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Survey Data and Subjective Beliefs in Business Cycle Models

WP 19-14

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Survey data and subjective beliefs in business cycle models

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Abstract

This paper develops a theory of subjective beliefs that departs from rational expectations, and shows that biases in household beliefs have quantitatively large effects on macroeconomic aggregates. The departures are formalized using model-consistent notions of *pessimism* and *optimism* and are disciplined by data on household forecasts. The role of subjective beliefs is quantified in a business cycle model with goods and labor market frictions. Consistent with the survey evidence, an increase in pessimism generates upward biases in unemployment and inflation forecasts and lowers economic activity. The underlying belief distortions reduce aggregate demand and propagate through frictional goods and labor markets. As a by-product of the analysis, solution techniques that preserve the effects of time-varying belief distortions in the class of linear solutions are developed.

JEL classification: E32, D84, E71

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1 Introduction

Survey data on households' expectations about future macroeconomic outcomes reveal significant systematic biases and comovement of these biases at business cycle frequencies. In this paper, we present a theory of subjective beliefs that departs from rational expectations and is disciplined using this survey evidence. Our theory formalizes these departures using model-consistent notions of *pessimism* and *optimism* and how they vary over the business cycle. Embedding this theory into a quantitative business cycle model, we show that fluctuations in the subjective belief biases drive a substantial share of movements in macroeconomic aggregates, particularly in the labor market.

We begin by documenting time-series and cross-sectional patterns in household forecasts for unemployment and inflation. Using the University of Michigan Surveys of Consumers, we show that household forecasts for unemployment and inflation are biased upward on average and both biases fluctuate significantly over the business cycle, increasing during recessions. Furthermore, in the cross section, households that forecast high inflation relative to the population also tend to forecast high unemployment. These results are corroborated by additional evidence from the Survey of Consumer Expectations conducted by the Federal Reserve Bank of New York.

We then develop a framework that delivers these deviations of households' beliefs from their rational expectations counterpart as an outcome of time-varying pessimism or optimism. We model pessimism (optimism) as agents overweighting the probability of future states that deliver low (high) continuation utilities, and require dynamically consistent decision rules for agents acting under their subjective beliefs. Since continuation utilities depend on agents' actions and equilibrium prices, the framework endogenously determines the subjective beliefs jointly with macroeconomic aggregates, providing a set of overidentifying restrictions. The forecast biases that we measure in the data are identified by the difference between the subjective and rational expectations forecasts in the model. This mapping between the theory and survey data provides us with moment restrictions that we use for calibration and estimation.

We show that time-varying pessimism and optimism is an important source of macroeconomic risk by applying our framework to a calibrated economy with nominal rigidities and a frictional labor market. The rational expectations version of the model reproduces the well-known unemployment volatility puzzle. On the other hand, once we include belief biases that are calibrated to match the survey data, the model generates the empirically observed large volatility of labor market variables.

The mechanism through which fluctuations in beliefs affect the macroeconomy is consistent with the empirical evidence—an increase in pessimism is contractionary and increases the belief biases in both inflation and unemployment forecasts. Pessimism raises households' subjective probability of lower productivity growth, tighter monetary policy, and further increases in pessimism because these outcomes are associated with low continuation values. More pessimistic consumers lower current demand because of consumption smoothing. Monopolistically competitive intermediate goods firms expect lower future productivity and hence higher marginal costs, which reduces incentives to lower prices. In the presence of labor market frictions, firms' pessimistic evaluation of future surpluses leads to lower match creation. In equilibrium, the decline in output and increase in unemployment due to an increase in pessimism is accompanied by a muted inflation response. Overall, agents are concerned about states with lower productivity, higher marginal costs, and tighter labor market conditions. This explains why our model generates countercyclical and positively correlated biases in inflation and unemployment forecasts.

Survey data provide an informative set of restrictions about the structure of the economy and sources of economic fluctuations. To illustrate these restrictions, we study two variants of the model. First, we consider a setting without TFP shocks. In this case, concerns about higher marginal costs are absent, and the model predicts a negative average inflation bias and a negative comovement between unemployment and inflation biases, both of which are counterfactual. The presence of uncertainty related to supply-type shocks is necessary to generate the correct sign and comovement of these biases.

Next, we study a variant with heterogeneous beliefs, in which we impose rational beliefs on the side of the firms. In this setting, an increase in pessimism consistent with the magnitude of fluctuations in unemployment biases still generates sizable responses of labor market variables, but the inflation bias is considerably attenuated compared with the benchmark. Firms with rational beliefs realize an increase in households' pessimism is contractionary, but similar to the case without TFP shocks, do not associate it with higher marginal costs, and inflation falls. Adverse states are therefore less correlated with high inflation, and pessimistic households overpredict inflation substantially less than in the data. The beliefs of firms therefore play an important role for the model to match the magnitude of the belief biases.

On the technical side, we develop a perturbation technique that incorporates the impact of time-varying belief biases in a first-order approximation of the model. The idea is to construct an appropriate scaling of the endogenously determined belief distortion that does not vanish as the perturbed economy approaches its deterministic limit. The approximation method leads to a tractable linear solution for the equilibrium dynamics with a role for subjective beliefs. The perturbation technique can be applied to a broader class of dynamic stochastic general equilibrium (DSGE) models, including settings in which agents have heterogeneous subjective beliefs. In our application, we use the heterogeneous belief setup to isolate the role of belief distortions of households and firms.

The paper contributes to the empirical and theoretical literatures that study deviations from full information rational expectations. A series of papers use household survey data to document empirical properties of forecast errors and test models of information frictions. For instance, see Carroll (2003), Mankiw et al. (2003), Coibion and Gorodnichenko (2012, 2015a), and Bordalo et al. (2018). In contrast to this literature, our focus is to build general equilibrium models disciplined by these survey data and study quantitative macroeconomic questions.[1] In addition, our theory

[1]Some notable exceptions that also utilize information from survey data in general equilibrium models are Jurado (2016), Baqaee (2019), Carroll et al. (2019), and Adam and Merkel (2019). A parallel literature studies empirical properties of survey forecasts on asset returns and embeds them in asset pricing models. See, for example, Amromin and Sharpe (2014), Greenwood and Shleifer (2014), Adam et al. (2017), Piazzesi et al. (2015), Szöke (2017), and Nagel and Xu (2019).

delivers a subjective measure for the *joint* distribution of outcomes and generates testable restrictions for the forecast errors across macroeconomic variables, which we confirm in the data. Models of information frictions are typically inconsistent with the large average biases that we document.

Our model of pessimism and optimism is also related to a stream of literature that builds quantitative models of business cycles with various forms of information processing (Mankiw and Reis (2007), Woodford (2013), Maćkowiak and Wiederholt (2015), Jurado (2016), Carroll et al. (2019)), extrapolative expectations (Adam and Merkel (2019)), fluctuations in confidence (Angeletos et al. (2018)), ambiguity (Bidder and Smith (2012), Ilut and Schneider (2014), Bianchi et al. (2018), Baqaee (2019)), and model misspecification (Molavi (2019)). In contrast to most of these papers, we use household survey forecasts to discipline departures from rational expectations. At the same time, we show that the common component in fluctuations in the subjective belief biases, which we measure in the Michigan Survey, closely resembles several qualitative proxies for consumer confidence, with the added benefit that survey data provide quantitative discipline on the magnitude of these belief biases. We elaborate on these connections in Section 6.

On the technical side, our modeling of subjective belief builds on the decision-theoretical foundations by Hansen and Sargent (2001a,b), Strzalecki (2011), Hansen and Sargent (2015), and others. In this framework, pessimistic subjective beliefs emerge from agents' concerns about model misspecification. Our contribution to this literature is the parsimonious modeling of time-varying pessimism that can easily be applied to a large class of DSGE models and the set of tools to compute and estimate equilibria with linear dynamics.

The paper is organized as follows. Section 2 describes key empirical findings from the survey data. Motivated by these findings, we introduce our theory of subjective beliefs in Section 3, link the implications of the theory to the belief biases in survey data, and develop a tractable solution technique for approximating the equilibrium dynamics. Section 4 is devoted to the construction and calibration of the structural business cycle model that embeds the subjective belief model. In Section 5, we discuss implications of the findings and the role of subjective beliefs in business cycle dynamics. Section 6 compares empirical predictions from our model to theories of information frictions and relates belief distortions implied by our model to related measures of confidence, sentiment, and disagreement used in the literature. Section 7 concludes. The appendix contains detailed derivations of the approximation method, description of the data, and further results and robustness checks.

2 Survey expectations

We start by analyzing data on households' expectations from the University of Michigan Surveys of Consumers (Michigan Survey). This survey collects answers to questions about households' own economic situation as well as their forecasts about the future state of the economy. We document large upward biases in average forecasts of future inflation and unemployment. These biases vary systematically over the business cycle and across individual households in the cross section.[2]

We define a *belief wedge* as the deviation of a survey response from the corresponding rational expectations forecast. This requires taking a stand on how to determine the probability measure that generates the data. To this end, we use a forecasting vector autoregression (VAR), described in Appendix C.3. As a robustness check, we also document patterns for the belief wedges constructed using responses in the Survey of Professional Forecasters (SPF) as the rational forecast.[3]

2.1 Time-series evidence

Figure 1 shows the differences between the Michigan Survey average household expectations and the rational forecasts for inflation and unemployment. The survey expectations are mean one-year-ahead expectations in the survey samples, constructed using quarterly data for the period 1982Q1–2015Q4. The unemployment rate survey forecast is inferred from categorical answers by fitting a time series of parametric distributions using the procedure from Carlson and Parkin (1975) and Mankiw et al. (2003).

The belief wedges in Figure 1 are large on average, vary over time, and have a strong common component that is correlated with the business cycle. Using the VAR as the rational forecast, the average inflation and unemployment wedges over the sample period are 1.25% and 0.58%, respectively. The wedges are also volatile, with standard deviations of 1.03% and 0.54% for inflation and unemployment, respectively. Finally, the wedges consistently increase during the shaded NBER recessions. This means not only that households overestimate unemployment and inflation relative to the VAR forecast, but also that these biases are larger when measures of business activity are low. The correlations of the inflation and unemployment wedges with the output gap are -0.37 and -0.54, respectively. Figure 1 shows that these patterns are robust to using the SPF forecasts as the rational benchmark. In Appendix C.4 we report the descriptive statistics using other ways of measuring the wedges, such as extending the data to a longer sample and using the median response across households.

We interpret these patterns as households expressing time-varying pessimism or optimism in their view of the aggregate economy. A pessimistic household overweights the probability of adverse future states relative to the data-generating measure. Unemployment is high in these adverse states, and households' unemployment forecasts hence exhibit a positive belief wedge. An increase in pessimism in a recession further increases this wedge.

The interpretation of high inflation biases as emerging from a pessimistic view of the economy also lines up with survey evidence on households' inflation attitudes. The Bank of England administers a quarterly Inflation Attitudes Survey in which households are asked, among other questions, what the impact of an increase in inflation would be on the United Kingdom economy.

[2] A detailed description of the construction of the data is provided in Appendix C.

[3] Some studies report modest biases in SPF forecasts, but these biases are an order of magnitude smaller than those we find in household surveys and are not robust to the chosen time period. See, for example, Elliott et al. (2008) and Capistrán and Timmermann (2009), who rationalize these biases by assuming forecasters have asymmetric loss functions.

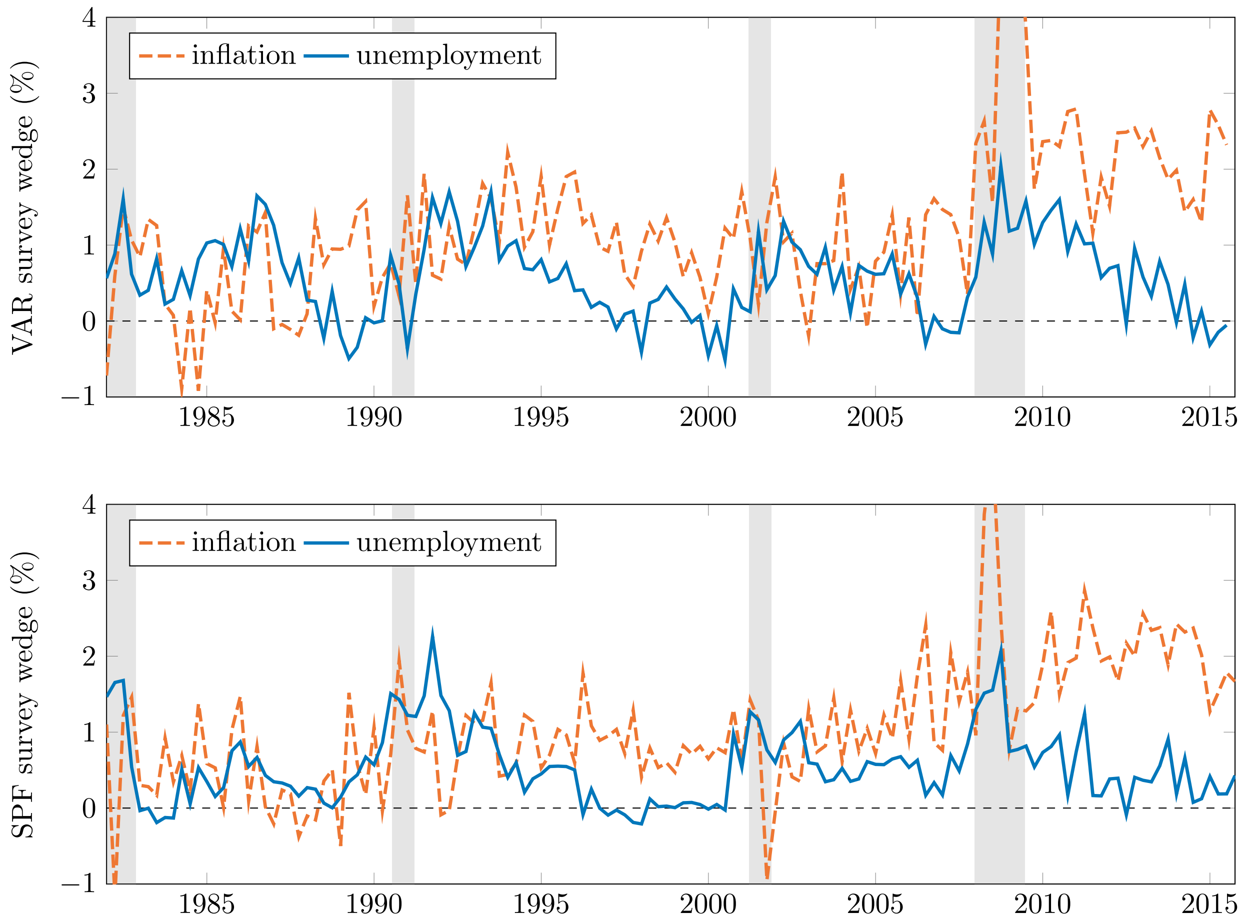

Figure 1: *Top panel:* Difference between the mean one-year-ahead forecasts from the Michigan Survey and corresponding statistical VAR forecasts. *Bottom panel:* Difference between the mean one-year-ahead forecasts from the Michigan Survey and corresponding SPF forecasts. Details on the construction of the data series are in Appendix C. NBER recessions are shaded.

Figure 2 shows that over the sample, between 50% and 80% of households responded that an increase in inflation would weaken the economy. Moreover, this fear of an adverse impact of higher inflation is highest during the Great Recession, and the correlation of this share of households with United Kingdom GDP growth is -0.51. The household median inflation forecast averaged over the 1999Q4–2017Q1 sample is 2.71%, while the realized inflation rate over this period averaged 2.05%. Therefore, United Kingdom households significantly overpredict inflation, associate high inflation with adverse economic outcomes, and tend to have larger biases during recessions.[4] That households associate high inflation with adverse outcomes is also confirmed and discussed by Shiller (1997).

These patterns are robust to alternative ways of measuring the wedges. A particularly insightful

[4]The large magnitude of the inflation wedge in household survey expectations is also consistent with the findings of Coibion and Gorodnichenko (2015b) for the United States, as well as with international evidence. For example, Coibion et al. (2018) find large positive inflation biases in household and firm surveys in New Zealand, and Vellekoop and Wiederholt (2017) document large and persistent positive biases in a long panel survey of households in the Netherlands.

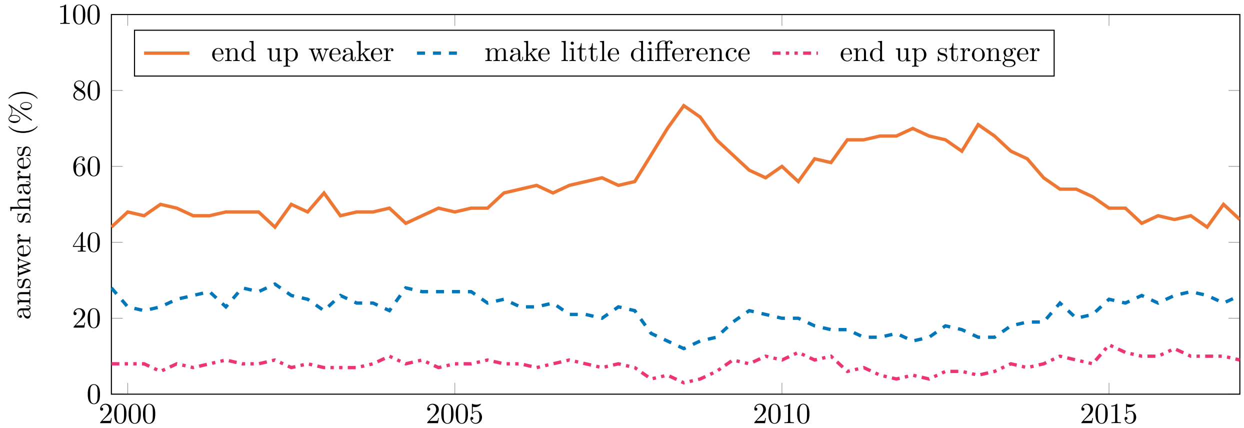

Figure 2: Bank of England Inflation Attitudes Survey, shares of answers to the question: "*If prices started to rise faster than they do now, do you think Britain's economy would ...*" Data sample 1999Q4–2017Q1.

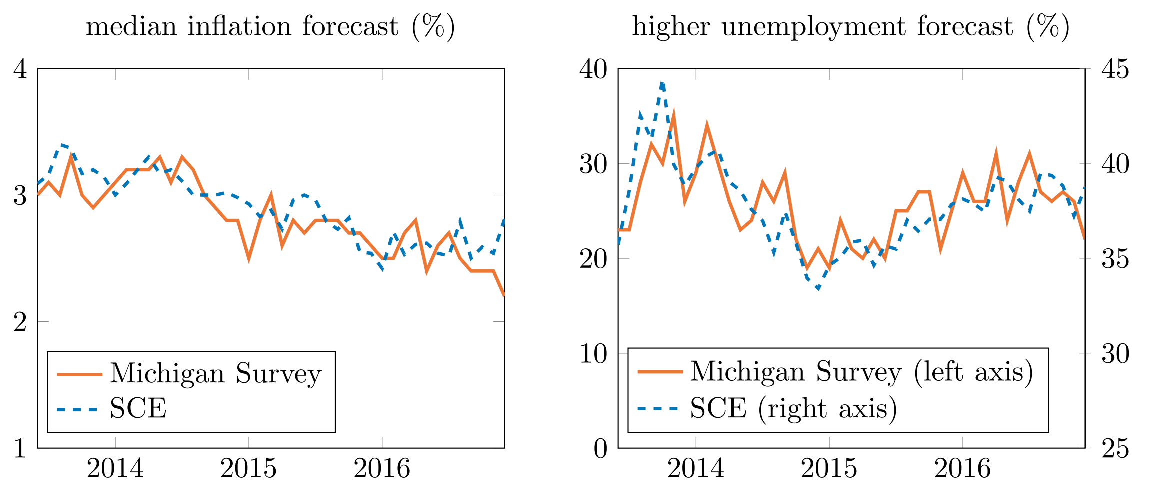

Figure 3: *Left panel:* Median inflation forecasts in the Michigan Survey and the SCE. *Right panel:* Share of respondents in the Michigan Survey stating that unemployment will be higher during the next 12 months and the mean probability that unemployment will be higher one year from now in the SCE. Details on the construction of the data series are in Appendix C.

check is a comparison of our results from the Michigan Survey with the Federal Reserve Bank of New York Survey of Consumer Expectations (SCE). The SCE contains a richer set of questions but only began in 2013. The left panel of Figure 3 shows that the median inflation forecasts from both surveys are very well aligned.[5] Since the two surveys do not ask the same questions about unemployment, the right panel shows two different sets of unemployment forecast statistics. We report the mean probability that unemployment will be higher one year from now from the SCE and the share of respondents who predict that unemployment will be higher in the next 12 months

[5] Armantier et al. (2013) and Manski (2017) advocate eliciting probabilistic forecasts from individual households. The Michigan Survey forecast is constructed by aggregating point forecasts of individual households, and we assume these to be the mean forecasts under the subjective distribution in the quantitative model. Households in the SCE report subjective distributions of the forecasted variables, which are then integrated to obtain mean forecasts at the household level. The alignment of the data from the two surveys justifies this assumption.

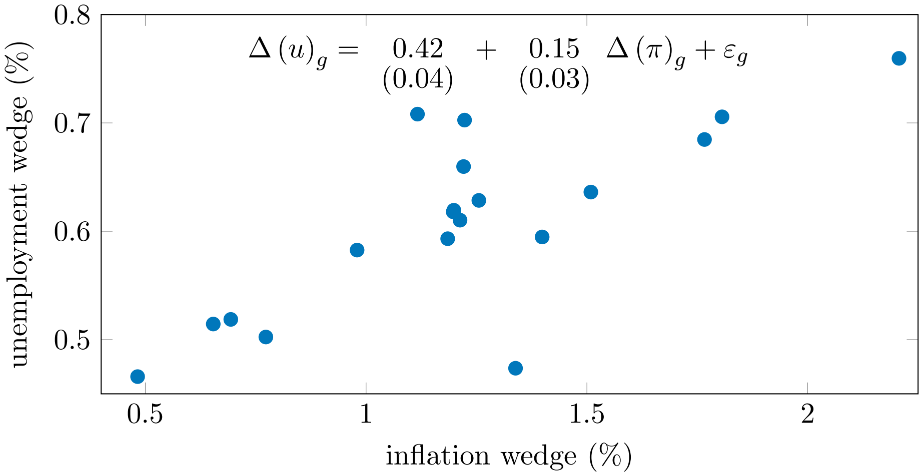

Figure 4: Relationship between the average inflation wedge $\Delta(\pi)_g$ of demographic group g and the corresponding average unemployment wedge $\Delta(u)_g$ in the Michigan Survey. Demographic groups are listed in Table 6 in Appendix C. Standard errors of regression coefficients are in parentheses.

from the Michigan Survey. The levels are not directly comparable, but the statistics comove over time.

2.2 Cross-sectional evidence

In addition to the time series, we also use household-level data to provide evidence for a positive cross-sectional correlation between the unemployment and inflation belief wedges and a strong comovement across time for disaggregated demographic groups. These patterns corroborate the idea that subjective beliefs about aggregate variables contain a common factor that reflects time-varying pessimism or optimism.

We begin by showing that the cross-sectional dispersion of the belief wedges exhibits systematic patterns across demographic groups and individual households—households with more upward-biased inflation forecasts also exhibit more substantial positive biases in unemployment forecasts. Figure 4 displays evidence at the level of demographic groups reported in the Michigan Survey for average wedges over the examined period 1982Q1–2015Q4.[6] Demographic groups with larger average inflation wedges also have larger unemployment wedges. Consistent with existing evidence, households with lower reported education and lower reported income levels make more biased forecasts, but these biases remain nontrivial even for high-education and high-income households. Kamdar (2018) also documents these cross-sectional patterns and interprets them as time variation in optimism and pessimism. We stress that these cross-sectional patterns are independent of the construction of the underlying rational forecast.[7]

[6]This demographic classification includes alternative age groups, geographical regions, quartiles of the income distribution, gender, and different levels of education. Table 6 in Appendix C provides additional details.

[7]In Appendix C.5, we show that this cross-sectional relationship is stable over time, holds at the level of individual households, and is robust to controlling for demographic composition. We also corroborate the cross-sectional patterns against those in the SCE, which contains a richer set of questions about aggregate and household-level variables (see

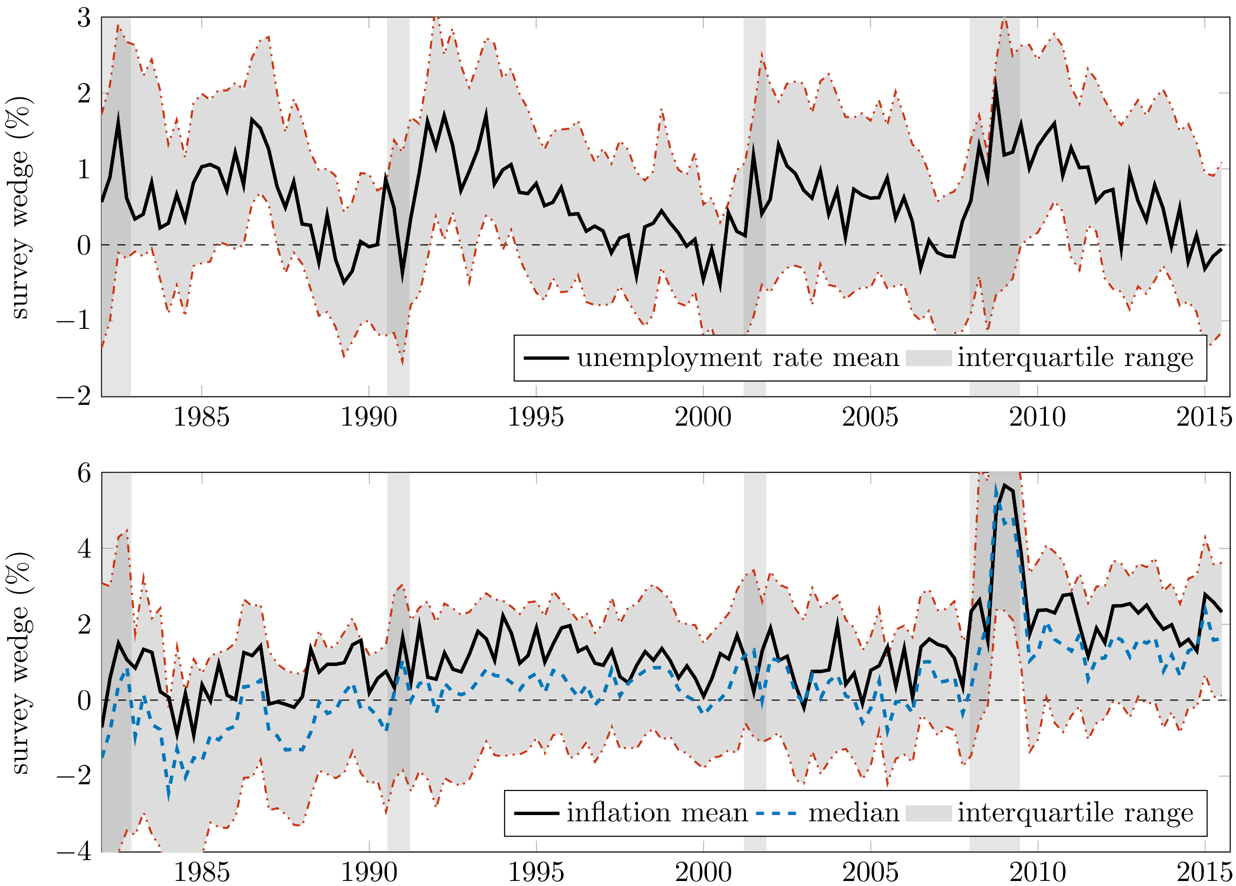

Figure 5: Dispersion in survey expectations in the Michigan Survey. The graphs show quantiles of the distribution of responses in the Michigan Survey, net of the mean VAR forecast. The top panel shows the unemployment responses, the bottom panel the inflation responses. Details on the construction of the data series are in Appendix C. NBER recessions are shaded.

We also plot the dispersion of the data from the Michigan Survey for the unemployment rate and inflation rate forecasts in Figure 5. For the inflation data, we have information on the quantiles of the cross-sectional distribution. For the unemployment rate forecast, we use the inferred distributions from categorical answers. The cross-sectional dispersion in the survey answers across individual households is substantial, but the interquartile range appears to be stable over time. The correlation between the mean and median inflation forecast is 0.94.

Overall, the time-series and cross-sectional evidence from the Michigan Survey paints a clear picture. Households on average expect higher unemployment and higher inflation relative to rational expectations, and these biases are larger in recessions.

Table 10 in Appendix C.5). Rozsypal and Schlafmann (2017) and Das et al. (2019) also study these cross-sectional forecast patterns in the Michigan Survey and document results consistent with ours.

3 Framework for subjective beliefs

Motivated by the empirical results from Section 2, we now introduce a framework for modeling deviations of agents' subjective beliefs from the data-generating probability measure. Denote the data-generating and subjective probability measures as P and $\widetilde{P}$, and the corresponding conditional expectations as $E_t[\cdot]$ and $\widetilde{E}_t[\cdot]$. The discrepancy between the two measures can be expressed using a strictly positive martingale with increments m_{t+1}.[8] The belief wedge for the one-period-ahead forecast of the macroeconomic variable z_t is then given by

$$\Delta_t^{(1)}(z) \doteq \widetilde{E}_t[z_{t+1}] - E_t[z_{t+1}] = E_t[m_{t+1}z_{t+1}] - E_t[z_{t+1}]. \tag{1}$$

The random variable m_{t+1}, which captures agents' subjective beliefs, acts as a weighting scheme on the distribution of time-$t+1$ outcomes.

A model-consistent notion of pessimism (optimism) is a weighting scheme that overweights (underweights) states that are adverse from the agents' viewpoint. To formalize this idea in a dynamically consistent environment, we extend the robust preference setting of Hansen and Sargent (2001a,b). Agents' preferences are represented using a concave period utility $u(\cdot)$ and the continuation value recursion

$$V_t = \min_{\substack{m_{t+1}>0 \\ E_t[m_{t+1}]=1}} u(x_t) + \beta E_t[m_{t+1}V_{t+1}] + \frac{\beta}{\theta_t} E_t[m_{t+1}\log m_{t+1}] \tag{2}$$

$$\theta_t = \overline{\theta} x_t, \tag{3}$$

$$x_{t+1} = \psi(x_t, w_{t+1}). \tag{4}$$

Here, x_t is an $n \times 1$ vector of stationary economic variables that follows the Markovian law of motion (4), $\overline{\theta}$ is a $1 \times n$ vector of parameters, and $w_{t+1} \sim N(0_k, I_{k\times k})$ is an independent and identically distributed (iid) vector of normally distributed shocks under the data-generating probability measure P. We take the function ψ as given for now but later derive it as a solution to a set of equilibrium conditions. The linear specification of θ_t allows for negative values, in which case the minimization problem in (2) turns into a maximization problem. In Section 4, we also endow the agent with a set of controls, which gives rise to a min–max specification of the recursion.

The minimization problem in (2) captures agents' concerns about model misspecification. The agent considers models that are difficult to distinguish statistically from the data-generating measure, and the degree of statistical similarity is controlled by the entropy penalty $E_t[m_{t+1}\log m_{t+1}]$, scaled by the penalty parameter θ_t. More pronounced statistical deviations that are easier to de-

[8] Formally, under the assumption that P and $\widetilde{P}$ are equivalent, there exists a strictly positive martingale M defined recursively as $M_{t+1} = m_{t+1}M_t$ with $M_0 = 1$ such that for any t and any time-$t+j$ measurable random variable z_{t+j}, $\widetilde{E}_t[z_{t+j}] = E_t[(M_{t+j}/M_t)\, z_{t+j}]$. The conditional distribution under $\widetilde{P}$ can therefore be fully characterized by specifying the process for m_t.

tect are represented by random variables m_{t+1} with a large dispersion that yields a large entropy. Optimal decisions and subjective beliefs that rationalize them are pinned down by the desire of the household to bound utility losses from potential model misspecification.

The solution to the minimization problem (2) satisfies

$$m_{t+1} = \frac{\exp\left(-\theta_t V_{t+1}\right)}{E_t\left[\exp\left(-\theta_t V_{t+1}\right)\right]}, \tag{5}$$

and m_{t+1} completely characterizes agents' subjective beliefs relative to the data-generating measure. Adverse outcomes are states with low continuation values V_{t+1}. The sign of θ_t captures whether the agent is pessimistic or optimistic, and the magnitude of θ_t controls the magnitude of the belief distortion. An increase in θ_t corresponds to an increase in pessimism. The value $\theta_t = 0$ corresponds to $m_{t+1} = 1$, in which case the one-period-ahead subjective belief coincides with the data-generating process.[9]

Agents endowed with preference formulation (2) act as dynamically consistent subjective expected utility agents with beliefs given by the probability measure $\widetilde{P}$. Since $\widetilde{P}$ rationalizes their actions, we impose the hypothesis that agents answer survey questions about economic forecasts according to the same $\widetilde{P}$ and relate the belief wedges from Section 2 to the difference between expectations under $\widetilde{P}$ and the data-generating measure P.

Two observations motivate this hypothesis. First, as we documented in Section 2 and consistent with the large literature on household survey expectations, household survey data on economic forecasts exhibit substantial and persistent biases characterized by fluctuations in pessimism and optimism that (5) formally captures. Second, subjective beliefs reported in surveys are found to be systematically related to real consumption behavior. Similar to us, Malmendier and Nagel (2016) use the Michigan Survey to substantiate a significant relationship between survey responses on subjective expectations of economic outcomes and individual consumer spending, borrowing, and lending decisions. Ichiue and Nishiguchi (2015) use household survey data from Japan to link inflation expectations and durable goods spending. Vellekoop and Wiederholt (2017) link households' portfolio choice to their inflation expectations in Dutch survey data. Giglio et al. (2019) relate investors' portfolio choice and surveyed return expectations. Gennaioli et al. (2015) show that subjective expectations of managers in the Duke University CFO Survey have predictive power for firm investment and production behavior. Tanaka et al. (2018) document that subjective GDP forecasts of Japanese firms predict their employment, investment, and output growth. Lastly, Crump et al. (2019) exploit the SCE to estimate agents' intertemporal elasticity of substitution using the relationship between subjective inflation expectations and expected spending behavior. All these findings support the rationale for associating the survey answers with the subjective beliefs that households use in their decision making.

[9]Substituting the solution for m_{t+1} into problem (2) yields the recursion $V_t = u\left(x_t\right) - \frac{\beta}{\theta_t}\log E_t\left[\exp\left(-\theta_t V_{t+1}\right)\right]$. When the period utility function is logarithmic, this is mathematically equivalent to Epstein and Zin (1989) preference under unitary elasticity of substitution with a time-varying risk aversion coefficient $\gamma_t = \theta_t + 1$ used, for example, in Dew-Becker (2014) or Alvarez and Atkeson (2017).

Combining equations (1) and (5) yields

$$\Delta_t^{(1)}(z) = Cov_t[m_{t+1}, z_{t+1}] = Cov_t\left[\frac{\exp(-\theta_t V_{t+1})}{E_t[\exp(-\theta_t V_{t+1})]}, z_{t+1}\right]. \tag{6}$$

The belief wedges associated with macroeconomic variables z_{t+1} thus depend on their covariance with agents' continuation value V_{t+1}. In the context of the empirical evidence from Section 2, when $\theta_t > 0$ and agents are pessimistic, they overpredict unemployment *because* unemployment is high in states that they perceive as adverse. Since continuation values $V_{t+1} = V(x_{t+1})$ and the law of motion (4) are endogenously determined, equation (6) combined with survey data yields a set of cross-equation restrictions for the equilibrium dynamics of the model.

3.1 General equilibrium and a solution method

We seek to incorporate the model of endogenous subjective beliefs in a large class of dynamic stochastic general equilibrium (DSGE) models. A wide range of DSGE models with subjective beliefs can be cast as a solution to a system of expectational difference equations,

$$0 = \widetilde{E}_t[g(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t)], \tag{7}$$

where $g_{t+1} = g(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t)$ is an $n \times 1$ vector function.[10] This vector of equations includes agents' Euler equations, which can be represented using subjective beliefs implied by m_{t+1}. Specifically, for the i-th equation of the system,

$$0 = \widetilde{E}_t\left[g_{t+1}^i\right] = E_t\left[m_{t+1} g_{t+1}^i\right].$$

The feedback between agents' subjective beliefs and the equilibrium law of motion requires jointly solving the system of equations (7) for the continuation value recursion (2), the law of motion (4), and the endogenously determined probability measure $\widetilde{P}$ defined through (5).

We develop a novel approximation technique for the equilibrium dynamics of x_t that builds on the series expansion method used in Borovička and Hansen (2014). The technique incorporates time variation in subjective beliefs in a tractable linear approximation of the equilibrium dynamics. Consider a class of models indexed by a perturbation parameter q that approximates the dynamics (4) by scaling the volatility of the innovations w_{t+1}:

$$x_{t+1}(\mathsf{q}) = \psi(x_t(\mathsf{q}), \mathsf{q} w_{t+1}, \mathsf{q}). \tag{8}$$

Hence, with each q, there is an associated state vector process $x_t(\mathsf{q})$ given by the law of motion (8), and $\mathsf{q} = 1$ recovers the original dynamics (4). The dynamics of $x_t(\mathsf{q})$ are approximated by

[10] Our solution method, fully described in Appendix B, is able to handle heterogeneous belief distortions for different forward-looking equations of the equilibrium system. We abstract from this heterogeneity in the main text to simplify notation but utilize this flexibility in Section 5 to disentangle the effect of belief distortions on the side of households and firms in our structural model.

constructing a first-order series expansion,

$$x_t(\mathsf{q}) \approx \bar{x} + \mathsf{q} x_{1t}, \tag{9}$$

where the 'first-derivative' process x_{1t} represents the local dynamics in the neighborhood of the steady state $\bar{x}$ and does not depend on q. The steady state $\bar{x}$ is the solution to (8) evaluated at $\mathsf{q} = 0$, given implicitly by $\bar{x} = \psi(\bar{x}, 0, 0)$. Assuming that the function $\psi(x, w, \mathsf{q})$ is sufficiently smooth, we obtain the dynamics of x_{1t} by differentiating (8) with respect to q, utilizing (9), and evaluating at $\mathsf{q} = 0$:

$$x_{1t+1} = \psi_q + \psi_x x_{1t} + \psi_w w_{t+1}, \tag{10}$$

where ψ_q, ψ_x, and ψ_w are conforming coefficient matrices representing the corresponding partial derivatives of $\psi(x, w, \mathsf{q})$ evaluated at the steady state. For example, $\psi_x \doteq \frac{\partial}{\partial x} \psi(x, w, \mathsf{q})\big|_{(\bar{x},0,0)}$.

The key innovation in our approach relative to the standard perturbation approximations in Sims (2002) or Schmitt-Grohé and Uribe (2004) is the approximation of the penalty parameter θ_t in the continuation value recursion. Substituting the belief distortion (5) into (2) and applying the perturbation argument to the stochastic processes V_t, x_t, and θ_t yields the perturbed continuation value recursion

$$V_t(\mathsf{q}) = u(x_t(\mathsf{q}), \mathsf{q}) - \frac{\beta}{\theta_t(\mathsf{q})} \log E_t \left[\exp\left(-\theta_t(\mathsf{q}) V_{t+1}(\mathsf{q})\right)\right]. \tag{11}$$

The usual expansion in the perturbation parameter leads to the following first-order approximation of the exponent in (11) and in the numerator of (5):

$$-\theta_t(\mathsf{q}) V_{t+1}(\mathsf{q}) \approx -\bar{\theta}(\bar{x} + \mathsf{q} x_{1t}) \left(\bar{V} + \mathsf{q} V_{1t+1}\right) \approx -\bar{\theta}\left(\bar{x} + \mathsf{q}\left(x_{1t}\bar{V} + \bar{x} V_{1t+1}\right)\right).$$

The scaling of the stochastic term by q indicates that as $\mathsf{q} \to 0$ (i.e., as the economy approaches its deterministic counterpart), the belief distortion in the perturbed model vanishes. Consequently, the usual first-order approximation of (11) is not affected by θ_t, a standard result arising from the smoothness of the certainty-equivalent transformation $\log E_t[\exp(\cdot)]$.[11]

Instead, we propose to use the perturbation

$$\theta_t(\mathsf{q}) = \overline{\theta} x_t(\mathsf{q}) \approx \frac{\overline{\theta}(\bar{x} + x_{1t})}{\mathsf{q}}. \tag{12}$$

Differentiating (11) with respect to q then yields a recursion for the first-derivative process V_{1t}:

$$V_{1t} = u_x x_{1t} + u_q - \beta \frac{1}{\overline{\theta}(\bar{x} + x_{1t})} \log E_t \left[\exp\left(-\overline{\theta}(\bar{x} + x_{1t}) V_{1t+1}\right)\right]. \tag{13}$$

This recursion is the first-order approximation of (11), and the nonlinearity stems from the perturbation choice (12). Using the guess

$$V_{1t} = V_x x_{1t} + V_q,$$

[11] The issue is analogous to the second-order nature of risk premia in small-noise approximations.

recursion (13) yields a pair of equations for coefficients V_x and V_q. The equation for V_x is a Riccati equation whose solution can be found iteratively (see Appendix B.2). As a result, the zeroth-order approximation of the belief distortion (5) (i.e., the evaluation of the expansion of (5) at $\mathsf{q} = 0$) takes the form

$$m_{0t+1} = \frac{\exp\left(-\overline{\theta}\left(\bar{x} + x_{1t}\right) V_x \psi_w w_{t+1}\right)}{E_t\left[\exp\left(-\overline{\theta}\left(\bar{x} + x_{1t}\right) V_x \psi_w w_{t+1}\right)\right]}. \tag{14}$$

This expression reveals the effect of the perturbation choice (12). The volatility of the shocks $\mathsf{q} w_{t+1}$ in the perturbed economy (8) vanishes with $\mathsf{q} \to 0$, but at the same time, the magnitude of agents' belief biases (12) scales up relative to the shock volatility. These two effects are constructed to offset each other such that in the economy that approaches its deterministic limit, the agents' subjective model remains nontrivially distinct from the data-generating process.

When we approximate agents' subjective model $\widetilde{P}$ using the zeroth-order term of the belief distortion (14), the vector of normally distributed innovations w_{t+1} in (4) under $\widetilde{P}$ has the distribution

$$w_{t+1} \sim N\left(-\overline{\theta}\left(\bar{x} + x_{1t}\right)\left(V_x \psi_w\right)', I_{k\times k}\right). \tag{15}$$

Instead of facing a vector of zero-mean shocks w_{t+1}, the agent perceives these shocks under her subjective beliefs as having a time-varying drift. The time variation is determined by a linear approximation to θ_t from equation (3), given by $\overline{\theta}\left(\bar{x} + x_{1t}\right)$. The relative magnitudes of the distortions of individual shocks are given by the sensitivity of the continuation value to the dynamics of the state vector, V_x, and the loadings of the state vector on individual shocks, ψ_w. An implication of (15) is the dynamics of the model (10) under the agents' subjective beliefs $\widetilde{P}$ that satisfy

$$\begin{aligned} x_{1t+1} &= \left[\psi_q - \psi_w \psi_w' V_x' \overline{\theta} \bar{x}\right] + \left[\psi_x - \psi_w \psi_w' V_x' \overline{\theta}\right] x_{1t} + \psi_w \widetilde{w}_{t+1} \\ &= \widetilde{\psi}_q + \widetilde{\psi}_x x_{1t} + \psi_w \widetilde{w}_{t+1}, \end{aligned} \tag{16}$$

where $\widetilde{w}_{t+1} \sim N\left(0_k, I_{k\times k}\right)$ is an iid vector of normally distributed shocks under the subjective probability measure $\widetilde{P}$.

3.2 Restrictions on subjective beliefs and mapping to survey data

Subjective beliefs alter both the conditional mean and the persistence of economic shocks. Moreover, variables that move θ_t and the continuation value in opposite directions exhibit a higher persistence under the subjective beliefs.[12] Assume that the forecasted variable z_t in (1) takes the form $z_t = \bar{z}' x_t$. Using the linearized dynamics under the data-generating measure (10) and under the subjective measure (16), we obtain an expression for the model-implied belief wedges:

$$\begin{aligned} \Delta_t^{(1)}(z) &= \widetilde{E}_t\left[z_{t+1}\right] - E_t\left[z_{t+1}\right] = \bar{z}' \psi_w \widetilde{E}_t\left[w_{t+1}\right] \\ &= -\overline{\theta}\left(\bar{x} + x_{1t}\right) \bar{z}'\left(\psi_w \psi_w'\right) V_x'. \end{aligned} \tag{17}$$

[12] This statement is precisely correct in the scalar case when $\psi_x^2 V_x \overline{\theta} < 0$.

Longer-horizon forecasts $\Delta_t^{(j)}(z) = \widetilde{E}_t[z_{t+j}] - E_t[z_{t+j}]$ are constructed correspondingly by iterating on the subjective dynamics (16).

The model predicts a one-factor structure in the dynamics of the belief wedges measured using the survey data. The relative distortions of survey forecasts of macroeconomic variables z_t are given by constant loadings $-\bar{z}'(\psi_w \psi_w') V_x'$, whereas the factor that measures the overall magnitude of the belief distortions, $\theta_t \approx \overline{\theta}(\bar{x} + x_{1t})$, varies over time. This one-factor structure is the key restriction that the linearized subjective beliefs model imposes on the joint dynamics of the survey answers and implies that the magnitudes of the belief wedges should comove over time, which is consistent with the evidence in Section 2.[13]

Equation (17) is the linearized version of formula (6). The vector of loadings $-\bar{z}'(\psi_w \psi_w') V_x'$, which is endogenously determined in equilibrium, is the negative of the covariance of the innovations to the value function, $V_x \psi_w$, with innovations $\bar{z}'\psi_w$ to the macroeconomic variable z_t for which we have survey data.

We therefore obtain a set of overidentifying restrictions, both for the relative belief wedges across forecasted variables and for their time-series properties. Importantly, free parameters that characterize subjective beliefs are restricted to the specification of the process θ_t irrespective of the number of state variables or exogenous shocks. The number of overidentifying restrictions thus grows with the number of variables for which we have forecast data.

4 Subjective beliefs in a structural business cycle model

In this section, we introduce the subjective beliefs framework into a calibrated version of an economy with nominal rigidities and a frictional labor market. In the absence of belief distortions, our environment is similar to Ravenna and Walsh (2008), Gertler et al. (2008), and Christiano et al. (2016). Our setup provides well-defined notions of unemployment and inflation, which directly map to the survey questions. We use this model to quantify the contribution of fluctuations in subjective beliefs to macroeconomic outcomes and to assess key channels in the propagation mechanism.

4.1 Model

The model economy is populated by a representative household with subjective beliefs described in Section 3, competitive producers of a homogeneous final good, and monopolistic producers of intermediate goods who employ workers hired in a frictional labor market. In the benchmark version of the model, all economic agents share the same subjective beliefs as the representative household. Alternative specifications that distinguish between the beliefs of households and firms are studied in Section 5.

[13] The first-order expansion generates linear dynamics with homoskedastic shocks. In a fully nonlinear solution, time variation in the belief wedges $\Delta_t^{(j)}(z)$ would also incorporate fluctuations in the dispersion of V_{t+1}, generated, for example, by stochastic volatility. We leave these considerations for future work.

4.1.1 Representative household

The preferences of the representative household are given by the recursion

$$V_t = \min_{\substack{m_{t+1}>0 \\ E_t[m_{t+1}]=1}} \max_{C_t, B_{t+1}} \log C_t + \beta E_t \left[m_{t+1} V_{t+1} \right] + \frac{\beta}{\theta_t} E_t \left[m_{t+1} \log m_{t+1} \right], \tag{18}$$

with time preference coefficient β and an AR(1) process for θ_t,

$$\theta_t = (1 - \rho_\theta)\bar{\theta} + \rho_\theta \theta_{t-1} + \sigma_\theta w_t^\theta. \tag{19}$$

The magnitude of the belief distortion is determined by fluctuations in θ_t specified exogenously in (19). However, equilibrium dynamics in the model endogenously determine the states that yield low continuation values V_{t+1}. These states are evaluated as adverse by the household and are then perceived as more likely under the subjective model. Naturally, the dynamics of the subjective beliefs then endogenously depend on the structure of other shocks in the model, which we describe in Section 4.1.4. We will refer to θ_t as the *belief shock*.

The household consists of a unit mass of workers who perfectly share consumption risk. A fraction L_t is employed and earns a real wage ξ_t. A fraction $1 - L_t$ is unemployed and collects unemployment benefits with real value D financed through lump-sum taxes. The household faces the nominal budget constraint

$$P_t C_t + B_{t+1} \leq \left(1 - L_t\right) P_t D + L_t P_t \xi_t + R_{t-1} B_t - T_t,$$

where P_t is the price of consumption goods, B_{t+1} denotes the one-period risk-free bonds purchased in period t with return R_t, and T_t are lump-sum taxes net of profits.

4.1.2 Labor market

At the end of period $t-1$, employed workers separate with probability $1-\rho$ and join the pool of unemployed, who search for jobs at the beginning of period t. The total number of searchers at the beginning of period t therefore is $1 - \rho L_{t-1}$. The law of motion for the mass of employed workers is given by

$$L_t = \rho L_{t-1} + \left(1 - \rho L_{t-1}\right) f_t = \left(\rho + h_t\right) L_{t-1},$$

where f_t is the endogenously determined job-finding probability and

$$h_t = \frac{f_t \left(1 - \rho L_{t-1}\right)}{L_{t-1}}$$

is the hiring rate. Measured unemployment is given by $u_t = 1 - L_t$, which includes people who do not rejoin employment after searching at the beginning of the period.

Firms in the labor market hire workers and sell labor services using a linear technology. At the beginning of period t, they post vacancies at rate v_t for a total number of vacancies $v_t L_{t-1}$. The labor market tightness is defined as the number of posted vacancies over the number of searchers,

$$\zeta_t = \frac{v_t L_{t-1}}{1 - \rho L_{t-1}}.$$

A Cobb–Douglas matching function with efficiency μ and curvature ν combines vacancies and workers to produce

$$M_t = \mu \left(v_t L_{t-1}\right)^{\nu} \left(1 - \rho L_{t-1}\right)^{1-\nu}$$

matches. The probability that a searching worker finds a job is then given by

$$f_t = \frac{M_t}{1 - \rho L_{t-1}} = \mu \zeta_t^{\nu},$$

and the vacancy-filling rate is equal to $q_t = f_t / \zeta_t$.

We now characterize workers' subjective valuations when they are employed and unemployed, and the subjective value of the firm in the labor market. Let $s_{t+1} = \beta C_t / C_{t+1}$ denote the marginal rate of substitution between consumption today and consumption tomorrow. The value of an unemployed worker U_t is given recursively as

$$U_t = D + \widetilde{E}_t \left[s_{t+1} \left(f_{t+1} J^w_{t+1} + \left(1 - f_{t+1}\right) U_{t+1} \right) \right],$$

where $\widetilde{E}_t [\cdot]$ represents the expectation under the subjective belief of the household, and J^w_t is the value of an employed worker. Similarly, the value of the employed worker satisfies the recursion

$$J^w_t = \xi_t + \widetilde{E}_t \left[s_{t+1} \left(\rho + \left(1 - \rho\right) f_{t+1} \right) J^w_{t+1} \right] + \widetilde{E}_t \left[s_{t+1} \left(1 - \rho\right) \left(1 - f_{t+1}\right) U_{t+1} \right].$$

Here, the term $\rho + (1 - \rho) f_{t+1}$ combines the probability ρ of continuing in the existing job and the probability $(1 - \rho) f_{t+1}$ of losing the job at the end of period t but immediately finding a new job at the beginning of period $t + 1$. Finally, the value of the worker to the firm is the present value of profits earned by the firm from the match, given by the difference between the worker's marginal product ϑ_t on the current job and the wage,

$$J_t = \vartheta_t - \xi_t + \rho \widetilde{E}_t \left[s_{t+1} J_{t+1} \right].$$

To close the labor market, we specify the free-entry condition and the wage-setting protocol. Let κ_v be the flow cost of posting a vacancy. The zero-profit condition for entering firms implies

$$J_t = \frac{\kappa_v}{q_t}.$$

We follow Shimer (2010) and use Nash bargaining with rigid wages. The firm and the worker bargain over a target wage ξ^*_t to split the match surplus according to

$$\eta \left(J_t + \xi_t - \xi^*_t \right) = \left(1 - \eta\right) \left(J^w_t - U_t + \xi^*_t - \xi_t \right),$$

where η is the bargaining power of the worker. The terms $J_t + \xi_t - \xi_t^*$ and $J_t^w - U_t + \xi_t^* - \xi_t$ are the surplus values to the firm and worker, respectively, of choosing the target wage ξ_t^* instead of the equilibrium wage ξ_t. The actual wage is a weighted average of last period's wage and the current target wage,

$$\xi_t = \lambda \xi_{t-1} + (1 - \lambda) \xi_t^*,$$

where λ is a wage rigidity parameter, with $\lambda = 0$ corresponding to flexible wages.

An important feature of the frictional labor market is the forward-looking nature of vacancy-posting decisions and bargaining. When evaluating the distribution of future states, workers inherit the beliefs of the representative household. Similarly, firms maximize profits using equilibrium state prices obtained from households' preferences and beliefs. This implies that fluctuations in θ_t directly affect the incentives of firms to post vacancies, through their effect on the valuation of the match surplus and equilibrium wages. This is a striking difference relative to the Walrasian spot market where workers are hired using only one-period employment contracts. In such an environment, fluctuations in subjective beliefs do not directly affect labor market decisions, since there is no uncertainty about economic conditions in the current period.

4.1.3 Production and market clearing

The frictional labor market is embedded in a New Keynesian framework with Calvo (1983) price setting. A homogeneous final good Y_t with price P_t is produced in a competitive market using the production technology

$$Y_t = \left[\int_0^1 (Y_{j,t})^{(\varepsilon-1)/\varepsilon} \, dj \right]^{\varepsilon/(\varepsilon-1)}, \qquad \varepsilon > 0,$$

where $Y_{j,t}$ are specialized inputs with prices $P_{j,t}$. Final good producers solve the static competitive problem

$$\max_{Y_{j,t}} P_t Y_t - \int_0^1 P_{j,t} Y_{j,t} dj,$$

leading to the first-order conditions

$$Y_{j,t} = \left(\frac{P_t}{P_{j,t}} \right)^{\varepsilon} Y_t, \qquad j \in [0, 1].$$

Specialized inputs are produced by monopolistic retailers indexed by j, using the production technology

$$Y_{j,t} = \exp(a_t) \, l_{j,t} - \phi,$$

where $l_{j,t}$ is the quantity of labor services used in production, a_t is the logarithm of the neutral technology level, and ϕ is a fixed cost of production. The retailer purchases labor services from labor market firms described in Section 4.1.2 at a competitive price ϑ_t.

The retailer is subject to Calvo-style price frictions and reoptimizes the price with probability $1 - \chi$. These infrequent adjustments imply that price setting is a dynamic problem affected by the belief distortions on the side of the firm.

Aggregate resources satisfy the constraint

$$C_t + \frac{\kappa_v}{q_t} h_t L_{t-1} = Y_t$$

and the market clearing condition for labor services is

$$\int_0^1 l_{j,t} dj = L_t.$$

4.1.4 Shock structure and monetary policy

We complete the model by specifying the remaining sources of exogenous variation to the economy. The monetary authority follows an interest rate rule

$$\log\left(R_t/\overline{R}\right) = \rho_r \log\left(R_{t-1}/\overline{R}\right) + (1-\rho_r)\left[r_\pi \log\left(\pi_t/\overline{\pi}\right) + r_y \log\left(Y_t/Y^*\right)\right] + \sigma_r w_t^r,$$

where w_t^R is an iid monetary policy shock, $\overline{\pi}$ an inflation target, and Y^* the steady-state value of Y_t. The neutral technology process a_t is specified as

$$a_{t+1} = \rho_a a_t + \sigma_a w_{t+1}^a.$$

The final source of exogenous variation is the belief shock θ_t specified in (19) that drives agents' subjective belief deviations from the data-generating process. We assume that all innovations are independent under the data-generating measure P:

$$\left(w_t^r, w_t^a, w_t^\theta\right)' \overset{iid}{\sim} N\left(0, I\right).$$

As we have seen in Section 3, this property does not carry over to the subjective measure where the joint distribution of future realizations of the innovations depends on the current level of θ_t.

4.2 Model solution and calibration

The equilibrium of the structural model from the previous section fits in the general framework that we developed in Section 3.[14] We apply the expansion methods from Section 3.1 to compute a linear approximation to the solution for the equilibrium dynamics. Most parameters are calibrated to discipline the steady state of the economy and its dynamic responses to technology and monetary policy shocks. Parameters for the TFP process are estimated using data from Fernald (2014), and the parameters of the process θ_t are set to make the model-implied belief wedges for inflation and unemployment consistent with the data from Section 2.

The calibrated parameter values are summarized in Table 1. The subjective discount factor $\beta = 0.994$ is set to target a steady state real return of 1% per year, and the intercept of the

[14] The full set of equilibrium equations of the model is stated in Appendix E.

Parameters		Value
β	Discount factor	0.994
ε	Elasticity of substitution intermediate goods	6.00
χ	Calvo price stickiness	0.75
λ	Wage rigidity	0.925
$\bar{\pi}$	Monetary policy rule: intercept	0.01
ρ_r	Monetary policy rule: smoothing	0.84
r_π	Monetary policy rule: loading on inflation	1.60
r_y	Monetary policy rule: loading on output	0.028
	Labor market	
ρ	Job survival probability	0.89
μ	Matching efficiency	0.67
ν	Curvature of matching function	0.72
η	Worker's bargaining weight	0.72
κ	Vacancy posting costs	0.09
D	Flow benefits of unemployment	0.57
	Shocks	
$\overline{\theta}$	Mean belief shock	5.64
ρ_θ	Persistence of belief shock	0.714
σ_θ	Volatility of belief shock	4.3
ρ_a	Persistence of TFP shock	0.84
$100\sigma_a$	Volatility of TFP shock	0.568
$100\sigma_r$	Volatility of monetary policy shock	0.078

Table 1: Benchmark parameter values. Model is calibrated at a quarterly frequency.

monetary policy rule $\bar{\pi} = 0.01$ to yield a steady-state annualized inflation of 2%.[15] The parameters $\varepsilon = 6$ and $\chi = 0.75$ governing nominal frictions are calibrated to match a markup of 20% and a frequency of price changes that corresponds to three quarters. We choose the job survival parameter $\rho = 0.89$ to target a quarterly separation rate of 11%. We follow Shimer (2005) and set the curvature of the matching function $\nu = 0.72$ and the worker's bargaining weight $\eta = 0.72$. The remaining labor market parameters—matching efficiency $\mu = 0.67$, unemployment benefits $D = 0.57$, and vacancy posting costs $\kappa = 0.09$—are calibrated to achieve an average job-finding rate of 0.67, a flow value of unemployment equal to 70% of wages, and a steady-state tightness normalized to one.

We use TFP data from Fernald (2014) to infer $\rho_a = 0.841$ and $\sigma_a = 0.568\%$. The smoothing parameter in the monetary policy rule $\rho_r = 0.84$ is taken from Christiano et al. (2016). The wage rigidity parameter $\lambda = 0.925$ and the monetary policy parameters $r_\pi = 1.60$, $r_y = 0.028$, and $\sigma_r = 0.078\%$ are chosen to make the impulse responses of inflation and unemployment to TFP and monetary policy shocks consistent with the VAR evidence in Dupor et al. (2009) and Christiano et al. (2016). As shown in equation (17), belief distortions in the model depend on the propagation

[15]The steady state of the linearized model is distorted by first-order effects of belief distortions captured by the term ψ_q in equation (10), so that the steady-state real return and inflation differ from $1/\beta - 1$ and $\bar{\pi}$, respectively.

Moment	Data	Model			
		benchmark	no θ_t	only θ_t	no TFP shocks
Mean of inflation wedge	1.25	0.90	0.00	0.00	−0.32
Mean of unemployment wedge	0.54	0.55	0.00	0.00	0.54
Volatility of inflation wedge	1.03	0.73	0.00	0.00	0.26
Volatility of unemployment wedge	0.45	0.45	0.00	0.00	0.43
Volatility of inflation	1.40	1.41	1.15	0.00	0.82
Volatility of output	2.32	2.22	1.55	0.00	1.09
Volatility of unemployment	1.65	1.39	0.55	0.00	0.87

Table 2: Data and model-implied theoretical moments for macroeconomic quantities and belief wedges. The sample period for the Data column is 1982Q1–2015Q4. Values in all columns are in percentages and annualized.

of fundamental shocks, and it is therefore crucial for the model to fit these responses. We focus on cumulative 10-quarter responses as a target. The model generates cumulative inflation responses of −2.79% (TFP) and −0.73% (monetary policy) and cumulative unemployment responses of −0.21% (TFP) and 0.74% (monetary policy), which are values that fall comfortably within the estimated 90% confidence intervals in both papers.

Finally, we calibrate the parameters of the process θ_t using the belief wedge data from Section 2. Our model predicts a one-factor structure of the belief wedges (17), and the first principle component explains 76% of the variation in the wedges. We set the autocorrelation coefficient ρ_θ to 0.714, which matches the autocorrelation of the first principal component extracted from the time series of unemployment and inflation wedges. The remaining two parameters for the mean and volatility of the belief shock, $\overline{\theta} = 5.64$ and $\sigma_\theta = 4.3$, are chosen to fit the means and volatilities of the unemployment and inflation wedges.

4.3 Model fit

The first two columns of Table 2 show the fit of the calibrated benchmark model to moments for inflation, unemployment, output, and the belief wedges. The model somewhat understates the mean and volatility of the inflation wedge and the volatility of unemployment but matches the remaining moments of macroeconomic variables and belief wedges well. To understand the role of fluctuations in θ_t in generating these results, we consider several versions of the model with alternative configurations of the shock processes. We report the fit of these alternatives in the remaining three columns of Table 2.

To highlight the role of belief shocks in matching the unemployment volatility observed in the data, the third column (no θ_t) reports statistics for the rational expectations version of the model ($\overline{\theta} = \sigma_\theta = 0$). In this case, the model generates the standard unemployment volatility puzzle, with TFP and monetary policy shocks being able to generate only a third of the empirically observed unemployment volatility. Belief wedges are zero by construction.

Figure 6 further emphasizes the importance of the θ_t shocks by comparing the model-implied

Figure 6: Data and model-implied time series for unemployment and belief wedges.

paths for unemployment and belief wedges against the data. We extract the innovations to the TFP process from the Fernald (2014) time series and innovations to the belief shock process from the time series for the first principal component of the belief wedges that we used in the calibration, and feed them into the model. We compare the model-implied path for unemployment to a counterfactual exercise in which we set $\theta_t = \overline{\theta}$ and hence shut down fluctuations in belief biases.

The top panel shows that adding belief shocks to the model moves the model-implied time series much closer to the data. In particular, without fluctuations in θ_t, the model overstates the unemployment during the dot-com boom in the late 1990s and understates the unemployment around the Great Recession. The model interprets the decrease in unemployment in the late

1990s as arising from relative optimism among households and attributes much of the increase in unemployment around 2008-2009 to an increase in pessimism. The middle and bottom panels of Figure 6 show that the time variation in θ_t generating this path for unemployment also implies belief wedges that closely match the data.

Time variation in θ_t plays a substantial role in explaining fluctuations in unemployment, but it does not act in isolation. For instance, when we shut down TFP and monetary policy shocks (setting $\sigma_a = \sigma_r = 0$), the belief wedges and macro aggregates exhibit no volatility (see the fourth column in Table 2, labeled "only θ_t"). In this equilibrium, households do not face any uncertainty about continuation values and do not form distorted expectations about the future path of the economy.[16]

The last column of Table 2 shows the results in the economy without TFP shocks ($\sigma_a = 0$) and reveals a key interaction between fluctuations in θ_t and other structural shocks that is required to rationalize the observed belief wedges. To make the economy comparable, we recalibrate the process for θ_t to fit the properties of the unemployment wedge from the benchmark model (setting $\overline{\theta} = 150$ and $\sigma_\theta = 117.7$). The model misses properties of the belief wedges despite generating a sizable amount of volatility in macroeconomic aggregates. The inflation wedge now has a substantially lower volatility, a negative mean, and a negative correlation with the unemployment wedge, all of which are inconsistent with the empirical evidence documented in Section 2.

In the absence of TFP shocks, pessimistic households are concerned about adverse realizations of monetary policy and belief shocks. Both of these shocks act as demand-type shocks, simultaneously lowering economic activity and inflation. Low realizations of households' continuation value are therefore associated with high unemployment and low inflation, moving the two respective belief wedges in opposite directions.

5 Understanding the role of subjective beliefs

In this section, we analyze two types of dynamic responses to innovations in θ_t: those under the data-generating measure P and those under the subjective belief $\widetilde{P}$. Together they clarify the mechanism through which fluctuations in subjective beliefs propagate in the economy.

5.1 Belief wedges and dynamic responses under subjective beliefs

Figure 7 depicts the impulse responses to an innovation w_t^θ under the data-generating measure. An increase in θ_t, depicted in the bottom right graph, is contractionary. A one standard deviation innovation to θ_t leads on impact to a fall of about 1% in output and a 1 percentage point increase in the unemployment rate. Inflation increases on impact for a short period but decreases afterward, with a 10-quarter cumulative response of approximately zero. The contractionary effects of an increase in θ_t are about two-thirds of the response to a typical productivity shock: a one standard

[16] The nonstochastic equilibrium is self-confirming in the sense of Fudenberg and Levine (1993). Households still distort the future distribution of the process θ_t, but that is irrelevant for their decisions.

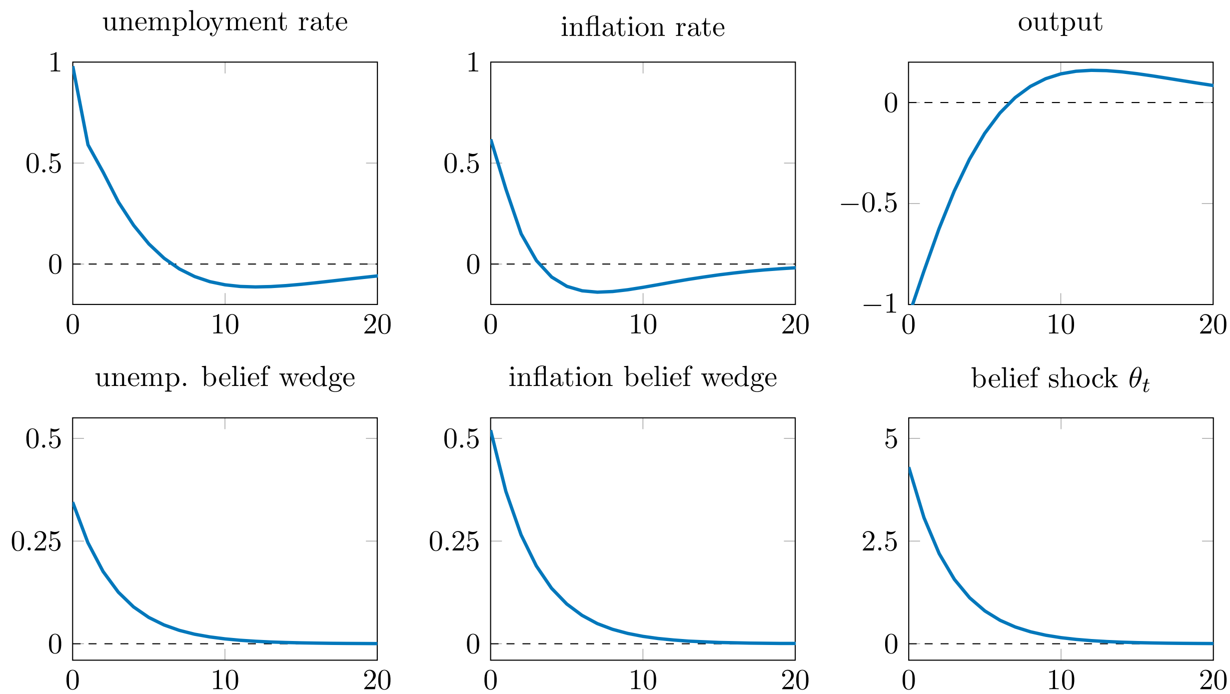

Figure 7: Impulse response functions to the belief shock innovation w^θ under the data-generating measure P. Output response is in percentages, and unemployment rate, inflation rate, and belief wedges are in percentage points. Inflation rate is annualized. Horizontal axis is in quarters.

deviation fall in productivity leads to a cumulative 10-quarter decrease of 1.2% in annual output, compared with a 0.8% decrease in the case of a belief shock. The bottom panels of Figure 7 show that households also increase their upward bias in inflation and unemployment forecasts relative to the data-generating measure, which is consistent with the survey data described in Section 2.

The dynamic responses of the exogenous shocks under households' subjective measure are shown in Figure 8. Under the data-generating measure P (dashed line), innovations to individual exogenous shock processes are uncorrelated and iid over time, and hence the technology and monetary policy shocks do not respond to w_t^θ. In contrast, under the households' subjective measure $\widetilde{P}$ (solid line), the shocks are correlated. Households associate an increase in θ_t with a negative productivity shock accompanied by a monetary tightening. They also forecast a further sequence of positive innovations to θ_t, hence increasing the subjective persistence of the belief shock. The particular correlation structure arises through the effect that these three innovations have on the continuation value V_t. The pessimistic household also distorts productivity shocks more than monetary policy shocks, reflecting a more adverse impact of the productivity shock on the continuation value.

The continuation value recursion (18) indicates that bad times must be generated by low levels of current and future consumption under the households' subjective model. The top left panel of Figure 9 confirms this intuition. Households facing an increase in θ_t forecast a large and very persistent drop in consumption (solid line) relative to the data-generating process (dashed line). A higher subjective persistence of bad times under the pessimistic belief is manifested in all macroeconomic

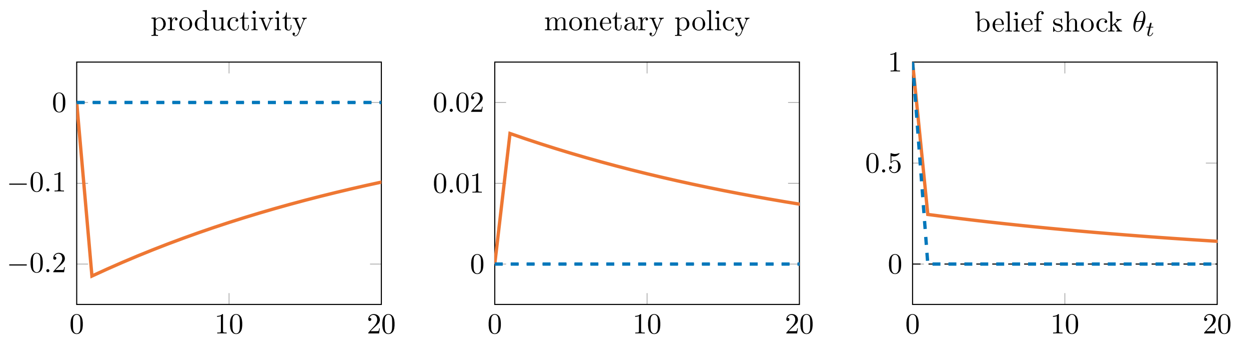

Figure 8: Impulse response functions of exogenous shocks to the belief shock innovation w^{θ} under the data-generating measure P (dashed line) and the subjective measure $\widetilde{P}$ (solid line). Horizontal axis is in quarters.

quantities.[17]

The equilibrium mapping from exogenous shocks to endogenous variables also explains why households forecast higher unemployment, lower output growth, and higher inflation relative to the data-generating process. When TFP shocks are sufficiently prominent, households' inflation expectations increase relative to the rational forecast because expectations of lower productivity imply higher marginal costs, which pushes prices upward through the optimal pricing behavior of firms. The top right panel of Figure 9 shows that households fear a persistently higher inflation in the future. As in the Bank of England Inflation Attitudes Survey depicted in Figure 2, they associate adverse states with high inflation.

Figure 9 also shows that the increase in θ_t has a particularly pronounced contractionary effect on labor market dynamics. Firm valuation J_t, given by the present discounted value of profits earned by the firm from a match with a worker,

$$J_t = \vartheta_t - \xi_t + \rho \widetilde{E}_t \left[s_{t+1} J_{t+1} \right],$$

decreases substantially under the more pessimistic belief in $\widetilde{E}_t[\cdot]$, which causes a large drop in vacancy-posting rates v_t and job-finding rates f_t. Again, adverse labor market conditions are expected to last significantly longer under the subjective belief.

The mechanism through which subjective beliefs alter forward-looking decisions in our model provides an interpretation of Euler equation wedges featured in a range of papers as a source of aggregate fluctuations. For example, Smets and Wouters (2007) introduce a "risk premium" shock that has been shown to play an important role in the post-2008 dynamics, Basu and Bundick (2017) and Hall (2017) include direct shocks to the discount rate, and Leduc and Liu (2016) use second-moment shocks to TFP. We share the idea that fluctuations in the consumption Euler equation wedge affect aggregate demand and that wedges in firms' Euler equations affect price setting and match creation. Our framework integrates these mechanisms through movements in subjective

[17] These findings are consistent with Piazzesi et al. (2015), who find higher persistence in the survey forecasts of interest rates in the Blue Chip Financial Forecasts data.

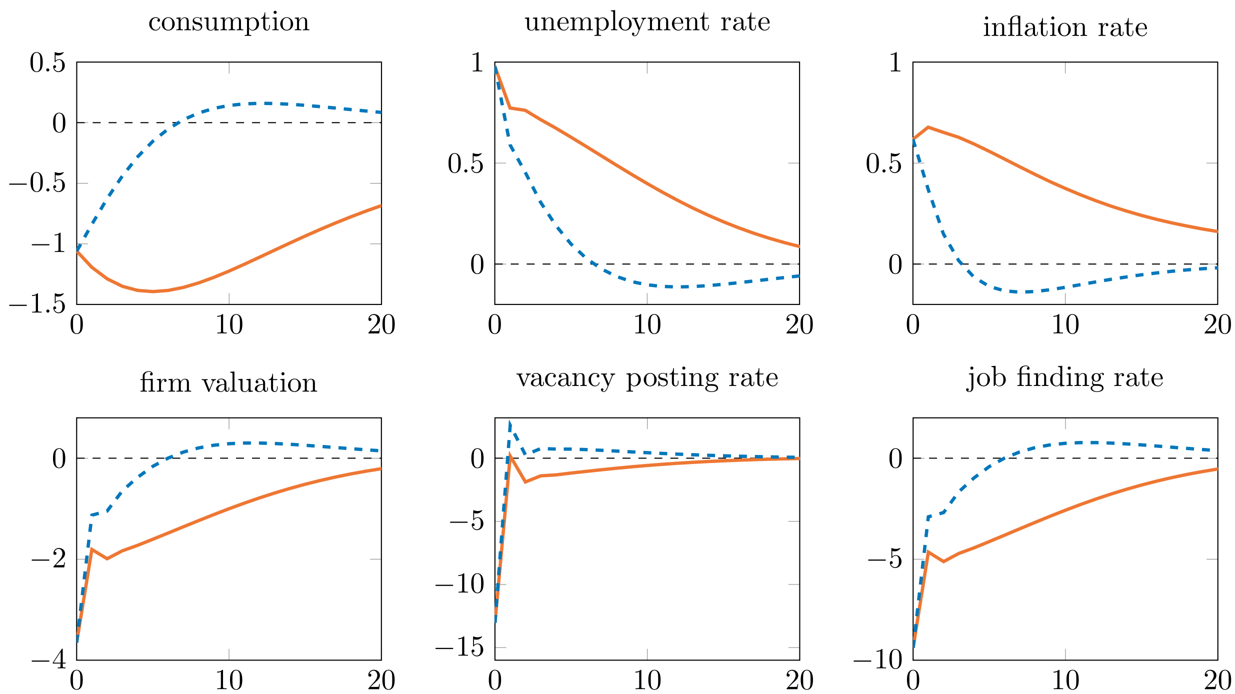

Figure 9: Impulse response functions to the belief shock innovation w^{θ} under the data-generating measure P (dashed line) and the subjective belief $\widetilde{P}$ (solid line). Responses are in percentages, except for the unemployment rate and inflation rate, which are in percentage points. Inflation rate is annualized. Horizontal axis is in quarters.

beliefs that are disciplined by survey data.

In addition, the importance of uncertainty about supply-type shocks that arises endogenously when we match survey evidence provides a rationale for the specification in Ilut and Schneider (2014). They use a model of exogenously specified time-varying ambiguity aversion about the TFP process based on the multiple-prior preferences of Gilboa and Schmeidler (1989) and Epstein and Schneider (2003). Agents in their model behave *as if* endowed with a subjective belief that exhibits time-varying pessimism about TFP shocks. We show that these types of belief distortions are needed to match survey data jointly with macroeconomic outcomes. More generally, our framework uses endogenous exposures of the continuation value as inputs to belief distortions and thereby avoids overparameterization. This approach is particularly useful in settings with multiple exogenous shocks.

5.2 Role of firms' subjective beliefs

The benchmark economy features homogeneous subjective beliefs imposed on all agents and their forward-looking decisions: the consumption-saving decision represented by the consumption Euler equation, the dynamic pricing behavior of intermediate goods producers that determines the New Keynesian Phillips curve, vacancy posting decisions of firms, and bargaining between firms and workers in the labor market, driven by valuation of firms' and workers' surpluses from created matches. To uncover the role played by the assumption that firms inherit the subjective beliefs of the households, we now study a variant of the model in which we impose rational expectations on firms while preserving subjective beliefs for the households.

To implement this variant, we exploit the tractability of our framework to solve for an equilibrium in which we "turn off" belief distortions on specific forward-looking equations. Formally, we look for the solution to the system of equations (7) modified as follows:

$$0 = E_t \left[\mathbb{M}_{t+1} g \left(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t \right) \right],$$

where g is, as before, the $n \times 1$ vector of functions that includes forward-looking Euler equations and other equilibrium conditions, and $\mathbb{M}_{t+1} \equiv \operatorname{diag} \left\{ m_{t+1}^{\sigma_1}, \ldots, m_{t+1}^{\sigma_n} \right\}$ are the separate belief distortions on each of the n equations. We consider two distinct belief distortions $\sigma_i \in \{0, 1\}$. The expression $m_{t+1}^0 \equiv 1$ denotes an undistorted equation under rational expectations, and

$$m_{t+1}^1 \equiv \frac{\exp \left(-\theta_t V_{t+1} \right)}{E_t \left[\exp \left(-\theta_t V_{t+1} \right) \right]}$$

denotes, as in (5), an equation under the subjective belief. For a given configuration of $\{\sigma_i\}$, we solve for new equilibrium dynamics and the associated continuation process V_{t+1}. To make the role of subjective beliefs in these economies comparable, we adjust the mean and volatility parameters of the exogenous process θ_t such that the mean and volatility of the unemployment wedge are unchanged. For details on implementation and a more general treatment of heterogeneous beliefs in this framework, see Appendix B.5.

Figure 10 depicts the dynamic responses in the variant with rational firms (dashed line) and compares them with the benchmark in which all agents have subjective beliefs (solid line). The model with rational firms produces similar fluctuations in unemployment as the benchmark but markedly different dynamics for inflation and wages—inflation is lower on impact, and wages fall by less. Intermediate goods firms with rational beliefs realize that an increase in θ_t is contractionary but, unlike their counterparts in the benchmark model, do not associate it with higher future marginal costs and therefore do not increase prices as much.[18]

The reduced fall in wages arises because of the asymmetry in beliefs of firms and workers in the labor market.[19] Under the Nash protocol, firms and workers split the surplus, which is the difference between firms' subjective valuation of output produced in the match and workers' subjective value of unemployment. When firms have rational beliefs, an increase in θ_t does not alter their valuation of the match output as much as it would have if they shared the pessimistic beliefs of the workers. As a result, the subjective surplus that is bargained over is higher, and the wage that implements the bargaining outcome falls by less.

[18]The distorted beliefs of the price-setting firms following a large increase in θ_t (see Figure 6) can thus also account for the "missing disinflation" during the Great Recession, discussed in Coibion and Gorodnichenko (2015b).

[19]In the bargaining process, firms and workers agree to disagree about their subjective valuation of the match, in the sense of Harrison and Kreps (1978) and Morris (1995).

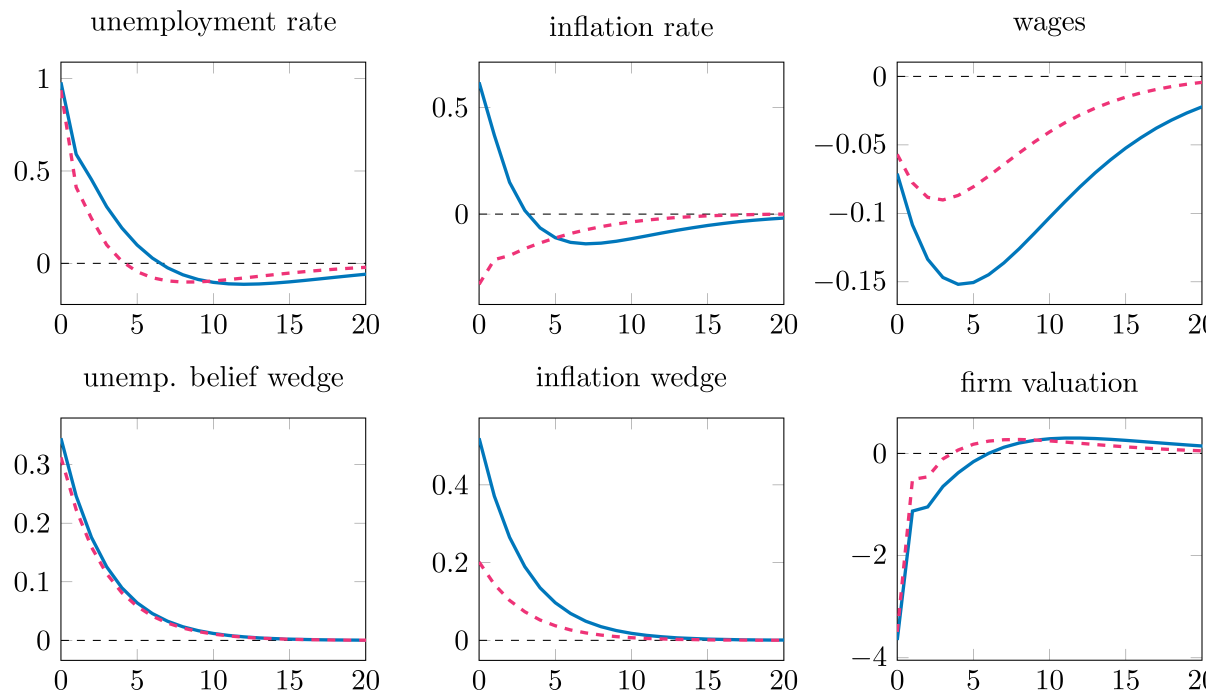

Figure 10: Impulse response functions to the belief shock innovation w^{θ} in the benchmark model with homogeneous subjective beliefs (solid line) and in an economy with rational firms (dashed line). Wage and firm valuation responses are in percentages, and unemployment rate, inflation rate, and belief wedges are in percentage points. Inflation rate is annualized. Horizontal axis is in quarters.

The drop in inflation in response to an increase in θ_t implies that inflation comoves less strongly with adverse states. As a result, even though we recalibrated the volatility of θ_t innovations to match the volatility of the unemployment wedge from the benchmark model, the model with rational firms predicts substantially lower volatility of the inflation wedge. Fluctuations in subjective beliefs on the side of the intermediate goods firms therefore constitute an essential ingredient to reconcile the inflation survey data and macroeconomic outcomes.

6 Alternative theories of subjective beliefs

Fluctuations in θ_t that generate time-varying pessimism resemble the notions of confidence or sentiment shocks, which have been used to generate comovement of macroeconomic variables at business cycle frequencies. Our framework explicitly links these fluctuations to forecast errors, which have also been extensively documented in the empirical literature on survey forecasts. In this section, we show that the time series for θ_t extracted from belief wedges is highly correlated with empirical proxies for consumer confidence and that the stochastic properties of the model-implied forecast errors are consistent with their empirical counterparts.

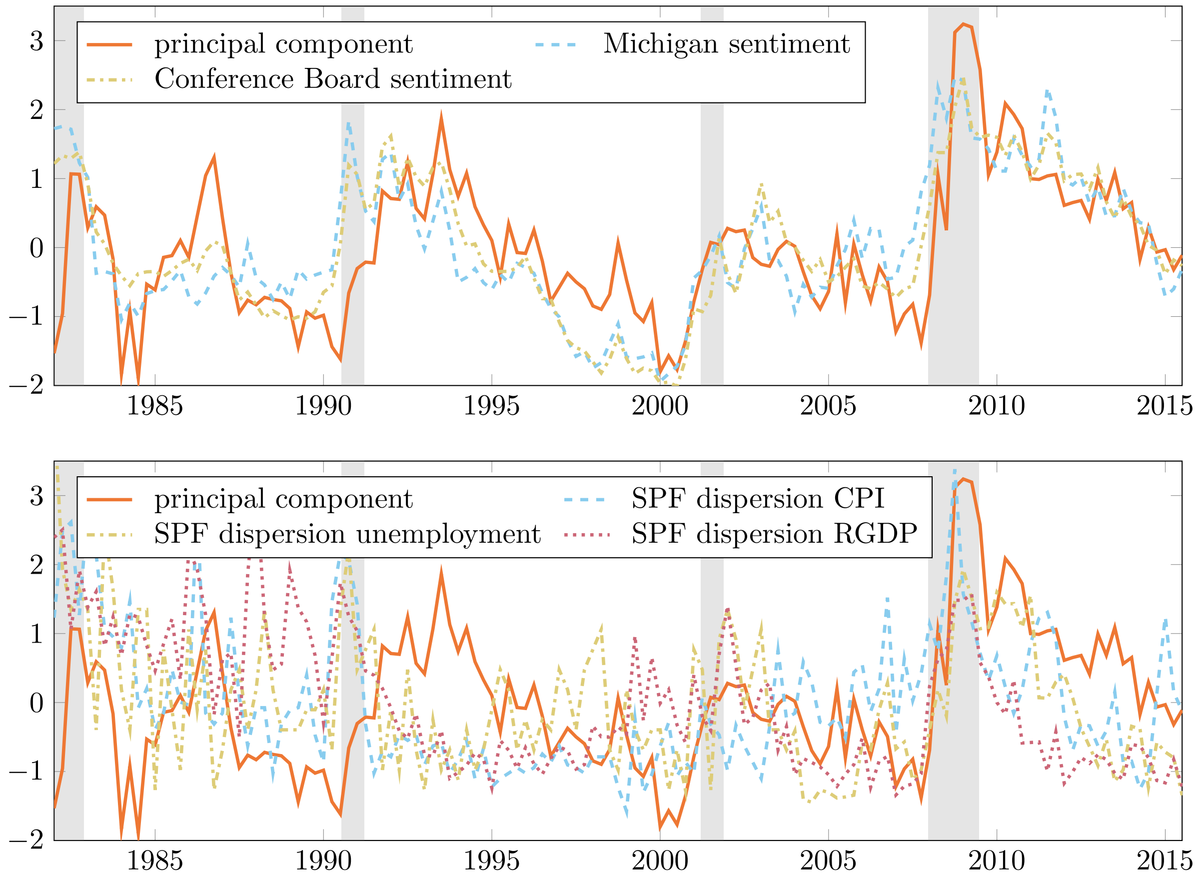

Figure 11: Comparison of the first principal component of the belief wedges with alternative measures of sentiment and disagreement. *Top panel:* negative of the Michigan Consumer Sentiment Index, negative of the Conference Board Consumer Confidence Index. *Bottom panel:* Interquartile dispersion in individual forecasts in Survey of Professional Forecasters of CPI inflation, unemployment, and real GDP growth. NBER recessions are shaded. Following Zhao (2017), we average dispersion in SPF forecasts over horizons from zero (nowcast) to three quarters to reduce noise.

6.1 Measures of sentiment and confidence

Existing work uses various empirical proxies for consumer sentiment as drivers of macroeconomic fluctuations.[20] Since these sentiment measures are procyclical, they may be capturing similar demand-driven forces as our mechanism. To understand how these alternative measures relate to our framework, we compare them in Figure 11 with our belief shock θ_t.

The top panel of Figure 11 shows a close connection between the belief shock and the Michigan Consumer Sentiment Index and the Conference Board Consumer Confidence Index. Both indices provide independent measures of sentiment that comove strongly with the belief shock, with correlations of –0.62 and –0.74, respectively. The comovement is consistent with our interpretation of

[20]For example, Barsky and Sims (2012) and Angeletos et al. (2018) use the Michigan Consumer Sentiment Index, and Leduc and Liu (2016) use the share of Michigan Survey households who report that it is not a good time to buy new cars because of an uncertain future. Angeletos et al. (2018) remark on issues with the qualitative nature of the Michigan Consumer Sentiment Index, which our approach addresses.

the belief wedges as arising from pessimism, as captured by negative movements in sentiment.

A shortcoming of using empirical measures of sentiment is the lack of their theoretical counterparts in quantitative models. The magnitude of fluctuations in these sentiments is calibrated indirectly to the volatility of macroeconomic quantities. Dominitz and Manski (2004), Coibion et al. (2017), and Manski (2017) therefore advocate the use of responses to quantitative survey questions in order to provide a tighter link between survey data and theoretical models. In line with this argument, we calibrate θ_t to match the level and volatility of the belief wedges, which we directly measure in survey data. In addition, our theory enforces additional restrictions on the relative magnitudes of the belief wedges, providing overidentifying restrictions that we verify empirically.

Instead of belief wedges, Ilut and Schneider (2014) use the dispersion of SPF forecasts of real GDP as a proxy for the degree of ambiguity on the side of households. In related work, Mankiw et al. (2003), Bachmann et al. (2013), and others use measures of cross-sectional forecast dispersion as a proxy for economic uncertainty, based on the assumption that higher dispersion is indicative of more difficulty in estimating the forecast distribution. The bottom panel of Figure 11 shows that measures of dispersion in SPF forecasts of inflation, unemployment, and real GDP growth are largely uncorrelated with the belief wedges, indicating that disagreement among SPF forecasters is distinct from pessimistic concerns of households. Although our framework and solution method allow for heterogeneity in beliefs that could explicitly model such disagreement in forecasts, we leave a quantitative analysis of such a model for future work.

6.2 Imperfect information and forecast errors

A large literature investigates properties of survey forecast errors through theories of imperfect information due to noisy signals and biases in information processing. Although encompassing the full range of alternatives is not possible, we focus on two prominent frameworks: a Bayesian model with frictions in information processing and a non-Bayesian model of overreaction.

Following the literature, we posit a univariate linear model for a variable of interest z_t and study various belief-updating schemes using regressions of the form

$$z_{t+j} - \widetilde{E}_t\left[z_{t+j}\right] = b_0 + b_z z_t + b_f \widetilde{E}_{t-1}\left[z_{t+j-1}\right] + \varepsilon_{t+j}, \tag{20}$$

where $z_t \in \{\pi_t, u_t\}$ and $\widetilde{E}_t\left[z_{t+j}\right]$ is the subjective forecast at time t for the realization j periods ahead.[21] Under full information rational expectations, we should expect the errors to be mean zero and unforecastable, $b_0 = b_z = b_f = 0$.

Coibion and Gorodnichenko (2012) show that various types of information sluggishness, including the sticky information model studied in Mankiw and Reis (2002) or Carroll (2003) or the noisy information model in the spirit of Lucas (1972), Sims (2003), and Woodford (2003b), imply $b_0 = 0$, $b_z \in (0, 1)$, and $b_f \in (-1, 0)$. Forecasts should be unconditionally unbiased and adjust slowly to

[21] Explicit assumptions and details of the framework that imply such an updating rule are provided in Appendix D.

	inflation				unemployment			
	b_0	b_z	b_f	R^2	b_0	b_z	b_f	R^2
data	-1.26	0.09	-0.55	0.074	-0.61	-0.35	0.13	0.180
	(0.12)	(0.11)	(0.19)		(0.06)	(0.18)	(0.17)	
model	-0.81	0.14	-0.45	0.091	-0.76	-0.42	-0.09	0.187

Table 3: Estimated coefficients for regression (20) in the data and its theoretical counterpart in the model. Standard errors for the empirical regression in parentheses.

new information in z_t. Coibion and Gorodnichenko (2012) document this sluggishness for inflation forecasts of households in the Michigan Survey, firms in the Livingston survey, and FOMC members in the Monetary Policy Reports. On the other hand, models of overshooting as in Bordalo et al. (2018) imply $b_z < 0$, as new information in z_t implies excessive updating of the forecast $\widetilde{E}_t [z_{t+j}]$ and hence a negatively related forecast error.[22] These models also imply that $b_0 = 0$.

Our model does not imply equation (20) as a structural relationship for belief updating, since the state space that determines the relationship between forecast errors and predictors involves multiple exogenous and endogenous variables. However, given the prevalence of such univariate predictive regressions in empirical work, it is still informative to run the reduced-form regressions on observed data and compare them with regressions using model-simulated data as a further validation of our mechanism.

We summarize the results in Table 3. First, as documented in Section 2, the data indicate that on average, households overpredict inflation and unemployment by a significant amount, hence the negative coefficients b_0 in the first row of the table. As we have already seen, our model is consistent with these large average biases, which cannot be rationalized by theories of information frictions or overreaction to news. Both of these theories postulate that households are correct on average. Our model also generates the right amount of predictability in inflation and unemployment forecast errors, reflected by very similar R^2 coefficients in the empirical and theoretical regressions.

The slope coefficients in the data row indicate that the evolution of inflation and unemployment forecasts is consistent with different belief updating mechanisms. The negative coefficient on the lagged forecast ($b_f = -0.55$) in the inflation regression suggests that households update inflation forecasts sluggishly. The positive coefficient on the current level of inflation ($b_z = 0.09$) is also consistent with this narrative, even though it is not statistically significant in our data. On the other hand, the slope coefficients in the unemployment regression have opposite signs. The negative coefficient on the current unemployment level ($b_z = -0.35$) suggests that households overreact to changes in unemployment.

Our model successfully reproduces the signs and magnitudes of the slope coefficients in the

[22] Bordalo et al. (2018) provide evidence of such overshooting in the *individual* forecasts in the SPF, although the *consensus* forecasts tend to respond sluggishly. They rationalize this finding using a model in which forecasters filter a signal contaminated with common and private noise, with a calibration where individual forecast errors are dominated by overreaction to the private noise component, while the average or consensus forecast is driven to sluggish adjustment to the common noise component. We focus on household forecasts in the Michigan Survey instead.

forecast error regressions for inflation and unemployment. As discussed in Section 5, an increase in pessimism increases the inflation and unemployment wedges while generating higher unemployment and a muted inflation response. The persistence of the belief shock means that high past inflation forecasts predict negative forecast errors and, as a result, generate a negative regression coefficient b_f. The muted inflation response implies little predictability of forecast errors by the level of inflation, which is manifested in a small coefficient b_z. On the other hand, households predict high unemployment following an increase in unemployment, which appears as an overreaction in unemployment forecasts and a negative coefficient b_z.

Our introduction of the belief shock to a standard DSGE model delivers both the mean biases and predictability patterns of households' inflation and unemployment forecasts as a joint outcome. Rather than updating beliefs about inflation and unemployment independently, the household understands the equilibrium relationship between these two variables but pessimistically biases their joint distribution. The learning models we have discussed are insufficient to explain the mean belief wedges, but we do not reject that they may contribute to the formation of subjective beliefs, and leave a combined model of pessimism and imperfect information for future work.

7 Conclusion

In this paper, we develop a framework in which agents' subjective beliefs depart from rational expectations and feature time-varying pessimism and optimism. Using survey data to discipline this departure, we show that subjective beliefs have an economically significant role in driving macroeconomic outcomes, especially labor market quantities.

Pessimistic agents in the model overweight outcomes that are associated with low continuation utilities. Systematic policy changes that alter the distribution of consumption will also affect the distribution of adverse states, and hence agents' subjective beliefs and decisions. We view the policy-invariant nature of the mapping between continuation utilities and beliefs as an extension of the rational expectations hypothesis that preserves immunity to the Lucas critique and makes our framework suitable for the study of normative questions.

A natural application is the conduct of monetary policy where managing private sector expectations stands at the forefront. Another direction is to exploit the cross-sectional differences in beliefs that we documented. With incomplete markets, heterogeneous exposures of continuation values to shocks will generate endogenous heterogeneity in beliefs and has implications for savings, portfolio choices, and labor market behavior. Such a modification of the framework can be applied to study the design of social insurance policies.

Appendix

A Subjective beliefs and belief wedges

In this section, we derive formulas for the belief distortions in the linearized version of the dynamic model described in Section 3, extended to include nonstationary shocks as in Appendix B.7. Let $(\Omega, \{\mathcal{F}_t\}_{t=0}^{\infty}, P)$ be the filtered probability space generated by the innovations $\{w_{t+1}\}_{t=0}^{\infty}$, with $w_{t+1} \sim N(0_k, I_{k\times k})$ iid. The subjective probability measure $\widetilde{P}$ is formally defined by specifying a strictly positive martingale M_{t+1} with one-period increments:

$$m_{t+1} = \frac{M_{t+1}}{M_t} = \exp\left(-\frac{1}{2}|\nu_t|^2 + \nu_t' w_{t+1}\right). \tag{21}$$

The conditional mean of the innovation vector under $\widetilde{P}$ then satisfies $\widetilde{E}_t[w_{t+1}] = \nu_t$. We consider linear model dynamics given by

$$\begin{aligned} x_t &= \widehat{x}_t + z_t \\ \widehat{x}_{t+1} &= \psi_q + \psi_x \widehat{x}_t + \psi_w w_{t+1} \\ z_{t+1} - z_t &= \phi_q + \phi_x \widehat{x}_t + \phi_q w_{t+1}. \end{aligned} \tag{22}$$

The vector x_t of economic variables therefore has a stationary component $\widehat{x}_t$ and a nonstationary component z_t that has a stationary growth rate.[23] We impose a restriction on the belief distortion (21):

$$\nu_t = \overline{H} + HF\widehat{x}_t,$$

where F is a $1 \times n$ vector and H, $\overline{H}$ are $k \times 1$ vectors. The belief distortion derived in the structural model is a special case of this restriction. In particular, in the case of the linear approximation of the stationary model developed in Section 3, we have $z_t \equiv 0$ and $x_{1t} = \widehat{x}_t$. Equation (15) implies that $\nu_t = -\overline{\theta}(\bar{x} + x_{1t})(V_x\psi_w)'$, and hence

$$\overline{H} = -\overline{\theta}\bar{x}(V_x\psi_w)' \qquad H = -(V_x\psi_w)' \qquad F = \overline{\theta}.$$

In the case of the nonstationary model from Appendix B.7, the expressions for $\overline{H}$, H, and F are given in equation (56).

Let $\zeta_t = Zx_t = Z(\widehat{x}_t + z_t)$ be an $m \times 1$ vector of variables for which we have observable data on households' expectations where Z is an $m \times n$ selection matrix. We are interested in τ-period-ahead belief wedges

$$\Delta_t^{(\tau)} = \widetilde{E}_t[\zeta_{t+\tau}] - E_t[\zeta_{t+\tau}].$$

Guess that

$$\begin{aligned} E_t[\zeta_{t+\tau} - \zeta_t] &= G_x^{(\tau)}\widehat{x}_t + G_0^{(\tau)} \\ \widetilde{E}_t[\zeta_{t+\tau} - \zeta_t] &= \widetilde{G}_x^{(\tau)}\widehat{x}_t + \widetilde{G}_0^{(\tau)}, \end{aligned}$$

where $G_x^{(\tau)}$, $G_0^{(\tau)}$, $\widetilde{G}_x^{(\tau)}$, and $\widetilde{G}_0^{(\tau)}$ are conformable matrix coefficients with initial conditions

$$G_0^{(\tau)} = \widetilde{G}_0^{(\tau)} = 0_{m\times 1} \qquad G_x^{(\tau)} = \widetilde{G}_x^{(\tau)} = 0_{m\times n}.$$

[23] The linear approximation of the model specified in Section 3 maps directly into this framework. We drop the subindices denoting first-order derivative processes for convenience.

We can then establish a recursive formula for the expectations under the data-generating measure

$$\begin{aligned}
G_x^{(\tau)}\widehat{x}_t + G_0^{(\tau)} &= E_t\left[\zeta_{t+\tau} - \zeta_t\right] = \\
&= E_t\left[Z\left(x_{t+1} - x_t\right) + G_x^{(\tau-1)}\widehat{x}_{t+1} + G_0^{(\tau-1)}\right] \\
&= G_0^{(\tau-1)} + Z\phi_q + \left(Z + G_x^{(\tau-1)}\right)\psi_q + \left[\left(Z + G_x^{(\tau-1)}\right)\psi_x + \left(Z\phi_x - Z\right)\right]\widehat{x}_t \\
&\quad + \left(\left(Z + G_x^{(\tau-1)}\right)\psi_w + Z\phi_w\right)E_t\left[w_{t+1}\right].
\end{aligned} \tag{23}$$

Since $E_t\left[w_{t+1}\right] = 0$, we obtain

$$\begin{aligned}
G_x^{(\tau)} &= \left(Z + G_x^{(\tau-1)}\right)\psi_x + \left(Z\phi_x - Z\right) \\
G_0^{(\tau)} &= G_0^{(\tau-1)} + Z\phi_q + \left(Z + G_x^{(\tau-1)}\right)\psi_q.
\end{aligned}$$

Under the subjective measure, the derivation is unchanged, except for the last line in (23), which now involves the subjective expectation $\widetilde{E}_t\left[w_{t+1}\right] = \overline{H} + HF\widehat{x}_t$. Then,

$$\begin{aligned}
\widetilde{G}_x^{(\tau)} &= \left(Z + \widetilde{G}_x^{(\tau-1)}\right)\psi_x + \left(Z\phi_x - Z\right) + \left(\left(Z + \widetilde{G}_x^{(\tau-1)}\right)\psi_w + Z\phi_w\right)HF \\
\widetilde{G}_0^{(\tau)} &= \widetilde{G}_0^{(\tau-1)} + Z\phi_q + \left(Z + \widetilde{G}_x^{(\tau-1)}\right)\psi_q + \left(\left(Z + \widetilde{G}_x^{(\tau-1)}\right)\psi_w + Z\phi_w\right)\overline{H}
\end{aligned}$$

Consequently,

$$\Delta_t^{(\tau)} = \left(\widetilde{G}_x^{(\tau)} - G_x^{(\tau)}\right)\widehat{x}_t + \widetilde{G}_0^{(\tau)} - G_0^{(\tau)}.$$

When the dynamics (22) are stationary and demeaned, $\overline{H}$, ϕ_q, ϕ_x, ϕ_w, and ϕ_q are all zero, and we get explicit expressions

$$\begin{aligned}
G_x^{(\tau)} &= Z\left(\psi_x\right)^\tau \\
G_0^{(\tau)} &= Z\sum_{i=0}^{\tau-1}\left(\psi_x\right)^i\psi_q = Z\left(I - \psi_x\right)^{-1}\left(I - \left(\psi_x\right)^\tau\right)\psi_q \\
\widetilde{G}_x^{(\tau)} &= Z\left(\psi_x + \psi_w HF\right)^\tau \\
\widetilde{G}_0^{(\tau)} &= Z\sum_{i=0}^{\tau-1}\left(\psi_x + \psi_w HF\right)^i\psi_q = Z\left(I - \left(\psi_x + \psi_w HF\right)\right)^{-1}\left(I - \left(\psi_x + \psi_w HF\right)^\tau\right)\psi_q.
\end{aligned}$$

B Linear approximation of models with robust preferences

The linear approximation in this paper builds on the series expansion method used in Holmes (1995), Lombardo (2010), and Borovička and Hansen (2014). The innovation in this paper consists of adapting the series expansion method to an approximation of models with robust preferences to derive a linear approximation that allows for endogenously determined time-varying belief distortions. The critical step in the expansion lies in the joint perturbation of the shock vector w_t and the penalty process θ_t.

B.1 Law of motion

We start with the approximation of the model for the law of motion (4) with a sufficiently smooth policy rule ψ. We consider a class of models indexed by the scalar perturbation parameter q that scales the volatility of the shock vector w_t

$$x_{t+1}(\mathsf{q}) = \psi\left(x_t(\mathsf{q}), \mathsf{q}w_{t+1}, \mathsf{q}\right) \tag{24}$$

and assume that there exists a series expansion of x_t around $\mathsf{q} = 0$:

$$x_t(\mathsf{q}) \approx \bar{x} + \mathsf{q}x_{1t} + \frac{\mathsf{q}^2}{2}x_{2t} + \ldots.$$

The processes x_{jt}, $j = 0, 1, \ldots$ can be viewed as derivatives of x_t with respect to the perturbation parameter, and their laws of motion can be inferred by differentiating (24) j times and evaluating the derivatives at $\mathsf{q} = 0$, assuming that ψ is sufficiently smooth. Here, we focus only on the approximation up to the first order:

$$\begin{aligned} \bar{x} &= \psi(\bar{x}, 0, 0) \\ x_{1t+1} &= \psi_q + \psi_x x_{1t} + \psi_w w_{t+1}. \end{aligned} \tag{25}$$

We begin with a case in which the equilibrium dynamics of x_t are stationary. Extensions to nonstationary environments are considered in Appendix B.7.

B.2 Continuation values

We now focus on the expansion of the continuation value recursion. Substituting the belief distortion (5) into the recursion (2) yields

$$V_t = u(x_t) - \frac{\beta}{\theta_t}\log E_t\left[\exp\left(-\theta_t V_{t+1}\right)\right]. \tag{26}$$

We are looking for an expansion of the continuation value

$$V_t(\mathsf{q}) \approx \bar{V} + \mathsf{q}V_{1t}. \tag{27}$$

To derive the solution of the continuation value, we are interested in expanding the following perturbation of the recursion:

$$V_t(\mathsf{q}) = u\left(x_t(\mathsf{q}), \mathsf{q}\right) - \beta\frac{\mathsf{q}}{\overline{\theta}\left(\bar{x} + x_{1t}\right)}\log E_t\left[\exp\left(-\frac{\overline{\theta}\left(\bar{x} + x_{1t}\right)}{\mathsf{q}}V_{t+1}(\mathsf{q})\right)\right]. \tag{28}$$

Here, we utilized the fact that $\theta_t = \overline{\theta}x_t \approx \overline{\theta}\left(\bar{x} + x_{1t}\right)$. More importantly, the perturbation scales jointly the volatility of the stochastic processes for V_t and $u(x_t)$ with the magnitude of the penalty parameter θ_t. In particular, the penalty parameter in the perturbation of equation (2) becomes $\mathsf{q}/\left[\overline{\theta}\left(\bar{x} + x_{1t}\right)\right]$ and decreases jointly with the volatility of the shock process. This assumption will imply that the benchmark and subjective models do not converge as $\mathsf{q} \to 0$, and the linear approximation around a deterministic steady state yields a nontrivial contribution from the subjective dynamics.

Using the expansion of the period utility function

$$u\left(x_t(\mathsf{q}), \mathsf{q}\right) \approx \bar{u} + \mathsf{q}u_{1t} = \bar{u} + \mathsf{q}\left(u_x x_{1t} + u_q\right),$$

we get the deterministic steady-state (zeroth-order) term by setting $\mathsf{q} = 0$:

$$\bar{V} = (1-\beta)^{-1}\bar{u}.$$

The first-order term in the expansion is derived by differentiating (28) with respect to q and is given by the recursion

$$V_{1t} = u_{1t} - \beta\frac{1}{\overline{\theta}\left(\bar{x}+x_{1t}\right)}\log E_t\left[\exp\left(-\overline{\theta}\left(\bar{x}+x_{1t}\right)V_{1t+1}\right)\right]. \tag{29}$$

Since $\bar{x}$ is constant and x_{1t} has linear dynamics (25), we hope to find linear dynamics for V_{1t} as well. Since $u_t = u\left(x_t\right)$, we can make the guess that $V_t^i\left(\mathsf{q}\right) = V^i\left(x_t\left(\mathsf{q}\right), \mathsf{q}\right)$, which leads to the following expressions for the derivative of V_t:

$$V_{1t} = V_x x_{1t} + V_q.$$

Using this guess and comparing coefficients, equation (29) leads to a pair of algebraic equations for the unknown coefficients V_x and V_q:

$$\begin{aligned} V_x &= u_x - \frac{\beta}{2}V_x\psi_w\psi_w' V_x'\overline{\theta} + \beta V_x\psi_x \\ V_q &= u_q - \frac{\beta}{2}\overline{\theta}\bar{x}V_x\psi_w\psi_w' V_x' + \beta V_x\psi_q + \beta V_q. \end{aligned}$$

The first from this pair of equations is a Riccati equation for V_x, which can be solved for given coefficients ψ_x and ψ_w.

B.3 Distortions and belief wedges

With the approximation of the continuation value at hand, we can derive the expansion of the one-period belief distortion m_{t+1} that defines the subjective model relative to the benchmark model. As in (28), we scale the penalty parameter θ_t jointly with the volatility of the underlying shocks:

$$m_{t+1}\left(\mathsf{q}\right) = \frac{\exp\left(-\frac{1}{\mathsf{q}}\theta_t V_{t+1}\left(\mathsf{q}\right)\right)}{E_t\left[\exp\left(-\frac{1}{\mathsf{q}}\theta_t V_{t+1}\left(\mathsf{q}\right)\right)\right]} \approx m_{0,t+1} + \mathsf{q}m_{1,t+1}.$$

It turns out that in order to derive the correct first-order expansion, we are required to consider a second-order expansion of the continuation value

$$V_t\left(\mathsf{q}\right) \approx \bar{V} + \mathsf{q}V_{1t} + \frac{\mathsf{q}}{2}V_{2t},$$

although the term V_{2t} will be inconsequential for subsequent analysis. Substituting in expression (27) and noting that $\bar{V}$ is a deterministic term, we can approximate m_{t+1} with

$$m_{t+1}\left(\mathsf{q}\right) \approx \frac{\exp\left(-\overline{\theta}\left(\bar{x}+x_{1t}\right)\left(V_{1t+1}+\frac{\mathsf{q}}{2}V_{2t+1}\right)\right)}{E_t\left[\exp\left(-\overline{\theta}\left(\bar{x}+x_{1t}\right)\left(V_{1t+1}+\frac{\mathsf{q}}{2}V_{2t+1}\right)\right)\right]}.$$

Differentiating with respect to q and evaluating at $\mathsf{q} = 0$, we obtain the expansion

$$\begin{aligned} m_{0t+1} &= \frac{\exp\left(-\overline{\theta}\left(\bar{x}+x_{1t}\right)V_{1t+1}\right)}{E_t\left[\exp\left(-\overline{\theta}\left(\bar{x}+x_{1t}\right)V_{1t+1}\right)\right]} \\ m_{1t+1} &= -\frac{1}{2\overline{\theta}\left(\bar{x}+x_{1t}\right)}m_{0t+1}\left[V_{2t+1} - E_t\left[m_{0t+1}V_{2t+1}\right]\right]. \end{aligned} \tag{30}$$

This expansion is distinctly different from the standard polynomial expansion familiar from the perturbation literature. First, observe that m_{0t+1} is not constant, as one would expect for a zeroth-order term, but nonlinear in V_{1t+1}. However, since $E_t[m_{0t+1}] = 1$, we can treat m_{0t+1} as a change of measure that will adjust the distribution of shocks that are correlated with m_{0t+1}. We will show that with Gaussian shocks, we can still preserve tractability. Further notice that $E_t[m_{1t+1}] = 0$.

The linear structure of V_{1t} also has an important implication for the subjective belief distortion constructed from m_{0t+1}. Substituting into (30) yields

$$m_{0t+1} = \frac{\exp\left(-\overline{\theta}\left(\bar{x} + x_{1t}\right) V_x \psi_w w_{t+1}\right)}{E_t\left[\exp\left(-\overline{\theta}\left(\bar{x} + x_{1t}\right) V_x \psi_w w_{t+1}\right)\right]}.$$

This implies that for a function $f(w_{t+1})$ with a shock vector $w_{t+1} \sim N(0, I)$, the first-order approximation is given by

$$\begin{aligned} \widetilde{E}_t\left[f\left(w_{t+1}\right)\right] &= E_t\left[m_{t+1} f\left(w_{t+1}\right)\right] \\ &\approx f_0\left(w_{t+1}\right) + E_t\left[m_{0t+1} f_1\left(w_{t+1}\right)\right]. \end{aligned} \tag{31}$$

The distortion generating the subjective measure $\widetilde{P}$ is therefore approximated by the zeroth-order term m_{0t+1}, and the vector w_{t+1} has the following distribution:

$$w_{t+1} \sim N\left(-\overline{\theta}\left(\bar{x} + x_{1t}\right)\left(V_x \psi_w\right)', I_k\right). \tag{32}$$

The mean of the shock is therefore time varying and depends on the linear process x_{1t}.

It follows that the belief wedges for the one-period-ahead forecast of the vector of variables x_t are given by

$$\Delta_t^{(1)} = \widetilde{E}_t\left[x_{t+1}\right] - E_t\left[x_{t+1}\right] = \psi_w \widetilde{E}_t\left[w_{t+1}\right] = -\overline{\theta}\left(\bar{x} + x_{1t}\right)\left(\psi_w \psi_w'\right) V_x'.$$

Belief wedges for longer-horizon forecasts are then computed using formulas from Appendix A, observing that we can set

$$F = \overline{\theta}, \qquad H = -\left(V_x \psi_w\right)', \qquad \overline{H} = -\overline{\theta}\bar{x}\left(V_x \psi_w\right)'.$$

The terms ψ_w and V_x are functions of structural parameters in the model solved in the following section.

B.4 Equilibrium conditions

We assume that equilibrium conditions in our framework can be written as

$$0 = E_t\left[\widetilde{g}\left(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t\right)\right], \tag{33}$$

where $\widetilde{g}$ is an $n \times 1$ vector function and the dynamics for x_t are implied by (4). This vector of equations includes expectational equations such as Euler equations of the robust household, which can be represented using subjective belief distortions m_{t+1}. We therefore assume that we can write the j-th component of $\widetilde{g}$ as

$$\widetilde{g}^j\left(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t\right) = m_{t+1}^{\sigma_j} g^j\left(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t\right),$$

where $\sigma_j \in \{0, 1\}$ captures whether the expectation in the j-th equation is under the household's subjective model. In particular, all nonexpectational equations and all equations not involving agents' preferences have $\sigma_j = 0$. System (33) can then be written as

$$0 = E_t\left[\mathbb{M}_{t+1} g\left(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t\right)\right], \tag{34}$$

where $\mathbb{M}_{t+1} = \text{diag}\left\{m_{t+1}^{\sigma_1}, \ldots, m_{t+1}^{\sigma_n}\right\}$ is a diagonal matrix of the belief distortions, and g is independent of θ_t. The zeroth-order and first-order expansions are

$$\begin{aligned}
0 &= E_t\left[\mathbb{M}_{0t+1} g_{0t+1}\right] = g_{0t+1} \\
0 &= E_t\left[\mathbb{M}_{0t+1} g_{1t+1}\right] + E_t\left[\mathbb{M}_{1t+1} g_{0t+1}\right] = E_t\left[\mathbb{M}_{0t+1} g_{1t+1}\right],
\end{aligned}$$

where the last equality follows from $E_t\left[m_{1t+1}\right] = 0$.

For the first-order derivative of the equilibrium conditions, we have

$$0 = E_t\left[\mathbb{M}_{0t+1} g_{1t+1}\right]. \tag{35}$$

The first-order term in the expansion of g_{t+1} is given by

$$\begin{aligned}
g_{1t+1} &= g_{x+} x_{1t+1} + g_x x_{1t} + g_{x-} x_{1t-1} + g_{w+} w_{t+1} + g_w w_t + g_q = \\
&= \left[\left(g_{x+}\psi_x + g_x\right)\psi_x + g_{x-}\right] x_{1t-1} + \left[\left(g_{x+}\psi_x + g_x\right)\psi_w + g_w\right] w_t + \\
&\quad + \left(g_{x+}\psi_x + g_{x+} + g_x\right)\psi_q + g_q + \left(g_{x+}\psi_w + g_{w+}\right) w_{t+1},
\end{aligned} \tag{36}$$

where symbols x_+, x, x_-, w_+, w, q represent partial derivatives with respect to $x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t$ and q, respectively. Given the subjective distribution of the shock vector (32), we can write

$$\widetilde{E}_t\left[w_{t+1}\right] = -\left(V_x\psi_w\right)' \overline{\theta}\left[\left(\bar{x} + \psi_q\right) + \psi_x x_{1t-1} + \psi_w w_t\right].$$

Let $[A]^i$ denote the i-th row of matrix A. Notice that

$$\left[g_{x+}\psi_w + g_{w+}\right]^i \left(V_x\psi_w\right)' \overline{\theta}$$

is a $1 \times n$ vector. Construct the $n \times n$ matrix $\mathbb{E}$ by stacking these row vectors for all equations $i = 1, \ldots, n$:

$$\mathbb{E} = \text{stack}\left\{\sigma_i\left[g_{x+}\psi_w + g_{w+}\right]^i \left(V_x\psi_w\right)' \overline{\theta}\right\},$$

which contains non-zero rows for expectational equations under the subjective model. Using matrix $\mathbb{E}$, we construct the conditional expectation of the last term in g_{1t+1} in (36). In particular,

$$\begin{aligned}
0 &= E_t\left[\mathbb{M}_{0t+1} g_{1t+1}\right] = \\
&= \left[\left(g_{x+}\psi_x + g_x\right)\psi_x + g_{x-}\right] x_{1t-1} + \left[\left(g_{x+}\psi_x + g_x\right)\psi_w + g_w\right] w_t + \\
&\quad + \left(g_{x+}\psi_x + g_{x+} + g_x\right)\psi_q + g_q - \mathbb{E}\left[\left(\bar{x} + \psi_q\right) + \psi_x x_{1t-1} + \psi_w w_t\right].
\end{aligned}$$

Equation (35) is thus a system of linear second-order stochastic difference equations. There are well-known results that discuss the conditions under which there exists a unique stable equilibrium path to this system (Blanchard and Kahn (1980), Sims (2002)). We assume that such conditions are satisfied. Comparing coefficients on x_{1t-1}, w_t, and the constant term implies that

$$0 = \left(g_{x+}\psi_x + g_x - \mathbb{E}\right)\psi_x + g_{x-} \tag{37}$$

$$0 = \left(g_{x+}\psi_x + g_x - \mathbb{E}\right)\psi_w + g_w \tag{38}$$

$$0 = \left(g_{x+}\psi_x + g_{x+} + g_x\right)\psi_q + g_q - \mathbb{E}\left(\bar{x} + \psi_q\right). \tag{39}$$

These equations need to be solved for ψ_x, ψ_w, ψ_q, and V_x where

$$V_x = u_x - \frac{\beta}{2} V_x \psi_w \psi_w' V_x' \overline{\theta} + \beta V_x \psi_x \tag{40}$$

and

$$\mathbb{E} = \text{stack} \left\{ \sigma_i \left[g_{x+} \psi_w + g_{w+} \right]^i (V_x \psi_w)' \, \overline{\theta} \right\}. \tag{41}$$

B.5 Multiple belief distortions

We proceeded with the derivation of the approximation under the assumption that there is only a single belief distortion affecting the equilibrium equations. This has been done for notational simplicity, and the extension to a framework with multiple agents endowed with heterogeneous belief distortions stemming from robust preferences is straightforward. Let us assume that there are J agents with alternative belief distortions characterized by $\left(V_t^j, m_{t+1}^j, \overline{\theta}^j\right)$, $j = 1, \ldots J$. The system of equilibrium conditions (34) given by

$$0 = E_t \left[\mathbb{M}_{t+1} g \left(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t \right) \right]$$

with $\mathbb{M}_{t+1} = \text{diag} \left\{ m_{t+1}^{\sigma_1}, \ldots, m_{t+1}^{\sigma_n} \right\}$ can then be extended to include alternative belief distortions indexed by $\sigma_i \in \{0, 1, \ldots, J\}$ where $m_{t+1}^0 \equiv 1$ denotes an undistorted equation. Subsequently, there are J distorted means of the innovations

$$\widetilde{E}_t^j \left[w_{t+1} \right] = - \left(V_x^j \psi_w \right)' \overline{\theta}^j \left[(\bar{x} + \psi_q) + \psi_x x_{1t-1} + \psi_w w_t \right]$$

that distort individual equations. Matrix $\mathbb{E}$ in (41) that collects the distortions of the equilibrium conditions then becomes

$$\mathbb{E} = \text{stack} \left\{ \left[g_{x+} \psi_w + g_{w+} \right]^i \left(V_x^{\sigma_i} \psi_w \right)' \overline{\theta}^{\sigma_i} \right\},$$

where $\sigma_i = 0$ corresponds to no distortion, and hence the i-th row is a row of zeros. The structure of the system (37)–(41) remains the same except that we now have J recursions for V_x^j in (40) and a modified matrix $\mathbb{E}$.

B.6 Special case: θ_t is an exogenous AR(1) process

In the application, we consider a special case that restricts θ_t to be an exogenous AR(1) process. With a slight abuse in notation, this restriction can be implemented by replacing the vector of variables x_t with $(x_t', f_t)'$ where f_t is a scalar AR(1) process representing the time variation in subjective beliefs as an exogenously specified shock:

$$f_{t+1} = (1 - \rho_f) \bar{f} + \rho_f f_t + \sigma_f w_{t+1}^f. \tag{42}$$

The dynamics of the model then satisfy

$$x_t = \psi \left(x_{t-1}, w_t, f_t \right) \tag{43}$$

with steady state $\left(\bar{x}', \bar{f}\right)'$. The vector $\overline{\theta}$ in (3) is then partitioned as $\overline{\theta}' = \left(\overline{\theta}_x', \overline{\theta}_f\right) = (0_{1\times n-1}, 1)$, and thus $\theta_t = f_t$. Constructing the first-order series expansion of (43), we obtain

$$\begin{pmatrix} x_{1t+1} \\ f_{1t+1} \end{pmatrix} = \begin{pmatrix} \psi_q \\ 0 \end{pmatrix} + \begin{pmatrix} \psi_x & \rho_f \psi_{xf} \\ 0 & \rho_f \end{pmatrix} \begin{pmatrix} x_{1t} \\ f_{1t} \end{pmatrix} + \begin{pmatrix} \psi_w & \sigma_f \psi_{xf} \\ 0 & \sigma_f \end{pmatrix} \begin{pmatrix} w_{t+1} \\ w^f_{t+1}, \end{pmatrix}$$

where w_{t+1} and w^f_{t+1} are uncorrelated innovations. The matrices ψ_x and ψ_w thus do not involve any direct impact of the dynamics of the belief shock f_{1t} and the matrix ψ_{xf} captures how the dynamics of f_{1t} influence the dynamics of endogenous state variables.

Let us further assume that the system (33) represents the equilibrium restrictions of the model except for equation (42). In this case, the function g does not directly depend on f. Repeating the expansion of the equilibrium conditions from Section B.4 and comparing coefficients on x_{t-1}, f_{t-1}, w_t, and the constant term yields the set of conditions for matrices ψ_x, ψ_w, ψ_{xf}, and ψ_q:

$$\begin{aligned} 0 &= \left(g_{x+}\psi_x + g_x\right)\psi_x + g_{x-} && (44)\\ 0 &= \left(g_{x+}\rho_f\psi_{xf} - \mathbb{E}\right) + \left(g_{x+}\psi_x + g_x\right)\psi_{xf} && (45)\\ 0 &= \left(g_{x+}\psi_x + g_x\right)\psi_w + g_w && (46)\\ 0 &= \left(g_{x+}\psi_x + g_{x+} + g_x\right)\psi_q + g_q - \mathbb{E}\bar{f} && (47) \end{aligned}$$

with

$$\begin{aligned} V_x &= u_x + \beta V_x \psi_x && (48)\\ V_f &= u_f - \frac{\beta\overline{\theta}}{2}\left(V_f^2\sigma_f^2 + 2V_x\psi_{xf}\sigma_f^2 V_f + V_x\left(\sigma_f^2\psi_{xf}\psi_{xf}' + \psi_w\psi_w'\right)V_x'\right) && (49)\\ &\quad +\beta\left(V_f\rho_f + V_x\psi_{xf}\rho_f\right) \\ \mathbb{E} &= \text{stack}\left\{\sigma^i\left[g_{x+}\psi_{xf}\sigma_f^2\left(V_f + V_x\psi_{xf}\right) + \left(g_{x+}\psi_w + g_{w+}\right)\psi_w' V_x'\right]^i\right\}\overline{\theta}. && (50) \end{aligned}$$

This set of equations is the counterpart of equations (37)–(41) and can be solved sequentially. First, notice that equations (44) and (46) can be solved for ψ_x and ψ_w, and these coefficients are not affected by the dynamics of f_t. But the equilibrium dynamics of x_t are affected by movements in f_t through the coefficient ψ_{xf}. The coefficient $\rho_f\psi_{xf}$ introduces an additional component in the time-varying drift of x_t, while $\sigma_f\psi_{xf}$ is an additional source of volatility arising from the shocks to household's subjective beliefs.

We solve this set of equations by backward induction. First, we use (37), (41), and (48) to find the rational expectations solution for ψ_x, ψ_w, V_x. Then we postulate that (43) is in fact a time-dependent law of motion

$$x_t = \psi^t\left(x_{t-1}, w_t, f_t\right)$$

with terminal condition at a distant date T

$$x_T = \psi^T\left(x_{T-1}, w_T, 0\right).$$

This corresponds to assuming that starting from date T, subjective belief distortions are absent in the model.

Plugging this guess into the set of equilibrium conditions, we obtain the set of algebraic equations

$$\begin{aligned}
0 &= \left(g_{x+}\psi_{xf}^{t+1}\rho_f - \mathbb{E}^{t+1}\right) + \left(g_{x+}\psi_x + g_x\right)\psi_{xf}^t && (51)\\
V_f^t &= u_f - \frac{\beta\overline{\theta}}{2}\left(\left(V_f^{t+1}\sigma_f\right)^2 + 2V_x\psi_{xf}^{t+1}\sigma_f^2 V_f^{t+1} + V_x\left(\sigma_f^2\psi_{xf}^{t+1}\left(\psi_{xf}^{t+1}\right)' + \psi_w\psi_w'\right)V_x'\right) && (52)\\
&\quad +\beta\rho_f\left(V_f^{t+1} + V_x\psi_{xf}^{t+1}\right)\\
\mathbb{E}^{t+1} &= \left[g_{x+}\psi_{xf}^{t+1}\left(V_f^{t+1} + V_x\psi_{xf}^{t+1}\right)\sigma_f^2 + \left(g_{x+}\psi_w + g_{w+}\right)\psi_w' V_x'\right]\overline{\theta}. && (53)
\end{aligned}$$

Equation (51) can then be solved for

$$\psi_{xf}^t = \left(g_{x+}\psi_x + g_x\right)^{-1}\left(\mathbb{E}^{t+1} - g_{x+}\psi_{xf}^{t+1}\rho_f\right) \tag{54}$$

Iterating backward on equations (52)–(54) until convergence yields the stationary solution of the economy with subjective beliefs as a long-horizon limit of an economy where these concerns vanish at a distant T. The system converges as long as its dynamics are stationary under the subjective model. Once we find the limit $\lim_{t\to-\infty}\mathbb{E}^t = \mathbb{E}$, we can also determine

$$\psi_q = \left(g_{x+}\psi_x + g_{x+} + g_x\right)^{-1}\left(\mathbb{E}\bar{f} - g_q\right).$$

B.7 Nonstationary models

For the purpose of applying the expansion method, we assumed that the state vector x_t is stationary. Our framework can, however, deal with deterministic or stochastic trends featured in macroeconomic models. Specifically, let us assume that there exists a vector-valued stochastic process z_t such that the dynamics of x_t can be written as

$$\begin{aligned}
x_t &= \widehat{x}_t + z_t && (55)\\
z_{t+1} - z_t &= \phi\left(\widehat{x}_t, w_{t+1}\right),
\end{aligned}$$

where $\widehat{x}_t$ is a stationary vector Markov process that replaces dynamics (4):

$$\widehat{x}_{t+1} = \psi\left(\widehat{x}_t, w_{t+1}\right).$$

The process z_t thus has stationary increments and x_t and z_t are cointegrated, element by element. A typical example of an element in z_t is a productivity process with a permanent component. Once we solve for the stationary dynamics of $\widehat{x}_t$, we can obtain the dynamics of x_t in a straightforward way using (55).

Assume that the period utility function can be written in the form

$$u\left(x_t\right) = \widehat{u}\left(\widehat{x}_t\right) + Z^u z_t,$$

where Z^u is a selection vector that selects the appropriate scaling from the vector z_t. For example,

$$u\left(x_t\right) = \log C_t = \log\left[\widehat{C}_t \exp\left(Z^u z_t\right)\right] = \log\widehat{C}_t + Z^u z_t,$$

where $Z^u z_t$ is the nonstationary component of the logarithm of consumption $\log C_t$, and $\widehat{C}_t = \widehat{C}\left(\widehat{x}_t\right)$ is the stationary part. It follows from equation (26) that we can write

$$V_t = \widehat{V}\left(\widehat{x}_t\right) + \left(1-\beta\right)^{-1} Z^u z_t,$$

and the stationary component of the continuation value $\widehat{V}\left(\widehat{x}_t\right)$ satisfies the recursion

$$\widehat{V}\left(\widehat{x}_t\right) = \widehat{u}\left(\widehat{x}_t\right) - \frac{\beta}{\theta_t} \log E_t\left[\exp\left(-\theta_t\left(\widehat{V}\left(\widehat{x}_{t+1}\right) + \left(1-\beta\right)^{-1} Z^u \phi\left(\widehat{x}_t, w_{t+1}\right)\right)\right)\right].$$

The first-order expansion of ϕ yields

$$\begin{aligned} \bar{z}_{t+1} - \bar{z}_t &= \phi\left(\bar{x}, 0\right) \\ z_{1t+1} - z_{1t} &= \phi_q + \phi_x \widehat{x}_{1t} + \phi_w w_{t+1}, \end{aligned}$$

where $\bar{x}$ is the steady state of $\widehat{x}_t$. We can now proceed as in the stationary case except using the expansion of functions $\widehat{u}$ and $\widehat{V}$. We have

$$\bar{V} = \left(1-\beta\right)^{-1}\left[\bar{u} + \beta\left(1-\beta\right)^{-1} Z^u \phi\left(\bar{x}, 0\right)\right]$$

and

$$\widehat{V}_{1t} = V_x \widehat{x}_{1t} + V_q$$

with

$$\begin{aligned} V_x &= u_x + \beta\left[V_x \psi_x + \left(1-\beta\right)^{-1} Z^u \phi_x\right] - \frac{\beta}{2}\left|V_x \psi_w + \left(1-\beta\right)^{-1} Z^u \phi_w\right|^2 \overline{\theta} \\ V_q &= u_q + \beta\left[V_q + V_x \psi_q + \left(1-\beta\right)^{-1} Z^u \phi_q\right] - \frac{\beta}{2}\overline{\theta}\bar{x}\left|V_x \psi_w + \left(1-\beta\right)^{-1} Z^u \phi_w\right|^2. \end{aligned}$$

The zeroth-order distortion is consequently given by

$$m_{0t+1} = \frac{\exp\left(-\overline{\theta}\left(\bar{x} + \widehat{x}_{1t}\right)\left(V_x \psi_w + \left(1-\beta\right)^{-1} Z^u \phi_w\right) w_{t+1}\right)}{E_t\left[\exp\left(-\overline{\theta}\left(\bar{x} + \widehat{x}_{1t}\right)\left(V_x \psi_w + \left(1-\beta\right)^{-1} Z^u \phi_w\right) w_{t+1}\right)\right]}$$

so that under the subjective belief,

$$w_{t+1} \sim N\left(-\overline{\theta}\left(\bar{x} + \widehat{x}_{1t}\right)\left(V_x \psi_w + \left(1-\beta\right)^{-1} Z^u \phi_w\right)', I_k\right).$$

Equation (16) then becomes

$$\begin{aligned} \widehat{x}_{1t+1} &= \psi_q - \overline{\theta}\bar{x}\psi_w\left(V_x \psi_w + \left(1-\beta\right)^{-1} Z^u \phi_w\right)' \\ &\quad + \left[\psi_x - \psi_w\left(V_x \psi_w + \left(1-\beta\right)^{-1} Z^u \phi_w\right)' \overline{\theta}\right] \widehat{x}_{1t} + \psi_w \widetilde{w}_{t+1} \\ &= \widetilde{\psi}_q + \widetilde{\psi}_x x_{1t} + \psi_w \widetilde{w}_{t+1}. \end{aligned}$$

Comparing these dynamics under the subjective belief with those under the data-generating process, we can again construct belief wedges for longer-horizon forecasts as in Section B.3. Under the nonstationary dynamics, these wedges $\Delta_t^{(j)} = \widetilde{E}_t\left[x_{t+j}\right] - E_t\left[x_{t+j}\right]$ are computed using the recursive calculations outlined in Appendix A, imposing

$$\begin{aligned} F &= \overline{\theta} \\ H &= -\left(V_x\psi_w + (1-\beta)^{-1} Z^u\phi_w\right)' \\ \overline{H} &= -\left(\overline{\theta}\bar{x}\right)\left(V_x\psi_w + (1-\beta)^{-1} Z^u\phi_w\right)'. \end{aligned} \tag{56}$$

To solve for the equilibrium dynamics, notice that we are still solving the set of equations (37)–(39) but now with V_x and $\mathbb{E}$ given by

$$\begin{aligned} V_x &= u_x + \beta\left[V_x\psi_x + (1-\beta)^{-1} Z^u\phi_x\right] - \frac{\beta}{2}\left|V_x\psi_w + (1-\beta)^{-1} Z^u\phi_w\right|^2 \overline{\theta} \\ \mathbb{E} &= \text{stack}\left\{\sigma_i\left[g_{x+}\psi_w + g_{w+}\right]^i \left(V_x\psi_w + (1-\beta)^{-1} Z^u\phi_w\right)'\overline{\theta}\right\}. \end{aligned}$$

In the special case described in Section B.6, the belief shock f_t is modeled as an exogenous AR(1) process. The first-order dynamics of the stochastic growth rate can be expressed as

$$z_{1t+1} - z_{1t} = \phi_q + \phi_x\widehat{x}_{1t} + \phi_{xf}f_{1t} + \phi_w w_{t+1} + \phi_{wf}w^f_{t+1}.$$

The only modifications appearing in the model solution are those related to the continuation value recursion and the shock distortion in $\mathbb{E}$. Specifically,

$$\begin{aligned} V_x &= u_x + \beta\left[V_x\psi_x + (1-\beta)^{-1} Z^u\phi_x\right] \\ V_f &= u_f + \beta\left(\rho_f V_f + \rho_f V_x\psi_{xf} + (1-\beta)^{-1} Z^u\phi_{xf}\right) \\ &\quad -\frac{\beta\overline{\theta}}{2}\left|V_x\psi_w + (1-\beta)^{-1} Z^u\phi_w\right|^2 - \frac{\beta\overline{\theta}}{2}\left|V_x\psi_{xf}\sigma_f + V_f\sigma_f + (1-\beta)^{-1} Z^u\phi_{wf}\right|^2 \\ \mathbb{E} &= \text{stack}\left\{\sigma^i\left[\left(g_{x+}\psi_w + g_{w+}\right)\left(V_x\psi_w + (1-\beta)^{-1} Z^u\phi_w\right)'\right]^i\right\}\overline{\theta} \\ &\quad +\text{stack}\left\{\sigma^i\left[g_{x+}\psi_{xf}\sigma_f\left(V_f\sigma_f + V_x\psi_{xf}\sigma_f + (1-\beta)^{-1} Z^u\phi_{wf}\right)\right]^i\right\}\overline{\theta}. \end{aligned}$$

In the recursive form, V_f and $\mathbb{E}$ can be solved by iterating on the pair of equations

$$\begin{aligned} V_f^t &= u_f + \beta\left(\rho_f V_f^{t+1} + \rho_f V_x\psi_{xf}^{t+1} + (1-\beta)^{-1} Z^u\phi_{xf}\right) \\ &\quad -\frac{\beta\overline{\theta}}{2}\left|V_x\psi_w + (1-\beta)^{-1} Z^u\phi_w\right|^2 - \frac{\beta\overline{\theta}}{2}\left|V_x\psi_{xf}^{t+1}\sigma_f + V_f^{t+1}\sigma_f + (1-\beta)^{-1} Z^u\phi_{wf}\right|^2 \\ \mathbb{E}^{t+1} &= \text{stack}\left\{\sigma^i\left[\left(g_{x+}\psi_w + g_{w+}\right)\left(V_x\psi_w + (1-\beta)^{-1} Z^u\phi_w\right)'\right]^i\right\}\overline{\theta} \\ &\quad +\text{stack}\left\{\sigma^i\left[g_{x+}\psi_{xf}^{t+1}\sigma_f\left(V_f^{t+1}\sigma_f + V_x\psi_{xf}^{t+1}\sigma_f + (1-\beta)^{-1} Z^u\phi_{wf}\right)\right]^i\right\}\overline{\theta} \end{aligned}$$

together with equation (54), which remains unchanged.

C Data and further empirical evidence

Macroeconomic data are collected from the Federal Reserve Bank of St. Louis database (FRED).[24] The data on households' expectations are obtained from the University of Michigan Surveys of Consumers.[25] We also use data from the Survey of Consumer Expectations administered by the Federal Reserve Bank of New York,[26] and data from the Survey of Professional Forecasters collected from the Federal Reserve Bank of Philadelphia website.[27] See Table 4 for details.

We use the consumer price index for all urban consumers: all items (CPIAUSCL in FRED) to compute the rate of inflation in the data. Computing the belief wedges using the personal consumption expenditures (PCE) index from the Bureau of Economic Analysis as an alternative would leave the cyclical component of the inflation wedge almost unchanged because the two series are highly correlated. However, the PCE series has a substantially lower mean (by 0.4% annually between 1982 and 2015), so using the PCE index as observations from the data-generating process would make households appear to overestimate inflation significantly more than in the case of the CPI. We prefer the CPI because its weighting is based on surveys of the composition of households' purchases, and is based on out-of-pocket expenditures, which are arguably more salient for the formation of households' beliefs.

For the rate of unemployment, we use the civilian unemployment rate (UNRATE in FRED) as the data counterpart. Since households in the Michigan Survey are asked about the *change* in the rate of unemployment, the potential issue with different average levels of alternative measures of unemployment that households could envision becomes irrelevant. We construct the level forecast as the realized UNRATE measure in the month when the forecast is made, plus the forecasted change in the unemployment rate from the Michigan Survey.

C.1 Survey data

For the inflation rate in the Michigan Survey, we record the cross-sectional mean, median, and quartile answers. The survey question on the unemployment rate only records up/same/down responses. We use the method from Carlson and Parkin (1975) and Mankiw et al. (2003) to fit a time series of normal distributions to these qualitative responses. Let q_t^u, q_t^s, and q_t^d be the fractions of survey answers up, same, down, respectively, recorded at time t. We assume that these categories are constructed from a continuous cross-sectional distribution of responses with normal density $N\left(\mu_t, \sigma_t^2\right)$. In particular, there exists a response threshold a such that an answer on the interval $[-a, a]$ is recorded as "same". This implies

$$q_t^d = \Phi\left(\frac{-a-\mu_t}{\sigma_t}\right) \qquad q_t^u = 1 - \Phi\left(\frac{a-\mu_t}{\sigma_t}\right),$$

and thus

$$-a-\mu_t = \sigma_t \Phi^{-1}\left(q_t^d\right) \qquad a-\mu_t = \sigma_t \Phi^{-1}\left(1-q_t^u\right),$$

[24]Federal Reserve Economic Data, Federal Reserve Bank of St. Louis, https://research.stlouisfed.org/fred2.

[25]Surveys of Consumers, University of Michigan, http://www.sca.isr.umich.edu/. See also Thomas (1999) for details on the survey methodology.

[26]Survey of Consumer Expectations, Center for Microeconomic Data, Federal Reserve Bank of New York, https://www.newyorkfed.org/microeconomics/sce.

[27]Survey of Professional Forecasters, Federal Reserve Bank of Philadelphia, https://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters/.

Households' expectations (Michigan Survey)

$\widetilde{E}_t\left[\sum_{j=1}^{4}\pi_{t+j}\right]$	Expected change in prices during the next year (Table 32, variable PX1), mean and median responses and quartiles of the cross-sectional distribution of individual answers. Questions: "*During the next 12 months, do you think that prices in general will go up, or go down, or stay where they are now?*" and "*By about what percent do you expect prices to go up, on the average, during the next 12 months?*"
$\widetilde{E}_t\left[\frac{1}{n}\sum_{j=1}^{4}u_{t+j}\right]$	Expected unemployment rate during the next year (Table 30, variable UMEX), construction of mean response and the dispersion detailed in the text. Question: "*How about people out of work during the coming 12 months – do you think there will be more unemployment than now, about the same, or less?*" We also report results interpreting answers to this question as expected unemployment in one year, $\widetilde{E}_t\left[u_{t+4}\right]$.

Households' expectations (SCE)

$\widetilde{E}_t\left[\sum_{j=1}^{4}\pi_{t+j}\right]$	Median one-year-ahead expected inflation rate (used in Figure 3). The time series is constructed by aggregating probabilistic responses to the question: "*In your view, what would you say is the percent chance that, over the next 12 months... the rate of inflation will be between x_i% and x_{i+1}%*" for a range of brackets across individual households. See Armantier et al. (2016) for details.
$\widetilde{P}_t[u_{t+4}]$	Probability of unemployment being higher in one year than today (used in Figure 3). Mean response to the question: "*What do you think is the percent chance that 12 months from now the unemployment rate in the U.S. will be higher than it is now?*"

Survey of Professional Forecasters

$E_t\left[\sum_{j=1}^{4}\pi_{t+j}\right]$	Forecasted CPI inflation rate, seasonally adjusted (CPI). Forecast at time t is constructed as the mean survey forecast made in second month of quarter $t+1$, for CPI inflation rate between quarters t and $t+4$.
$E_t[\frac{1}{n}\sum_{j=1}^{4}u_{t+j}]$	Forecasted unemployment rate, seasonally adjusted (UNEMP). Forecast at time t is constructed as the mean survey forecast made in second month of quarter $t+1$, for the average unemployment rate in quarters $t+1$ to $t+4$.

Macroeconomic variables (FRED)

π_t	Consumer price index for all urban consumers: all items, seasonally adjusted (CPIAUCSL). Quarterly logarithmic growth rate, last month to last month of quarter.
u_t	Civilian unemployment rate, quarterly, seasonally adjusted (UNRATE).
$\log\left(\mathcal{Y}_t/\mathcal{Y}_{t-1}\right)$	Real gross domestic product, quarterly, seasonally adjusted annual rate (GDPC96). Quarterly logarithmic growth rate.
$\log\left(\mathcal{Y}_t/\bar{\mathcal{Y}}_t\right)$	Output gap. Difference between real gross domestic product, quarterly, seasonally adjusted annual rate (GDPC96) and real potential output (GDPPOT).

Table 4: Data definitions for key macroeconomic and survey variables.

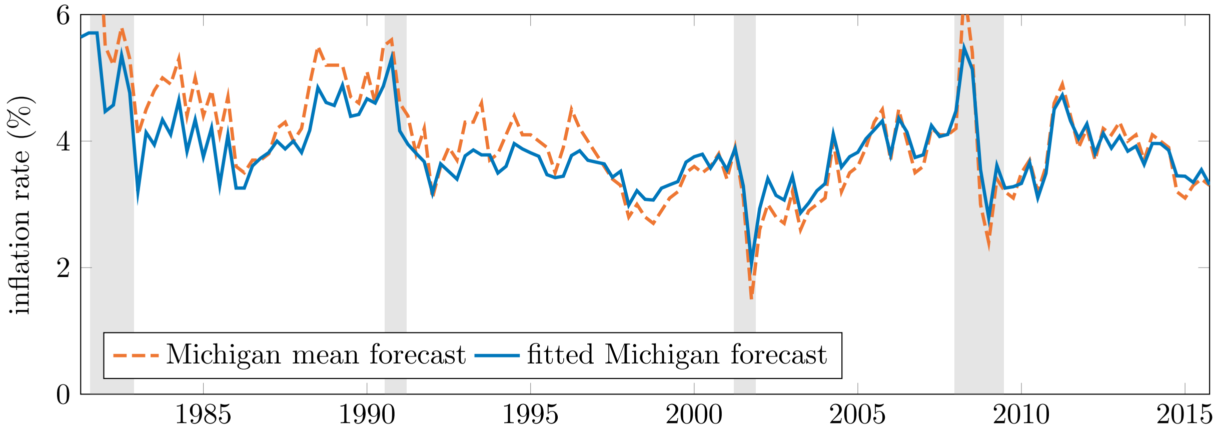

Figure 12: Mean one-year-ahead inflation rate forecast in the Michigan Survey (dashed line) and the fitted mean forecast (solid line) constructed using the Carlson and Parkin (1975) and Mankiw et al. (2003) method from categorical data. NBER recessions are shaded.

and therefore

$$\sigma_t = \frac{2a}{\Phi^{-1}\left(1-q_t^u\right)-\Phi^{-1}\left(q_t^d\right)}$$
$$\mu_t = a-\sigma_t\Phi^{-1}\left(1-q_t^u\right).$$

The constant a is then determined so that the time-series average of the cross-sectional dispersions σ_t divided by the observed average cross-sectional dispersion for the SPF forecast corresponds to the analogous ratio for the inflation responses, for which we have dispersion data readily available. We use the resulting means μ_t as the time series of mean unemployment rate forecasts.

To verify that the obtained time series μ_t provides a meaningful fit to the actual mean forecast, we verify the methodology using the inflation forecast data. We categorize individual numerical inflation forecast responses in each period into three bins, $< 3\%$, $3-5\%$, and $> 5\%$, and then fit a time series of normal distributions as described above, using the three time series of answer shares in each of the bins as input. Figure 12 compares the time series of actual mean forecasts with the time series of fitted means constructed using categorical data. The correlation between the two series is 92.8%, and the time-series averages differ only by 0.12%, providing strong support for the methodology as a plausible approximation of the actual mean forecast.

C.2 Information sets

The construction of belief wedges requires taking a stance on how to align information sets available to surveyed households and the econometrician. We use a quarterly VAR for our benchmark forecast under the data-generating (rational) measure. The Michigan Survey contains aggregated data at the quarterly frequency starting from 1960. We use these quarterly time series for the time period 1960Q3–2015Q4 in Figure 1 and Panel A of Table 5. We use the responses reported during quarter $t+1$ as those made with information available to the households at the end of quarter t. The forecasting horizon is assumed to span quarters $t+1$ to $t+4$.

We use monthly data from the Michigan Survey and available micro data from the monthly cross sections of the survey for the period 1982Q1–2015Q4. When computing the belief wedges relative to the VAR forecast, we use responses from the first month of quarter $t+1$ as those made by households with information available at the end of quarter t. Time-series moments for the wedges in this sample are summarized in Panel B of Table 5.

The SPF is administered during the second month of each quarter. To compute the belief wedge relative to the SPF forecast, we therefore use Michigan Survey responses from the second month of each quarter as well to align information sets for the two forecasts. We again use the time period 1982Q1–2015Q4. Forecasts made in the second month of quarter $t+1$ are assumed to span quarters $t+1$ to $t+4$ in the quarterly analysis. Panel C of Table 5 summarizes the data.

C.3 Forecasting VAR

We use a standard quarterly forecasting VAR to compute the forecasts of inflation and unemployment under the data-generating measure. All time series are downloaded from FRED for the period 1960Q1–2015Q4: CPI inflation (CPIAUCSL, percentage change to a year ago), real GDP (nominal series GDP divided by GDP deflator GDPCTPI, annualized percentage quarterly change), unemployment rate (UNRATE), log change in the relative price of investment goods (PIRIC), capital utilization rate (CUMFNS), hours worked (HOANBS), consumption rate ((PCDG+PCEND+PCESV)/GDP), investment rate (GPDI/GDP), and the federal funds rate (FEDFUNDS). The VAR is estimated with two lags. These choices for the forecasting VAR are similar to those made in Christiano et al. (2005), Del Negro et al. (2007), or Christiano et al. (2011). We experimented by increasing the lag number up to four, and by adding labor market variables as in Christiano et al. (2016), and all these choices do not materially change the results.

C.4 Further time-series evidence on the belief wedges

Figure 1 in the main text and Panel A from Table 5 contain time-series characteristics of the belief wedges from the Michigan Survey constructed using survey data for the period 1982Q1–2015Q4, net of the corresponding VAR forecasts. This is our preferred time period because the Michigan Survey for this period contains better-quality disaggregated survey data at the monthly frequency that allow us to better align information sets (Appendix C.2), study the cross-sectional patterns between the belief wedges, and compare the Michigan Survey responses with available SPF forecasts.

We use the Michigan Survey responses aggregated at the quarterly frequency for the period 1960Q1–2015Q4 as a robustness check. These results are reported in Panel B of Table 5. The patterns in the data are largely unchanged (information on the median inflation forecast is not available in the Michigan Survey for this time period). The belief wedges continue to be large, volatile, and countercyclical. The mean inflation wedge is somewhat smaller than in Panel A, and the lower correlation between the output gap and GDP growth implies that the wedges continue to be strongly countercyclical when using the output gap as the measure of economic activity, but the relationship with GDP growth is weaker.

Finally, we also construct the belief wedges using the responses from the Survey of Professional Forecasters as a measure of forecasts under the data-generating measure. Panel C from Table 5 provides the time-series characteristic for these wedges. As in the previous cases, we obtain large and volatile belief wedges that are highly negatively correlated with the business cycle.

In all three panels, we report alternative specifications for the wedges. For the inflation wedge, we show the results for the mean and median inflation forecast for the Michigan Survey. For the unemployment wedge, we produced two wedges based on alternative interpretations of the relevant question in the Michigan Survey. The wedge $\Delta_t^{(4)}(u)$ is the wedge for the forecast of the unemployment rate four quarters ahead. The

Panel A: 1982Q1–2015Q4, VAR forecast			correlation matrix					
	mean	std	(1)	(2)	(3)	(4)	(5)	(6)
(1) Unemployment wedge $\Delta_t^{(4)}(u)$	0.58	0.54	1.00	0.87	0.23	0.21	−0.54	−0.32
(2) Unemployment wedge $\overline{\Delta}_t^{(4)}(u)$	0.54	0.45		1.00	0.20	0.22	−0.29	−0.43
(3) Mean inflation wedge $\Delta_t^{(4)}(\pi)$	1.25	1.03			1.00	0.94	−0.37	−0.53
(4) Median inflation wedge $\Delta_t^{(4)}(\pi)$	0.43	1.14				1.00	−0.32	−0.60
(5) Output gap $\log\left(\mathcal{Y}_t/\bar{\mathcal{Y}}_t\right)$	−1.75	1.93					1.00	0.61
(6) GDP growth $\log\left(\mathcal{Y}_t/\mathcal{Y}_{t-4}\right)$	2.67	2.03						1.00

Panel B: 1960Q2–2015Q4, VAR forecast			correlation matrix					
	mean	std	(1)	(2)	(3)	(4)	(5)	(6)
(1) Unemployment wedge $\Delta_t^{(4)}(u)$	0.43	0.63	1.00	0.89	0.17	—	−0.49	0.00
(2) Unemployment wedge $\overline{\Delta}_t^{(4)}(u)$	0.43	0.44		1.00	0.24	—	−0.40	−0.16
(3) Mean inflation wedge $\Delta_t^{(4)}(\pi)$	0.78	1.17			1.00	—	−0.49	−0.56
(4) Median inflation wedge $\Delta_t^{(4)}(\pi)$	—	—				—	—	—
(5) Output gap $\log\left(\mathcal{Y}_t/\bar{\mathcal{Y}}_t\right)$	−1.00	2.29					1.00	0.32
(6) GDP growth $\log\left(\mathcal{Y}_t/\mathcal{Y}_{t-4}\right)$	2.97	3.30						1.00

Panel C: 1982Q1–2015Q4, SPF forecast			correlation matrix					
	mean	std	(1)	(2)	(3)	(4)	(5)	(6)
(1) Unemployment wedge $\Delta_t^{(4)}(u)$	0.55	0.49	1.00	0.97	0.18	0.21	−0.38	−0.60
(2) Unemployment wedge $\overline{\Delta}_t^{(4)}(u)$	0.48	0.47		1.00	0.16	0.22	−0.18	−0.53
(3) Mean inflation wedge $\Delta_t^{(4)}(\pi)$	1.07	0.85			1.00	0.94	−0.14	−0.29
(4) Median inflation wedge $\Delta_t^{(4)}(\pi)$	0.43	1.14				1.00	−0.32	−0.60
(5) Output gap $\log\left(\mathcal{Y}_t/\bar{\mathcal{Y}}_t\right)$	−1.75	1.93					1.00	0.61
(6) GDP growth $\log\left(\mathcal{Y}_t/\mathcal{Y}_{t-4}\right)$	2.67	2.03						1.00

Table 5: Time-series and business cycle statistics for the belief wedges. *Panel A*: Belief wedge relative to a VAR forecast, time period 1982Q1–2015Q4. *Panel B*: Belief wedge relative to a VAR forecast, time period 1960Q2–2015Q4 (median inflation forecast not available for this period). *Panel C*: Belief wedge relative to the SPF forecast, time period 1982Q1–2015Q4. For details, see Appendix C.4.

wedge $\overline{\Delta}_t^{(4)}(u)$ is the wedge for the forecast of the average unemployment rate during the next four quarters.

C.5 Further cross-sectional evidence on the belief wedges

In this section, we provide further evidence on the cross-sectional relationship between household-level survey answer biases for alternative questions, documented in the Michigan Survey and the SCE.

In the cross-sectional analysis (except for Table 6), we do not convert unemployment responses using the procedure described in Appendix C.1, but encode categorical household-level responses on the forecasted change in the unemployment rate {down, same (or don't know), up} for household i in demographic group g and month t as $\widetilde{u}_{i,g,t} \in \{-1, 0, 1\}$. We drop respondents aged 65 and above and those with missing responses. Population and group-level averages $\widetilde{u}_t$ and $\widetilde{u}_{g,t}$ then represent the share of respondents who forecast an increase in unemployment minus the share that forecasts a decrease. For the inflation responses, we drop households who indicate "don't know," have a missing response, or have extreme forecasts (above 20% or below −10%). The results are robust to keeping the extreme forecasts.

	actual	SPF	all	18-34	35-44	45-54	55-64	W	NC	NE	S
π	2.69	2.91	3.96	4.03	3.94	3.91	3.81	3.87	3.90	3.91	4.09
u	6.22	6.29	6.83	6.69	6.85	6.92	6.93	6.81	6.83	6.88	6.81
u share	—	—	16.5	12.0	17.0	19.5	19.3	15.7	16.8	18.0	16.0
		male	female	bottom	2nd Q	3rd Q	top	HS	SC	COL	GS
π		3.38	4.46	4.90	4.20	3.67	3.17	4.50	3.89	3.46	3.34
u		6.74	6.90	6.98	6.85	6.80	6.68	6.92	6.83	6.72	6.73
u share		12.9	19.6	21.3	17.6	15.6	11.2	20.0	16.9	12.7	12.7

Table 6: Demographic characteristics of households' expectations on inflation (π) and the unemployment rate. The line labeled "u share" is the percentage share of responses that the unemployment rate will increase minus the percentage share stating that the unemployment rate will decrease. The line labeled "u" is the average fitted unemployment rate forecast computed as in Appendix C.1. Time-series averages, all values are annualized and in percentages, time period 1982Q1–2015Q4. *Actual*: actual average inflation and unemployment rate; *SPF*: average SPF forecast; *all*: average household forecast; *18-34 etc*: age groups; *W*: West region; *NC*: North-Central; *NE*: North-East; *S*: South; *bottom, 2nd Q, 3rd Q, top*: income quartiles; *HS*: high school education; *SC*: some college; *COL*: college degree; *GS*: graduate studies.

Table 6 reports the conditional time-series averages of the households' forecasts for different demographic groups in the Michigan Survey, displayed in Figure 4. More educated respondents and respondents with higher incomes overpredict inflation and unemployment less on average, but all demographic groups still overpredict both quantities. Moreover, demographic groups that on average overpredict inflation relatively more also overpredict unemployment relatively more.

Tables 7–9 provide further details at the level of demographic groups and individual households. First, we ask whether in times when demographic group g on average overpredicts inflation more relative to population, the group also overpredicts unemployment more relative to population. Table 7 summarizes the regression coefficients in time-series regressions of the form

$$\widetilde{u}_{g,t} - \widetilde{u}_t = \alpha_g + \beta_g \left[\widetilde{\pi}_{g,t} - \widetilde{\pi}_t\right] + \varepsilon_{g,t}, \tag{57}$$

where $\widetilde{u}_{g,t}$, $\widetilde{\pi}_{g,t}$ are the average forecasts of demographic group g in month t, and $\widetilde{u}_t$, $\widetilde{\pi}_t$ are the average forecasts in month t for the whole population. The estimated regression coefficients $\widehat{\beta}_g$ are all positive, and most of them are highly statistically significant.

Next, we investigate whether in times when *individual households* i overpredict inflation more relative to the population, they also overpredict unemployment relatively more. The regression on the pooled sample with demographic controls is

$$\widetilde{u}_{i,g,t} - \widetilde{u}_t = \alpha + \beta \left[\widetilde{\pi}_{i,g,t} - \widetilde{\pi}_t\right] + \delta' D_{i,g,t} + \varepsilon_{i,g,t},$$

where $\widetilde{u}_{i,g,t}$, $\widetilde{\pi}_{i,g,t}$ are the forecasts of household i belonging to demographic group g in month t and $D_{i,g,t}$ is the vector of demographic group dummies. The estimated slope coefficient is $\widehat{\beta} = 2.08$ with a standard error of 0.04. We also run pooled regressions using differences between individual household forecasts and the group-specific average in the given month:

$$\widetilde{u}_{i,g,t} - \widetilde{u}_{g,t} = \alpha_c + \beta_c \left[\widetilde{\pi}_{i,g,t} - \widetilde{\pi}_{g,t}\right] + \varepsilon_{i,g,t} \tag{58}$$

for different demographic categorizations $c \in \{$pooled population, education, income, region, age, sex$\}$. Ta-

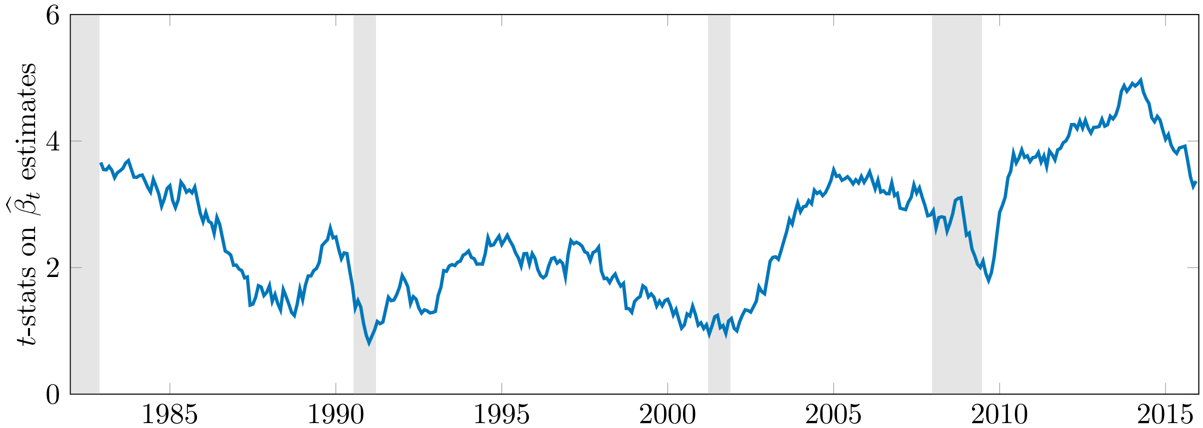

Figure 13: Smoothed (12-month moving average) t-statistics on the estimates $\widehat{\beta}_t$ in regression (59) for the case when the demographic sorting c corresponds to the pooled population (i.e., $\widetilde{u}_{g,t} = \widetilde{u}_t$.)

ble 8 reports the estimates of regression coefficients $\widehat{\beta}_c$.

To show that these cross-sectional relationships are stable over time, we run the regressions month by month and for each demographic sorting c:

$$\widetilde{u}_{i,g,t} - \widetilde{u}_{g,t} = \alpha_{c,t} + \beta_{c,t} \left[\widetilde{\pi}_{i,g,t} - \widetilde{\pi}_{g,t}\right] + \varepsilon_{i,g,t}. \tag{59}$$

Table 9 shows the mean and standard deviation of the *distribution* of estimated coefficients $\widehat{\beta}_{c,t}$ for each of the categorizations. Regardless of the demographic categorization, around 95% of all the estimated coefficients $\widehat{\beta}_{c,t}$ are positive, and about two-thirds of them have a t-statistic larger than 1.96. Figure 13 plots the smoothed time series of the coefficients for the pooled population case and documents that the significantly positive cross-sectional relationship between the belief wedges is not specific to a particular subperiod in the data.

Finally, we also corroborate the cross-sectional patterns with those in the SCE. Table 10 reports the cross-sectional correlations for the following survey questions about forecasts of aggregate and household-level variables (for more details on the survey design and questions, see Armantier et al. (2016)):

(1) expected rate of inflation over the next 12 months;

(2) percent chance that 12 months from now, the unemployment rate in the United States will be higher than it is now;

(3) percent chance that 12 months from now, on average, stock prices in the United States stock market will be higher than they are now;

(4) expected percent increase in individual earnings over the next 12 months conditional on staying in the same job;

(5) expected percent increase of total household income over the next 12 months;

(6) percent chance of losing job in the next 12 months;

(7) percent chance of finding a job in next three months conditional on losing job today.

Variables that are positively correlated with the notion of "good times" are taken with opposite signs. The first three variables represent forecasts of macroeconomic variables, and the remaining four refer to

			18-34	35-44	45-54	55-65	W	NC	NE	S
$100 \times \widehat{\beta}_g$			3.37	2.32	2.16	2.92	2.89	1.57	1.98	4.67
std. err.			0.92	0.82	0.81	0.74	0.94	0.91	0.86	0.89
	male	female	bottom	2nd Q	3rd Q	top	HS	SC	COL	GS
$100 \times \widehat{\beta}_g$	3.95	4.52	0.56	0.72	2.97	0.85	4.41	5.50	2.60	5.50
std. err.	1.16	1.18	0.83	0.87	0.88	1.09	0.92	0.85	0.92	1.08

Table 7: Regression coefficients in regression (57) run separately for alternative demographic groups g, listed in the caption of Table 6. $100 \times \widehat{\beta}_g$ scales the left-hand side in the regression to percentage shares.

	population	education	income	region	age	sex
$100 \times \widehat{\beta}_c$	2.19	2.15	2.14	2.19	2.20	2.12
std. err.	0.04	0.04	0.04	0.04	0.04	0.04

Table 8: Regression coefficients in pooled regression (58) for alternative demographic categorizations c. $100 \times \widehat{\beta}_c$ scales the left-hand side in the regression to percentage shares.

	population	education	income	region	age	sex
average $100 \times \widehat{\beta}_{c,t}$	2.32	2.27	2.27	2.32	2.33	2.26
std. dev. $100 \times \widehat{\beta}_{c,t}$	1.39	1.33	1.37	1.38	1.38	1.39
months	408	408	408	408	408	408
$\#\ t > 0$	392	393	396	394	392	389
$\#\ t > 1.96$	266	265	260	269	270	260

Table 9: Regression coefficients in regression (59) for alternative demographic categorizations c. "Months" indicates the number of monthly regressions we run in each case, and $\#\ t > 0$ and $\#\ t > 1.96$ indicate the number of regressions from that sample in which the estimate $\widehat{\beta}_{c,t}$ has a t-statistic larger than zero or 1.96, respectively. $100 \times \widehat{\beta}_c$ scales the left-hand side in the regression to percentage shares.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
(1) inflation	1.00	0.27	0.78	0.87	0.85	0.56	0.70
(2) unemployment		1.00	0.21	0.29	0.11	0.49	0.22
(3) stock prices (−)			1.00	0.90	0.64	0.44	0.66
(4) earnings growth (−)				1.00	0.82	0.50	0.80
(5) income growth (−)					1.00	0.45	0.85
(6) job loss						1.00	0.58
(7) job finding (−)							1.00

Table 10: Cross-sectional correlations for responses in the pooled sample from the SCE, 2013M06–2016M09. See text for details on individual questions. (−) denotes variable taken with a negative sign.

households' individual outcomes. Table 10 shows that the correlations are all positive and mostly economically large, indicating that households who forecast higher inflation are also generally more pessimistic about aggregate and individual outcomes. Notice that stock prices, earnings, and income growth are nominal variables, so the pessimism about real quantities for households who forecast higher inflation is even stronger. These results confirm and extend our findings from the cross-sectional analysis of the Michigan Survey.

D Alternative models of belief updating

In this section, we provide a theoretical justification for regression (20) using two models of information processing. The first is a sticky information model in the spirit of Mankiw and Reis (2002). Assume that the forecasted variable z_t follows an AR(1) process

$$z_t = \rho z_{t-1} + w_t$$

with iid innovations w_t and $\rho \in [0,1]$. Under full information, the j-period-ahead forecast is $E_t\left[z_{t+j}\right] = \rho^j z_t$. Under sticky information, each agent updates her information about the current state with probability $1-\lambda \in (0,1]$. At every time t, a fraction $(1-\lambda)\lambda^k$ of agents last observed the state of the process at time $t-k$ (the case $\lambda = 0$ thus corresponds to the full information model). The cross-sectional average of the individual forecasts at time t, which plays the role of the aggregate forecast in (20), is therefore given by

$$\widetilde{E}_t\left[z_{t+j}\right] = (1-\lambda)\sum_{k=0}^{\infty}\lambda^k E_{t-k}\left[z_{t+j}\right] = (1-\lambda)\sum_{k=0}^{\infty}\lambda^k\rho^{j+k}z_{t-k},$$

where $E_{t-k}\left[z_{t+j}\right]$ is the time-t forecast of an agent who has last updated her information at time $t-k$. Since

$$z_{t-k} = \sum_{m=0}^{\infty}\rho^m w_{t-k-m},$$

we get

$$\widetilde{E}_t\left[z_{t+j}\right] = (1-\lambda)\sum_{k=0}^{\infty}\lambda^k\rho^{j+k}\sum_{m=0}^{\infty}\rho^m w_{t-k-m} = \sum_{k=0}^{\infty}\left(1-\lambda^{k+1}\right)\rho^{j+k}w_{t-k},$$

which can be represented recursively as

$$\widetilde{E}_t\left[z_{t+j}\right] = (1-\lambda)\rho^j z_t + \lambda\rho\widetilde{E}_{t-1}\left[z_{t-1+j}\right].$$

This yields the expression for forecast errors of the average forecast

$$z_{t+j} - \widetilde{E}_t\left[z_{t+j}\right] = \lambda\rho^j z_t - \lambda\rho\widetilde{E}_{t-1}\left[z_{t-1+j}\right] + \sum_{m=0}^{j-1}\rho^m w_{t+j-m}.$$

This corresponds to regression (20) with $b_0 = 0$, $b_z = \lambda\rho^j \in [0,1)$, and $b_f = -\lambda\rho \in (-1,0]$. The regression coefficients reduce to $b_z = b_f = 0$ in the absence of information frictions ($\lambda = 0$).

The second model is a noisy information model motivated by Lucas (1972), Sims (2003), and Woodford (2003a). Specifically, z_t follows again an AR(1) process but is not observable. Instead, each agent i receives a combination of a public signal $y_t = z_t + \chi_t$, $\chi_t \sim N\left(0,\sigma_\chi^2\right)$ that is common for everybody and an idiosyncratic private signal $y_{it} = z_t + \eta_{it}$, $\eta_{it} \sim N\left(0, \sigma^2_\eta\right)$. The state space system can then be written as

$$\begin{aligned} z_t &= \rho z_{t-1} + w_t \qquad w_t \sim N\left(0, \Sigma_w\right) \\ s_{it} &= h z_t + v_{it} \qquad v_{it} \sim N\left(0, \Sigma_v\right) \end{aligned}$$

where $s_{it} = \left(y_t, y_{it}\right)'$, $h = (1,1)'$, $v_{it} = \left(\chi_t, \eta_{it}\right)'$, and Σ_v is diagonal with elements σ^2_χ and σ^2_η. The standard steady-state Kalman filter solution to the filtering problem implies that agent i's time-t forecast of z_{t+j} follows the law of motion

$$\widetilde{E}^i_t\left[z_{t+j}\right] = \rho \widetilde{E}^i_{t-1}\left[z_{t-1+j}\right] + K\left(\rho^{j-1} s_{it} - h\widetilde{E}^i_{t-1}\left[z_{t-1+j}\right]\right),$$

where K is the Kalman gain parameter, given by

$$\begin{aligned} K &= \rho\Sigma h'\left(h\Sigma h' + \Sigma_v\right)^{-1} \\ \Sigma &= \rho^2\Sigma - \rho^2\Sigma h'\left(h\Sigma h' + \Sigma_v\right)^{-1} h\Sigma + \Sigma_w. \end{aligned}$$

Denoting $\widetilde{E}_t\left[z_{t+j}\right]$ the cross-sectional average of the individual forecasts, we obtain

$$\widetilde{E}_t\left[z_{t+j}\right] = \rho \widetilde{E}_{t-1}\left[z_{t-1+j}\right] + K\left(\rho^{j-1} s_t - h\widetilde{E}_{t-1}\left[z_{t-1+j}\right]\right),$$

where $s_t = \left(z_t + \chi_t, z_t\right)'$. The law of motion for the average forecast can therefore be written as

$$\widetilde{E}_t\left[z_{t+j}\right] = \rho^{j-1} K h z_t + (\rho - Kh)\,\widetilde{E}_{t-1}\left[z_{t-1+j}\right] + \rho^{j-1} K_2 \chi_t,$$

where K_2 is the second element of K. Writing this forecast-updating equation in terms of forecast errors, we get

$$z_{t+j} - \widetilde{E}_t\left[z_{t+j}\right] = \left(\rho^j - \rho^{j-1} K h\right) z_t - (\rho - Kh)\,\widetilde{E}_{t-1}\left[z_{t-1+j}\right] - \rho^{j-1} K_2 \chi_t + \sum_{m=0}^{j-1} \rho^m w_{t+j-m}.$$

As in the sticky information model, this corresponds to regression (20) with $b_0 = 0$, $b_z = \left(\rho^j - \rho^{j-1} K h\right) \in [0,1)$, and $b_f = -(\rho - Kh) \in (-1, 0]$. The regression coefficients reduce to $b_z = b_f = 0$ in the absence of signal noise ($\Sigma_v = 0$).

E Equilibrium equations of the structural model

In this section, we summarize the full set of equilibrium conditions for the model described in Section 4.

E.1 Representative household

Value function recursion:

$$V_t = (1-\beta)\log\left(C_t\right) - \beta\theta_t \log E_t\left[\exp\left(-\frac{1}{\theta_t} V_{t+1}\right)\right]$$

Budget constraint:

$$P_t C_t + B_{t+1} \leq \left(1 - L_t\right) P_t D + L_t P_t \xi_t + R_{t-1} B_t - T_t$$

Stochastic discount factor:

$$s_{t+1} = \beta \frac{C_t}{C_{t+1}}$$

Euler equation for bond purchases:

$$1 = R_t \widetilde{E}_t \left[s_{t+1} \right]$$

E.2 Labor market

Law of motion for employment:

$$L_t = \left(\rho + h_t \right) L_{t-1}$$

Hiring rate:

$$h_t = \frac{f_t \left(1 - \rho L_{t-1} \right)}{L_{t-1}}$$

Vacancy-filling rate:

$$q_t = \frac{h_t}{v_t}$$

Labor market tightness:

$$\zeta_t = \frac{v_t L_{t-1}}{1 - \rho L_{t-1}}$$

Matching technology:

$$f_t = \mu \zeta_t^{\nu}$$

Present value of real wages (conditional on the job existing):

$$\xi_t^p = \xi_t + \rho \widetilde{E}_t \left[s_{t+1} \xi_{t+1}^p \right]$$

Present value of marginal revenue (conditional on the job existing):

$$\vartheta_t^p = \vartheta_t + \rho \widetilde{E}_t \left[s_{t+1} \vartheta_{t+1}^p \right]$$

The value of a job to the worker:

$$J_t^w = \xi_t^p + A_t$$

Outside benefits of being on a job:

$$A_t = \left(1 - \rho \right) \widetilde{E}_t \left[s_{t+1} \left(f_{t+1} J_{t+1}^w + \left(1 - f_{t+1} \right) U_{t+1} \right) \right] + \rho \widetilde{E}_t \left[s_{t+1} A_{t+1} \right]$$

Present value of unemployment:

$$U_t = D_t + \widetilde{E}_t \left[s_{t+1} \left(f_{t+1} J_{t+1}^w + \left(1 - f_{t+1} \right) U_{t+1} \right) \right]$$

Present value of the worker to the firm:

$$J_t = \vartheta_t^p - \xi_t^p$$

Free-entry condition:

$$J_t = \frac{\kappa_t^v}{q_t} + \kappa_t^h$$

Nash bargaining surplus sharing rule for target wage:

$$\eta \left(J_t + \xi_t - \xi_t^*\right) = (1-\eta)\left(J_t^w - U_t + \xi_t^* - \xi_t\right)$$

Actual wage:

$$\xi_t = \lambda \xi_{t-1} + (1-\lambda)\,\xi_t^*$$

E.3 Production

Optimal price setting:

$$\begin{aligned}
K_t &= \lambda \vartheta_t \frac{Y_t}{z_t A_t} + \xi \widetilde{E}_t\left[s_{t+1}\pi_{t+1}^{\varepsilon} K_{t+1}\right] \\
F_t &= Y_t + \xi \widetilde{E}_t\left[s_{t+1}\pi_{t+1}^{\varepsilon-1} F_{t+1}\right] \\
1 - \xi \pi_t^{\varepsilon-1} &= (1-\xi)\left(\frac{K_t}{F_t}\right)^{1-\varepsilon}
\end{aligned}$$

E.4 Shock processes and resource constraint

θ_t process:

$$\theta_t = (1-\rho_\theta)\bar{\theta} + \rho_\theta \theta_{t-1} + \sigma_\theta w_t^\theta$$

Technology process:

$$a_{t+1} = \rho_a a_t + \sigma_a w_{t+1}^a$$

Monetary policy rule:

$$\log\left(R_t/\overline{R}\right) = \rho_r \log\left(R_{t-1}/\overline{R}\right) + (1-\rho_r)\left[r_\pi \log\left(\pi_t/\overline{\pi}\right) + r_y \log\left(Y_t/Y^*\right)\right] + \sigma_r w_t^r$$

Aggregate resource constraint:

$$C_t + \frac{\kappa_v}{q_t} h_t L_{t-1} = Y_t$$

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