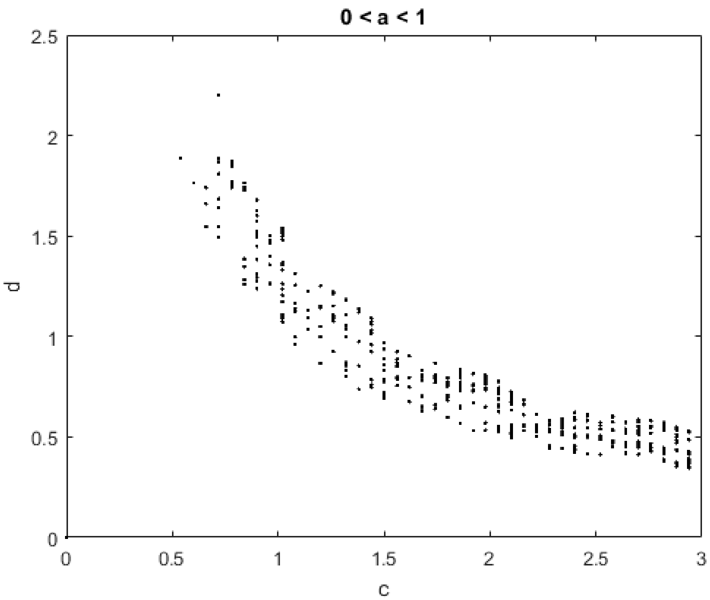

Figure 7: Period-3 Points for Ranges of $0 < a < 1$, $\xi = 0.5$

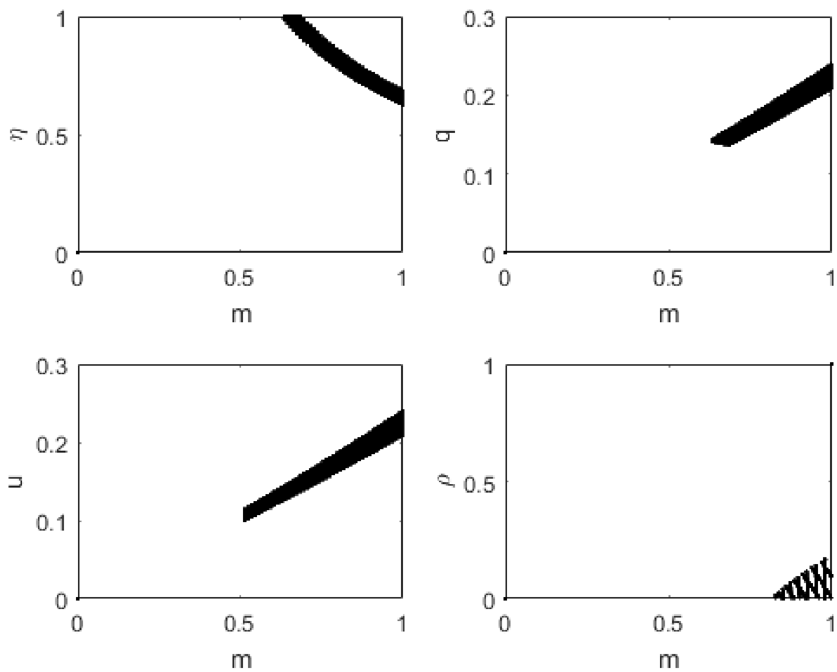

Figure 8: Global Dynamics: Chaos Regions

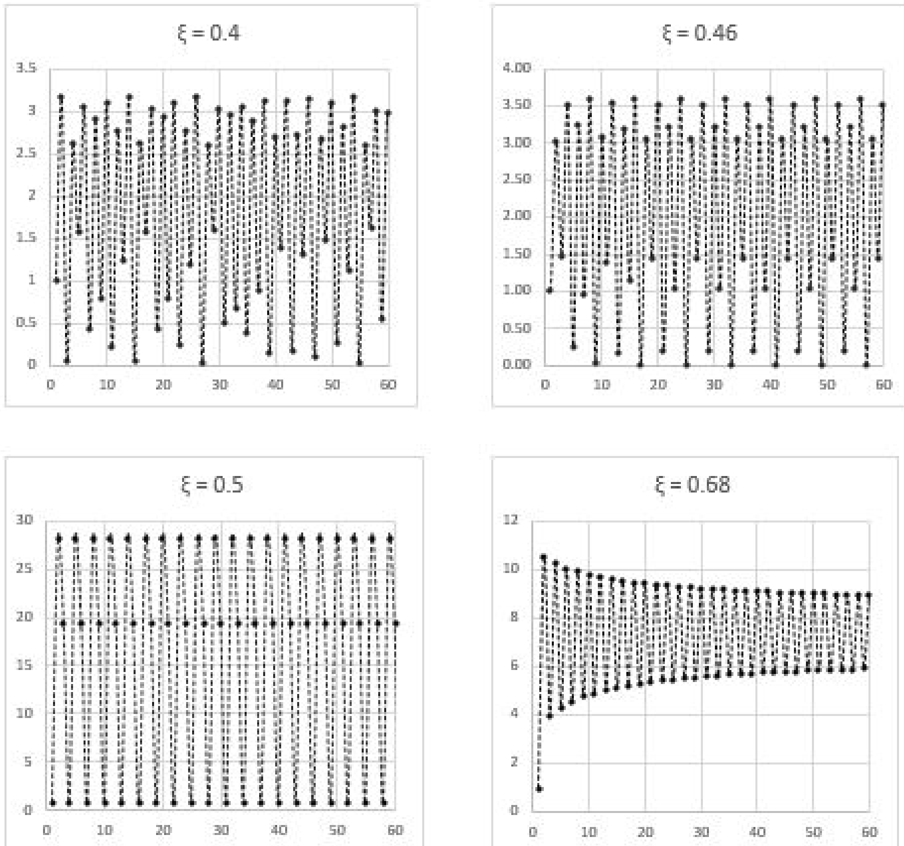

Figure 9: Cycles and Chaos for Various Values of ξ

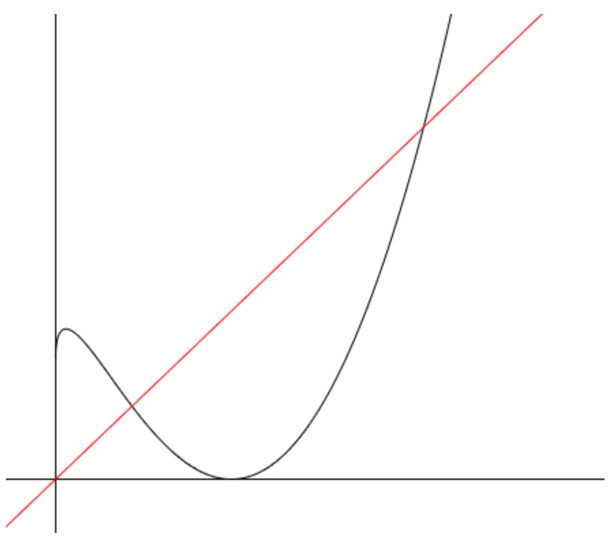

Figure 10: Multiple Steady States of Map g at $a = 0.8$, $c = 0.7$, $d = 2$ and $\xi = 0.5$