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Regressive Welfare Effects of Housing Bubbles

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Abstract

We analyze the welfare effects of asset bubbles in a model with income inequality and financial friction. We show that a bubble that emerges in the value of housing, a durable asset that is fundamentally useful for everyone, has regressive welfare effects. By raising the housing price, the bubble benefits high-income savers but negatively affects low-income borrowers. The key intuition is that, by creating a bubble in the market price, savers' demand for the housing asset for investment purposes imposes a negative externality on borrowers, who only demand the housing asset for utility purposes. The model also implies a feedback loop: high income inequality depresses the interest rates, facilitating the existence of housing bubbles, which in turn have regressive welfare effects.

Keywords: rational bubble; inequality; housing; financial friction
JEL classifications: E10; E21; E44

1 Introduction

Many countries have experienced episodes of bubble-like booms in asset prices. Examples include the real estate booms in Japan in the 1980s, Southeast Asia in the 1990s, the U.S. in the 2000s, and more recently in China and Vietnam.[1] In general, when there is a high demand for savings but limited investment outlet, the rates of returns from investment are depressed and real estate investment can serve as a prominent store of value. Thus, a low

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[1] See, e.g., Hunter (2005), Mian and Sufi (2014), Fang et al. (2015).

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interest rate environment, as seen in the recent decade, provides a fertile ground for the emergence of asset bubbles, including those in real estate. Given the prevalence of bubble episodes, a central question arises for academics and policymakers: What are the welfare effects of asset bubbles?

In this paper, we highlight the nuanced welfare effects of asset bubbles, especially those in real estate prices. We develop a simple overlapping generations (OLG) model of bubbles with intra-generation heterogeneity and financial friction. As described in Section 2 of the paper, households have identical preferences over a perishable consumption good and a durable and perfectly divisible housing asset in fixed supply. Young agents receive endowments, and a fraction of them are *savers*, who are born with high endowments, and the remaining fraction are *borrowers*, who are born with low endowments. Young borrowers, given their low endowment, want to borrow to purchase the desired amount of housing that maximizes their utility. In contrast, young savers, given their high endowment, would instead like to save for old age. Thus, for savers, housing not only yields utility dividend, but also serves as a savings or investment vehicle. To highlight the difference between these two motives, we assume that the utility over housing has a satiation level $\bar{h}$. Any additional unit of housing above the satiation level yields no additional utility and thus purely serves as an investment vehicle.

In an economy without financial friction, households can achieve their first-best allocations by borrowing and lending in the credit market. However, in the presence of financial friction, such as imperfect contract enforcement, young borrowers face a binding credit constraint, modeled as an exogenous limit on borrowers' debt capacity. In equilibrium, the constraint effectively limits how much savers can store their income by investing in the credit market. As we show in Section 3, in an economy with high income inequality, there is a shortage of storage for savers, which can lead to an equilibrium interest rate that is below the economy's growth rate. The low interest rate environment in turn facilitates the emergence of asset bubbles.

In Section 4, the main part of our paper, we study housing bubbles. In a housing bubble equilibrium, young savers acquire housing in excess of the satiation level, because they would like to use the housing asset as an investment vehicle to save for old age. This "speculative" demand for the housing asset by savers is similar to the demand of a bubble asset in a standard rational bubbles model: an agent purchases an asset because he or she expects to be able to sell it to someone else in the future.

We then show that the housing bubble has opposite effects on borrowers and savers. The housing bubble increases the return from real estate investment for high-income savers, who demand storage of value, and hence increases their welfare (relative to the bubbleless benchmark). In contrast, the housing bubble *reduces* the welfare of borrowers, because it raises the price of housing and the speculative demand of savers for the housing asset crowds out the allocation of housing to borrowers, who in equilibrium have a relatively higher marginal utility from housing. By positively affecting high-income savers and negatively affecting low-income borrowers, the housing bubble thus has *regressive welfare effects*. Overall, our results imply a feedback loop on inequality: high income inequality depresses the interest rate, thereby facilitating the existence of housing bubbles, which in turn have regressive welfare effects. The key insight is that, by creating a bubble in the market price of housing, savers' demand for the housing asset for investment purposes imposes a negative pecuniary externality on borrowers, who only demand the housing asset for utility purposes.

In comparison, Section 5.1 shows that the regressive welfare implications are lessened if the model considers pure bubbles, which are widely used in the rational bubble literature for their simplicity. A pure bubble is an asset that has no fundamental value, but which is traded at a positive price (such as fiat money or unbacked public debt). The pure bubble provides an additional and separate investment vehicle for savers: besides lending and investing in the housing market, savers can invest in the pure bubble asset (by purchasing the asset when young and reselling it when old). Unlike the housing bubble equilibrium, the pure bubble equilibrium is characterized by an *endogenous segmentation* in the bubble market. This is because only savers purchase the bubble asset for investment purposes, while credit-constrained borrowers have no demand for the asset. Furthermore, the option to invest in the pure bubble asset is a substitute for the option to use the housing asset purely as an investment vehicle. As a consequence, the crowding effect on housing that was prevalent in the housing bubble case is absent in the presence of a pure bubble. Consequently, the negative externality on borrowers' welfare is absent in the pure bubble equilibrium. Furthermore, because the presence of the pure bubble asset effectively enriches the available menu of investment vehicles, we can show that the allocations in the pure bubble equilibrium Pareto-dominates the allocations in the housing bubble equilibrium.

Related literature. Our paper is related to the rational bubble literature, which has a long heritage dating back to Samuelson (1958), Diamond (1965), and Tirole (1985). For a survey of this literature, see Miao (2014) and Martin and Ventura (2017).[2] Much of the literature has focused on a positive analysis of bubbles. A common theme in this literature is that rational bubbles emerge to reduce some inefficiency in the financial market, such as an aggregate shortage of assets for storage or a credit market imperfection, as in Miao and

[2] There is a complementary literature that focuses on the role of heterogeneous information in coordinating agents' actions to purchase and sell bubbles. See, inter alia, Abreu and Brunnermeier (2003), Doblas-Madrid (2012), Doblas-Madrid and Lansing (2014), and Barlevy (2014).

Wang (2011), Martin and Ventura (2012), and Hirano and Yanagawa (2017), Ikeda and Phan (2018). By departing from the pure bubble assumption and modeling a bubble as attached to a fundamentally useful durable asset such as housing, our paper is related to Arce and López-Salido (2011), Miao and Wang (2012), Wang and Wen (2012), Hillebrand and Kikuchi (2015), Zhao (2015), Kikuchi and Thepmongkol (2015), and Basco (2016).

To the best of our knowledge, among papers that analyze the welfare effects of bubbles, ours is the first to document regressive welfare effects of a housing bubble. Saint-Paul (1992), Grossman and Yanagawa (1993), and King and Ferguson (1993) show that if there is a positive externality in the accumulation of capital, the emergence of bubbles on an unproductive asset would inefficiently divert resources from investment. Similarly, Hirano et al. (2015) show that oversized bubbles inefficiently crowd out productive investment. On the other hand, Miao et al. (2015) show that bubbles can crowd in too much investment. Caballero and Krishnamurthy (2006) show that bubbles can marginally crowd out domestic savings and cause a shortage of liquid international assets in a small open economy framework. Focusing instead on risk, Ikeda and Phan (2016) show that rational bubbles financed by credit can be excessively risky. The regressive welfare effects that we highlight are complementary to the welfare effects highlighted by these papers.

2 Model

Consider an endowment economy with overlapping generations of agents who live for two periods. Time is discrete and infinite, with dates denoted by $t = 0, 1, 2, \ldots$. The population of young households in each period is constant with population $L_t = 1$ for all t. There is a consumption good and a housing asset. The consumption good is perishable and cannot be stored. The housing asset is durable and perfectly divisible. The supply of housing is fixed to one. The consumption good is the numeraire and the market price of a unit of the housing asset is denoted by p_t.

Heterogeneity. Each generation consists of two types of households – high-income savers and low-income borrowers (or debtors) – denoted by $i \in \{s, d\}$ correspondingly. Each group has an equal unit measure population. Each young household is endowed with e^i units of the consumption good, where $e^s > e^d$. In addition, each household receives an endowment of $e > 0$ when old.[3] Without loss of generality, we normalize $e^d = 1$. Thus, any increase in e^s leads to an increase in (within-generation) income inequality.

[3]We thus focus on the heterogeneity of endowments in young age. One can interpret the young-age endowment as wage income and the old-age endowment as payment from, e.g., social security. Furthermore, we have effectively set the economic growth rate to be zero. It is straightforward to extend our framework with exogenous endowment growth.

Preferences. Households derive utility from the housing asset and from the consumption good, consumed both when young and old. They have a separable utility function of the following form

$$U(c^i_{t,y}, c^i_{t+1,o}, h^i_t) \equiv u(c^i_{t,y}) + \beta u(c^i_{t+1,o}) + v(\min\{h^i_t, \bar{h}\}),$$

where $c^i_{t,y}$ and $c^i_{t+1,o}$ denote consumption in young and old age of a household of type $i \in \{s, d\}$ born in period t, β is the discount factor, h^i_t denotes the housing, and $\bar{h} \in (\frac{1}{2}, 1)$ is a constant that denotes the satiation level of housing consumption. The utility functions u and $v : (0, \infty) \to \mathbf{R}$ satisfy the usual Inada conditions ($u', v' > 0, u'', v'' < 0, \lim_{c \to 0+} u'(c) = \lim_{h \to 0+} v'(h) = \infty$).

The key assumption behind the expression $v(\min\{h^i_t, \bar{h}\})$ is that households receives no additional marginal utility beyond $\bar{h}$. This assumption helps clearly illustrate the speculative motive: if a household only purchases housing for utility, then it should never purchase more than $\bar{h}$; however, if a household additionally wants to use housing as an investment vehicle, then it may purchase in excess of $\bar{h}$.

Credit market and credit friction. Households can borrow and lend to each other via a credit market. Let R_t denote the gross interest rate for debt between period t and $t + 1$. As in Bewley (1977), Huggett (1993), and Aiyagari (1994), we model credit friction in the simplest possible way: an agent can commit to repay at most $\bar{d}$ units of the consumption good, where $\bar{d} \geq 0$ is an exogenous debt limit. This imperfection in the financial market will lead to a constraint on households' ability to borrow, as manifested in the optimization problem below.

Remark 1. The presence of the credit friction is important for the existence of asset price bubbles, which requires dynamic inefficiency. If the credit market were frictionless (e.g., $\bar{d} := \infty$), then the economy would be dynamically efficient and bubbles cannot arise (see Section 4).

Optimization. A household purchases housing, consumes, and borrows or lends when young, and then sells their housing asset and consumes when old. The optimization problem of a young household of type $i \in \{s, d\}$ born in period t consists of choosing housing asset position h^i_t, net financial asset position a^i_t, and old-age consumption c^i_{t+1} to maximize lifetime utility:

$$\max_{h^i_t, c^i_{t,y}, c^i_{t+1,o}, a^i_t} U(c^i_{t,y}, c^i_{t+1,o}, h^i_t) \tag{1}$$

subject to a budget constraint in young age:

$$p_t h_t^i + \frac{1}{R_t} a_t^i + c_{t,y}^i = e^i, \tag{2}$$

a budget constraint in old age:

$$c_{t+1,o}^i = p_{t+1} h_t^i + a_t^i + e, \tag{3}$$

nonnegativity constraints on housing and consumption:[4]

$$h_t^i, c_{t,y}^i, c_{t+1,o}^i \geq 0,$$

and the credit constraint:

$$a_t^i \geq -\bar{d}. \tag{4}$$

Throughout most of the paper, we assume for simplicity that:

$$\bar{d} = 0.$$

This assumption allows us to focus on the housing bubble's effect on the housing market rather on the credit market. We relax this assumption in Section 5.2 and show that, as long as $\bar{d}$ is sufficiently small so that the credit constraint binds for borrowers, our main results carry through.

Finally, to close the model, without loss of generality assume that the old savers own the entire supply of housing in the initial period $t = 0$. We define an equilibrium as follows:

Definition 1. An competitive equilibrium consists of allocation $\{h_t^i, c_{t,y}^i, c_{t+1,o}^i, a_t^i\}_{t\geq 0}$ and positive prices $\{p_t, R_t\}_{t\geq 0}$ such that:

1. Given prices, the allocations solve the optimization problem (1) for all $i \in \{s, d\}$ and $t \geq 0$.

2. The consumption good market clears:

$$c_{t,y}^s + c_{t,y}^d + p_t(h_t^d + h_t^s) = e^s + e^d + e, \forall t \geq 0;$$

3. The credit market clears:

$$a_t^s + a_t^d = 0, \forall t \geq 0;$$

[4] Because of the Inada conditions, these constraints do not bind in equilibrium.

4. And the housing market clears:

$$h_t^s + h_t^d = 1, \forall t \geq 0.$$

We will focus on steady-state (or stationary) equilibria, where quantities and prices are time-invariant.

For the analyses in subsequent sections, it is convenient to label a *constrained economy* as an economy where all households face an additional constraint $h^i \leq \bar{h}$, which effectively prevents them from purchasing units of the housing asset purely as an investment vehicle.[5]

As mentioned earlier, given the preferences over housing, if a household purchases more than $\bar{h}$ in housing, the excess amount $h - \bar{h}$ – which yields no additional housing utility to the household – must be purchased purely for investment. Thus, we label a *housing bubble equilibrium* as an equilibrium where at least one type of households purchases more housing than $\bar{h}$, i.e., $\max\{h^s, h^d\} > \bar{h}$. As shown in Section 4, this equilibrium can be mapped to a rational bubble equilibrium, where the bubble arises in the market value of housing. In contrast, we define a *bubbleless equilibrium* as an equilibrium where no type of household purchases housing in excess of $\bar{h}$, i.e., $\max\{h^s, h^d\} \leq \bar{h}$.

3 Bubbleless equilibrium

We start with the case of a bubbleless equilibrium, where households only purchase housing for utility and $h^i \leq \bar{h}$ for both types of households. For simplicity, we focus on the most interesting parameter space in which there is sufficient inequality (e^s sufficiently large) such that in equilibrium savers can satiate the utility for housing ($h^s = \bar{h}$), while borrowers are credit constrained and cannot achieve the satiation level ($h^d = 1 - \bar{h} < \bar{h}$).

Given $h^s = \bar{h}$, the conjectured equilibrium can be uniquely characterized by two key variables: the price of housing p and the interest rate R. Specifically, from budget constraints (2) and (3), the consumption profile c_y^i, c_o^i for $i \in \{s, d\}$ are functions of R and p. In turn, R and p are uniquely determined by a system of two first-order conditions, the first of which is the unconstrained savers' intertemporal Euler equation, and the second is the borrowers'

[5] This could be because the government imposes a regulation against speculative investment in housing. Note that recently several governments, including the Chinese government, have expressed intentions to curb speculation in the property market (e.g., Bloomberg, 2016).

interior ($0 < h^d < \bar{h}$) first-order condition with respect to housing:[6]

$$
\begin{aligned}
u'(c_y^s) &= \beta R\, u'(c_o^s), && (5)\\
p\, u'(c_y^d) &= v'(h^d) + \beta p\, u'(c_o^d), && (6)
\end{aligned}
$$

where the consumption allocations . The first condition equates the marginal cost (in terms of utility for a saver) of saving a unit of consumption to the marginal gain. The second equates the marginal cost (in terms of utility for a borrower) of investing in one unit of housing to the marginal gain, which consists of the marginal utility dividend v' and the marginal return from subsequently reselling the housing unit. We denote R_n and p_n as the solution to this system of equations (the subscript n refers to "no bubble"). Similarly, $\{c^i_{y,n}, c^i_{o,n}\}_{i\in\{s,d\}}$ and U^i_n denote the associated consumption profiles and lifetime utility. It is straightforward to show that all else equal, R_n is decreasing and p^s is increasing the endowment of savers e^s.

Figure 1 illustrates the determination of the interest rate R_n and housing price p_n. The blue curve labeled "foc savers" plots the lending first-order condition (5) of savers, while the red curve labeled "foc borrowers" plots condition (6) of borrowers.[7] Their intersection determines R_n and p_n. The dashed line plots what happens when there is an exogenous increase in the endowment e^s of savers while all other parameters stay the same (and thus an increase in inequality). The curve associated with the Euler equation of savers shifts to the left, leading to a decrease in the interest rate R_n (as there is an increase in the demand for storage from savers). Thus, higher inequality is associated with a lower interest rate.

For convenience, we implicitly define $\bar{e}$ as the savers' endowment threshold such that

$$
R_n|_{e^s=\bar{e}} = 1. \tag{7}
$$

It is then immediate that $R_n \geq 1$ if and only if $e^s \leq \bar{e}$. In other words, the interest rate R_n is at least as large as the growth rate of the economy if and only if there is not too much inequality.

For the conjectured prices and allocations characterized above to indeed constitute a bubbleless equilibrium, savers must find it optimal to not invest in housing in excess of $\bar{h}$. When is this the case? By purchasing each marginal unit of housing in excess of $\bar{h}$ at price p_t and reselling it at price p_{t+1} in the subsequent period, the individual gets a return rate of

[6] In the conjectured allocation, savers are consuming housing at the kink $h^s = \bar{h}$, where the housing utility term $v(\min\{h^s, \bar{h}\})$ is not differentiable. Hence, we do not have a first-order condition for housing for savers that corresponds to (6).

[7] The foc borrowers curve is flat because of the assumption $\bar{d} = 0$, which makes equation (6) independent of R. When $\bar{d} > 0$, the foc borrowers curve will be downward sloping, and all of our analysis will continue to hold. See Section A.8 and Figure 6 for more details.

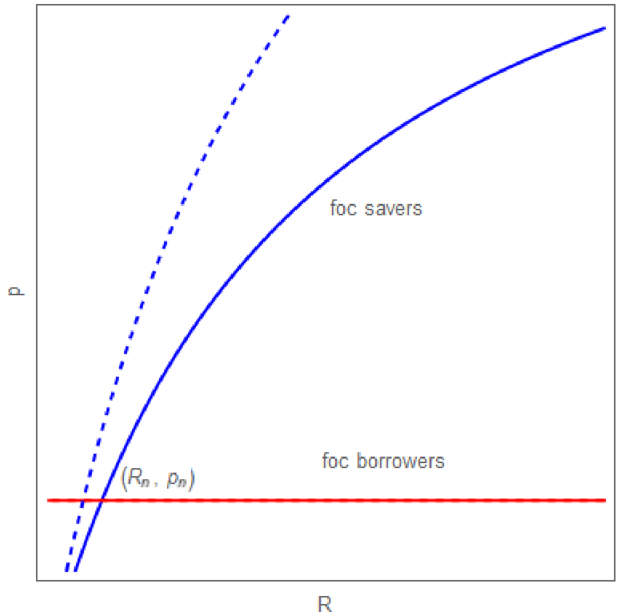

Figure 1: Determination of bubbleless steady state's interest rate and housing price

p_{t+1}/p_t, which is equal to 1 in steady state. The alternative method of saving via the credit market yields the return rate of R. Hence, for each individual saver to not have an incentive to use housing as an investment vehicle, it must be that $R_n \geq 1$, i.e., the interest rate R_n is at least as large as the growth rate of the economy. The following lemma formalizes this insight:

Lemma 1 (Bubbleless steady state)**.** *The bubbleless steady state exists if and only if $R_n \geq 1$, or equivalently, $e^s \leq \bar{e}$, where $\bar{e}$ is implicitly defined by (7). This steady state is characterized by interest rate R_n and housing price p_n, which are implicitly defined by (5) and (6) with housing allocation $h^s = \bar{h}$. All else equal, R_n is strictly decreasing in e^s.*

Proof. Appendix A.1. □

Remark 2. Note that the bubbleless equilibrium as described above is the only equilibrium in the constrained economy (recall Definition 1). This is because even when $e^s > \bar{e}$ and $R_n < 1$, the no-speculation constraint $h^i \leq \bar{h}$ binds for savers, which prevents them from making a profit from investing in the housing asset in excess of $\bar{h}$.

What happens when $e^s > \bar{e}$ and $R_n < 1$? As we show in the next section, the low interest rate environment provides a fertile ground for a bubble to arise in the housing market.

4 Housing bubble equilibrium

When $R_n < 1$, as we showed in the previous section, the bubbleless equilibrium does not exist, as savers find it attractive to invest in housing even if they do not derive additional utility. This section characterizes the housing bubble equilibrium that will instead arise.

In this equilibrium, savers will purchase housing in excess of $\bar{h}$. In the region $h^s > \bar{h}$, the housing utility function is differentiable. Hence, unlike in the bubbleless case, the first-order condition with respect to housing of savers must hold with equality, leading to the following equation:

$$pu'(c_y^s) = v'(h^s) + \beta pu'(c_o^s).$$

Combining the savers' housing first-order condition above with their lending first-order condition (5) yields a familiar asset pricing equation:

$$p = \underbrace{\frac{v'(h^s)}{u'(c_y^s)}}_{\text{marginal utility dividend}} + \underbrace{\frac{p}{R}}_{\text{resale value}}. \tag{8}$$

Equation (8) states that the price of each unit of housing is equal to the marginal housing dividend, captured by the marginal rate of substitution $\frac{v'(h^s)}{u'(c_y^s)}$, plus the resale value discounted by the interest rate p/R. Alternatively, using time subscript on p_t, equation (8) can be written as $p_t = \frac{v'(h^s)}{u'(c_y^s)} + \frac{p_{t+1}}{R}$, whose recursion leads to a standard forward-looking asset pricing equation $p_t = \sum_{j\geq 0} \frac{v'(h^s)/u'(c_y^s)}{R^j} + \lim_{j\to\infty} \frac{p_{t+j+1}}{R^j}$, where the first term on the right-hand side captures the discounted value of the marginal utility dividend stream, and the second term captures the "bubble component."

Since any amount in excess of $\bar{h}$ yields no marginal housing utility, the term $v'(h^s)/u'(c_y^s)$ in equation (8) is equal to zero, leading to:

$$p = \frac{p}{R}, \tag{9}$$

i.e., the price of housing must grow at the interest rate. Because $p > 0$, it then necessarily follows from (9) that $R = 1$ in the housing bubble equilibrium.

There is another intuitive way to understand the identity $R = 1$ as a no-arbitrage condition for savers. For these agents, the investment in each additional unit of housing in excess of $\bar{h}$ yields a marginal return rate of 1 in steady state, as savers purchase each unit of housing and subsequently resell it for a return rate of $p/p = 1$. In equilibrium, savers must be indifferent between investing in housing and lending. For this to be the case, the interest rate on lending R must be equal to 1.

In summary, the conjectured housing bubble equilibrium can be characterized as follows. Savers purchase $h^s > \bar{h}$ (instead of $h^s = \bar{h}$ as in the bubbleless case). The interest rate is determined by savers' no-arbitrage condition $R_b = 1$. Other variables are uniquely characterized by the housing allocation h^s and the housing price p via budget constraints (2) and (3). In turn, h^s and p are determined by two equations: savers' intertemporal Euler (5), which given $R = 1$ is simplified to

$$u'(c_y^s) = \beta u'(c_0^s), \tag{10}$$

and borrowers' housing first-order condition (6). We denote by h_b^s and p_b the solution to this system of equations. Similarly, $\{c_{y,b}^i, c_{o,b}^i\}_{i\in\{s,d\}}$ and U_b^i denote the associated consumption profiles and lifetime utility.

Figure 2 illustrates the determination of h_b^s and p_b. Here, the curve labeled "foc borrowers" plots the housing first-order condition (6) of borrowers on the $h^s \times p$ plane, while the curve labeled "foc savers" plots the lending first-order condition (10) of savers. Their intersection determines h_b^s and p_b.

Furthermore, as in Figure 1, the dashed line in Figure 2 plots what happens when there is an exogenous increase in e^s while all other parameters stay the same. The curve associated with the Euler equation of savers shift to the right on the $h^s \times p$ plane, leading to an increase in the housing price p_b and in h_b^s. This is because savers have more resources to invest into the housing market.

As formalized in the following lemma, the conjectured allocations and prices characterized above indeed constitute a housing bubble steady state if and only if $e^s > \bar{e}$:

Lemma 2 (Housing bubble steady state)**.** *The housing bubble steady state exists if and only if $R_n < 1$, or equivalently $e^s > \bar{e}$. This steady state is characterized by interest rate $R_b = 1$, housing allocation $h^s > \bar{h}$, and housing price p_b, which are implicitly defined by (6) and (10). All else equal, h_b^s and p_b are strictly increasing in e^s.*

Proof. Appendix A.2. □

Remark 3. An interesting feature of our model that differs from the standard rational bubbles models (e.g., Samuelson, 1958; Diamond, 1965; Tirole, 1985) is that the housing asset is *not* a pure bubbly asset – defined as an asset that does not have any fundamental value but is traded at a positive price (see Section 5.1). In our framework, the housing asset is fundamentally useful as it yields utility to households. However, the pricing equation (8) of the housing asset is similar to that of a pure bubbly asset in the sense that, from the perspective of savers, each *marginal* unit of additional investment in the housing asset beyond the satiation point $\bar{h}$ does not yield any additional *marginal* utility.

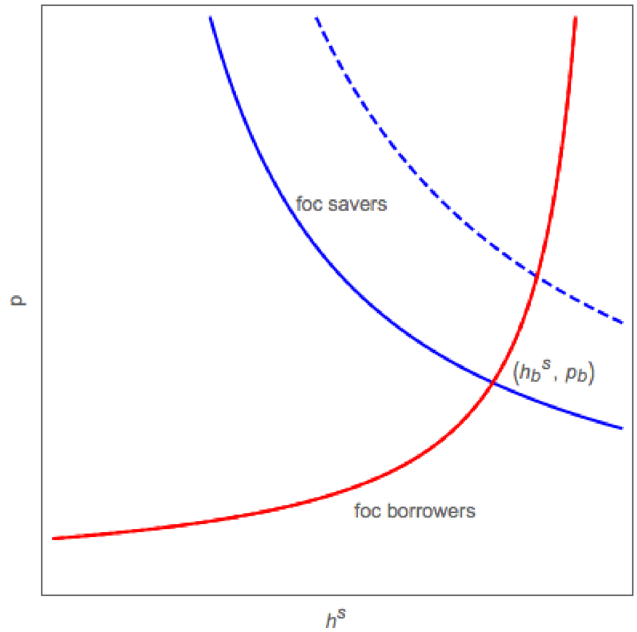

Figure 2: Determination of housing bubble steady state's interest rate and housing price

Remark 4. As is standard in the rational bubbles literature, the housing bubble steady state is saddle-path stable, as shown in Appendix A.7.

Remark 5. The low interest rate condition ($R_n < 1$) for the existence of bubbles is related to the dynamic inefficiency condition in Diamond (1965) and Tirole (1985). There has been an active debate over the empirical validity of this condition. For instance, Abel et al. (1989) argue that the U.S. (between 1929 and 1985) and six other advanced economies (between 1960 and 1985) satisfy a sufficient condition for dynamic efficiency that aggregate investment falls short of capital income. However, using more updated data on mixed income and land rents, Geerolf (2017) finds evidence to the contrary, providing strong support for the hypothesis that these major economies are dynamically inefficient.[8] Recent theoretical models have also pointed out that dynamic inefficiency is not always a necessary condition for the existence of rational bubbles if there are financial frictions (e.g., Miao and Wang, 2011; Farhi and Tirole, 2012; Martin and Ventura, 2012) or international capital inflows (e.g., Ikeda and Phan, 2018).

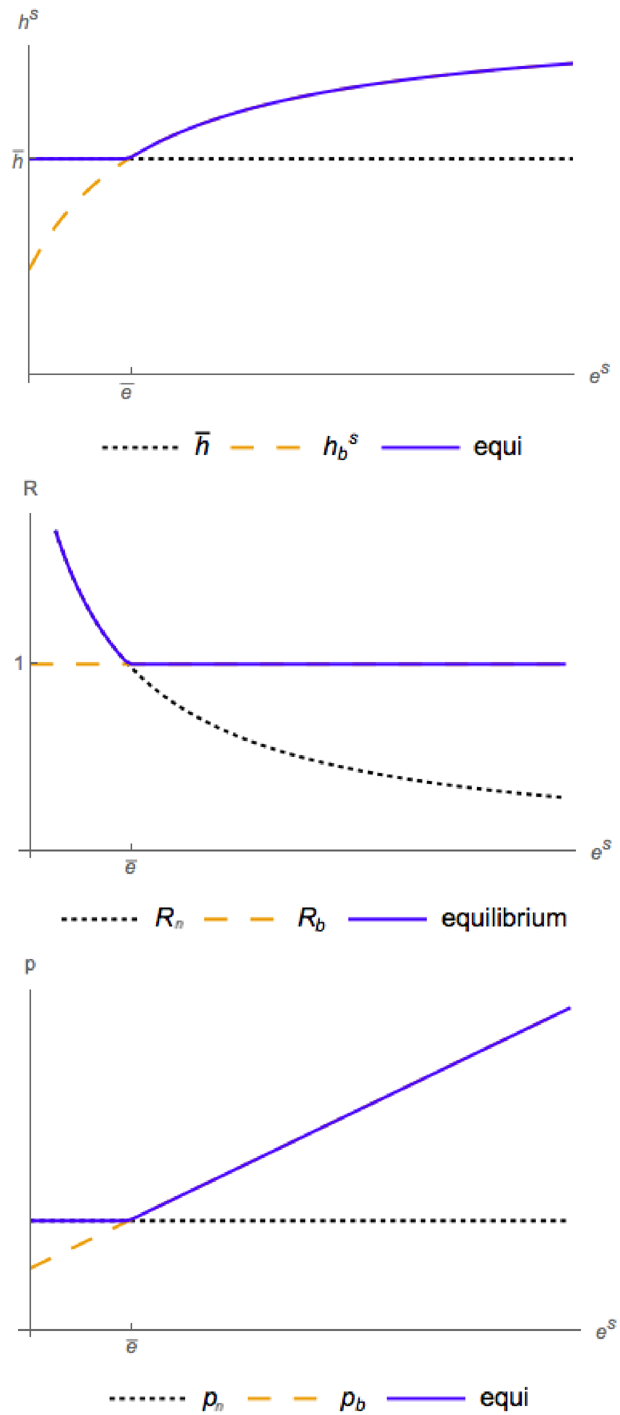

Figure 3: Equilibrium housing allocation, interest rate, and housing price as functions of savers' endowment

4.1 Comparative Statics

We now conduct a comparative statics exercise and compare the bubble and the bubbleless equilibria. Throughout, we vary the endowment of savers e^s while keeping all other parameters fixed. Recall that an increase in e^s leads to an increase in the income of the top earners (savers) relative to the bottom earners (borrowers).

Figure 3 plots the equilibrium interest rate R, housing price p, and housing allocation h^s as functions of e^s.[9] It also plots $h_n = \bar{h}, R_n, p_n$ (the dotted lines) and h_b^s, R_b, p_b (the dashed lines). From Lemma 1 and Lemma 2, we know that when $e^s \leq \bar{e}$, the equilibrium is bubbleless and savers only purchase housing up to the satiation level $h^s = \bar{h}$ (the top panel). The interest rate and housing price are determined by R_n and p_n (the middle and bottom panels). As shown and as consistent with Lemma 1, R_n is decreasing while p_n is increasing in e^s.

When $e^s > \bar{e}$, we know from the lemmas that the housing bubble equilibrium arises and replaces the bubbleless equilibrium, explaining the kinks in the functions for the equilibrium quantity and prices. Because of the investment motive, savers purchase housing beyond the satiation level $\bar{h}$ (the top panel). The interest rate is equal to the marginal return rate of buying and selling a unit of housing: $R_b = 1$ (the middle panel). Consistent with Lemma 2, the housing price p_b and housing allocation h_b^s are increasing in e^s (the bottom and top panels).

Most importantly, note that in the region $e^s > \bar{e}$, not only do savers purchase more housing ($h^s > \bar{h}$), but also the equilibrium interest rate R_b is higher than R_n (the middle panel) and the equilibrium housing price p_b is higher than p_n (the bottom panel). In other words, the presence of the bubble in the housing market when $e^s > \bar{e}$ raises the equilibrium interest rate and the housing price. The following lemma summarizes this observation:

Lemma 3 (Housing bubble prices)**.** *If $e^s > \bar{e}$ (so that the housing bubble equilibrium exists), then $p_b > p_n$ and $R_b > R_n$.*

Proof. See Appendix A.3. □

Intuitively, by providing an additional investment vehicle for savers, the housing bubble raises savers' demand for the housing asset, leading to an increase in its price. Furthermore,

[8] See also Giglio et al. (2016), who find no evidence of bubbles that violate the transversality condition in housing markets in the U.K. and Singapore. Also see Engsted et al. (2016) who provide econometric evidence for explosive housing bubbles in OECD countries.

[9] To be consistent with our analysis, the minimum value of e^s in the figure is set to be sufficiently large (relative to e^d, which was normalized to one) so that borrowers are credit constrained and they cannot achieve the satiation housing level $\bar{h}$ in equilibrium.

the investment by savers in the bubbly housing market crowds out their lending in the credit market, raising the interest rate.

4.2 Welfare Analysis

We can now establish the main result of the paper that the presence of the housing bubble has regressive welfare effects and exacerbates welfare inequality. Recall that $U_n^i, i \in \{s, d\}$, denotes the lifetime utility of households from bubbleless consumption profile $\{c_{y,n}^i, c_{o,n}^i\}$ and housing allocation $h^s = 1 - h^d = \bar{h}$ derived in Section 3.[10] Similarly, U_b^i denotes the lifetime utility from consumption profile $\{c_{y,b}^i, c_{o,b}^i\}$ and housing allocation $h^s = 1 - h^d = h_b$ derived in Section 4. We will compare U_n^i and U_b^i.

The presence of the housing bubble has heterogeneous effects on savers and borrowers. For savers, who want to save for old age, the housing bubble improves their welfare by improving the return from investing in the housing asset. In contrast, the housing bubble has a *negative* effect on the welfare of borrowers. This is because it increases the price of housing, hence reducing the amount of housing that borrowers purchase and consequently their housing utility. The following proposition summarizes these regressive welfare effects of the bubble and the main result of our paper:

Proposition 4 (Housing bubble benefits savers but not borrowers)**.** *If $e^s > \bar{e}$ (so that the housing bubble equilibrium exists), then $U_b^s > U_n^s$ but $U_b^d < U_n^d$.*

Proof. Appendix A.4 □

Figure 4 illustrates the welfare effects of the housing bubble. It plots U_b^i and U_n^i as functions of e^s. When $e^s > \bar{e}$, the housing bubble arises and it raises the welfare of savers (reflected by the fact that the solid line representing savers' lifetime utility U_b^s lies above the dotted line representing U_n^s) but reduces the welfare of borrowers (reflected by the fact that the solid line representing borrowers' lifetime utility U_b^d lies below the dotted line representing U_n^d).

Furthermore, the figure also illustrates that the housing bubble exacerbates the welfare inequality in the economy. In the low-inequality region $e^s \leq \bar{e}$, a higher value of e^s is (obviously) associated with a higher welfare level for savers in equilibrium, as reflected by an upward sloping solid curve representing U_n^s. However, in this region, in the absence of speculative demand for the housing asset, an increase in e^s does not lead to a higher demand

[10] Also recall that when $e^s > \bar{e}$, even though the bubbleless equilibrium no longer exists, U_n^i can still be thought of as the lifetime utility in the constrained economy where households cannot use housing for speculation.

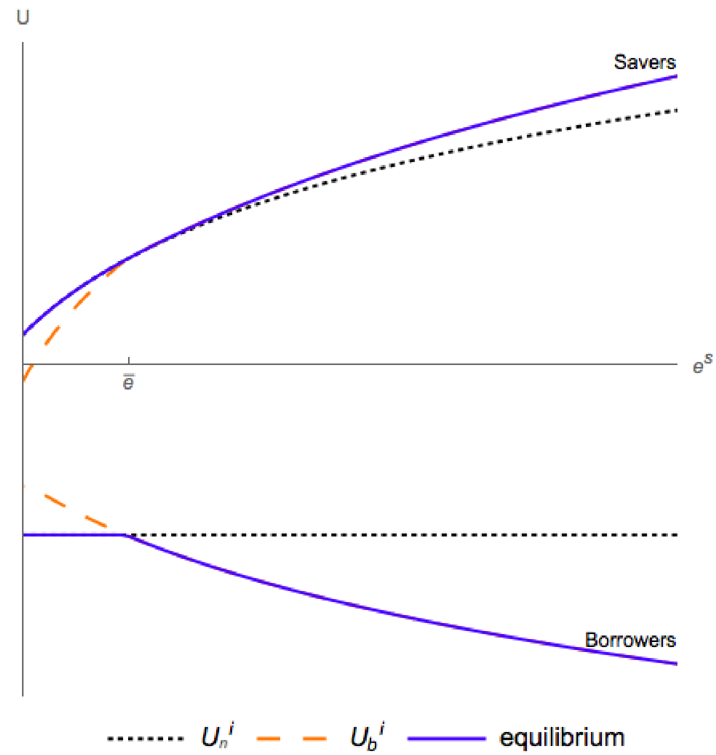

Figure 4: Comparative statics on welfare

for housing, nor a higher housing price. Thus, borrowers are not affected, as reflected by a flat solid line representing U_n^d.

In the high-inequality region $e^s > \bar{e}$, the equilibrium features a housing bubble and the equilibrium lifetime utility switch from U_n^i to U_b^i (represented by the kinks at $\bar{e}$ in the solid curves). In this region, an increase in e^s raises savers' speculative demand for and the price of the housing asset, crowding out the allocation of housing to borrowers. Hence, borrowers' lifetime utility decreases, as reflected by the downward sloping portion of the solid curve representing U_b^d. Therefore, an interesting implication arises on the feedback loop between inequality and bubble: high income inequality facilitates the existence of a housing bubble, which in turn has regressive welfare effects and exacerbates welfare inequality.

5 Discussions

5.1 Comparison with a pure bubble

To appreciate the welfare results established in the previous section, we compare them against the welfare effects of a pure bubble, which is an asset that pays no dividend but has a positive market price and which has been analyzed extensively in the literature (e.g., Samuelson, 1958; Diamond, 1965; Sargent and Wallace, 1982; Tirole, 1985). The pure bubble asset can be useful as a savings instrument, however, unlike housing, the pure bubble asset does *not* give households any direct utility. As a consequence, there will be an *endogenous segmentation* of the pure bubble market, as only savers purchase the asset as an investment vehicle. Consequently, the bubble will have much less effect on borrowers, as we show below.

5.1.1 Housing asset does not yield any utility

First, let us consider the benchmark where households do not derive utility from housing, that is, $v(h) \equiv 0$. Then the housing asset provides no fundamental value. The model is then similar to the models of pure rational bubbles (e.g., fiat money or unbacked government debt) of Samuelson (1958), Diamond (1965), and Sargent and Wallace (1982).

As is standard and well known in this environment, there is always a bubbleless equilibrium where the housing asset is not traded. The lifetime utility in this equilibrium is given by $U_n^s = u(e^s) + \beta u(e)$ and $U_n^d = u(e^d) + \beta u(e)$. As usual, the bubbleless interest rate is determined by the Euler equation of savers: $R_n = \frac{u'(c_y^s)}{\beta u'(c_o^s)} = \frac{u'(e^s)}{\beta u'(e)}$.

Furthermore, when $R_n < 1$, the economy is dynamically inefficient and there exists another bubble equilibrium, where the housing asset is traded at a positive price. The key difference compared to the equilibrium in Section 4 is that, because here households do not derive utility from housing, borrowers will *not* have any incentive to purchase the housing asset. Thus, the equilibrium features an endogenous segmentation in the housing market: $h^d = 0$ and $h^s = 1$. As investing in the bubble allows savers to transfer more resources from young age to old age, it improves their welfare: savers' lifetime utility with the bubble is $U_b^s = u(e^s - p_b) + \beta u(e + p_b) > U_n^s$, where p_b is the bubble equilibrium price of the housing asset. As borrowers do not participate in the housing market, the rise in housing price due to the bubble does *not* affect them: borrowers' lifetime utility with the bubble is also $U_b^d = u(e^d) + \beta u(e) = U_n^d$.

In summary, when households do not derive utility from housing ($v \equiv 0$), the negative effect on borrowers of the increase in the price of housing due to the bubble is absent. The endogenous segmentation of the housing market effectively shields borrowers from the effect of the housing price bubble.

5.1.2 Pure bubble as another asset

In this subsection, we will show that the insight from the previous subsection also carries through to an alternative environment where households still derive utility from housing (i.e., v satisfies the same properties as in Section 2) but an additional pure bubble asset is introduced.

For clarity, in this section we focus on the constrained economy where households face a constraint $h^i \leq \bar{h}$ for $i \in \{s, d\}$, which prevents them from using units of the housing asset purely as an investment vehicle. This assumption effectively prevents the possibility of a housing bubble coexisting with a pure bubble in equilibrium and thus allows us to clearly distinguish between the housing bubble equilibrium and the pure bubble equilibrium to be derived below.

Formally, assume there is an asset in fixed unit supply that pays no dividend but is traded at price b_t per unit. Given prices, each household of type i chooses its holding $x_t^i \geq 0$ of the bubble asset. Their optimization problem is:

$$\max_{h_t^i, c_{t,y}^i, c_{t+1,o}^i, x_t^i, a_t^i} \quad U(h_t^i, c_{t,y}^i, c_{t+1,o}^i), \tag{11}$$

subject to budget constraints:

$$\begin{aligned} p_t h_t^i + b_t x_t^i + a_t^i \frac{1}{R_t} + c_{y,t}^i &= e^i, & (12) \\ c_{o,t+1}^i &= p_{t+1} h_t^i + b_{t+1} x_t^i + a_t^i + e, & (13) \end{aligned}$$

the nonnegativity constraints:

$$x_t^i, h_t^i, c_{t,y}^i, c_{t+1,o}^i \geq 0,$$

the credit constraint (4), and the constraint on housing investment:

$$h_t^i \leq \bar{h}.$$

To close the model, assume that old savers own the entire supply of housing and the bubble in the initial period $t = 0$.

The definition of a pure bubble equilibrium is similar to Definition 1, except that we have an additional condition that the pure bubble price is positive:

$$b_t > 0, \forall t \geq 0,$$

and an additional market clearing condition of the bubble asset:

$$x_t^s + x_t^d = 1, \forall t \geq 0.$$

As before, we focus on steady-state equilibria where the quantities and prices are time-invariant.

Existence and Characteristics. Similar to the housing bubble case, savers in equilibrium must be indifferent between investing in the pure bubble asset and lending. The former yields a return of $b/b = 1$ in steady state, and the latter yields R. Hence, the interest rate in the pure bubble steady state must also be $R = 1$ as in the housing bubble case. It is also straightforward to show, as in Lemma 2, that the pure bubble exists if and only if $R_n < 1$, or equivalently, $e^s > \bar{e}$ (see Appendix A.5). Furthermore, given $e^s > \bar{e}$, it is straightforward to show that the constraint $h^s \leq \bar{h}$ binds for savers, leading to allocation $h^s = 1 - h^d = \bar{h}$.

Given that borrowers are credit constrained, and that the return from the pure bubble asset is equal to the interest rate, it is not optimal for borrowers to invest in the pure bubble asset. Thus, the equilibrium will feature an endogenous segmentation of the market into participants (savers) and nonparticipants (borrowers), i.e., $x^s = 1$ and $x^b = 0$.

Given $h^s = \bar{h}$ and $x^s = 1$, other variables in the conjectured equilibrium are uniquely characterized by two key variables: the price of housing p and the price of the pure bubble asset b. From budget constraints (12) and (13), the consumption profile c_y^i, c_o^i are functions of p and b. In turn, p and b are determined by the first-order conditions (6) and (10). The following lemma summarizes our discussion above. It also shows that the price of housing in the pure bubble equilibrium is the same as the price of housing in the bubbleless equilibrium:

Lemma 5 (Pure bubble equilibrium)**.** *The pure bubble steady state exists if and only if $e^s > \bar{e}$. The steady state is characterized by interest rate $R = 1$, housing allocation $h^s = \bar{h}$, pure bubble asset allocation $x^s = 1$, along with housing price $p = p_n$ and pure bubble price b that is implicitly determined by the Euler equation of savers:*

$$u'(e^s - p_n\bar{h} - b) = \beta u'(e + p_n\bar{h} + b) \tag{14}$$

Proof. Appendix A.5. □

Welfare Analysis. Assume $e^s > \bar{e}$ so that the pure bubble equilibrium exists. Are the welfare implications of a pure bubble different from those of a housing bubble? Let U_x^i denote the lifetime utility in the pure bubble equilibrium.

Like the housing bubble, the pure bubble allows savers to store their income into old age more efficiently and hence improves their welfare relative to the bubbleless case, i.e., $U_x^s > U_n^s$.

However, as the pure bubble market absorbs savers' demand for storage, savers no longer need to use the housing asset for an investment purpose. Thus, the pure bubble does not affect the housing price (as stated in Lemma 5). As a consequence, it does not affect the welfare of borrowers, i.e., $U_x^d = U_n^d$. In summary, the negative externality of savers' investment in the housing market is absent in the pure bubble equilibrium.

Furthermore, we can compare welfare across the pure bubble equilibrium and the housing bubble equilibrium. On the one hand, because the negative externality on borrowers is absent in the pure bubble case, it is straightforward to see that borrowers are better off in this case: $U_x^d > U_b^d$. On the other hand, because the pure bubble is similar to the housing bubble in providing storage for savers, it can be shown that savers get the same welfare in the two equilibria: $U_x^s = U_b^d$. The intuition above is summarized by the following result:

Corollary 6. *The pure bubble steady state Pareto-dominates the housing bubble steady state:* $U_x^d > U_b^d$ *and* $U_x^s = U_b^s$.

Proof. Appendix A.6 □

5.2 Role of the credit constraint

So far, we have assumed that the credit limit is $\bar{d} = 0$. However, our main result (in particular Proposition 4 of the regressive welfare effects) carries through if $\bar{d} > 0$, as long as we continue to assume e^s is sufficiently large so that the credit constraint (4) binds for borrowers in equilibrium. In fact, the negative effect of the housing bubble on borrowers will be strengthened when $\bar{d} > 0$. This is because the bubble exerts an additional negative effect on borrowers through the credit market. To see this, recall that the budget constraint of credit-constrained young borrowers is given by

$$c_y^d + ph^d = e^d + \frac{\bar{d}}{R}.$$

Since the bubble raises the interest rate R (relative to the bubbleless rate), it reduces the total resources available for young borrowers $e^d + \frac{\bar{d}}{R}$ and thus reduces their ability to consume and purchase housing. In short, the housing bubble not only raises the cost of housing, but also raises the cost of borrowing for credit-constrained borrowers. On the flip side, by raising the interest rate on lending, the bubble provides an additional benefit to savers who want save for old age. Thus, the presence of a positive binding credit constraint $\bar{d}$ amplifies the regressive welfare effect of the housing bubble. Appendix A.8 provides more details of the analysis.

5.3 Policy discussions and rental market

Given the externality of savers' demand in housing for investment purposes, policy interventions may be warranted. In fact, recent policy debates in China and other emerging economies (e.g., Bloomberg, 2016; Nikkei Asian Review, 2017; The Economist, 2018), can be discussed within the context of our model.[11] The well-documented shortage of assets for storage of wealth, such as safe government bonds, in these economies could be a reason why bubbles tend to arise in the real estate markets. A restriction on speculative investment in housing could prevent housing bubbles but would entail trade-offs, because it would help low-income households but hurt high-income households, as highlighted in our welfare analysis.

Furthermore, we have so far abstracted away from the presence of a rental market. In Appendix A.9, we extend the model to relax this assumption. We show that if the rental market is frictionless, then borrowers can rent from savers, reducing the demand for borrowing and thus raising the interest rates. As a consequence, the housing bubble may not arise. However, the housing bubble will arise again just as in the benchmark model if there are sufficient frictions in the rental market. In practice, many problems, including the presence of moral hazard and adverse selection, are prevalent in the rental markets. It has been documented that such markets are quite underdeveloped in emerging economies, including even in major cities in China (The Economist, 2018). An implication of the model is that the development of a functioning rental market, as discussed in recent proposals by the Chinese government to develop a market for good-quality rental housing, could help mitigate the negative externality of the housing bubble on low-income households.

6 Concluding remarks

We have shown that a housing bubble, or, more generally, a bubble attached to a fundamentally useful asset, has heterogeneous welfare effects on households, depending on their demand for savings and borrowing. By providing an additional investment vehicle, it raises the returns from investment for savers and thus improves their welfare. However, by raising the interest rate on debt and raising the housing price, the housing bubble negatively affects the welfare of borrowers, who need debt to finance their purchase of housing. Our model also implies a feedback loop on inequality: high income inequality leads to an environment with low interest rates, which facilitate housing bubbles, which in turn have regressive welfare effects. The key insight is that savers' demand for the housing asset for investment purposes imposes a negative externality on borrowers, who only demand the housing asset for utility purposes, by creating a bubble in the market price.

[11] As an anecdote, the Chinese leader Xi Jinping said in his address at the 19th Party Congress in Beijing that "Houses are built to be inhabited, not for speculation" (Bloomberg, 2017).

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A Appendix

A.1 Proof of Lemma 1

Recall that R_n and p_n solve the two first-order conditions (5) and (6) with $h^s = \bar{h}$, which can be rewritten as:

$$u'(\underbrace{e^s - p_n\bar{h}}_{c^s_{y,n}}) = \beta R_n u'(\underbrace{e + p_n\bar{h}}_{c^s_{o,n}}) \tag{15}$$

$$p_n u'(\underbrace{e^d - p_n(1-\bar{h})}_{c^d_{y,n}}) = v'(1-\bar{h}) + \beta p_n u'(\underbrace{e + p_n(1-\bar{h})}_{c^d_{o,n}}). \tag{16}$$

Equation (16) uniquely determines p_n. To see this, define the difference between the two sides of (16) as:

$$\Delta(p) \equiv p_n u'\left(e^d - p(1-\bar{h})\right) - v'(1-\bar{h}) - \beta p u'\left(e + p(1-\bar{h})\right).$$

Because u is strictly concave, this function is strictly increasing in p. Also, $\Delta(0) < 0$. Furthermore, since $\lim_{c\to 0+} u'(c) = \infty$, it follows that $\lim_{p\to e^d/(1-\bar{h})-} \Delta(p) = \infty$. Hence, by continuity, there exists a unique $p_n \in (0, e^d/(1-\bar{h}))$ such that $\Delta(p_n) = 0$. Then, given p_n, equation (15) uniquely determines $R_n = \frac{u'\left(e^s - p_n\bar{h}\right)}{\beta u'\left(e + p_n\bar{h}\right)}$.

We now show that all else equal, R_n is increasing in e^s. Since an increase in e^s does not affect equation (16), it follows that p_n is not affected by a change in e^s. As a consequence, an increase in e^s lowers $R_n = \frac{u'(e^s - p_n h^s)}{\beta u'(e + p_n h^s)}$.

Finally, we show that the bubbleless steady state exists if and only if $e^s \leq \bar{e}$. First, suppose on the contrary that a bubbleless steady state exists but $e^s > \bar{e}$. Because the interest rate R_n is decreasing in e^s, by the definition of $\bar{e}$, it follows that $R_n < 1$. However, this would imply that a saver would prefer to unilaterally deviate from purchasing $h^s = \bar{h}$ to purchasing $h^s > \bar{h}$, because investing in additional housing yields a marginal return rate of 1 that dominates the return R_n from the credit market. Thus, the existence of a bubbleless steady state requires $e^s \leq \bar{e}$.

Second, suppose $e^s \leq \bar{e}$. Then $R_n \geq 1$. Thus, no saver would have an incentive to unilaterally deviate from purchasing $h^s = \bar{h}$ to purchasing $h^s > \bar{h}$, because investing in additional housing yields a marginal return rate of 1 that is weakly dominated by the return R_n from the credit market. It then straightforwardly follows that the allocations and prices associated with $h^s = \bar{h}$, R_n, and p_n satisfy the market clearing and individual optimization problems and thus constitute a bubbleless steady state.

A.2 Proof of Lemma 2

Recall that h^s_b and p_b solve (10) and (6), which can be rewritten respectively as:

$$u'(\underbrace{e^s - ph^s}_{c^s_{y,b}}) = \beta u'(\underbrace{e + ph^s}_{c^s_{o,b}}) \tag{17}$$

$$pu'(\underbrace{e^d - p(1-h^s)}_{c^d_{y,b}}) = v'(1-h^s) + \beta p u'(\underbrace{e + p(1-h^s)}_{c^d_{o,b}}). \tag{18}$$

Using similar arguments to those used in Section A.1, it is straightforward to show that h^s_b and p_b are unique.

Now, we show h_b^s and p_b are increasing in e^s. Consider two arbitrary endowment levels $e_1^s > e_2^s$, with the corresponding solutions $(h_{b,1}^s, p_{b,1})$ and $(h_{b,2}^s, p_{b,2})$. We need to show that $h_{b,1}^s > h_{b,2}^s$ and $p_{b,1} > p_{b,2}$.

Because of the concavity of u, equation (17) determines ph^s as an increasing function of e^s, implying $p_{b,1}h_{b,1}^s > p_{b,2}h_{b,2}^s$. Equation (18) can be further rewritten as:

$$p\left[u'\left(e^d - p + ph^s\right)\right) - \beta u'\left(e + p - ph^s\right)\right] = v'(1 - h^s).$$

Thus,

$$\frac{p_{b,1}\left[u'\left(e^d - p_{b,1} + p_{b,1}h_{b,1}^s\right)\right) - \beta u'\left(e + p_{b,1} - p_{b,1}h_{b,1}^s\right)\right)\right]}{p_{b,2}\left[u'\left(e^d - p_{b,2} + p_{b,2}h_{b,2}^s\right)\right) - \beta u'\left(e + p_{b,2} - p_{b,2}h_{b,2}^s\right)\right)\right]} = \frac{v'(1 - h_{b,1}^s)}{v'(1 - h_{b,2}^s)}. \tag{19}$$

Suppose on the contrary that $p_{b,1} \leq p_{b,2}$. Then, because $p_{b,1}h_{b,1}^s > p_{b,2}h_{b,2}^s$, it must be that $h_{b,1}^s > h_{b,2}^s$. Then the left-hand side of (19) is smaller than 1, while the right-hand side is larger than 1, leading to a contradiction. Therefore, $p_{b,1} > p_{b,2}$. Similarly, $h_{b,1}^s > h_{b,2}^s$. In other words, both h_b^s and p_b as implicitly determined by (17) and (18).

Now we show that the housing bubble steady state exists if and only if $e^s > \bar{e}$. By definition, the conjectured allocations and prices constitute a housing bubble steady state if and only if $h_b^s > \bar{h}$. Note that when $e^s = \bar{e}$, the systems (15-16) and (17-18) yield the same solution: $R_n = R_b = 1$, $h_n^s = h_b^s = \bar{h}$, and $p_n = p_b$. Furthermore, we already proved that h_b^s is increasing in e^s. Hence, $h_b^s > \bar{h}$ if and only if $e^s > \bar{e}$. This completes the proof.

A.3 Proof of Lemma 3

The fact that if $e^s > \bar{e}$, then $R_b > R_n$ immediately follows from Lemma 1 and Section A.1. From Lemma 2 and the proof in Section A.2, we know that p_b is increasing in e^s, and that $p_b = p_n$ when $e^s = \bar{e}$. Hence, it immediately follows that if $e^s > \bar{e}$, then $p_b > p_n$.

A.4 Proof of Proposition 4

We start by showing that $U_b^s > U_n^s$. Recall that

$$U_b^s = u(\underbrace{e^s - p_b h_b^s}_{c_{y,b}^s}) + \beta u(\underbrace{e + p_b h_b^s}_{c_{o,b}^s}) + v(\bar{h}),$$

where the three terms on the right-hand side correspond to the utility over consumption in young age, the discounted utility over consumption in old age, and the utility over (satiated) housing consumption, respectively. Similarly, recall that

$$U_n^s = u(\underbrace{e^s - p_n \bar{h}}_{c_{y,n}^s}) + \beta u(\underbrace{e + p_n \bar{h}}_{c_{o,n}^s}) + v(\bar{h}).$$

Thus, after canceling out the $v(\bar{h})$ term (as savers satiate their housing utility in both steady-state allocations), it suffices to show that

$$u(e^s - p_b h_b^s) + \beta u(e + p_b h_b^s) > u(e^s - p_n \bar{h}) + \beta u(e + p_n \bar{h}),$$

or equivalently

$$F(p_b h_b^s) > F(p_n \bar{h}), \tag{20}$$

where $F(x) \equiv u(e^s - x) + \beta u(e + x)$. Note that the solution to $\max_x F(x)$ is the solution to the first-order condition $u'(e^s - x) = \beta u'(e + x)$. From (17), it follows that $x = p_b h_b^s$ solves $\max_x F(x)$. From (15) and the fact that $R_n < 1$, it follows that $x = p_n \bar{h}$ does *not* solve $\max_x F(x)$. Therefore, (20) automatically follows. This completes the proof of $U_b^s > U_n^s$.

Now we show $U_n^d > U_b^d$. Recall that

$$U_b^d = u(\underbrace{e^d - p_b h_b^d}_{c_{y,b}^d}) + \beta u(\underbrace{e + p_b h_b^d}_{c_{o,b}^d}) + v(h_b^d),$$

and

$$U_n^d = u(\underbrace{e^d - p_n h_n^d}_{c_{y,n}^d}) + \beta u(\underbrace{e + p_n h_n^d}_{c_{o,n}^d}) + v(h_n^d),$$

where $h_b^d = 1 - h_b^s$ and $h_n^d = 1 - \bar{h} < \bar{h}$. Also recall that $h_b^d < h_n^d$. Hence, under the bubbleless steady-state prices, the housing allocation h_b^d (along with the net asset position $a_b^d = 0$) is feasible for borrowers. Thus, by the definition of h_n^d and $a_n^d = 0$ as the solution to the optimization problem of borrowers given the bubbleless steady-state prices, it follows that borrowers must be at least better off with the bubbleless steady-state allocations than with the bubble steady-state allocations:

$$U_n^d \geq u(e^d - p_n h_b^d) + \beta u(e + p_n h_b^d) + v(h_b^d). \tag{21}$$

Furthermore, combining the fact that $p_n h_b^d < p_b h_b^d$ (because the housing price is higher in the housing bubble steady state) and the fact that $u'(e^d - p_b h_b^d) > \beta u'(e + p_b h_b^d)$ (borrowers in the housing bubble steady state are constrained in their ability to tilt consumption forward) yields

$$u(e^d - p_n h_b^d) + \beta u(e + p_n h_b^d) > u(e^d - p_b h_b^d) + \beta u(e + p_b h_b^d). \tag{22}$$

Inequalities (21) and (22) then imply $U_n^d > U_b^d$, as desired.

A.5 Proof of Lemma 5

It is straightforward that given the conjectured prices, the conjectured allocations solve the optimization problems of individual borrowers and savers. Thus, by definition, to verify whether the allocations and prices specified in the lemma constitute a pure bubble steady state, it is equivalent to verify that $b > 0$ (the pure bubble has a positive price). Recall that b solves equation (14). Because u is strictly concave, it follows from this equation that $b > 0$ if and only if

$$u'(e^s - p_n \bar{h}) < \beta u'(e + p_n \bar{h}).$$

Because of equation (15), this inequality is equivalent to

$$R_n < 1,$$

as desired.

A.6 Proof of Corollary 6

Recall from Proposition 4 that $U_n^d > U_b^d$. Furthermore, from Lemmas 1 and 5, we know that the bubbleless and pure bubble steady states yield the same allocation to borrowers, and hence $U_x^d = U_n^d$. Thus, it immediately follows that $U_x^d > U_b^d$.

It remains to show that $U_x^s = U_b^s$. Since the interest rate is $R = 1$ in both the housing bubble and the pure bubble steady states, the Euler equation for savers in both steady states is

$$u'(c_y^s) = \beta u'(c_o^s).$$

Furthermore, in both cases, we have

$$c_y^s + c_o^s = e^s + e.$$

In other words, both $(c_{y,b}^s, c_{o,b}^s)$ and $(c_{y,x}^s, c_{o,x}^s)$ are solutions to the system of two equations above. As u is strictly concave, the system has only one solution. Thus the two consumption allocations are the same. Furthermore, savers achieve the same housing utility $v(\bar{h})$ in both steady states. Hence, $U_x^s = U_b^s$, as desired.

A.7 Stability of the housing bubble steady state

The housing bubble steady state values (p_b, h_b, R_b) constitute the fixed point of the following dynamic system, which can be summarized by two difference equations in the price of bubble p_t and the allocation of housing to savers h_t^s:

$$p_t = \frac{v'(1-h_t^s)}{u'(e^d - p_t(1-h_t^s))} + \beta \frac{u'(e + p_{t+1}(1-h_t^s))}{u'(e^d - p_t(1-h_t^s))} p_{t+1} \tag{focD}$$

$$p_t = \beta \frac{u'(e + p_{t+1}h_t^s)}{u'(e^s - p_t h_t^s)} p_{t+1}, \tag{focS}$$

along with the following Euler equation that determines the interest rate R_t:

$$R_t = \frac{u'(e^s - p_t h_t^s)}{\beta u'(e + p_{t+1}h_t^s)}.$$

Equations (focD) and (focS) are simply the first-order conditions of borrowers and savers with respect to housing, where we have substituted the interest rate R_t by the previous Euler equation. Given the state variable p_t, these two equations implicitly define h_t^s and p_{t+1}, denoted by $\mathcal{H}(p_t)$ and $\mathcal{P}(p_t)$, respectively. From h_t^s and p_{t+1}, we also obtain R_t as a function $\mathcal{R}(p_t)$ of p_t. Let $z_t := (p_t, h_t^s, R_t)$. Then these functions define a dynamical system $z_{t+1} = \Phi(z_t) \equiv (\mathcal{P}(p_t), \mathcal{H}(\mathcal{P}(p_t)), \mathcal{R}(\mathcal{P}(p_t)))$. It can be verified from the assumptions on the utility functions that $\mathcal{P}$ and $\mathcal{R}$ are increasing functions of p_t. Figure 5 illustrates these two equations. Fixing p_t at an arbitrary value, the blue and red solid lines plot the curves corresponding to equations (focD) and (focS), respectively, which determine p_{t+1} and h_t^s for a given p_t. The dashed lines plot how the curves shift when the state variable p_t increases.

We now show that the housing bubble steady state is saddle-path stable. Suppose $p_0 > p_b$. Then, because of the monotonicity of $\mathcal{R}$, we have $R_0 = \mathcal{R}(p_0) > \mathcal{R}(p_b) = 1$. It then follows that $p_1 = p_0 R_0 > p_0$. Hence $p_1 > p_0 > p_b$. By recursion, we get $p_t > \cdots > p_1 > p_0 > p_b$, for all $t > 0$. Hence, $\{p_t\}$ does not converge to p_b. Similarly, suppose $p_0 < p_b$. Then $p_t < \cdots < p_1 < p_0 < p_b$, for all $t > 0$ and $\{p_t\}$ again does not converge to p_b. Only when $p_0 = p_b$ does the system converge to the housing bubble steady state. Hence, the housing bubble steady state is saddle-path stable. This saddle-path stability is standard in the rational bubbles literature (e.g., Tirole, 1985).

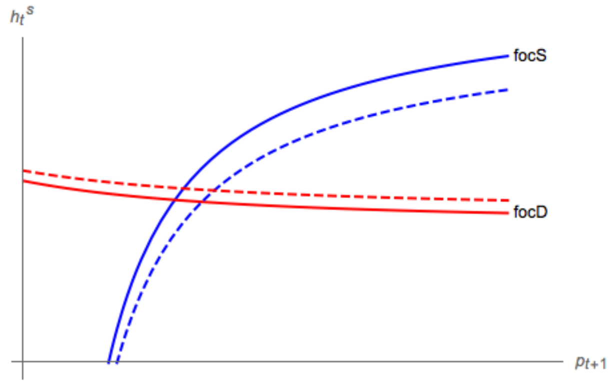

Figure 5: Illustrations of (focD) and (focS).

A.8 Derivation for Section 5.2

With a strictly positive debt limit $\bar{d}$ that is binding for borrowers, the bubbleless steady-state equations (15) and (16) that determine p_n and R_n are instead given by:

$$\begin{aligned} u'\left(e^s - \frac{\bar{d}}{R_n} - p_n\bar{h}\right) &= \beta R_n u'\left(e + \bar{d} + p_n\bar{h}\right) \\ p_n u'\left(e^d + \frac{\bar{d}}{R_n} - p_n(1-\bar{h})\right) &= v'(1-\bar{h}) + \beta p u'\left(e - \bar{d} + p_n(1-\bar{h})\right). \end{aligned}$$

Figure 1 is replaced by a similar Figure 6, where the horizontal line representing the first-order condition for borrowers in the former figure is replaced by the downward sloping line in the latter. It is straightforward to show that the comparative statics result that R_n is decreasing in e^s continues to hold.

Similarly, the housing bubble steady-state equations (17) and (18) that determine h_b^s and p_b are instead given by:

$$\begin{aligned} u'\left(e^s - \frac{\bar{d}}{R_b} - p_b h^s\right) &= \beta u'\left(e + \bar{d} + p_b h_b^s\right) \\ p_b u'\left(e^d + \frac{\bar{d}}{R_b} - p(1-h_b^s)\right) &= v'(1-h_b^s) + \beta p u'\left(e - \bar{d} + p_b(1-h_b^s)\right). \end{aligned}$$

It is straightforward to extend the proof of Lemma 2 to show that h_b^s is increasing in e^s, and thus $h_b^s > \bar{h}$ if and only if $e^s > \bar{e}$, and we skip the details for brevity. It follows that a housing bubble steady state exists if and only if $e^s > \bar{e}$, or equivalently $R_n < 1$, as in Lemma 2. Figure 2 also does not qualitatively change with $\bar{d} > 0$.

We now show $U_b^d < U_n^d$ (i.e., the housing bubble reduces lifetime utility of borrowers). Since $h_b^d = 1 - h_b^s < 1 - \bar{h} = h_n^d$, it follows that under the bubbleless steady-state prices, the housing allocation h_b^d (along with the net asset position $a_b^d = a_n^d = -\bar{d}$) is feasible for borrowers. Thus, by the definition of h_n^d and a_n^d as the solution to the optimization problem of borrowers given the bubbleless steady-state prices, it follows that borrowers must be at least better off with the bubbleless steady-state allocations:

$$U_n^d \geq u(e^d + \bar{d}R_n - p_n h_b^d) + \beta u(e - \bar{d} + p_n h_b^d) + v(h_b^d). \tag{23}$$

Furthermore, combining the fact that not only $p_n h_b^d < p_b h_b^d$ (because the housing price is higher in the housing bubble steady state), *but also* $\frac{\bar{d}}{R_n} < \frac{\bar{d}}{R_b}$ *(because the interest rate is higher in the*

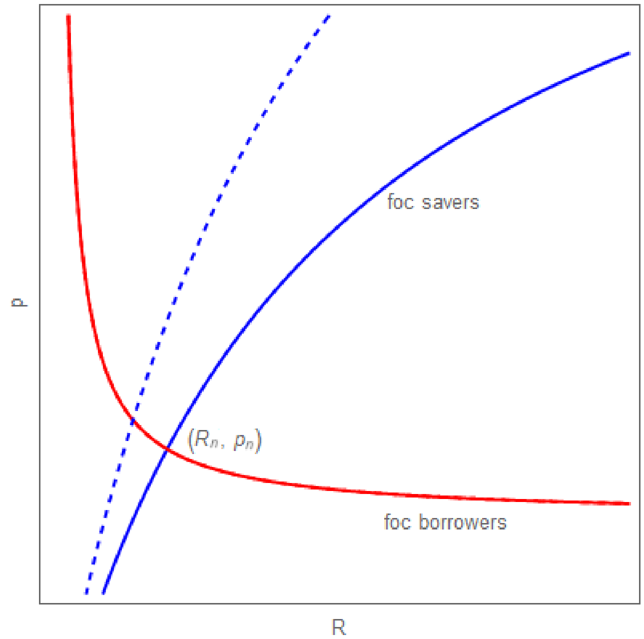

Figure 6: Determination of bubbleless steady state's interest rate and housing price with $\bar{d} > 0$.

housing bubble steady state, negatively affecting the ability of borrowers to raise debt) and the fact that $u'(e^d + \bar{d}R_b - p_b h_b^d) > \beta u'(e - \bar{d} + p_b h_b^d)$ (borrowers in the housing bubble steady state are constrained in their ability to tilt consumption forward) yields

$$u(e^d + \frac{\bar{d}}{R_n} - p_n h_b^d) + \beta u(e - \bar{d} + p_n h_b^d) > u(e^d + \frac{\bar{d}}{R_b} - p_b h_b^d) + \beta u(e - \bar{d} + p_b h_b^d). \tag{24}$$

Inequalities (23) and (24) then imply $U_n^d > U_b^d$, as desired.

Similarly, we can show $U_b^s > U_n^s$ (that the housing bubble benefits savers).

A.9 Extension with rental market

Frictionless rental market

In this extension, we introduce a perfectly competitive and frictionless rental market to the model in the main text. Let $h^i \geq 0$ continue to denote a household of type i's purchase of the housing asset, and let $\hat{h}^i$ denote the household's net rental housing, where a positive position means the household is a renter and a negative position means the household is a landlord. The purchasing and renting prices of housing are denoted by p and $\hat{p}$, respectively.

Taking (steady-state) prices as given, the optimization problem of a representative household of type i is to maximize $U(c_y^i, c_o^i, h^i + \hat{h}^i)$ subject to the credit constraint $a^i \geq -\bar{d}$, to nonnegative constraints $c_y^i, c_o^i, h^i, h^i + \hat{h}^i \geq 0$, and to the following budget constraints:

$$\begin{aligned} ph^i + \hat{p}\hat{h}^i + \frac{a^i}{R} + c_y^i &= e_y^i \\ c_o^i &= ph^i + a^s + e. \end{aligned}$$

In an equilibrium with the competitive rental market, the rental market must clear: $\hat{h}^s + \hat{h}^d = 0$. We again focus on the parameter region where e^s is sufficiently high such that in equilibrium, borrowers will be credit constrained.

The first-order condition with respect to rental housing of a representative borrower yields

$$\hat{p} = v'(h^d + \hat{h}^d)/u'(c_y^d). \tag{25}$$

Intuitively, the price of housing must be equal to the marginal rate of substitution between housing services and consumption for borrowers.

As in the main model, the first-order conditions with respect to borrowing/lending of the unconstrained savers and the constrained borrowers yield:

$$\frac{1}{R} = \frac{\beta u'(c_o^s)}{u'(c_y^s)} > \frac{\beta u'(c_o^d)}{u'(c_y^d)}. \tag{26}$$

Furthermore, by letting $h_c^s \equiv h^s + \hat{h}^s$ denote the consumption of housing services of a representative saver (the amount of housing that enters the utility function), we can rewrite the optimization problem of savers as maximizing $U(c_y^s, c_o^s, h_c^s)$ subject to the following budget constraints

$$\begin{aligned} ph^s + \frac{a^s}{R} + c_y^s &= e_y^s + \hat{p}(h^s - h_c^s) \\ c_o^s &= ph^s + a^s + e, \end{aligned}$$

and the nonnegativity constraints $c_y^s, c_o^s, h^s, h_c^s \geq 0$. The first-order condition with respect to h^s of this problem then yields:

$$(p - \hat{p})u'(c_y^s) = \beta p u'(c_o^s),$$

or equivalently:

$$p = \hat{p} + \frac{p}{R}. \tag{27}$$

Intuitively, the price of housing is equal to the dividend, measured by the rental price $\hat{p}$, plus the resale value $\frac{p}{R}$. In summary, the equilibrium prices $p, \hat{p}$, and R must satisfy (25), (26), and (27).

Equations (25), (26), and (27) imply two results. First, the fact that the credit constraint binds for borrowers implies that borrowers do not buy housing and are net renters. To see this, let λ^d be the Lagrange multiplier associated with the nonnegativity constraint $h^d \geq 0$. Then the first-order condition with respect to h^d yields:

$$\begin{aligned} p &= \frac{v'(h^d + \hat{h}^d)}{u'(c_y^d)} + p\frac{\beta u'(c_o^d)}{u'(c_y^d)} + \lambda^d \\ &= \hat{p} + p\frac{\beta u'(c_o^d)}{u'(c_y^d)} + \lambda^d. \end{aligned}$$

Combined with the inequality in (26) due to the binding credit constraint, we get:

$$p < \hat{p} + \frac{p}{R} + \lambda^d.$$

Combined with asset pricing equation (27), we get

$$0 < \lambda^d,$$

i.e., the constraint $h^d \geq 0$ binds. Thus, in equilibrium $h^d = 0$ and $h^s = 1$. Because v satisfies the Inada conditions, it follow that $\hat{d} > 0$, i.e., borrowers will rent in equilibrium.

Second, since $\hat{p}$ and p must be positive in any equilibrium with the rental market, equation (27) implies that $R > 1$. Therefore, the presence of the rental market effectively rules out the possibility of bubbles, including housing bubbles. In fact, equation (27) implies that in equilibrium the price of housing is simply equal to the present value of rental dividends:

$$p = \frac{\hat{p}}{1 - \frac{1}{R}},$$

i.e., the housing price is equal to the fundamental value of housing, and the bubble component is necessarily zero.

Frictional rental market

The previous subsection assumes a frictionless rental market. However, in practice, rental markets tend to have many frictions, especially in developing economies. In this subsection, we show a simple way to further extend the model to capture, in a reduced-form, rental market frictions.

Specifically, assume that there is a transaction (or maintenance) cost Ψ associated with renting, and the budget constraint of young households would then become:

$$ph^i + \left(1 + \Psi(\hat{h}^i)\right)\hat{p}\hat{h}^i + c_y^i = e_y^i,$$

where for simplicity we assume $\Psi(h) \equiv \frac{\psi}{2}h^2$, with $\psi \geq 0$ being an exogenous constant. Also assume for simplicity that the maintenance cost is a dead-weight loss to the economy (i.e., the cost is not paid to anyone).

When $\psi = 0$, the model collapses to the frictionless rental market in the previous subsection. Obviously, when $\psi \to \infty$, the cost of renting is too high and the model collapses to the main model with no rental market in the main text. However, we can find a tighter upper bound for ψ. If there were an equilibrium with rental market clearing, then from households' first-order conditions, the asset pricing equation of housing would become:

$$p = (1 - \psi\hat{h}^d)\hat{p} + \frac{p}{R}. \tag{28}$$

For the equilibrium price p to be positive, it is necessary that the dividend term $(1 - \psi\hat{h}^d)\hat{p}$ in equation (28) above is positive. It is then straightforward to show that for ψ sufficiently large (a sufficient condition is $\psi > 1/(1 - \bar{h})$), an equilibrium with a rental market does not exist.

In summary, if the rental market is sufficiently frictional, then the equilibrium with rental described in the previous subsection breaks down, and we are back to the model without rental in the main text.