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Idea Diffusion and Property Rights

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Idea Diffusion and Property Rights

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Abstract

We study the innovation and diffusion of technology at the industry level. We derive an industry's transition from birth to maturity, and we characterize how diffusion affects the incentive to innovate. Socially optimal compensation for innovators should internalize industry-level learning and matching externalities, and we find the compensation should be higher if imitators are allowed to resell ideas. Our model also shows that enhancing idea diffusion is socially beneficial and can generate industry overtaking patterns endogenously. We apply the model to the early histories of the U.S. automobile and personal computer industries and quantify the theoretical predictions.

Keywords: Innovation, Industry Dynamics, Intellectual Property Rights

JEL Classification: L0, O3

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1 Introduction

Innovation and diffusion are fundamental drivers of technological progress and long-run growth. An innovation cannot fulfill its potential without being widely adopted, but rapid diffusion and imitation may reduce the incentive to innovate. In this paper, we study the interplay between innovation and diffusion in a competitive industry setting, and discuss welfare and policy implications.

The model features an industry with a downward-sloping demand curve for a homogeneous product and a group of zero measure potential producers. An innovation or "idea" enables an agent to produce the good at zero cost subject to a capacity constraint. At the outset, agents decide whether to pay a sunk cost to innovate. Some will do so immediately; others may consider innovating later, or wait to imitate an innovation.

Imitation occurs in random matching between those who have the idea and those who do not. Without intellectual property (IP) protection, imitation is free; otherwise the imitator may have to pay a fee to the idea seller and the fee is determined by the latter's bargaining share.

We study two IP protection regimes concerning payments for ideas. In Regime 1, imitators are prohibited from reselling ideas to other imitators. A potential adopter may copy an idea from an imitator, but the payment goes to the idea's original innovator rather than the imitator. This scenario is relevant for patent licensing (or transfer of non-patented know-how) that forbids reselling.[1] In contrast, Regime 2 allows imitators to resell ideas to other imitators and retain the proceeds, a situation that permits sublicensing.

Our model leads to the following findings. First, under free imitation or under either IP protection regime, innovators enter the industry only at the beginning and the number of imitators then follows an S-shaped logistic diffusion curve over time. More innovators enter in Regime 1 than in Regime 2 or when idea sellers' bargaining share is larger, resulting in faster industry growth.

Second, socially optimal compensation for innovators should internalize two externalities: positive knowledge spillovers and a negative matching externality. Moreover,

[1] Non-patented know-how in the U.S. is protected through trade secrets laws, confidentiality agreements, licensing and employment contracts. Firms take those measures to restrict employees, contractors and business partners from imitating or leaking know-how, and to define terms under which know-how can be used or shared.

the socially optimal bargaining share of idea sellers should be larger in Regime 2 where innovators collect the payoff of ideas partly indirectly. These findings shed new light on optimal licensing policies as we discuss in Section 4.2.

Third, differences in idea diffusion speed can explain industry overtaking patterns. Specifically, our model shows that a lower diffusion speed may encourage entry of innovators and raise initial industry capacity, but that it lowers imitation and leads to slower growth of capacity. In reality, diffusion speed may vary across locations due to differences in laws and regulations restricting entry of imitators. In the U.S., for example, some states enforce non-compete contracts that restrict labor turnover and idea diffusion, while other states do not; Saxenian (1994), Gilson (1999), and Franco and Mitchell (2008) point out that the enforcement of non-compete contracts in Massachusetts but not in California accounted for the overtaking of venture activity on Route 128 by that in Silicon Valley. Our model generates such overtaking and shows that restricting diffusion reduces welfare.

We fit the model to the early histories of the U.S. automobile and personal computer industries, both of which show S-shaped growth in the number of producers in the period before the shakeout, a pattern shared by many industries. We thus add to the literature on industry life cycles – e.g., Gort and Klepper (1982), Jovanovic and MacDonald (1994), Klepper (1996), Filson (2001), and Hayashi *et al.* (2017).

Those studies focus on explaining the shakeout of firms, while our study explains the expansion of firm numbers prior to the shakeout. We find both auto and PC industries face highly elastic demands, under which entry of imitators can drive prices down only slowly. Because this encourages innovation and exacerbates the matching externality, the socially optimal bargaining share of idea sellers should be low for both industries, and lower for the more price elastic PC. In both industries, restricting diffusion would reduce too much free imitation and would reduce social welfare.

In our model, random matching between agents who have ideas and those who do not gives rise to a logistic diffusion process. This is consistent with prior work that features logistic diffusion in the technology diffusion literature (e.g., Griliches 1957, Mansfield 1961, Bass 1969, 2004, Young 2009), as well as in the epidemics studies (e.g., Atkeson 2020, Garibaldi, Moen, and Pissarides 2020, among many applications of the SIR model to the spread of the COVID-19 disease).[2] And the quadratic

[2]Our model focuses on the diffusion process driven by information spillovers. There are also models where diffusion is driven by falling prices of inputs; e.g., David (1969) and Manuelli and Seshadri (2014).

matching function underlying the logistic diffusion was recently studied by Lauermann, Nöldeke, and Tröger (2020).

We add to several other strands of the literature. First, the work on competitive innovation. Boldrin and Levine (2008) among others provide evidence that competitive innovation is pervasive in history and in modern day markets. They point out that in both theory and practice, limited supply due to capacity constraints would keep the price of a new good above the marginal cost of production over a sustained period of time. This generates competitive rents and hence provides incentives for innovation. They consider a single innovator's entry decision in a market where the number of imitators grows at a constant rate. By contrast, our model endogenizes the entry number of innovators and generates S-shaped growth in the number of imitators. By using logistic diffusion, our model implies a matching externality that is missing in those studies, so our policy implications are different.

Second, our results on licensing apply out of steady state. Benhabib, Perla, and Tonetti (2021) show that the growth-maximizing bargaining share of innovators depends on how imitation affects the incentive to innovate and thus affects the movement in the distribution of idea quality, and Hopenhayn and Shi (2020) show that such bargaining share also depends on the parameters in the matching function. These papers study steady-state growth of aggregate distribution whereas we study transitional dynamics of a specific industry. Our model thus directly connects with early industry life-cycle patterns of price, output, and S-shaped growth of firm numbers and implies industry-specific licensing policies. We also find that the socially optimal licensing rate depends on the type of licensing: It should be higher if imitators can resell ideas to other imitators.

In the model, owners of ideas use them to compete in the product market and thus the flow value of an idea depends on how many others use it. Manea (2021) also assumes ideas are sold in bilateral meetings and uses bargaining to allocate rents, but in his model, the flow value of an idea to its user does not depend on how many others have it or use it.

The paper is organized as follows. Section 2 lays out and solves the baseline model where imitation is free. Section 3 introduces two IP protection regimes and characterizes the resulting equilibria. Section 4 conducts welfare and policy analysis, and Section 5 applies the model to data from the U.S. automobile and personal computer industries. Section 6 provides additional discussion, and Section 7 concludes.

2 Baseline model: free imitation

Consider a competitive market in continuous time. There is a measure N of potential producers. At date 0, a measure k_0 who we call "innovators," invest an amount c each in an innovation that results in the ability to produce one unit of a new good each period at zero cost. They start producing immediately. After that, the innovation spreads to others. At any date $t \geq 0$ the measure of producers (as well as market output) is k_t, and the remaining $N - k_t$ agents are "outsiders." We normalize outsiders' earnings to zero and denote by u_t an outsider's date-t option value of entering the industry in the future.

The total output of the homogeneous good is k_t, and the product price is

$$p_t = Ak_t^{-\beta}, \tag{1}$$

where A is a market size parameter and $\beta > 0$ the inverse demand elasticity.

Two types of producers.—All producers have an idea and all are equally productive,[3] but some are "innovators" while the others are "imitators." An innovator has invested a cost c to invent an idea for production, and an imitator has copied a producer's idea. In the baseline model, we assume no IP protection so copying an idea from others is free, and imitators incur no other cost to enter the industry.[4] We shall then introduce IP protection and explore socially optimal policies.

2.1 Diffusion process

Ideas spread via a classic logistic process, in which the conditional probability for an outsider to imitate an idea is a linear function of k_t:

$$\frac{dk_t/dt}{N - k_t} = \gamma k_t. \tag{2}$$

The parameter $\gamma > 0$ captures the positive influence of existing idea adopters on future imitators, which could occur either directly (e.g., through meetings between incumbents and outsiders) or indirectly (e.g., outsiders may reverse engineer ideas through observing existing outputs in the market).[5]

[3] Section 6.2 extends the analysis to allow firms to have heterogeneous production capacity.

[4] Section 5.4.3 extends the model to allow for a positive entry cost $c^m < c$ for imitators. Our analysis and findings continue to hold, but the solutions become more complicated.

[5] Based on survey results from 130 lines of business, Levine *et al.* (1987) show that reverse engineering, conversations with employees of innovating firms, learning from publications, technical

An outsider can also enter as an innovator after date 0, but Proposition 1 will show that no one will choose to do so. Thus, for $t > 0$ imitation is the only way that agents become producers, and Eq. (2) then implies that the mass of producers evolves as

$$\frac{dk_t}{dt} = \gamma k_t (N - k_t). \tag{3}$$

The solution to (3) is

$$k_t = \frac{Ne^{\gamma Nt}}{e^{\gamma Nt} + \frac{N}{k_0} - 1}. \tag{4}$$

Figure 1 shows that the estimated logistic diffusion model (cf. Eq. (4)) matches the time paths of firm numbers well for the U.S. automobile and PC industries.

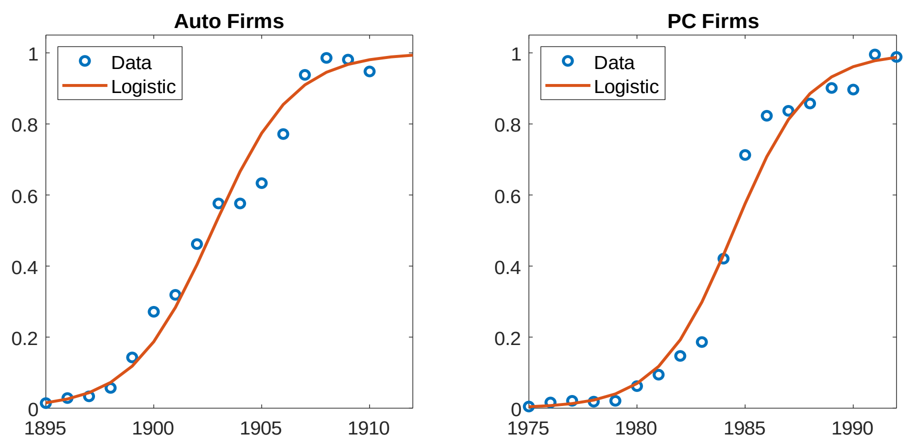

Fig. 1. Logistic Diffusion: Fitting Time Paths of Firm Numbers

Note: Figure 1 plots the time path of firm numbers in the U.S. automobile and PC industries as a fraction of total potential entrants, with the latter proxied by the peak number of firms observed in each industry. See Section 5.1 for the data sources and industry definition.

2.2 Value functions

A producer receives revenue p_t for selling the good at date t, so his value v_t satisfies

$$rv_t = p_t + \frac{dv_t}{dt}, \tag{5}$$

where r is the interest rate.

meetings or patent disclosure, and licensing technology are main alternative methods of learning and imitating new processes and products.

An outsider's hazard rate for learning an idea to become a producer is γk_t. Accordingly, his value u_t at date t satisfies

$$ru_t = \gamma k_t (v_t - u_t) + \frac{du_t}{dt}. \tag{6}$$

The free entry condition requires that

$$v_t - u_t = c \tag{7}$$

for $t = 0$.

2.3 Equilibrium outcome

In what follows, we shall assume that innovation is costly enough to prevent the N agents from all innovating at date 0:

$$c > \frac{AN^{-\beta}}{r + \gamma N}. \tag{8}$$

Proposition 1 *Given condition (8), innovators enter only at date 0 and $k_0 \in (0, N)$ solves*

$$\underbrace{\frac{1}{N - k_0}\int_0^\infty e^{-rt}\left(\frac{N}{k_t} - 1\right) A k_t^{1-\beta} dt}_{v_0 - u_0} \;=\; c, \tag{9}$$

where k_t is given by Eq. (4).

Proof. See Internet Appendix IA-A1. ■

In Proposition 1, condition (8) guarantees the existence of interior solution for k_0: The isoelastic demand in Eq. (1) implies that $k_0 > 0$, otherwise p_0 would be infinite, and condition (8) ensures that $k_0 < N$.[6] The equilibrium price of the good declines over time, and so does the value of the idea. As a result, $v_t - u_t < c$ for all $t > 0$ so that innovators enter only at date 0. Equation (9) yields the solution for k_0 which increases with A but decreases with c, r and γ.

[6] Note that if all agents were to innovate at date 0 (i.e., $k_0 = N$), product price would be fixed at $AN^{-\beta}$ and each firm's present value would be $v_0 = \frac{AN^{-\beta}}{r}$. However, $c > v_0 = \frac{AN^{-\beta}}{r}$ would be sufficient but not necessary for preventing the N agents from all innovating at date 0. In fact, Eq. (6) implies that when $k_0 = N$, a marginal outsider has the option value $u_0 = \frac{\gamma N}{r(r+\gamma N)} AN^{-\beta} > 0$ to become an imitator. Condition (8) hence eliminates the equilibrium in which $k_0 = N$ by requiring $c > v_0 - u_0$ for $k_0 = N$. See Internet Appendix IA-A1 for the formal proof.

3 Intellectual property rights and licensing

We now introduce IP protection and assume idea sellers negotiate with imitators and collect a licensing fee $\alpha\omega_t$, where ω_t is an imitator's present value at t. The parameter $\alpha \in [0, 1]$ is an idea seller's compensation share or licensing rate, which nests our baseline model (i.e., $\alpha = 0$).[7] We shall analyze two regimes that differ in how much revenue innovators get from idea sales.

3.1 Regime 1: Imitators cannot resell ideas

In Regime 1, the original innovators receive all of their ideas' sale revenue. An imitator may learn the idea from any incumbent producer or by reverse engineering, but then he has to pay the idea's *original* innovator.

Since an imitator cannot resell the innovation, his only revenue comes from selling the good, and his value ω_t satisfies

$$r\omega_t = p_t + \frac{d\omega_t}{dt}. \tag{10}$$

An innovator receives revenue from selling both the good and the idea. The number of ideas sold at t is $\gamma k_t (N - k_t)$, and the total date-t revenue from these sales is $\gamma k_t (N - k_t) \alpha\omega_t$. Proposition 2 will show that at equilibrium, innovators enter only at date 0. In that case, the total date-t revenue is divided equally among the k_0 innovators. Therefore, the date-t value v_t of an innovator satisfies

$$rv_t = p_t + \frac{\gamma k_t (N - k_t)}{k_0}\alpha\omega_t + \frac{dv_t}{dt}. \tag{11}$$

An outsider's hazard rate for meeting a producer is γk_t. Therefore, his value at date t, u_t, satisfies

$$ru_t = \gamma k_t [(1 - \alpha)\omega_t - u_t] + \frac{du_t}{dt}, \tag{12}$$

with k_0 again satisfying (7).

[7] This bargaining protocol differs from Nash bargaining in which the innovator and imitator would split the joint surplus $\omega_t - u_t$ from the idea transfer. This alternative bargaining is easier to enforce than a Nash bargaining because the courts need to know only ω_t and not the imitators' outside options. Section 6.4 shows that α does coincide with the Nash bargaining share in a limiting version of the model in which $N \to \infty$.

3.2 Regime 2: Imitators can resell ideas

In Regime 2, imitators can keep the proceeds from the ideas that they resell. An incoming idea buyer pays the agent from whom he copies the idea.

All producers now have the same value, i.e., $v_t = \omega_t$. Proposition 2 will show that innovators will again enter only at date 0. The revenue from a single idea sale is αv_t, and the total revenue from idea sales, $\gamma k_t (N - k_t) \alpha v_t$, is now shared by all the k_t producers. Therefore, v_t now satisfies

$$rv_t = p_t + \gamma (N - k_t) \alpha v_t + \frac{dv_t}{dt}. \tag{13}$$

The value of an outsider, u_t, now satisfies

$$ru_t = \gamma k_t ((1 - \alpha)v_t - u_t) + \frac{du_t}{dt}. \tag{14}$$

Motivation for the two regimes.—Both regimes reflect practice. Regime 2 removes the burden of enforcing the no-reselling constraint, but imitators need to pay for a licensing fee that incorporates their future revenues from reselling, which may be infeasible if they are financially constrained. By contrast, Regime 1 requires a smaller licensing payment, but the no-reselling constraint on idea buyers may be hard to enforce.

3.3 Equilibrium outcomes

Proposition 2 will show that in both regimes innovators enter only $t = 0$, so that the time path of firm numbers is given by Eq. (4). Thus the diffusion process is the same in both regimes, but k_0 will generally differ because in Eq. (7) both v_0 and u_0 depend on the regime.

Intuitively, in Regime 1 where imitators cannot resell ideas, the value of an innovator depends on its entry date. If a measure-0 outsider were to deviate from the equilibrium and enter as an innovator at date $\tau > 0$, its value at date $t \geq \tau$, denoted as v_t^τ, would satisfy

$$rv_t^\tau = p_t + \frac{\gamma k_t (N - k_t)}{k_\tau} \alpha \omega_t + \frac{dv_t^\tau}{dt}. \tag{15}$$

This differs from v_t in Eq. (11) because this deviating entrant would only have a chance $1/k_\tau$ (where k_τ is the number of all incumbent producers at his entry date τ)

instead of $1/k_0$ to share the industry's total idea sale revenues from new imitators. The later an innovator enters the market, the less the offspring this innovator would have at any date $t \geq \tau$ to disseminate his idea and allow him to collect idea sale revenues. Since the number of firms continues to rise over time, it does not pay an innovator to enter after date 0.

By contrast, in Regime 2 where imitators can resell ideas, the value of an innovator at any date t does not depend on his entry date. If any innovator were to enter the industry after date 0, he would share the same value v_t as any incumbent, be it an innovator or an imitator. The free entry condition requires that $v_t - u_t = c$ for $t = 0$ and we can formally verify that at equilibrium $v_t - u_t < c$ for any $t > 0$ so that even in Regime 2, innovators enter only at date 0.

In what follows, we shall assume that innovation is costly enough to prevent N agents from all innovating at date 0. As Internet Appendixes IA-A2 and IA-A3 show, this requires that

$$c > \frac{r + \alpha\gamma N}{r(r + \gamma N)} A N^{-\beta}. \tag{16}$$

To distinguish Regimes 1 and 2, we use the superscripts **I** and **II**, and so the masses of date-0 innovators will be denoted by $k_0^{\mathbf{I}}$ and $k_0^{\mathbf{II}}$. Proposition 2 states the conditions that $k_0^{\mathbf{I}}$ and $k_0^{\mathbf{II}}$ satisfy:

Proposition 2 *Given condition (16), market equilibrium yields:*

(A) In Regime 1, innovators enter only at date 0 and $k_0^{\mathbf{I}} \in (0, N)$ solves

$$\underbrace{\frac{1}{N - k_0} \int_0^\infty e^{-rt} \left(\alpha \frac{N}{k_0} + (1 - \alpha) \frac{N}{k_t} - 1 \right) A k_t^{1-\beta} dt}_{v_0 - u_0} = c; \tag{17}$$

(B) In Regime 2, innovators enter only at date 0 and $k_0^{\mathbf{II}} \in (0, N)$ solves

$$\underbrace{\frac{1}{N - k_0} \int_0^\infty e^{-rt} \left(\left(\frac{N}{k_0} \right)^{\alpha} \left(\frac{N}{k_t} \right)^{1-\alpha} - 1 \right) A k_t^{1-\beta} dt}_{v_0 - u_0} = c; \tag{18}$$

where, in each regime, k_t is given by Eq. (4).

Proof. See Internet Appendixes IA-A2 and IA-A3. ■

3.4 Properties of equilibrium

Equations (17) and (18) pin down the solutions for $k_0^{\mathbf{I}}$ and $k_0^{\mathbf{II}}$, respectively. These implicit functions can be simplified under unit price elasticity ($\beta = 1$) in the following two examples: $\alpha = 0$ and $\alpha = 1$. In these two cases, Eqs. (17) and (18) are equivalent:

$$k_0^{\mathbf{I}} = k_0^{\mathbf{II}} = \frac{A}{c(r + \gamma N)} \qquad \text{(when } \alpha = 0\text{)}, \tag{19}$$

$$k_0^{\mathbf{I}} = k_0^{\mathbf{II}} = \frac{A}{rc} \qquad \text{(when } \alpha = 1\text{)}, \tag{20}$$

i.e., Regimes 1 and 2 generate the same number of innovators in each case. This follows because, when idea sellers are not compensated at all ($\alpha = 0$) or when they are fully compensated ($\alpha = 1$), innovators' payoffs then do not depend on whether they are paid for their ideas directly (in Regime 1) or indirectly (in Regime 2). Moreover, Eqs. (19) and (20) suggest that a larger market size A encourages innovation, while a bigger innovation cost c or a higher interest rate r does the opposite. Across these two cases, k_0 is larger when idea sellers are fully compensated, which suggests α has a positive effect on innovation. The effect of the diffusion rate γ is more involved. In Eq. (19), γ shows a negative effect on k_0, but it plays no role in Eq. (20), which suggests the effect may depend on the values of α and β.[8]

The intuition from the special cases generalizes. Assuming condition (16) holds, comparing Eqs. (17) and (18) yields the findings in Propositions 3 and 4:

Proposition 3

$$\text{(A)} \quad k_0^{\mathbf{I}} \;\; \textit{and} \;\; k_0^{\mathbf{II}} \begin{cases} \textit{increase with } \alpha \textit{ and } A, \\ \textit{decrease with } c \textit{ and } r. \end{cases} \tag{21}$$

(B) All parameters being equal across the two regimes,

$$\frac{k_0^{\mathbf{I}}}{k_0^{\mathbf{II}}} = \begin{cases} 1 & \textit{for } \alpha \in \{0, 1\} \\ > 1 & \textit{for } \alpha \in (0, 1) \end{cases} . \tag{22}$$

Proof. See Internet Appendix IA-A4. ■

[8]Note when $\beta = 1$, industry revenue becomes constant over time, so the diffusion rate γ would not affect the present value of industry revenue which is the value of innovators when $\alpha = 1$.

Proposition 3 shows that the intuition from the special examples holds generally. It also shows that for $\alpha \in (0,1)$, all else equal, fewer innovators enter in Regime 2 than in Regime 1. This is because innovators' revenues get discounted when they collect the payoff of ideas indirectly in Regime 2. Since the two regimes share the same diffusion process, this implies that industry output is higher for all t under Regime 1 due to its larger entry of innovators at date 0.

Next, we ask how the diffusion rate γ affects innovation. The effect of γ on $k_0^{\mathbf{I}}$ and $k_0^{\mathbf{II}}$ hinges on the values of α and β as follows:

Proposition 4 *(A) For inelastic or unit elastic demand (i.e., $\beta \geq 1$),*

$$k_0^{\mathbf{I}} \textit{ and } k_0^{\mathbf{II}} \begin{cases} \textit{decrease with } \gamma \textit{ when } \beta \geq 1 > \alpha, \\ \textit{do not vary with } \gamma \textit{ when } \beta = \alpha = 1. \end{cases}$$

(B) For elastic demand (i.e., $\beta < 1$),

$$k_0^{\mathbf{I}} \textit{ and } k_0^{\mathbf{II}} \begin{cases} \textit{decreases with } \gamma \textit{ if } \alpha \textit{ is sufficiently low}, \\ \textit{increases with } \gamma \textit{ if } \alpha \textit{ is sufficiently high}. \end{cases}$$

Proof. See Internet Appendix IA-A5. ■

The intuition for these results is as follows: There are two channels through which diffusion affects innovation. One is the negative price effect, captured by β – the faster the diffusion, the lower the revenue from selling the good. The other is the positive idea-selling effect, captured by α – the faster the diffusion, the larger the revenue from selling the idea. When demand is inelastic ($\beta > 1$), a faster inflow of imitators would reduce the industry revenue stream, so the price effect dominates and the innovators' value at date 0 would drop even with the highest bargaining share ($\alpha = 1$). This is also true for the unit demand elasticity case when $\alpha < \beta = 1$. When demand is elastic ($\beta < 1$), a faster inflow of imitators would increase the industry revenue stream. If α is sufficiently high compared with β, the idea-selling effect dominates, which raises the incentive to innovate. Otherwise, the price effect dominates, which dampens innovation.

Figure 2 illustrates Proposition 4(B): Under elastic demand (i.e., $\beta < 1$), a higher diffusion rate γ reduces innovation k_0 (i.e., $\partial k_0 / \partial \gamma < 0$) if β is sufficiently high or α is sufficiently low in Regimes 1 and 2. Comparing regimes, a higher γ is more likely to reduce innovation in Regime 2 for a given value of α because innovators are paid indirectly.

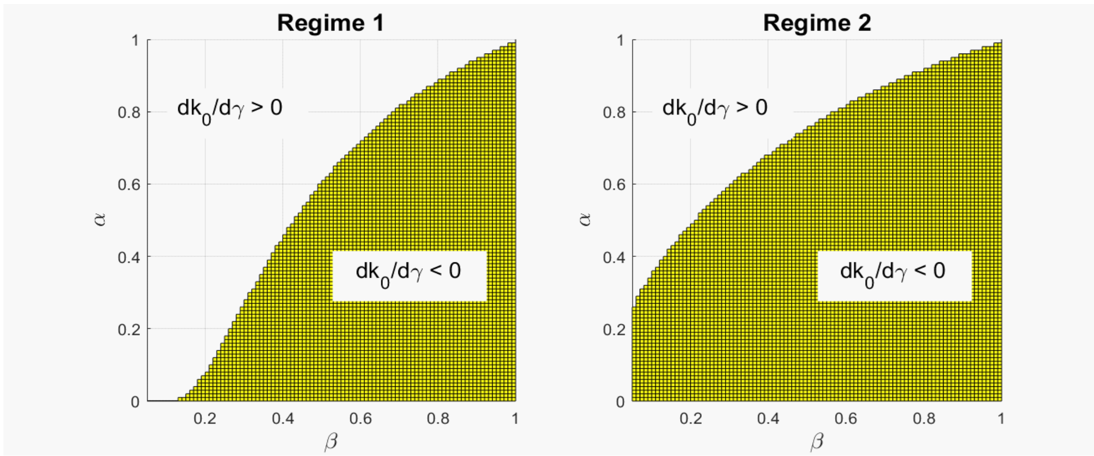

Fig. 2. Effect of γ on Innovation under elastic Demand

Note: Figure 2 numerically solves $dk_0/d\gamma$ for the full parameter space of $\alpha \in [0,1]$ when $\beta < 1$. The simulation assumes A=1, N=1, r=0.05, c=30, and γ= 0.55.

The effect of the diffusion rate γ on innovation has direct implications for industry growth patterns. If a rise in γ raises k_0, it would raise output at all dates. Proposition 4(B) shows this can happen for a subset of parameter values when demand is elastic (i.e., $\beta < 1$) and idea sellers' bargaining share α is high, as illustrated by the unshaded areas in Fig. 2. For the remaining parameter values (as shown by the shaded areas in Fig. 2) or when demand is inelastic or unit elastic (i.e., $\beta \geq 1$), a rise in γ would reduce k_0. Proposition 5 proves that the higher-γ trajectory would then overtake the lower-γ one:

Proposition 5 *(Industry overtaking) In both Regimes 1 and 2, whenever a larger diffusion rate $\gamma^i(> \gamma^j)$ leads to a smaller $k_0^i(< k_0^j)$, there exists a date t' where*

$$t' = \frac{1}{(\gamma^i - \gamma^j)N} \ln \left(\frac{k_0^j\left(N - k_0^i\right)}{k_0^i\left(N - k_0^j\right)} \right) \tag{23}$$

such that $k_t^i > k_t^j$ for all $t > t'$.

Proof. Eq. (4) states that $k_t^l = \frac{Ne^{\gamma^l Nt}}{e^{\gamma^l Nt} + \frac{N}{k_0^l} - 1}$ for $l \in \{i, j\}$. Therefore, $k_t^i > k_t^j \iff \frac{k_t^i}{N - k_t^i} > \frac{k_t^j}{N - k^j} \iff \frac{e^{\gamma^i Nt}}{\frac{N}{k_0^i} - 1} > \frac{e^{\gamma^j Nt}}{\frac{N}{k_0^j} - 1} \iff t > t'$, where t' is given by Eq. (23). ∎

Thus the model generates overtaking when two sectors or two locations face the same environment (i.e., same A, α, c and N) except that γ is higher in one than in the other. For example, a high-tech sector may use technology based on ideas that spread faster than they do in other sectors. As a result, high-tech sectors would have a higher γ and would start smaller but grow faster.

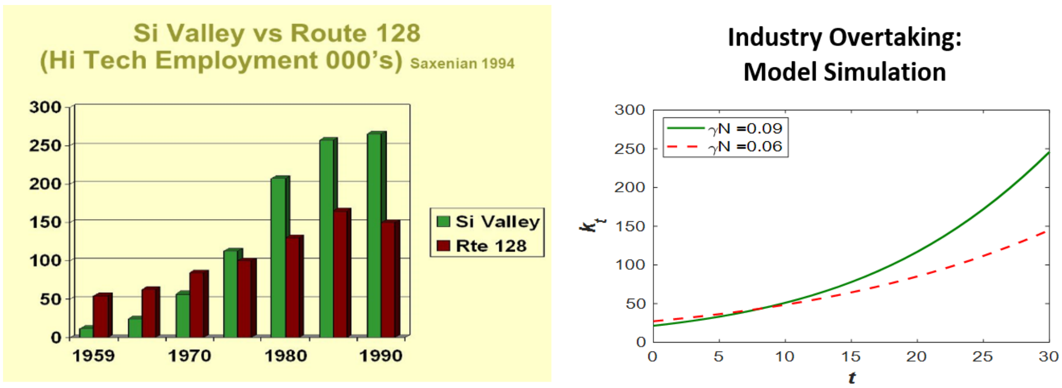

Fig. 3. Industry Overtaking: Data and Model

Note: Figure 3 illustrates that our model (right) can generate the overtaking pattern in Saxenian (1994)'s data (left). The simulation assumes $\alpha = 0$, $\beta = 1$, $A/c = 3$, $N = 1000$, $r = 0.05$, and plots k_t in two locations with high *vs.* low diffusion rate γN $\big(0.09$ vs. $0.06\big)$.

There also could be legal or regulatory reasons why γ differs over regions. For example, California bans non-compete contracts and therefore indirectly encourages labor turnover and spin-offs[9] whereas Massachusetts enforces those contracts. As Saxenian (1994), Gilson (1999), and Franco and Mitchell (2008) argue, this may help explain why Silicon Valley has overtaken Massachusetts' Route 128 in developing high-tech industry. Our model generates such a pattern if γ is higher in California than in Massachusetts. Under conditions in Proposition 4, Route 128 would then have a higher initial entry rate of firms (i.e., a higher k_0) than Silicon Valley. Thus our model would predict the type of overtaking portrayed in Fig. 3.[10]

[9]Spin-offs are firms founded by former employees of incumbent firms to conduct businesses in the same industry. While our model does not include labor in production, employees could work in the same company but not be involved in directly producing the new product. And then they could learn about the idea internally.

[10]Figure 3 is not a quantitative analysis, only an illustrative example. We conduct a quantitative analysis on noncompetes and industry overtaking in the context of the early automobile industry in Section 5.3.2.

4 Welfare and policy analysis

We now study the welfare implications of the model. Consumers' utility from consuming output k is the integral under the demand curve. For $\beta \in (0,1)$, aggregate utility at output k is

$$U(k) = \int_0^k A s^{-\beta} ds = \frac{A}{1-\beta} k^{1-\beta}. \tag{24}$$

For $\beta \geq 1$ the above integral is infinite; to ensure consumer surplus is finite, we put a maximum, $A\varepsilon^{-\beta}$, on the willingness to pay. Let $D(s) = \min\left(A\varepsilon^{-\beta}, As^{-\beta}\right)$ and define aggregate utility as $U(k) = \int_0^k D(s)\, ds = A\left(\int_0^\varepsilon \varepsilon^{-\beta} ds + \int_\varepsilon^k s^{-\beta} ds\right)$ where $\varepsilon \ll k$. Accordingly, for $\beta = 1$ we have

$$U(k) = A(\ln k + 1 - \ln \varepsilon), \tag{25}$$

and for $\beta > 1$, we have

$$U(k) = \frac{\beta A}{\beta - 1}\varepsilon^{1-\beta} + \frac{A}{1-\beta} k^{1-\beta}. \tag{26}$$

4.1 Planner's problem

The planner would like to maximize social welfare W_0 given by

$$W_0 = \int_0^\infty e^{-rt} U(k_t)\, dt - ck_0, \tag{27}$$

where k_t follows Eq. (4).

Given the logistic diffusion process, a matching externality arises when N is finite because an innovator's matching rate $\frac{dk_t/dt}{k_t} = \gamma(N - k_t)$ falls with k_t while an imitator's matching rate $\frac{dk_t/dt}{N-k_t} = \gamma k_t$ rises with k_t. Section 6.4 shows that if N grows but γ shrinks so that $\gamma N \to \lambda > 0$ as $N \to \infty$, Eq. (3) converges to $\frac{dk_t}{dt} = \lambda k_t$ so that $k_t = k_0 e^{\lambda t}$, which coincides with exponential diffusion. The externality then disappears because neither matching rate depends on k_t since $\frac{dk_t/dt}{k_t} \to \lambda$ for innovators and $\frac{dk_t/dt}{N-k_t} \to 0$ for imitators. However, exponential diffusion does not match general industry growth patterns and the policy implications also differ.

In what follows we assume that Eq. (8) holds so that innovation is costly enough to keep the socially optimal entry mass of innovators k_0^* to be an interior solution (i.e., $k_0^* < N$) at date 0. We then have the following proposition.

Proposition 6 *Given condition (8), social optimum requires that innovators enter only at date 0 and k_0^* solves*

$$\underbrace{\int_0^\infty e^{-(r+\gamma N)t}\left(\frac{k_t}{k_0}\right)^2 Ak_t^{-\beta}dt}_{\text{marginal social return to } k_0} \;=\; c. \tag{28}$$

Proof. See Internet Appendix IA-A6. ■

The claim of Proposition 6 is intuitive. As of date $\tau \geq 0$, the social return to innovation is

$$SR_\tau = \int_\tau^\infty e^{-r(t-\tau)}U(k_t)\,dt$$

and one can verify that the marginal social return $\partial SR_\tau/\partial k_\tau$ is strictly decreasing in k_τ. So if k_0^* is chosen so that $\partial SR_0/\partial k_0 = c$ at date 0, thereafter $\partial SR_\tau/\partial k_\tau < c$ for any $\tau > 0$. Hence, it is socially optimal to innovate only at date 0. And the condition $\partial SR_0/\partial k_0 = c$ yields Eq. (28). Finally, welfare given by Eq. (27) is strictly concave in k_0, so for $k_0^* < N$ to hold, one needs $\frac{dW_0}{dk_0}|_{k_0=N} < 0$, which yields condition (8). Henceforth, we assume that condition (8) always holds.

Denote the socially optimal welfare by W_0^*. We have the following comparative-static findings: Both k_0^* and W_0^* increase in market size A but decrease in innovation cost c and interest rate r. Moreover, while k_0^* decreases with the diffusion rate γ if the demand is not too elastic, W_0^* always increases with γ (see Internet Appendix IA-A6 for the proofs).

Next, we discuss policy implications of our model. We show that the planner can achieve k_0^* by enforcing a socially optimal licensing rate α^*,[11] and that policy interventions that raise the diffusion speed γ are socially desirable.

4.2 Socially optimal licensing

For any $\alpha \in [0, 1]$, the interior solution condition (16) faced by market participants relates to the interior solution condition (8) faced by the social planner by the in-

[11]The planner can also achieve k_0^* by providing an innovation subsidy (or tax) s^* (see Internet Appendix IA-A7 for the analysis).

equality

$$\underbrace{\frac{r+\alpha\gamma N}{r(r+\gamma N)}AN^{-\beta}}_{\text{RHS of (16)}} \geq \underbrace{\frac{AN^{-\beta}}{r+\gamma N}}_{\text{RHS of (8)}}.$$

The two are identical if $\alpha = 0$.

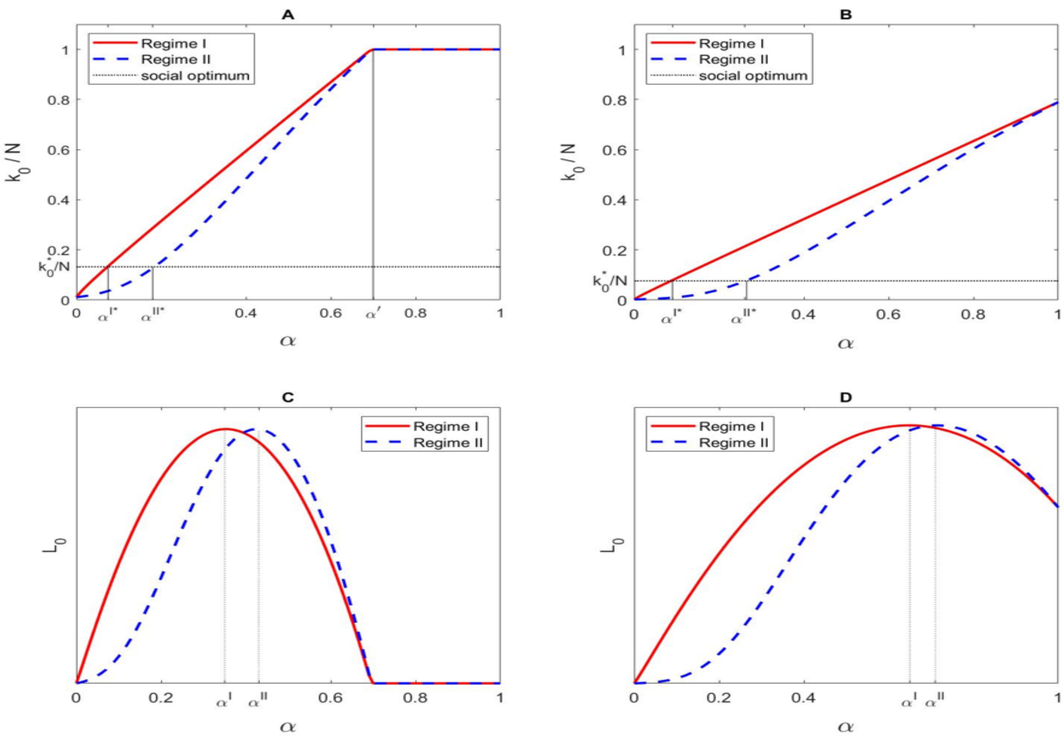

Fig. 4. Welfare-Maximizing vs. Licensing-Revenue-Maximizing α

Denote the socially optimal bargaining shares for Regimes 1 and 2 by $\alpha^{\mathbf{I}*}$ and $\alpha^{\mathbf{II}*}$, respectively. Assume that condition (8) always holds. If there exists an $\alpha' \in (0, 1]$ such that the inequality (16) becomes an equality (i.e., $c = \frac{r+\alpha'\gamma N}{r(r+\gamma N)}AN^{-\beta}$), one can prove that the planner could choose the optimal bargaining shares $\alpha^{\mathbf{I}*}$ and $\alpha^{\mathbf{II}*}$ such that $0 < \alpha^{\mathbf{I}*} < \alpha^{\mathbf{II}*} < \alpha'$ to achieve the socially optimal k^* in Regimes 1 and 2, respectively. This is illustrated by Fig. 4(A). Alternatively, if the inequality (16) holds for any $\alpha \leq 1$, one can prove that $0 < \alpha^{\mathbf{I}*} < \alpha^{\mathbf{II}*} < 1$, as illustrated by Fig. 4(B). Note that $\alpha^{\mathbf{I}*} < \alpha^{\mathbf{II}*}$ results from Proposition 3: If there is a value of α that leads to $k_0^{\mathbf{I}} = k^*$ in Regime 1, the same α would lead to $k_0^{\mathbf{II}} < k^*$ in Regime 2, so a larger value of α is needed to achieve k^* in Regime 2.

The above findings are stated formally in Proposition 7.

Proposition 7

$$0 < \alpha^{\mathbf{I}*} < \alpha^{\mathbf{II}*} < 1. \tag{29}$$

Proof. See Internet Appendix IA-A8. ■

Proposition 7 holds because if $\alpha^{\mathbf{I}} = \alpha^{\mathbf{II}} = 0$, no innovator would internalize knowledge spillovers they create for imitators, so fewer innovators enter than is socially optimal. On the other hand, if $\alpha^{\mathbf{I}} = \alpha^{\mathbf{II}} = 1$, innovators would not fully internalize the negative matching externality they impose on one another, so more innovators would enter than is socially optimal.

Licensing-revenue-maximizing α.—In general, the licensing rate that maximizes licensing revenue does not maximize social welfare. To see this, we plot the licensing revenues in Fig. 4(C) and 4(D), corresponding to Fig. 4(A) and 4(B) respectively. The present value of total licensing revenue paid to innovators is

$$L_0^{\mathbf{I}} = \int_0^{\infty} e^{-rt} \alpha (k_t - k_0) A k_t^{-\beta} dt \tag{30}$$

in Regime 1, and

$$L_0^{\mathbf{II}} = \int_0^{\infty} e^{-rt} \left(\left(\frac{k_t}{k_0} \right)^{\alpha} k_0 - k_0 \right) A k_t^{-\beta} dt \tag{31}$$

in Regime 2 (See Internet Appendix IA-A9 for the proof).

Figure 4 compares the socially optimal α and the licensing-revenue-maximizing α for the two regimes. Figures 4(A) and 4(C) which are plotted using the same parameter values, show the case in which all the agents would innovate at date 0 when α is sufficiently large. As a function of α, licensing revenue has an inverted-U shape: When $\alpha = 0$, no imitator pays the licensing fee, so licensing revenue is zero; when α is sufficiently large, no one enters as an imitator, so licensing revenue is again zero. Regimes 1 and 2 each have a unique α that maximizes licensing revenue, and each is larger than the socially optimal α. In fact, because the innovation cost c is ignored in the licensing revenue maximization, the value of α that maximizes licensing revenue would induce too many agents to innovate at date 0 and in so doing, overspend the innovation cost c. Figures 4(B) and 4(D) (that also use the same parameter values) show a similar finding in the case where not all the agents would innovate at date 0 for any α. The licensing revenue again displays an inverted-U curve with regard to α and remains positive at $\alpha = 1$.[12]

Policy implications.—Our analysis offers new insights into licensing policies. In practice, licensing rates in each industry are usually determined by negotiations between an IP owner and a licensee for a split of the latter's profit. In an IP violation, the court evaluates and grants the IP owner damages also based on a hypothetical negotiation between the two parties (see e.g., Razgaitis, 1999, Goldscheider, 2011). Historically, the "25% rule" has commonly been used in licensing practice (see Internet Appendix IA-B1),[13] which prescribes that the licensor should receive 25% of the licensee's profit because the latter takes on more responsibility of producing and marketing the good and should keep a larger share. The rule was also upheld by courts until 2011, after which courts have relied on more comprehensive factors—specifically, the Georgia-Pacific factors, named after a landmark 1970 court case—to reconstruct hypothetical negotiations and determine reasonable licensing rates.

Our analysis, however, suggests that neither the fixed 25% rule nor bilateral bargaining is likely to yield the socially optimal licensing rate. Rather, the optimal rate should address industry-level learning and matching externalities that lie beyond the profit considerations of individual market participants, and it should reflect market factors (e.g., demand elasticity, diffusion speed, and innovation cost) and the type of licensing (e.g., resale permitted or not).

4.3 Socially optimal diffusion

Consider our baseline model in which incumbents are not compensated by imitators (i.e., $\alpha = 0$). In this case, imitation causes the maximal disincentive for innovation.

[12] Benhabib, Perla, and Tonetti (2021) also derive an inverted-U curve of licensing revenue with regard to the licensing rate. When the licensing rate is low, increasing it would not deter much imitation and hence would raise licensing revenue. However, if licensing rate is high, increasing it further would deter too much imitation and would ultimately generate less licensing revenue.

[13] By matching the reported royalty rates with licensees' operating profit margins in a sample of 347 companies in 15 industries between 1990-2000, Goldscheider *et al.* (2002) find that the median royalty rate conforms to the 25% rule. Using more recent data, Kemmerer and Lu (2012) show such a result holds broadly in a sample of 3,887 companies in 14 industries based on 3,015 patent licensing transactions collected over a 21-year period prior to 2007.

From the social welfare point of view, should the planner reduce the diffusion speed γ (e.g., by restricting entry of imitators) to enhance incentives for innovation?

Note that when $\alpha = 0$, Proposition 4 shows that the entry of innovators decreases in γ. Therefore, a policy that reduces the diffusion rate γ would boost the entry of innovators. Such policy, however, would not necessarily raise welfare. Internet Appendix IA-A10 shows that for the unit demand elasticity case (i.e., $\beta = 1$), social welfare always increases in the diffusion rate γ. This suggests that in the numerical example above (cf. Fig. 3), a higher value of γ not only helps Silicon Valley overtake Route 128 in industry size but also yields higher social welfare.[14] In Section 5, we empirically analyze the U.S. auto and PC industries where demands are price elastic (i.e., $\beta < 1$). We find that welfare increases in γ in both industries and we evaluate the impact quantitatively.

5 Empirical applications

In this section, we apply our model to industry data. We consider two historically important industries: automobile and personal computer, where idea diffusion played an important role in the industries' development.[15] Using model calibration and counterfactual exercises, we evaluate and quantify our theoretical predictions.

5.1 Parameter estimation

We first estimate the model parameters using auto and PC industry data. The data come from the following sources:

Auto.—Smith (1970) lists every make of passenger cars produced commercially in the United States from 1895-1969. Smith's list of car makes is used to derive the number of auto firms each year. Thomas (1977) provides annual data of average car price and output from 1900-1929. Cabral *et al.* (2018) identify spinoff firms in the car industry.

[14] A comprehensive numerical analysis shows that welfare always increases in γ for different values of $\alpha \in [0, 1]$ and $\beta > 0$ for both Regimes 1 and 2.

[15] E.g., Klepper (2010) documents how the spawning of employee spin-offs and entry by firms in related industries drove the development of the automobile and the semiconductor industries.

PC.—Firm numbers are from Stavins (1995) and the *Thomas Register of American Manufacturers*, which include desktop and portable computers. Price and quantity information is from the *Information Technology Industry Data Book*. The import value data of computers and accessories are from the U.S. Census Bureau.

In addition, Williamson (2020) provides annual data of U.S. population, real GDP, and the GDP deflator.

5.1.1 Auto industry

The U.S. automobile industry started in 1890s and grew from a small infant industry to a major sector of the economy in a few decades. Starting with 3 firms in 1895, the number of auto producers exceeded 200 around 1910. A shakeout then followed when a major process innovation, the assembly line, was introduced in the early 1910s. As a result, the number of firms declined sharply while the industry output expanded tremendously. Eventually, only 24 firms survived into the 1930s. Figure 5 plots the number of firms and output per firm in the U.S. auto industry from 1895-1929.

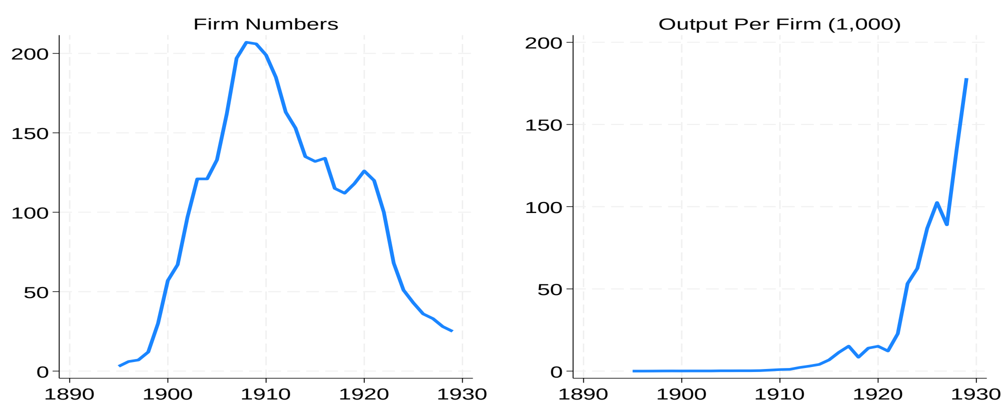

Fig. 5. Auto Firm Numbers and Output Per Firm

Our model describes the auto industry development well for the pre-shakeout period (1895-1910). As shown in Fig. 5, during that period, the time path of firm numbers followed an S-shaped curve and the average output per firm stayed flat which reflects firms' production capacity constraint. To calibrate the model, we focus on the pre-shakeout era. We assume the shakeout to be an unexpected shock in the benchmark analysis, and we then extend the model to incorporate the shakeout as an anticipated shock in Section 5.4.2.

Diffusion estimation We first use the data of firm numbers in the pre-shakeout period, 1895-1910, to estimate the diffusion parameters. In doing so, we rewrite Eq. (4) to estimate the diffusion process of k_t as follows:

$$\ln \frac{k_t}{N - k_t} = z + \lambda t, \tag{32}$$

where $z = \ln \frac{k_0}{N-k_0}$, and $\lambda = \gamma N$.[16]

We assume that the shakeout started after almost all the potential firms had entered the industry. Accordingly, we set $N = 210$ and run the regression model.[17] The result shows that

$$\ln \frac{k_t}{N - k_t} = \underset{(0.28)^{***}}{-4.17} + \underset{(0.03)^{***}}{0.54}\, t, \tag{33}$$

and the standard errors are reported in parentheses. The estimates of z and λ are both statistically significant at 1% (denoted by three stars) and the adjusted $R^2 = 0.95$. The fit is shown in Fig. 6. Based on the estimates of diffusion parameters, we calibrate $\gamma N = 0.54$ and $k_0 = 3.20$ (i.e., $\ln \frac{k_0}{N-k_0} = -4.17$).

For robustness checks, we re-estimated the diffusion process by using a subsample and by using the matching function (3) that allows differencing the data. The regression results are consistent with the estimates above (see Internet Appendix IA-B2).

Demand estimation We then estimate the auto demand function using annual data on real auto price p_t (in 2012 prices) and industry output Q_t from 1900–1929. Equation (1) implies a simple log-log demand function:

$$\ln(Q_t) = a - \phi \ln(p_t).$$

To address potential endogeneity of the price variable, we use output per firm lagged by a year as an instrumental variable to estimate the demand elasticity parameter ϕ in a two-stage least-squares regression. Output per firm, while assumed fixed in our theory, did grow over the long term in data due to technological progress.

[16] Note that Eq. (4) implies $\frac{k_t}{N-k_t} = \frac{e^{\gamma N t}}{\frac{N}{k_0}-1}$, which leads to Eq. (32).

[17] We consider an alternative assumption for N in Section 5.4.1 as a robustness check.

If unobserved demand shocks are not serially correlated, lagged output per firm can serve as a valid supply shifter to trace out the demand curve.[18]

The first-stage regression result (adj. $R^2 = 0.87$) is

$$\ln(p_t) = \underset{(0.14)^{***}}{11.37} - \underset{(0.02)^{***}}{0.24} \times \ln(\textit{output per firm})_{t-1},$$

and the second-stage regression result ($R^2 = 0.82$) is

$$\ln(Q_t) = \underset{(2.74)^{***}}{47.04} - \underset{(0.29)^{***}}{3.61} \times \ln(p_t). \tag{34}$$

All the estimates are statistically significant at 1%. The fit is shown in Fig. 6.

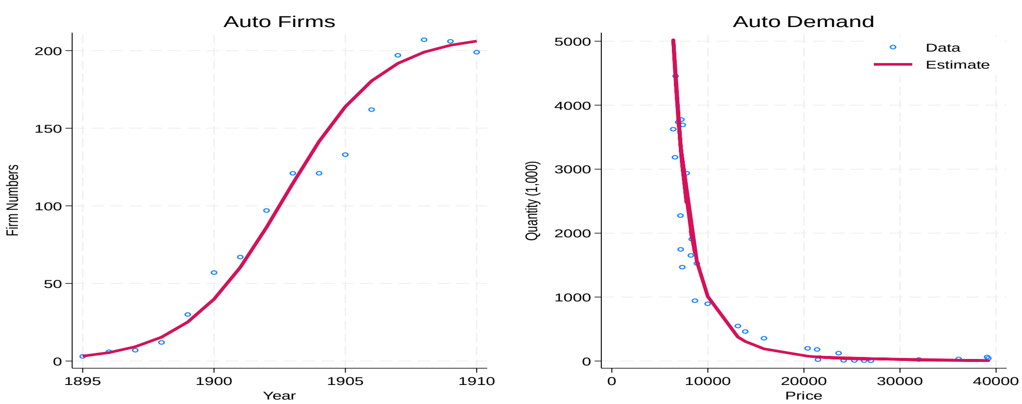

Fig. 6. Auto Diffusion and Demand Estimates

The IV estimation gives $\phi = 3.61$ and $a = 47.04$. Because our model specifies an inverse demand function (1) that implies

$$\ln Q_t = \frac{1}{\beta} \ln \tilde{A} - \frac{1}{\beta} \ln p_t,$$

[18]A primary challenge in demand estimation is the endogeneity of prices. A classic solution is to identify demand parameters through supply shifters that are independent of demand. We take this approach by choosing reasonable instruments and spell out the identifying assumptions. However, there can still be other complications of demand estimation that require additional treatments, e.g., influence from substitutes or complementary goods. Berry and Haile (2021) provide a comprehensive review of challenges and solutions in demand estimation.

this yields that $\beta = 0.28$ (i.e., $\frac{1}{\beta} = \phi = 3.61$) and $\tilde{A} = 456,102$ (i.e., $\frac{1}{\beta}\ln\tilde{A} = 47.04$).[19] For robustness checks, we also re-ran the IV regressions by controlling changes of population and per capita income over time, and the results are very similar (see Internet Appendix IA-B3).

Moreover, to check if lagged output per firm is a valid instrument, we follow Cabral *et al.* (2018) and use a different instrumental variable, the share of spin-off firms in the auto industry. Empirical studies show that the founders of spin-off firms are more experienced, so spin-off firms tend to perform better than *de novo* firms (Klepper, 2010). Therefore, the share of spin-off firms in the industry shifts supply and can serve as an instrument to trace out the demand curve. Using this alternative instrument and with the same sample range (1900-1929), the first-stage regression result (adj. $R^2 = 0.84$) is

$$\ln(p_t) = \underset{(0.09)^{***}}{10.52} - \underset{(0.38)^{***}}{4.77} \times (\textit{share of spin-off firms})_{t-1},$$

and the second-stage regression result ($R^2 = 0.82$) is

$$\ln(Q_t) = \underset{(2.79)^{***}}{47.28} - \underset{(0.29)^{***}}{3.63} \times \ln(p_t).$$

The estimates are all statistically significant at 1% and very close to our estimates above.

5.1.2 PC industry

The personal computer industry was developed 80 years later than the automobile industry, but the industry evolution was similar. Starting with two firms in 1975, the number of PC producers exceeded 430 in 1992. A shakeout then started when the number of firms fell sharply while the industry output continued to expand.

Figure 7 plots the number of firms and output per firm in the U.S. PC industry from 1975-1999. Like in the auto industry case, our model describes the pre-shakeout period (1975-1992) of the PC industry well. As shown in Fig. 7, during that period, the time path of firm numbers followed an S-shaped curve and the average output per firm stayed flat.

[19] In the model, we normalize a firm's output to 1, so $Q_t = k_t$ and the inverse demand function is $p_t = Ak_t^{-\beta}$. In the empirical analysis, we denote a firm's output by q, so $Q_t = qk_t$ and the corresponding inverse demand function becomes $p_t = \tilde{A}Q_t^{-\beta}$.

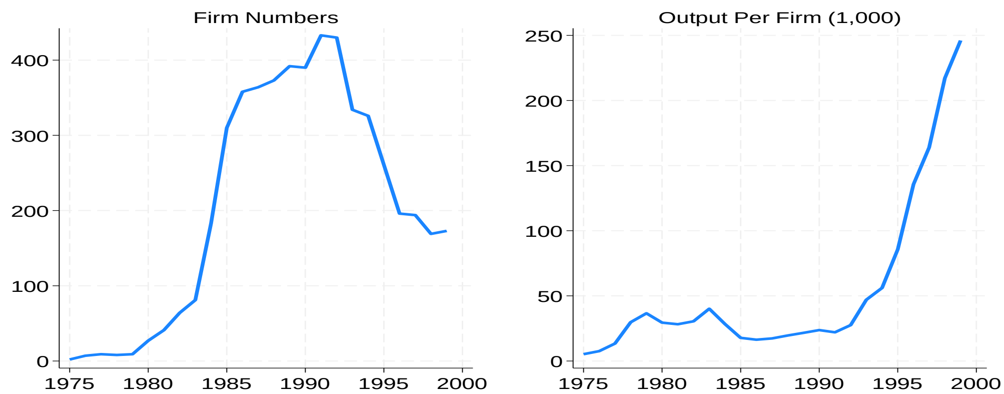

Fig. 7. PC Firm Numbers and Output Per Firm

Diffusion estimation We first use the data of firm numbers in the pre-shakeout period, 1975-1992, to estimate the diffusion parameters. We assume the shakeout started after almost all the potential PC firms had entered the industry. Accordingly, we set $N = 435$ and run the following regression model (35).[20] The result shows that

$$\ln \frac{k_t}{N - k_t} = \underset{(0.29)^{***}}{-5.49} + \underset{(0.03)^{***}}{0.58}\ t, \tag{35}$$

with the standard errors reported in parentheses. The coefficient estimates are statistically significant at 1% and the adjusted $R^2 = 0.96$. The fit is shown in Fig. 8. Based on the estimates of diffusion parameters, we calibrate $\gamma N = 0.58$, and $k_0 = 1.78$ (i.e., $\ln \frac{k_0}{N-k_0} = -5.49$).

As for autos, to check robustness, we again re-estimated the diffusion process by using a subsample and by using the matching function Eq. (3) that allows differencing the data. The regression results are consistent with the estimates above (see Internet Appendix IA-B4).

Demand estimation We then estimate the PC demand function using annual data on real PC price p_t (in 2012 prices) and industry output Q_t from 1975-1992. As before, in order to address potential endogeneity of the price variable, we use output per firm lagged by a year as an instrument to estimate the demand elasticity ϕ.

[20] We consider an alternative assumption for N in Section 5.4.1 as a robustness check.

The first-stage regression result (adj. $R^2 = 0.23$) is

$$\ln(p_t) = \underset{(0.50)^{***}}{9.62} - \underset{(0.05)^{**}}{0.12} \times \ln(\textit{output per firm})_{t-1},$$

and the second-stage regression result ($R^2 = 0.94$) is

$$\ln(Q_t) = \underset{(12.52)^{***}}{137.15} - \underset{(1.49)^{***}}{14.58} \times \ln(p_t). \tag{36}$$

Standard errors are in parentheses, with two and three stars indicating statistical significance at 5% and 1%, respectively. The fit is shown in Fig. 8.

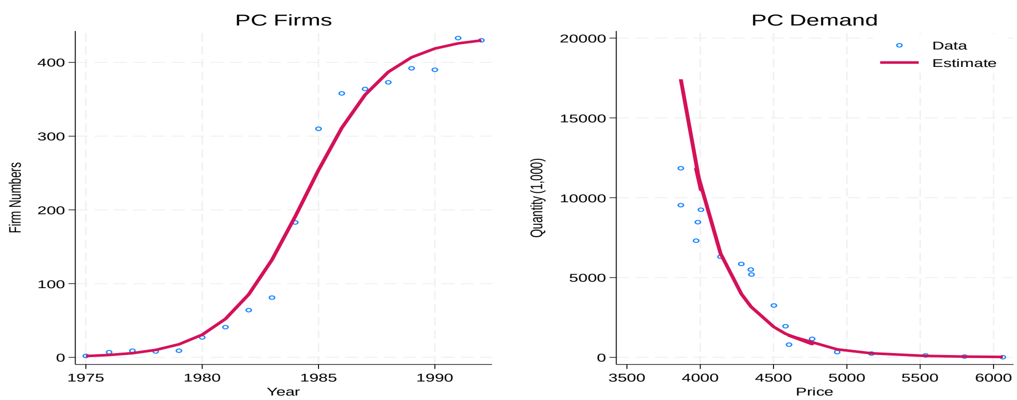

Fig. 8. PC Diffusion and Demand Estimates

The IV estimation gives $\phi = 14.58$ and $a = 137.15$. This yields $\beta = 0.07$ (i.e., $\frac{1}{\beta} = \phi = 14.58$) and $\tilde{A} = 12,170$ (i.e., $\frac{1}{\beta} \ln \tilde{A} = a = 137.15$). For robustness checks, we also re-ran the IV regressions by controlling for changes of population and per capita income over time, and the results are very similar (see Internet Appendix IA-B5).

Furthermore, to check if lagged output per firm is a valid instrument for the PC industry, we use the real import value of computers and accessories (in 2012 prices) as an alternative instrument. Increasing imports shifted the supply of the emerging PC industry and can serve as an instrument to trace out the demand curve. Using this alternative instrument and with the same sample range (1975-1992), the first-stage regression result (adj. $R^2 = 0.91$) is

$$\ln(p_t) = \underset{(0.02)^{***}}{8.68} - \underset{(0.01)^{***}}{0.11} \times \ln(\textit{value of imports})_t,$$

and the second-stage regression result ($R^2 = 0.96$) is

$$\ln(Q_t) = \underset{(6.54)^{***}}{137.58} - \underset{(0.78)^{***}}{14.64} \times \ln(p_t).$$

The estimates are all statistically significant at 1% and very close to our estimates above.

5.2 Model calibration

To calibrate the model, we first pick values for N, γN and k_0 from the diffusion estimation for the auto and the PC industries, respectively. We then pick values for β and A from the demand estimation. In the model, a firm's output is normalized to 1 per period. While this does not affect the theoretical analysis, we account for a firm's production size in the empirical applications. In doing so, we denote by q a firm's output and by Q the industry output, so $Q_t = qk_t$ at date t. Accordingly, we revise Eqs. (17), (18) and (28) as follows by replacing A with $\tilde{A}q^{1-\beta}$ (where $\tilde{A}$ and β are from the demand function estimation above):

$$\text{Regime 1:} \quad \frac{1}{N-k_0}\int_0^\infty e^{-rt}\left(\alpha\frac{N}{k_0} + (1-\alpha)\frac{N}{k_t} - 1\right)\tilde{A}q^{1-\beta}k_t^{1-\beta}dt = c; \tag{37}$$

$$\text{Regime 2:} \quad \frac{1}{N-k_0}\int_0^\infty e^{-rt}\left(\left(\frac{N}{k_0}\right)^\alpha\left(\frac{N}{k_t}\right)^{1-\alpha} - 1\right)\tilde{A}q^{1-\beta}k_t^{1-\beta}dt = c; \tag{38}$$

$$\text{Social optimum:} \quad \int_0^\infty e^{-(r+\gamma N)t}\left(\frac{k_t}{k_0}\right)^2\tilde{A}q^{1-\beta}k_t^{-\beta}dt = c. \tag{39}$$

In the auto case, a firm on average produced less than 1,000 cars a year up to 1910, and we calibrate $q = 900$ based on output per firm in 1910 and $\tilde{A}q^{1-\beta} = 61.11$ (million). In the PC case, using output per firm in 1992, we calibrate $q = 27,500$ and $\tilde{A}q^{1-\beta} = 163.63$ (million). We also set the annual interest rate $r = 0.05$.

Licensing played little role in the creation of auto and PC firms in the early life cycle of the two industries due to difficulties of authenticating original innovations and enforcing IP protection.[21] Thus we set $\alpha = 0$ for both industries in the calibration.

[21] The IP protection system did not function properly in the early life cycle of the two industries and led to many controversies. The Selden patent is one famous example in the early auto industry. It created major litigation among firms but did not really affect the industry development, and

When $\alpha = 0$, Regimes 1 and 2 both reduce to the free-imitation baseline given by Eq. (9). We then use

$$\text{Free imitation:} \quad \frac{1}{N-k_0}\int_0^\infty e^{-rt}\left(\frac{N}{k_t}-1\right)\tilde{A}q^{1-\beta}k_t^{1-\beta}dt = c \tag{40}$$

to solve for c. Table 1 summarizes the benchmark parameter values calibrated for the auto and the PC industries, which we shall use to study the socially optimal licensing rate $\alpha^* > 0$ for a counterfactually successful IP protection system.[22]

Table 1. Model Calibration

	α	r	N	γN	k_0	β	$\tilde{A}q^{1-\beta}$	implied c
Auto	0	0.05	210	0.54	3.20	0.28	61.11	172.70
PC	0	0.05	435	0.58	1.78	0.07	163.63	986.87

Figure 9 plots the calibrated model dynamics for the auto industry.

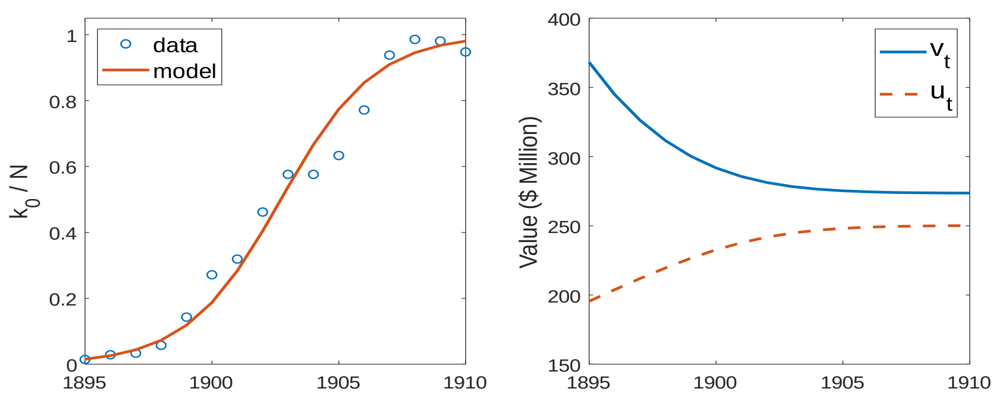

Fig. 9. Model Calibration: Auto

The number of firms k_t grows along a logistic curve. Meanwhile, v_t decreases while u_t increases over time. The initial difference $v_0 - u_0$ equals the model-implied

the patent was later overturned by the court (see e.g., Welsh 1948, Howells and Katznelson 2024). The PC industry had a similar experience. Famous examples include that the original computer invention failed to be patented (Kalat, 2021), Texas Instruments created patent wars (Pollack, 1990), and Compaq's and other clone computers skirted IBM's IP (Mitchell, 2017).

[22] We consider alternative model parameter values in Section 5.4 for robustness checks.

innovation cost $c_{\text{Auto}} = \$172.70$ million (in 2012 prices). By 1910, the value of a producer v_t falls to \$274 million and the value of a future imitator rises to \$250 million. Because almost all the potential entrants N have entered the industry by then, the total value of firms $v_{1910}k_{1910}$ is very close to the present value of the industry revenue $p_{1910}Q_{1910}/r$.

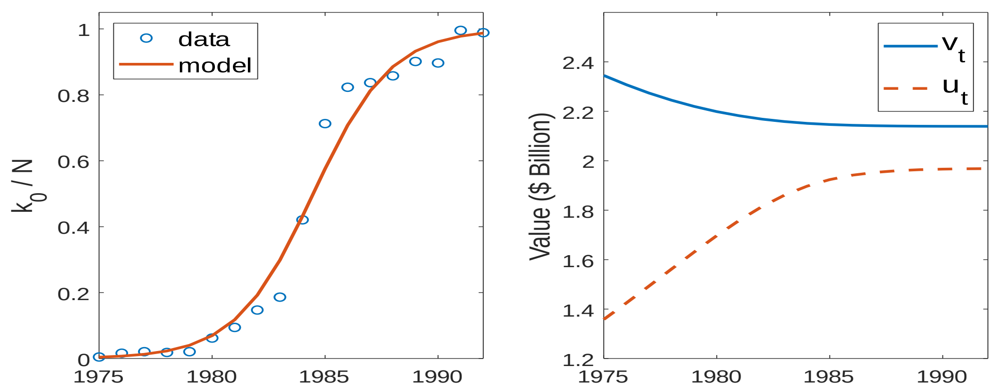

Fig. 10. Model Calibration: PC

Figure 10 plots the calibration results for the PC industry. Again, the number of firms k_t grows along a logistic curve, and v_t decreases while u_t increases over time. The initial difference $v_0 - u_0$ equals the model-implied innovation cost $c_{\text{PC}} = \$\ 986.87$ million (in 2012 prices). By 1992, the value of a producer v_t falls to \$2.14 billion and the value of a future imitator rises to \$1.97 billion. Because almost all the potential entrants have entered the industry by then, the total value of firms $v_{1992}k_{1992}$ is very close to the present value of the industry revenue $p_{1992}Q_{1992}/r$.

5.3 Counterfactual analysis

Given the calibrated model parameter values, we then conduct counterfactual analysis and evaluate welfare.

5.3.1 Socially optimal licensing rate

We first evaluate the effect of the compensation share α in Regimes 1 and 2, and start with the auto industry.

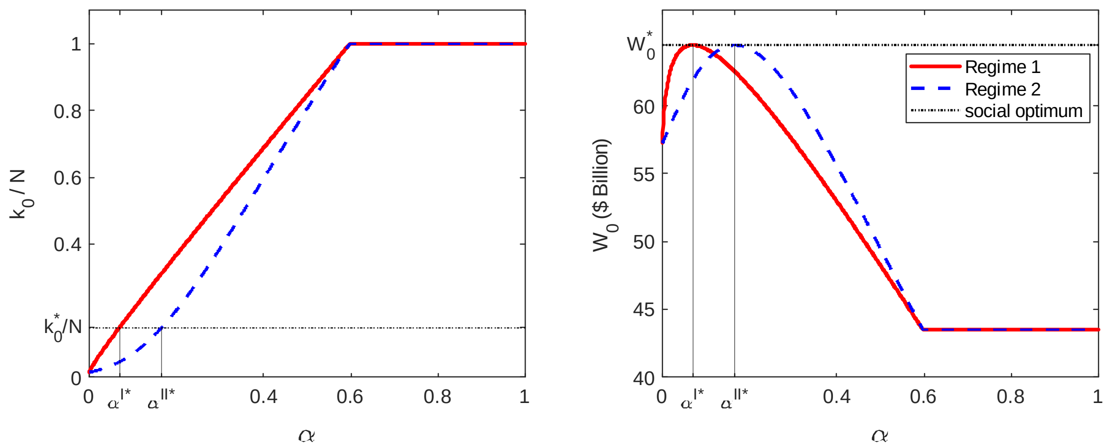

Fig. 11. Effect of α : Auto

Given the innovation cost c_{Auto} derived from the model calibration, we solve the equilibrium industry dynamics for each counterfactual value of $\alpha \in (0, 1]$. Particularly, Eqs. (37) and (38) allow us to pin down the counterfactual entry number of innovators k_0 at date 0. Figure 11 shows that k_0 strictly increases with α for both Regimes 1 and 2 when $0 < \alpha < 0.60$ and that Regime 1 has a higher value of k_0 than Regime 2. For $\alpha \geq 0.60$, the values of k_0 in both regimes reach the corner solution $k_0 = N$. Equation (39) pins down the socially optimal entry number of innovators k_0^*/N to be 0.149, which can be achieved by choosing $\alpha^{\mathbf{I}*}_{\text{Auto}} = 0.069$ in Regime 1 and $\alpha^{\mathbf{II}*}_{\text{Auto}} = 0.166$ in Regime 2. The social optimum yields a social surplus $W^*_{0,\text{Auto}} = \$64.46$ billion (in 2012 prices).

We then look into the PC industry. Given the innovation cost c_{PC} derived from the model calibration, Eqs. (37) and (38) pin down the entry number of innovators k_0 for each counterfactual value of $\alpha \in (0, 1]$. Figure 12 shows that k_0 strictly increases with α for both Regimes 1 and 2 when $0 < \alpha < 0.42$ and that Regime 1 has a higher value of k_0 than Regime 2. For $\alpha \geq 0.42$, the values of k_0 in both regimes reach the corner solution $k_0 = N$. The socially optimal entry number of innovators k_0^*/N is 0.164, which can be achieved by choosing $\alpha^{\mathbf{I}*}_{\text{PC}} = 0.055$ in Regime 1 and $\alpha^{\mathbf{II}*}_{\text{PC}} = 0.135$ in Regime 2. The social optimum yields a social surplus $W^*_{0,\text{PC}} = \$798.9$ billion (in 2012 prices).

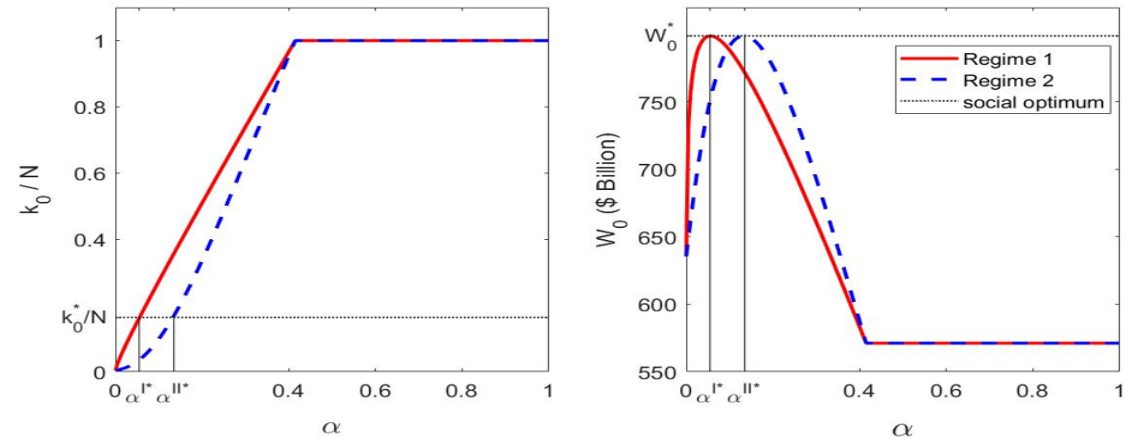

Fig. 12. Effect of α : PC

Comparative statics for α^*.—Figure 13 plots comparative statics for the socially optimal compensation share α^* under Regimes 1 and 2 based on the auto calibration. The results show the following:

- *α^* increases with β.*—A higher β means a lower price elasticity, which leads price to decline faster which discourages k_0. This makes the matching externality less of a concern, so α^* rises.

- *α^* decreases with γ (holding N fixed, when γ is sufficiently large).*—A higher γ implies a better imitation technology, so the planner would need less innovation when γ is sufficiently large and so α^* falls.

- *α^* rises with c but falls with $\bar{A}$ ($\equiv \tilde{A}q^{1-\beta}$).*—A higher c or a lower $\bar{A}$ discourages k_0. This makes the matching externality less of a concern, so α^* rises.

- *α^* rises with N (holding $\gamma N = \lambda$ fixed).*—A higher N leads to faster price decline which discourages k_0. This, together with a larger pool of potential adopters N, makes the matching externality less of a concern, so α^* rises.

- *Comparison of Regimes 1 and 2.*—α^* is higher under Regime 2 than under Regime 1, and the difference rises with β, γ, $c/\bar{A}$, and N.

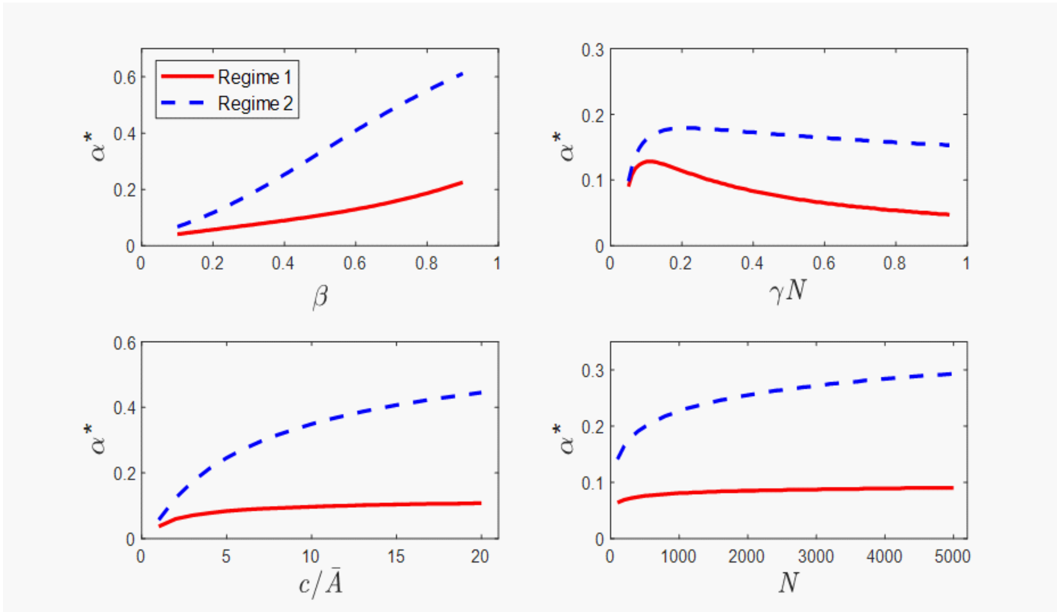

Fig. 13. Comparative Statics for α^* under Regimes 1 and 2

The comparative statics help explain the difference in α^* between the auto and the PC. Compared with the auto, the PC industry has a smaller β and a larger γN, and these two dominate the offsetting forces of the larger $c/\bar{A}$ and larger N and hence $\alpha^*_{\text{PC}} < \alpha^*_{\text{Auto}}$ under each regime. Quantitatively, by comparing counterfactuals that let one industry take on the other industry's parameter values, we find that the smaller β (i.e., the higher price elasticity) accounts most for the smaller α^*_{PC}.

5.3.2 Socially optimal diffusion rate

We can similarly evaluate the effects of varying the diffusion rate γ (holding N fixed). Consider again the scenario where innovators are not compensated by imitators, so $\alpha = 0$. Should the planner slow down the diffusion?

Figure 14 shows that for both the auto and the PC industries, k_0 decreases with γ while W_0 increases with γ. Therefore, if the planner were to reduce γ, the entry of innovators k_0 would rise but social welfare would fall. The intuition is that while slowing down diffusion could encourage entry of innovators, it would forego too much free imitation and the welfare effect of the latter dominates.

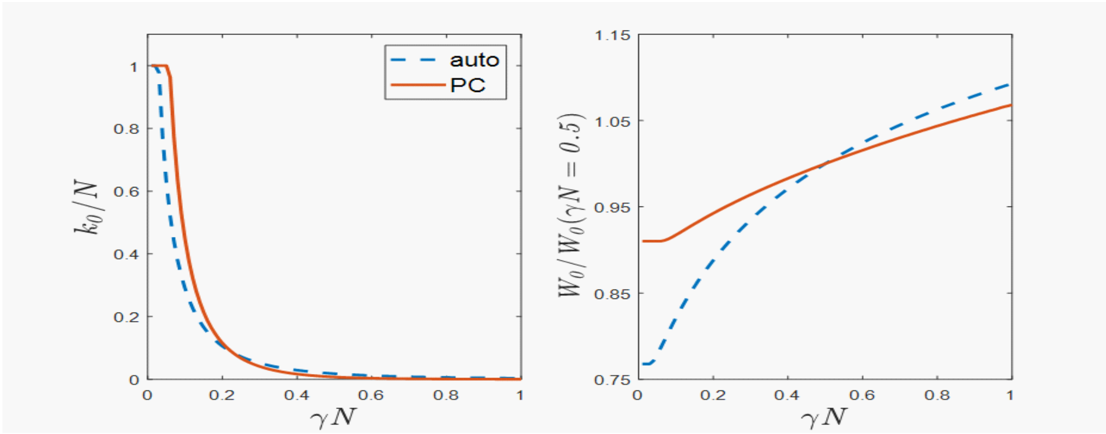

Fig. 14. Effect of the Diffusion Rate γ

To further study how policy affects diffusion and welfare, we split the auto data by region according to regulatory regimes. Specifically, we compare the diffusion of auto producers in Michigan to that in the rest of the country from 1895-1910. The result shows that for Michigan, the diffusion estimates are

$$\ln \frac{k_t}{N - k_t} = \underset{(0.89)^{***}}{-6.03} + \underset{(0.09)^{***}}{0.68}\ t,$$

and the adjusted $R^2 = 0.86$. For the rest of country, the diffusion estimates are

$$\ln \frac{k_t}{N - k_t} = \underset{(0.23)^{***}}{-3.72} + \underset{(0.03)^{***}}{0.46}\ t,$$

and the adjusted $R^2 = 0.95$. Compared to the estimate $\gamma N = 0.54$ from Eq. (33) based on the national sample, the estimated diffusion rate for Michigan $\gamma N^{\text{Michigan}} = 0.68$ is higher, while that for the rest of the country $\gamma N^{\text{rest}} = 0.46$ is lower.

A key reason for the higher diffusion rate in Michigan was its policy on noncompetes. In the early years of the auto industry, Michigan had a legal and political culture that prioritized worker freedom over employer protection, which culminated in its 1905 Anti-Trust Act that banned non-compete agreements outright. This set Michigan apart from most other states where noncompetes were widely enforced. As a result, the auto firms in Michigan had a much higher spinoff rate than other places (Cabral *et al.*, 2018). Counterfactually, suppose that all other states were to adopt policies similar to Michigan's and achieve the higher diffusion rate $\gamma N = 0.68$ nationwide. Our model can then assess the welfare impact: As shown in Fig. 14, raising the diffusion rate from 0.54 to 0.68 nationwide would have led to a lower initial number of innovators (falling from $k_0 = 3.20$ to $k_0 = 1.87$) but a 3.1% higher social welfare (rising from $W_0 = \$57.23$ billion to $W_0 = \$58.98$ billion).[23]

5.4 Robustness checks

For robustness checks, we re-do our analysis under alternative assumptions regarding the pool of potential entrants, anticipated shakeout, and imitators' entry costs. The results are consistent with our previous findings; restricting diffusion speed γ always reduces social welfare in both industries. In what follows, we show the sensitivity of the socially optimal licensing rate α^* to alternative model assumptions.

5.4.1 Pool of potential entrants

In the benchmark analysis, we assumed that the shakeout started after almost all the potential firms had entered the industry. Alternatively, one could consider that the shakeout started in the middle of the diffusion process, so there might be a larger pool of potential entrants. For example, we may assume $N = 1,000$ (instead of 210) for the auto case and $N = 2,000$ (instead of 425) for the PC case. In each case, we then obtain a smaller γN from the diffusion estimation. The new estimates imply that it would take 30 years for each industry to reach 99% adoption rate among potential producers had the shakeout not happened, doubling what is assumed in the benchmark calibration.

We then re-do the calibration and counterfactual exercises with the alternative N. Regarding the socially optimal licensing rate, we now find for the auto case, $\alpha^{\mathbf{I}*}_{\text{Auto}} = 0.134$ under Regime 1 or $\alpha^{\mathbf{II}*}_{\text{Auto}} = 0.309$ under Regime 2, while for the PC case, $\alpha^{\mathbf{I}*}_{\text{PC}} = 0.095$ under Regime 1 or $\alpha^{\mathbf{II}*}_{\text{PC}} = 0.215$ under Regime 2. These estimates of α^* are larger than those found in the benchmark analysis, due to the larger N and smaller γN and higher c from the re-calibrated models, which is consistent with the

[23] Related to our analysis, Marx *et al.* (2009) explore an exogenous reversal of the ban on employee noncompete agreements in Michigan in 1985 as a natural experiment. Using micro data and a differences-in-differences approach, and controlling for changes in the auto industry central to Michigan's economy, they find that the enforcement of noncompetes attenuates mobility, most sharply for inventors with firm-specific skills, and for those who specialize in narrow technical fields.

prediction of our comparative-statics analysis (cf. Fig. 13).

5.4.2 Anticipated shakeout

Our model can be extended to allow the shakeout to be anticipated. Specifically, we could assume that the industry expects a disruptive innovation to arrive at the Poisson rate ρ. This innovation would make obsolete existing technologies and drive firm values to zero.[24]

Accordingly, the value of an incumbent firm under Regime 2 satisfies

$$rv_t = p_t + \gamma\,(N - k_t)\,\alpha v_t - \rho v_t + \frac{dv_t}{dt},$$

i.e.,

$$(r + \rho)\,v_t = p_t + \gamma\,(N - k_t)\,\alpha v_t + \frac{dv_t}{dt}. \tag{41}$$

Including $\rho > 0$ in Eq. (41) is equivalent to raising r to $r + \rho$ in Eq. (13). Similarly, we can revise the value function conditions for outsiders as well as for Regime 1 and for the social planner's problem. The original functional forms of our model hold, except that r becomes $r + \rho$.

Considering that the shakeout occurred in the 16th year for the auto industry and in the 18th year for the PC industry, we take the average and calibrate $\rho = 1/17 = 0.06$. Accordingly, we set $r + \rho = 0.05 + 0.06 = 0.11$ and redo the model calibration and counterfactual analysis.

Regarding the socially optimal licensing rate, we now find in the auto case, $\alpha^{\mathbf{I}*}_{\text{Auto}} = 0.145$ under Regime 1 or $\alpha^{\mathbf{II}*}_{\text{Auto}} = 0.295$ under Regime 2, while in the PC case, $\alpha^{\mathbf{I}*}_{\text{PC}} = 0.115$ or $\alpha^{\mathbf{II}*}_{\text{PC}} = 0.235$. The values of α^* are larger than the benchmark analysis due to the higher discount $r + \rho$ in spite of the lower c implied by the re-calibrated model.

[24]For example, an industry may expect a disruptive innovation (e.g., the assembly line in the auto case) to arrive at the Poisson rate ρ. This innovation would require an incumbent firm to incur a big capital investment to produce a newly designed product at a massive scale. When that innovation did arrive, the new (and lower) equilibrium price could support the capital investment made by only a few firms and the rest would have to exit. As a result, the present value of an investing firm (net of its investment costs) would be zero, and the value of an exiting firm would also be zero.

5.4.3 Imitators' entry cost

So far we have assumed that except for paying the licensing fee, imitators incur no entry cost. This extreme case offers imitators the greatest advantage in competing against innovators and also simplifies the solution.

We now extend the model to include a positive entry cost c^m for imitators, where $c^m < c$. One interpretation is that imitators may have to pay such a cost to absorb the new technology and set up the production. With a positive c^m, the net present value for an outsider to become an imitator at date t changes to $\omega_t - c^m$ instead of ω_t as in the original model.

As a result, the value functions for imitators, innovators and outsiders in the free-imitation baseline model become

$$\omega_t = v_t, \qquad rv_t = p_t + \frac{dv_t}{dt}, \tag{42}$$

$$\text{and} \quad ru_t = \gamma k_t \left(v_t - c^m - u_t\right) + \frac{du_t}{dt}. \tag{43}$$

The counterparts of (42) and (43) in Regime 1 become

$$r\omega_t = p_t + \frac{d\omega_t}{dt}, \qquad rv_t = p_t + \frac{\gamma k_t \left(N - k_t\right)}{k_0}\alpha(\omega_t - c^m) + \frac{dv_t}{dt}, \tag{44}$$

$$\text{and} \quad ru_t = \gamma k_t \left[(1-\alpha)(\omega_t - c^m) - u_t\right] + \frac{du_t}{dt}; \tag{45}$$

and those in Regime 2 become

$$\omega_t = v_t, \qquad rv_t = p_t + \gamma \left(N - k_t\right) \alpha(v_t - c^m) + \frac{dv_t}{dt}, \tag{46}$$

$$\text{and} \quad ru_t = \gamma k_t \left((1-\alpha)(v_t - c^m) - u_t\right) + \frac{du_t}{dt}. \tag{47}$$

The planner also takes c^m into account when maximizing social welfare

$$W_0 = \int_0^\infty e^{-rt} \left[U\left(k_t\right) - \gamma k_t \left(N - k_t\right) c^m\right] dt - ck_0,$$

where $\gamma k_t \left(N - k_t\right) c^m$ is the entry cost incurred by the inflow of imitators.

We then calibrate the extended model to the U.S. auto and PC industries by assuming that c^m is a fraction of c (see Internet Appendix IA-A11 for the analytical solution to the extended model). All else equal, adding an entry cost for imitators incentivizes more agents to innovate. Then, for the extended model to match the observed number of innovators in the data, the calibrated innovation cost c needs to be higher. Despite the higher c, the entry cost of imitators c^m makes it socially desirable to incentivize more innovation comparing to the case $c^m = 0$. As a result, the socially optimal licensing rate α^* becomes higher. Comparing across industries, since the PC is more price elastic than the auto, the former continues to warrant a smaller socially optimal licensing rate than the latter.

Table 2. The Effect of Imitators' Entry Cost

	$\alpha^{\mathbf{I}^*}_{\text{Auto}}$	$\alpha^{\mathbf{II}^*}_{\text{Auto}}$	$\alpha^{\mathbf{I}^*}_{\text{PC}}$	$\alpha^{\mathbf{II}^*}_{\text{Pc}}$
$c^m = 0$	0.069	0.166	0.055	0.135
$c^m = 0.25c$	0.073	0.179	0.057	0.136
$c^m = 0.5c$	0.084	0.212	0.058	0.143

Table 2 compares the model findings for $c^m = 0$, $c^m = 0.25c$, and $c^m = 0.5c$. The results confirm that the higher c^m is as a fraction of c, the higher is the value of α^* in Regimes 1 and 2 for both industries, and that $\alpha^{\mathbf{I}^*}_{\text{PC}} < \alpha^{\mathbf{I}^*}_{\text{Auto}}$ and $\alpha^{\mathbf{II}^*}_{\text{Pc}} < \alpha^{\mathbf{II}^*}_{\text{Auto}}$ continue to hold. Moreover, comparing with our benchmark analysis where $c^m = 0$, the increase in the model-implied socially optimal licensing rate α^* is quantitatively small when $c^m = 0.25c$ or $c^m = 0.5c$ due to the higher value of c implied by the re-calibrated model.

6 Additional discussion

6.1 Matching function specification

In our model, the matching function (3) features increasing returns to scale. However, the assumption on returns to scale is inessential for our analysis. To see why, let us generalize Eq. (3) to

$$\frac{dk_t}{dt} = \hat{\gamma} k_t (N - k_t) \qquad \text{where} \qquad \hat{\gamma} = \frac{\gamma}{N^{\psi}}.$$

The solution of k_t then becomes

$$k_t = \frac{N e^{\hat{\gamma} N t}}{e^{\hat{\gamma} N t} + \frac{N}{k_0} - 1} = \frac{N e^{\gamma N^{1-\psi} t}}{e^{\gamma N^{1-\psi} t} + \frac{N}{k_0} - 1}.$$

By rescaling the diffusion parameter γ with a constant $\frac{1}{N^\psi}$, the matching function features increasing returns to scale if $\psi < 1$, constant returns if $\psi = 1$, and decreasing returns if $\psi > 1$. In the model, we assumed that $\psi = 0$, but our analysis and findings would hold for any ψ because a time series study takes N and $\frac{\gamma}{N^\psi}$ as given; the value of ψ plays no role except in a counterfactual that would involve changing the value of N.

Also, the labor search literature often assumes a Cobb-Douglas matching function:

$$\frac{dk_t}{dt} = \gamma k_t^\theta \left(N - k_t\right)^{1-\theta},$$

where $0 < \theta < 1$. However, the Cobb-Douglas formulation does not appear to fit data better, and more importantly, it does not have a closed-form solution for the time path of k_t. Therefore the logistic formulation we use has analytical advantages.

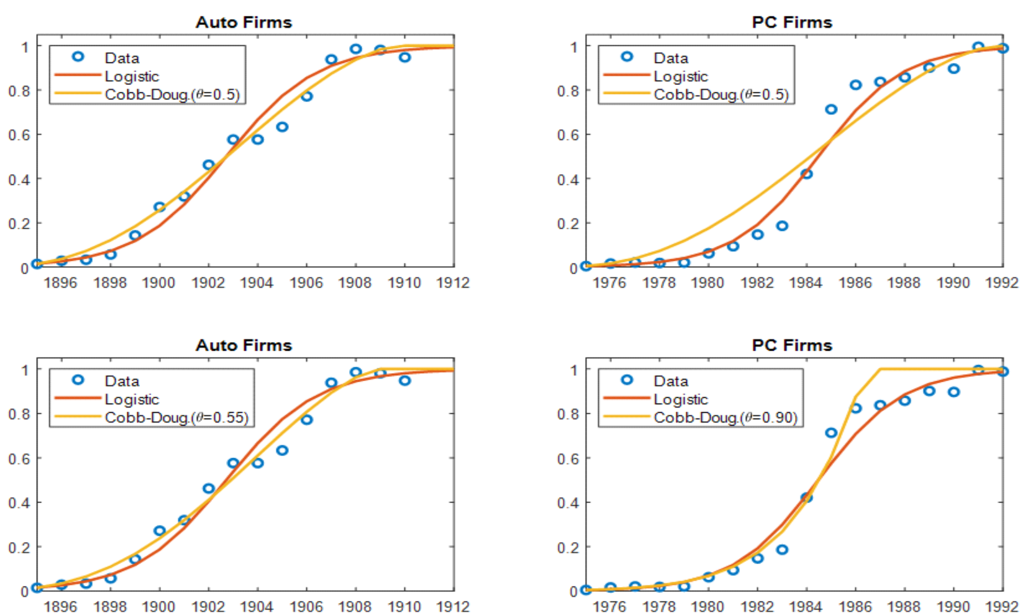

Fig. 15. Diffusion Models: Fitting Time Paths of Firm Numbers

Figure 15 shows that the estimated logistic diffusion model (cf. Eq. (4)) matches the time paths of firm numbers well for U.S. automobile and PC industries. Compared to the symmetric Cobb-Douglas counterpart (i.e., $\theta = 0.5$), logistic diffusion shows a more pronounced inflection point and fits better for the PC industry, as shown in the top panels. In the bottom panels, we compare logistic diffusion with the best fitting Cobb-Douglas formulation for each industry without restricting θ. The former still fits better for the PC industry.[25]

6.2 Heterogeneous production capacity

The model assumes that all firms have the same constant production capacity. The constancy of output per firm prior to the shakeout shown in Fig. 5 and 7 suggests that this is not unreasonable in the pre-shakeout period. Still, the model can accommodate some output heterogeneity with the following minor adjustment: If k_t is the number of firms at date t, suppose the production capacity of innovator i is s_i, a random variable with CDF $H(s_i)$ with support $(0, \bar{s})$ and a mean equal to unity.

Among innovators we assume that the variables s_i are random, independent over i, but are realized after the innovator has paid the cost c. We also assume that matching is undirected so that H pertains to the imitators, too. Industry output would be

$$Q_t = k_t \int_0^{\bar{s}} s dH(s) = k_t.$$

Then $Q_t = k_t$, and p_t would remain the same as before. Ex ante expected revenue of firm i would be $p_t E(s_i) = p_t$. The rest of the analysis would stay unchanged.

A subset of the values of s in the support of H could then develop into "dominant designs" in the terminology of Utterback and Abernathy (1975) and survive the shakeout.

6.3 Industry growth factors

Our model can also be extended to incorporate growing market demand and firm productivity. In doing so, one may extend the model setup by allowing the market demand parameter A_t and each firm's production capacity q_t to increase over time.

[25]By maximizing the fit with a Cobb-Douglas function without restricting θ, we estimate that $\theta^{\text{auto}} = 0.55$ and $\theta^{\text{PC}} = 0.90$. Figure 15 shows that Cobb-Douglas curves fit slightly better in the auto case ($R^2 = 0.986$ when $\theta = 0.5$ and $R^2 = 0.987$ when $\theta = 0.55$) than the logistic curve ($R^2 = 0.974$), but the logistic curve fits better in the PC case ($R^2 = 0.981$) than the Cobb-Douglas curves ($R^2 = 0.936$ when $\theta = 0.5$ and $R^2 = 0.969$ when $\theta = 0.90$).

As a result, the total industry output is $Q_t = q_t k_t$ at date $t > 0$, and the product price is

$$p_t = A_t Q_t^{-\beta} = A_t \left(q_t k_t\right)^{-\beta}.$$

As long as profit per firm (i.e., $p_t q_t = A_t q_t^{1-\beta} k_t^{-\beta}$) does not increase in t, we find that at equilibrium, innovators still enter the industry only at date 0. In Regime 1, the number of innovators $k_0^{\mathbf{I}} \in (0, N)$ is determined by

$$\underbrace{\frac{1}{N-k_0}\int_0^{\infty} e^{-rt}\left(\alpha\frac{N}{k_0} + (1-\alpha)\frac{N}{k_t} - 1\right) A_t \left(q_t k_t\right)^{1-\beta} dt}_{v_0 - u_0} \;=\; c,$$

and in Regime 2, the number of innovators $k_0^{\mathbf{II}} \in (0, N)$ is determined by

$$\underbrace{\frac{1}{N-k_0}\int_0^{\infty} e^{-rt}\left(\left(\frac{N}{k_0}\right)^{\alpha}\left(\frac{N}{k_t}\right)^{1-\alpha} - 1\right) A_t \left(q_t k_t\right)^{1-\beta} dt}_{v_0 - u_0} \;=\; c,$$

where, in each regime, k_t follows the logistic diffusion process given by Eq. (4) (see Internet Appendix IA-A12 for the proof).

The intuition is that despite the growth in market demand and firm productivity, the entry of imitators may continue to bring down the competitive rents enjoyed by earlier entrants. As a result, innovators enter only at date 0, and the private and social trade-offs that we have studied between innovation and imitation continue to exist. To apply the model extension to data, one needs to specify the laws of motion for A_t and q_t explicitly. Given that our model with time-invariant A_t and q_t fits the U.S. auto and PC industry data well, we leave the empirical application of this extension for future research.[26]

[26] There may be exceptional cases where market demand or firm capacity expands so much that profit per firm increases over time. If so, some agents could enter as innovators after date 0. That scenario might be relevant for some products or industries, but from a modeling point of view, it blurs the tradeoff between innovation and imitation. We run Bass model regressions using both auto and PC industry data by allowing some firms to enter the industry as independent innovators after date 0, but the effect is statistically insignificant (see Internet Appendixes IA-B2 and IA-B4).

6.4 The $N \to \infty$ limit

The special case where $N \to \infty$ does not fit the industry data well, but the solutions are simpler and may help illustrate the findings of our model. In this subsection, we study a limiting version of our model with $N \to \infty$. All proofs are provided in Internet Appendix IA-A13.

Let N get large but at the same time reduce γ so that $\gamma N \to \lambda > 0$, a constant. The logistic diffusion process (3) then converges to $\frac{dk_t}{dt} = \lambda k_t$, and then (4) becomes $k_t = k_0 e^{\lambda t}$. This is essentially the exponential diffusion process considered by the models of competitive innovation (i.e., Boldrin and Levine, 2002, 2008, and Quah, 2002). Those studies embed such a diffusion process in a growth model, and we now incorporate it into our industry dynamic model.

Assuming that $\lambda < r$ so that welfare is bounded, we prove that, at equilibrium, innovators only enter at date 0, and the number of innovators is

$$k_0^{\mathbf{I}} = \left(\frac{A(r + (\beta - 1)\lambda + \alpha\lambda)}{c\,(r + \beta\lambda)\,(r + (\beta - 1)\lambda)} \right)^{\frac{1}{\beta}}, \tag{48}$$

$$k_0^{\mathbf{II}} = \left(\frac{A}{c(r + (\beta - \alpha)\lambda)} \right)^{\frac{1}{\beta}} \tag{49}$$

in Regime 1 and Regime 2, respectively.

Equations (48) and (49) imply that $k_0^{\mathbf{I}} = k_0^{\mathbf{II}} > 0$ for $\alpha \in \{0, 1\}$ and $k_0^{\mathbf{I}} > k_0^{\mathbf{II}} > 0$ for $\alpha \in (0, 1)$. Hence, innovation occurs even when $\alpha = 0$, and Regime 1 yields more innovation unless $\alpha \in \{0, 1\}$.

Socially optimal licensing rate.—While $\alpha = 0$ is compatible with positive innovation, it is not optimal. We prove that it is socially optimal to innovate only at date 0 and that the corresponding number of innovators is

$$k_0^* = \left(\frac{A}{(r + (\beta - 1)\lambda)c} \right)^{\frac{1}{\beta}}. \tag{50}$$

Comparing Eq. (50) to Eqs. (48) and (49) shows that $k_0^* = k_0^{\mathbf{I}} = k_0^{\mathbf{II}}$ iff $\alpha = 1$. I.e., $\alpha^* = 1$ for both Regimes 1 and 2.

Why does the limiting model yield an optimal licensing rate different from that when N is finite? The key is that when N is finite, there is a matching externality that an innovator creates and ignores, which reduces other agents' innovation payoff.

In the limiting model, however, there is no such externality – an innovator's matching rate is fixed at $\frac{dk_t/dt}{k_t} = \lambda$ while an imitator's matching rate is fixed at 0 given that $N \to \infty$. Therefore, the finite-N version not only fits the industry evolution pattern better but also incorporates the important matching externality.

Socially optimal diffusion rate.—Parallel to the logistic diffusion case where the planner does not want to reduce γ, here the planner does not want to reduce λ under the exponential diffusion:

$$\frac{\partial W_0}{\partial \lambda} > 0 \quad \text{for } \alpha \in [0,1], \text{ and } \beta > 0 \tag{51}$$

for both Regimes 1 and 2.

7 Conclusion

We modeled an innovation and its diffusion in one industry and calibrated the model to data of the U.S. automobile and personal computer industries. Though starting nearly one century apart, the two industries shared the basic feature of an S-shaped diffusion prior to the shakeout. Our empirical findings match well the expansion of firm numbers prior to the shakeout in each industry and quantify the theoretical predictions of the model.

Our analysis offers several results regarding welfare and policy. First, capacity constraints imply that licensing raises the revenues of innovators and, to a degree, that licensing is socially beneficial. Second, socially optimal compensation for innovators should reflect both the learning externality and the matching externality that may arise when idea transfers take place. Third, socially optimal licensing rates should be higher if imitators can resell ideas to other imitators.

Finally, a policy restricting diffusion always reduces welfare. And when demand is elastic and innovators' compensation share is high, slowing down diffusion reduces output at all dates. In other cases, slowing down diffusion may encourage innovation and raise initial capacity, but it would lower imitation so that capacity grows more slowly. As a result, the mechanism can generate industry overtaking.

References

[1] Atkeson, A. (2020). "What will be the economic impact of Covid-19 in the US? Rough estimates of disease scenarios." NBER Working Paper 26867.

[2] Bass, F. (1969). "A new product growth for model consumer durables." *Management Science*, 15(5), 215–227.

[3] Bass, F. (2004). "Comments on "A new product growth for model consumer durables": The Bass model." *Management Science*, 50(12), 1833–1840.

[4] Benhabib, J., Perla, J., & Tonetti, C. (2021). "Reconciling models of diffusion and innovation: A theory of the productivity distribution and technology frontier." *Econometrica*, 89(5), 2261–2301.

[5] Berry, S., & Haile, P. (2021). "Foundations of demand estimation." NBER Working Paper No. 29305.

[6] Boldrin, M., & Levine, D. K. (2008). "Perfectly competitive innovation." *Journal of Monetary Economics*, 55(3), 435-453.

[7] Boldrin, M., & Levine, D. K. (2002). "Perfectly competitive innovation." Staff Report 303, Federal Reserve Bank of Minneapolis.

[8] Cabral, L., Wang, Z., & Xu, D. Y. (2018). "Competitors, complementors, parents and places: Explaining regional agglomeration in the U.S. Auto Industry." *Review of Economic Dynamics*, 30, 1–29.

[9] David, P. A. (1969). "A contribution to the theory of diffusion." Stanford University Research Center in Economic Growth Memorandum 71.

[10] Filson, D. (2001). "The nature and effects of technological change over the industry life cycle." *Review of Economic Dynamics*, 4(2), 460–494.

[11] Franco, A. M., & Mitchell, M. F. (2008). "Covenants not to compete, labor mobility, and industry dynamics." *Journal of Economics & Management Strategy*, 17(3), 581–606.

[12] Garibaldi, P., Moen, E. R., & Pissarides, C. A. (2020). "Modelling contacts and transitions in the SIR epidemics model." *Covid Economics, Vetted and Real-Time Papers*, 5, 1–20.

[13] Gilson, R. J. (1999). "The legal infrastructure of high technology industrial districts: Silicon Valley, Route 128, and Covenants Not to Compete." *New York University Law Review*, 74, 575–629.

[14] Goldscheider, R. (2011). "The classic 25% rule and the art of intellectual property licensing." *Duke Law & Technology Review*, 10, 1-23.

[15] Goldscheider, R., Jarosz, J. & Mulhern, C. (2002). "Use of the 25 per cent rule in valuing IP," *les Nouvelles*, December, 123-133.

[16] Gort, M., & Klepper, S. (1982). "Time paths in the diffusion of product innovations." *Economic Journal*, 92, 630–653.

[17] Griliches, Z. (1957). "Hybrid corn: An exploration in the economics of technical change." *Econometrica* 25(4), 501–522.

[18] Hayashi, F., Li, B. G., & Wang, Z. (2017). "Innovation, deregulation, and the life cycle of a financial service industry." *Review of Economic Dynamics*, 26, 180-203.

[19] Hopenhayn, H., & Shi, L. (2020). "Knowledge creation and diffusion with limited appropriation." Working Paper.

[20] Howells, J. & Katznelson, R. D. (2024). "Freedom to operate analysis as competitive necessity—the Selden automobile patent case revisited." *Journal of Intellectual Property Law & Practice*, 19 (6), 493-507.

[21] *Information Technology Industry Data Book.* (1998). Washington, DC: Information Technology Industry Council.

[22] Jovanovic, B., & MacDonald, G. (1994). "The life cycle of a competitive industry." *Journal of Political Economy*, 102 (2), 322–347.

[23] Kalat, D. (2021). "Why computers aren't patented." *ThinkSet Magazine*, April 26, Berkeley Research Group, LLC.

[24] Kemmerer, J., & Lu, J. (2012). "Profitability and royalty rates across industries." KPMG International.

[25] Klepper, S. (1996). "Entry, exit, growth, and innovation over the product life cycle." *American Economic Review*, 86, 562–583.

[26] Klepper, S. (2010). "The origin and growth of industry clusters: The making of Silicon Valley and Detroit." *Journal of Urban Economics*, 67(1), 15–32.

[27] Lauermann, S., Nöldeke, G., & Tröger, T. (2020). "The balance condition in search-and-matching models." *Econometrica*, 88(2), 595–618.

[28] Levin, R., Klevorick, A., Nelson, R., Winter, S., Gilbert, R., & Griliches, Z. (1987). "Appropriating the returns from industrial research and development." *Brookings Papers on Economic Activity* , Vol. 1987, No. 3, 783-831.

[29] Manea, M. (2021). "Bottleneck links, essential intermediaries and competing paths of diffusion in networks." *Theoretical Economics*, 16(3), 1017–1053.

[30] Mansfield, E. (1961). "Technical change and the rate of imitation." *Econometrica*, 29(4), 741–766.

[31] Manuelli, R., & Seshadri, A. (2014). "Frictionless technology diffusion: The case of tractors." *American Economic Review*, 104(4), 1368-91.

[32] Marx, M., Strumsky, D., & Fleming, L. (2009). "Mobility, skills, and the Michigan non-compete experiment." *Management Science,* 55(6), 875-889.

[33] Mitchell, R. (2017). "Tales from 80s tech: How Compaq's clone computers skirted IBM's IP and gave rise to EISA." *News*, August 14, All about Circuits.

[34] Pollack, A. (1990). "A chip maker's profit on patents." *The New York Times*, Oct. 16.

[35] Quah, D. (2002). "24/7 competitive innovation." London School of Economics Centre for Economic Performance Working Paper 1218.

[36] Razgaitis, R. (1999). *Early-Stage Technologies: Valuation and Pricing*. John Wiley & Sons, Inc.

[37] Saxenian, A. (1994). *Regional advantage: Culture and competition in Silicon Valley and Route 128*. Cambridge, MA: Harvard University Press.

[38] Smith, P. H. (1970). *Wheels within wheels: A short history of American motor car manufacturing*. Funk and Wagnalls, New York.

[39] Stavins, J. (1995). "Model entry and exit in a differentiated-product industry: The personal computer market." *Review of Economics and Statistics* 77(4), 571–584.

[40] *Thomas Register of American Manufacturers (1958–1999)*. Springfield, IL: Thomas.

[41] Thomas, R. P. (1977). *An analysis of the pattern of growth of the automobile industry 1895–1929*. Arno Press, New York.

[42] Utterback, J., & Abernathy, W. J. (1975). "A dynamic model of process and product innovation." *Omega*, 3(6), 639-656.

[43] Welsh, C. A. (1948). "Patents and competition in the automobile industry." *Law and Contemporary Problems*, 13(2), 260-277

[44] Williamson, S. H. (2020). "What was the U.S. GDP then?" MeasuringWorth.

[45] Young, H. P. (2009). "Innovation diffusion in heterogeneous populations: Contagion, social influence, and social learning." *American Economic Review*, 99, 1899-1924.

Internet Appendix to "Idea Diffusion and Property Rights"

Boyan Jovanovic and Zhu Wang

IA-A. Proofs

IA-A1. Proof of Proposition 1

Proof. We first conjecture that no agent enters as an innovator after date 0 and $k_0 < N$, so the time path of firm numbers follows logistic diffusion:

$$k_t = \frac{Ne^{\gamma Nt}}{e^{\gamma Nt} + \frac{N}{k_0} - 1}. \quad \text{(A.1)}$$

The value of a producer v_t is determined by

$$rv_t = p_t + \frac{dv_t}{dt}. \quad \text{(A.2)}$$

Solving the differential equation (A.2) yields

$$v_t = \int_t^\infty e^{-r(s-t)} p_s ds = \int_t^\infty e^{-r(s-t)} A k_s^{-\beta} ds. \quad \text{(A.3)}$$

Because k_s increases with s, v_t decreases with t.

Next, we show that $v_t - u_t$ decreases with t. Recall that the value of an outsider, u_t, is determined by

$$ru_t = \gamma k_t (v_t - u_t) + \frac{du_t}{dt}, \quad \text{(A.4)}$$

which, together with Eq. (A.2), implies that

$$\frac{d(v_t - u_t)}{dt} = (r + \gamma k_t)(v_t - u_t) - p_t. \quad \text{(A.5)}$$

Let $\psi_t \equiv v_t - u_t$. Then Eq. (A.5) reads

$$\frac{d\psi_t}{dt} - (r + \gamma k_t)\psi_t = -p_t.$$

Define $z_t = \exp \int -(r + \gamma k_t)dt$. We then have

$$\frac{d(z_t \psi_t)}{dt} = -z_t p_t,$$

which yields the general solution

$$\psi_t = z_t^{-1} \int -z_t p_t dt + z_t^{-1} C,$$

where C is the constant of integration.

Given that $k_t = \frac{Ne^{\gamma Nt}}{e^{\gamma Nt}+\frac{N}{k_0}-1}$, we can solve for z_t:

$$z_t = e^{-rt}\frac{k_0}{N-k_0+k_0e^{\gamma Nt}}.$$

Since ψ_t needs to be bounded as $t\to\infty$, we have $C=0$. We then have

$$\begin{aligned}\psi_t &= z_t^{-1}\int -z_tp_tdt \\ &= e^{rt}\frac{N-k_0+k_0e^{\gamma Nt}}{k_0}\int -e^{-rt}\frac{k_0}{N-k_0+k_0e^{\gamma Nt}}p_tdt \\ &= e^{rt}\frac{N-k_0+k_0e^{\gamma Nt}}{k_0}\int_t^\infty e^{-rs}\frac{k_0}{N-k_0+k_0e^{\gamma Ns}}p_sds \\ &= \int_t^\infty e^{-r(s-t)}\left(1-\frac{e^{\gamma N(s-t)}-1}{(\frac{N-k_0}{k_0})e^{-\gamma Nt}+e^{\gamma N(s-t)}}\right)p_sds. \end{aligned} \tag{A.6}$$

Define $j=s-t$. We can rewrite Eq. (A.6) as

$$\psi_t = \int_0^\infty e^{-rj}\left(1-\frac{e^{\gamma Nj}-1}{(\frac{N-k_0}{k_0})e^{-\gamma Nt}+e^{\gamma Nj}}\right)p_{t+j}dj.$$

In the integral, the terms $\left(1-\frac{e^{\gamma Nj}-1}{(\frac{N-k_0}{k_0})e^{-\gamma Nt}+e^{\gamma Nj}}\right)$ and p_{t+j} both decrease with t, hence $\psi_t = v_t - u_t$ strictly decreases with t. Given the free entry condition $v_0-u_0=c$, we have $v_t-u_t<c$ for any $t>0$, and so no agent would want to enter as an innovator after date 0.

Eq. (A.3) implies that

$$v_0 = \int_0^\infty e^{-rt}Ak_t^{-\beta}dt. \tag{A.7}$$

At date 0, the total industry discounted revenue, $\int_0^\infty e^{-rt}Ak_t^{1-\beta}dt$, is shared by the two groups – the initial entrants k_0 and the outsiders $N-k_0$. With the free entry condition $v_0-c=u_0$, we have

$$\int_0^\infty e^{-rt}Ak_t^{1-\beta}dt = v_0k_0+u_0\left(N-k_0\right) = v_0N-c\left(N-k_0\right). \tag{A.8}$$

Plugging Eq. (A.7) into Eq. (A.8) yields Eq. (A.9):

$$\frac{\int_0^\infty e^{-rt}\left(\frac{N}{k_t}-1\right)Ak_t^{1-\beta}dt}{N-k_0} = c. \tag{A.9}$$

Interior solution (i.e., $k_0<N$).—We have shown that k_0 is determined by Eq. (A.9). Note that as $k_0\to N$, we have $k_t\to N$. Hence,the numerator and the

denominator on the left hand side of Eq. (A.9) both converge to 0 as $k_0 \to N$. Applying L'Hôpital's rule, the left hand side of Eq. (A.9) converges to $\frac{AN^{-\beta}}{r+\gamma N}$ as $k_0 \to N$. We can further prove that Eq. (A.9) yields $dk_0/dc < 0$, and so the baseline model has an interior solution $k_0 < N$ iff condition (A.10) holds, i.e., iff

$$c > \frac{AN^{-\beta}}{r+\gamma N}. \tag{A.10}$$

■

IA-A2. Proof of Proposition 2(A)

Proof. In Regime 1, potential adopters can copy an idea from an imitator but the fee goes to the idea's original innovator. We first assume that, at equilibrium, innovators enter only at date 0 and $k_0 < N$, so that the time path of firm numbers is determined by Eq. (A.1). We then ask if any agent would want to deviate by entering as an innovator at a date $\tau > 0$.

The entry of a measure-zero innovator at $\tau > 0$ would not change the industry quantity and price through Eq. (A.1). Upon entry, the value of this innovator comes from two sources: One is that he will get a fraction $1/k_t$ of the total industry revenue $Ak_t^{1-\beta}$ at each date $t \geq \tau$ by selling goods; the other is that he will get a chance $1/k_\tau$ to collect idea-sale revenues from new imitators at each date $t \geq \tau$ (note that k_τ is the number of all incumbent firms at his entry date τ, so $1/k_\tau$ is the probability for new imitators at each date $t \geq \tau$ to trace him as the original innovator of the idea they copy). At each date $t \geq \tau$, a fraction $\frac{k_t-k_\tau}{k_t}$ of firms in the industry are imitators who enter between date τ and date t, so this new innovator at his entry date τ expects to have $1/k_\tau$ chance to receive the discounted sum of the fraction $\alpha\,(k_t - k_\tau)\,/k_t$ of the total industry revenue $Ak_t^{1-\beta}$ as his idea-sale revenue starting from date τ.

Therefore, the value of this new innovator at his entry date τ, denoted by v_τ^τ, is

$$v_\tau^\tau = \int_\tau^\infty e^{-r(t-\tau)}\left(\frac{1}{k_t} + \frac{\alpha}{k_\tau}(\frac{k_t-k_\tau}{k_t})\right)Ak_t^{1-\beta}dt \tag{A.11}$$

Note that v_τ^τ varies by entry date τ because the number of existing firms k_τ increases with τ. By contrast, the value of an innovator who entered at date 0 would have the date-τ value

$$v_\tau^0 = \int_\tau^\infty e^{-r(t-\tau)}\left(\frac{1}{k_t} + \frac{\alpha}{k_0}(\frac{k_t-k_\tau}{k_t})\right)Ak_t^{1-\beta}dt, \tag{A.12}$$

so $v_\tau^\tau < v_\tau^0$ for any $\tau > 0$, and $v_\tau^\tau = v_\tau^0$ for $\tau = 0$.

Equation (A.1) implies that for any date $t \geq \tau$,

$$k_t = \frac{Ne^{\gamma N(t-\tau)}}{e^{\gamma N(t-\tau)} + (\frac{N}{k_\tau} - 1)}, \tag{A.13}$$

and that

$$\frac{k_t}{k_\tau} = \frac{Ne^{\gamma N(t-\tau)}}{N + (e^{\gamma N(t-\tau)} - 1)k_\tau}. \tag{A.14}$$

We can rewrite Eq. (A.11) as

$$v_\tau^\tau = \int_\tau^\infty e^{-r(t-\tau)} \left(1 - \alpha + \alpha \frac{k_t}{k_\tau}\right) Ak_t^{-\beta} dt. \tag{A.15}$$

Defining $s = t - \tau$, Eq. (A.15) becomes

$$v_\tau^\tau = \int_0^\infty e^{-rs} \left(1 - \alpha + \alpha \frac{k_{\tau+s}}{k_\tau}\right) Ak_{\tau+s}^{-\beta} ds. \tag{A.16}$$

Note that Eqs. (A.13) and (A.14) imply that

$$\frac{k_{\tau+s}}{k_\tau} = \frac{Ne^{\gamma Ns}}{N + (e^{\gamma Ns} - 1)k_\tau}, \quad k_{\tau+s}^{-\beta} = \left(\frac{Ne^{\gamma Ns}}{e^{\gamma Ns} + (\frac{N}{k_\tau} - 1)}\right)^{-\beta},$$

which both decrease in k_τ. In Eq. (A.16), because k_τ increases in τ, $\frac{k_{\tau+s}}{k_\tau}$ and $k_{\tau+s}^{-\beta}$ decrease in τ, and hence v_τ^τ decreases in τ.

Similarly, because an imitator can keep $(1-\alpha)$ share of his output, the total value of outsiders $u_\tau(N - k_\tau)$ at date τ equals the imitators' share of the total discounted industry revenues from date τ and onward. Therefore, we have

$$u_\tau(N - k_\tau) = \int_\tau^\infty e^{-r(t-\tau)} \left(\frac{(1-\alpha)(k_t - k_\tau)}{k_t}\right) Ak_t^{1-\beta} dt,$$

which implies

$$u_\tau = \int_\tau^\infty e^{-r(t-\tau)} \left(\frac{(1-\alpha)(k_t - k_\tau)}{k_t(N - k_\tau)})\right) Ak_t^{1-\beta} dt. \tag{A.17}$$

Inserting Eq. (A.14) into Eq. (A.17), we derive

$$u_\tau = \int_\tau^\infty e^{-r(t-\tau)} \left(\frac{(1-\alpha)\left(e^{\gamma N(t-\tau)} - 1\right)}{Ne^{\gamma N(t-\tau)}}\right) Ak_t^{1-\beta} dt. \tag{A.18}$$

Again, defining $s = t - \tau$, Eq. (A.18) becomes

$$u_\tau = \int_0^\infty e^{-rs} \left(\frac{(1-\alpha)\left(e^{\gamma Ns} - 1\right)}{Ne^{\gamma Ns}}\right) Ak_{\tau+s}^{1-\beta} ds. \tag{A.19}$$

Equation (A.19) implies that if $\beta = 1$, u_τ is a constant that does not vary with τ; if $\beta > 1$, $k_{\tau+s}^{1-\beta}$ decreases with k_τ so u_τ decreases with τ; and if $\beta < 1$, $k_{\tau+s}^{1-\beta}$ increases with k_τ so u_τ increases with τ. Moreover, combining Eqs. (A.16) and (A.19), we have

$$v_\tau^\tau - u_\tau = \int_0^\infty e^{-rs}\left(1-\alpha+\alpha\frac{k_{\tau+s}}{k_\tau} - \frac{(1-\alpha)\left(e^{\gamma Ns}-1\right)}{Ne^{\gamma Ns}}k_{\tau+s}\right)Ak_{\tau+s}^{-\beta}ds. \quad \text{(A.20)}$$

Within the integral of Eq. (A.20), both terms $\left(1-\alpha+\alpha\frac{k_{\tau+s}}{k_\tau} - \frac{(1-\alpha)\left(e^{\gamma Ns}-1\right)}{Ne^{\gamma Ns}}k_{\tau+s}\right)$ and $k_{\tau+s}^{-\beta}$ decrease in τ, so $v_\tau^\tau - u_\tau$ decreases with τ. Therefore, given the free entry condition $v_0^0 - u_0 = c$, we have $v_\tau^\tau - u_\tau < c$ for any $\tau > 0$, so no innovator would enter the industry after date 0.

Let $v_0 = v_0^0$. Equations (A.12) and (A.17) yield

$$v_0 - u_0 = \frac{1}{N-k_0}\int_0^\infty e^{-rt}\left(\alpha\frac{N}{k_0} + (1-\alpha)\frac{N}{k_t} - 1\right)Ak_t^{1-\beta}dt. \quad \text{(A.21)}$$

The free entry condition $v_0 - u_0 = c$ then pins down the mass of innovators k_0 at date 0, as shown by Eq. (A.22):

$$\frac{\int_0^\infty e^{-rt}\left(\alpha\frac{N}{k_0} + (1-\alpha)\frac{N}{k_t} - 1\right)Ak_t^{1-\beta}dt}{N-k_0} = c. \quad \text{(A.22)}$$

Interior solution (i.e., $k_0^{\mathbf{I}} < N$).—We have shown above that $k_0^{\mathbf{I}}$ is determined by Eq. (A.22). Note that as $k_0 \to N$, we have $k_t \to N$. Hence, the numerator and the denominator of the left hand side of Eq. (A.22) both go to 0 as $k_0 \to N$. Applying L'Hôpital's rule, the left hand side of Eq. (A.22) converges to $\frac{r+\alpha\gamma N}{r(r+\gamma N)}AN^{-\beta}$ as $k_0 \to N$. Proposition 3(A) shows that $dk_0^{\mathbf{I}}/dc < 0$. Therefore, the model has an interior solution $k_0^{\mathbf{I}} < N$ in Regime 1 iff condition (A.23) holds, i.e., iff

$$c > \frac{r+\alpha\gamma N}{r(r+\gamma N)}AN^{-\beta}. \quad \text{(A.23)}$$

Additional properties of the dynamic path.—The proof above confirms that innovators only enter at date 0. The time path of firm numbers is given by Eq. (A.1), and the time path of u_t is solved by Eq. (A.19). Following that, the dynamic paths of ω_t, v_t^τ and v_t for any $t \geq 0$ can also be derived. Recall that

$$r\omega_t = p_t + \frac{d\omega_t}{dt},$$

which yields

$$\omega_t = \int_t^\infty e^{-r(s-t)}p_s ds = \int_t^\infty e^{-r(s-t)}Ak_s^{-\beta}ds.$$

Because k_s increases in s, ω_t declines in t.

Counterfactually, suppose a marginal innovator did enter at date $\tau > 0$. For any $t \geq \tau$, he would collect $Ak_s^{-\beta}$ in each period $s \geq t$ by selling goods, and collect a fraction $\frac{\alpha(k_s - k_t)}{k_s k_\tau}$ of the total industry revenues $Ak_s^{1-\beta}$ from new entrants after date t by selling ideas. Therefore, his value at date t would be determined by

$$v_t^\tau = A \int_t^\infty e^{-r(s-t)} k_s^{-\beta} ds + \frac{A}{k_\tau} \int_t^\infty e^{-r(s-t)} \alpha (1 - \frac{k_t}{k_s}) k_s^{1-\beta} ds. \tag{A.24}$$

Because k_τ increases in τ, v_t^τ declines in τ.

Finally, Eq. (A.24) suggests that an innovator who had entered at date 0 would have a value v_t for any $t \geq 0$:

$$v_t = v_t^0 = A \int_t^\infty e^{-r(s-t)} k_s^{-\beta} ds + \frac{A}{k_0} \int_t^\infty e^{-r(s-t)} \alpha (1 - \frac{k_t}{k_s}) k_s^{1-\beta} ds. \tag{A.25}$$

Equation (A.25) suggests that the time path of v_t depends on parameter values. For example, v_t may decrease in t when $\alpha = 0$ or when $\beta \geq 1$, or v_t may initially increase and later decrease in t if α is close to 1, β is close to zero, and c is sufficiently large.
■

IA-A3. Proof of Proposition 2(B)

Proof. In Regime 2, all firms at date-t share the same value v_t regardless of their entry date or type. We first conjecture that no agent would enter as an innovator after date 0 and $k_0 < N$, so that the time path of firm numbers is determined by Eq. (A.1).

Recall in Regime 2, v_t is determined by

$$rv_t = p_t + \gamma (N - k_t) \alpha v_t + \frac{dv_t}{dt} \tag{A.26}$$

$\Rightarrow$

$$\frac{dv_t}{dt} - [r - \gamma (N - k_t) \alpha] v_t = -p_t. \tag{A.27}$$

Defining $z_t = \exp \left(\int - [r - \gamma (N - k_t) \alpha] dt \right)$, we can rewrite Eq. (A.27) as

$$\frac{d}{dt} (z_t v_t) = -z_t p_t,$$

which yields the general solution

$$v_t = z_t^{-1} \int -z_t p_t dt + z_t^{-1} C,$$

where C is the constant of integration.

Given that $k_t = \frac{Ne^{\gamma Nt}}{e^{\gamma Nt}+\frac{N}{k_0}-1}$, we can solve for z_t:

$$z_t = \exp\left(\int -\left[r-\gamma\left(N-k_t\right)\alpha\right]dt\right) = e^{-rt}\left(\frac{N-k_0}{k_0}e^{-\gamma Nt}+1\right)^{-\alpha}.$$

Accordingly,

$$\begin{aligned} v_t &= e^{rt}\left(\frac{N-k_0}{k_0}e^{-\gamma Nt}+1\right)^{\alpha}\int -e^{-rt}\left(\frac{N-k_0}{k_0}e^{-\gamma Nt}+1\right)^{-\alpha}p_t dt \\ &\quad +e^{rt}\left(\frac{N-k_0}{k_0}e^{-\gamma Nt}+1\right)^{\alpha}C, \end{aligned}$$

which requires that $C=0$ because v_t needs to be bounded as $t\to\infty$. We then solve for v_t as follows:

$$\begin{aligned} v_t &= e^{rt}\left(\frac{N-k_0}{k_0}e^{-\gamma Nt}+1\right)^{\alpha}\int -e^{-rt}\left(\frac{N-k_0}{k_0}e^{-\gamma Nt}+1\right)^{-\alpha}p_t dt \\ &= e^{rt}\left(\frac{N-k_0}{k_0}e^{-\gamma Nt}+1\right)^{\alpha}\int_t^{\infty} e^{-rs}\left(\frac{N-k_0}{k_0}e^{-\gamma Ns}+1\right)^{-\alpha}p_s ds \\ &= \int_t^{\infty} e^{-r(s-t)}\left(\frac{\frac{N-k_0}{k_0}e^{-\gamma Ns}+1}{\frac{N-k_0}{k_0}e^{-\gamma Nt}+1}\right)^{-\alpha}p_s ds \\ &= \int_t^{\infty} e^{-r(s-t)}\left(1-\frac{1-e^{-\gamma N(s-t)}}{\frac{k_0}{N-k_0}e^{\gamma Nt}+1}\right)^{-\alpha}p_s ds. \end{aligned} \tag{A.28}$$

Defining $j=s-t$, we can rewrite Eq. (A.28) as

$$v_t = \int_0^{\infty} e^{-rj}\left(1-\frac{1-e^{-\gamma Nj}}{\frac{k_0}{N-k_0}e^{\gamma Nt}+1}\right)^{-\alpha}p_{t+j}dj. \tag{A.29}$$

In the integral, both terms $\left(1-\frac{1-e^{-\gamma Nj}}{\frac{k_0}{N-k_0}e^{\gamma Nt}+1}\right)^{-\alpha}$ and $p_{t+j}=Ak_{t+j}^{-\beta}$ decrease in t. Therefore, v_t decreases with t.

Next, we show v_t-u_t decreases with t. Recall in Regime 2, u_t is determined by

$$ru_t = \gamma k_t\left((1-\alpha)v_t-u_t\right)+\frac{du_t}{dt}, \tag{A.30}$$

which, together with Eq. (A.26), implies that

$$\frac{d\left(v_t-u_t\right)}{dt} = (r+\gamma k_t)(v_t-u_t)-(p_t+\gamma N\alpha v_t). \tag{A.31}$$

Defining $\psi_t \equiv v_t - u_t$, we can rewrite Eq. (A.31) as

$$\frac{d\psi_t}{dt} - (r + \gamma k_t)\psi_t = -(p_t + \gamma N\alpha v_t).$$

Define $z_t = \exp \int -(r + \gamma k_t)dt$. We then have

$$\frac{d(z_t\psi_t)}{dt} = -z_t(p_t + \gamma N\alpha v_t),$$

which yields the general solution

$$\psi_t = z_t^{-1} \int -z_t(p_t + \gamma N\alpha v_t)dt + z_t^{-1}C,$$

where C is the constant of integration.

Given that $k_t = \frac{Ne^{\gamma Nt}}{e^{\gamma Nt} + \frac{N}{k_0} - 1}$, we can solve z_t :

$$z_t = e^{-rt} \frac{k_0}{N - k_0 + k_0 e^{\gamma Nt}}.$$

Again, ψ_t needs to be bounded as $t \to \infty$, so $C = 0$. We then have

$$\begin{aligned} \psi_t &= z_t^{-1} \int -z_t(p_t + \gamma N\alpha v_t)dt \\ &= e^{rt} \frac{N - k_0 + k_0 e^{\gamma Nt}}{k_0} \int -e^{-rt} \frac{k_0}{N - k_0 + k_0 e^{\gamma Nt}} (p_t + \gamma N\alpha v_t)dt \\ &= e^{rt} \frac{N - k_0 + k_0 e^{\gamma Nt}}{k_0} \int_t^\infty e^{-rs} \frac{k_0}{N - k_0 + k_0 e^{\gamma Ns}} (p_s + \gamma N\alpha v_s)ds \\ &= \int_t^\infty e^{-r(s-t)} \left(1 - \frac{e^{\gamma N(s-t)} - 1}{(\frac{N-k_0}{k_0})e^{-\gamma Nt} + e^{\gamma N(s-t)}}\right) (p_s + \gamma N\alpha v_s)ds. \qquad \text{(A.32)} \end{aligned}$$

Define $j = s - t$. Then Eq. (A.32) reads

$$\psi_t = \int_0^\infty e^{-rj} \left(1 - \frac{e^{\gamma Nj} - 1}{(\frac{N-k_0}{k_0})e^{-\gamma Nt} + e^{\gamma Nj}}\right) (p_{t+j} + \gamma N\alpha v_{t+j})dj. \qquad \text{(A.33)}$$

Note that in the integral, the terms $\left(1 - \frac{e^{\gamma Nj}-1}{(\frac{N-k_0}{k_0})e^{-\gamma Nt}+e^{\gamma Nj}}\right)$ and $(p_{t+j} + \gamma N\alpha v_{t+j})$ both decrease in t, hence $\psi_t = v_t - u_t$ strictly decreases with t. Given the free entry condition that $v_0 - u_0 = c$ at date 0, we know $v_t - u_t < c$ at any date $t > 0$, so no agent would enter as an innovator after date 0.

Note that Eq. (A.28) implies that

$$v_t = \int_t^\infty e^{-r(s-t)} \left(\frac{k_s}{k_t}\right)^\alpha p_s ds,$$

so that

$$v_0 = \int_0^\infty e^{-rt} \left(\frac{k_t}{k_0}\right)^\alpha A k_t^{-\beta} dt. \tag{A.34}$$

At date 0, the total industry discounted revenue, $\int_0^\infty e^{-rt} A k_t^{1-\beta} dt$, is shared by the two groups – the initial incumbents k_0 and the outsiders $N - k_0$. With the free entry condition $v_0 - c = u_0$ we have

$$\int_0^\infty e^{-rt} A k_t^{1-\beta} dt = v_0 k_0 + u_0 (N - k_0) = v_0 N - c (N - k_0). \tag{A.35}$$

Plugging Eq. (A.34) into Eq. (A.35) yields Eq. (A.36):

$$\frac{\int_0^\infty e^{-rt} \left(\left(\frac{N}{k_0}\right)^\alpha \left(\frac{N}{k_t}\right)^{1-\alpha} - 1\right) A k_t^{1-\beta} dt}{N - k_0} = c. \tag{A.36}$$

Interior solution (i.e., $k_0^{\mathbf{II}} < N$).—We have shown that $k_0^{\mathbf{II}}$ is determined by Eq. (A.36). Note that as $k_0 \to N$, we have $k_t \to N$. Hence, the numerator and the denominator of the left hand side of Eq. (A.36) both go to 0 as $k_0 \to N$. Applying L'Hôpital's rule, the left hand side of Eq. (A.36) converges to $\frac{r+\alpha\gamma N}{r(r+\gamma N)} A N^{-\beta}$ as $k_0 \to N$. Proposition 3(A) shows that $dk_0^{\mathbf{II}}/dc < 0$. Therefore, the model has an interior solution $k_0^{\mathbf{II}} < N$ in Regime 2 iff condition (A.23) holds, i.e., iff

$$c > \frac{r + \alpha\gamma N}{r(r + \gamma N)} A N^{-\beta}.$$

■

IA-A4. Proof of Proposition 3

Proof. (A) We first prove that $k_0^{\mathbf{I}}$ increases with α and A, but decreases with c and r. Rewrite Eq. (A.22) as

$$G = \int_0^\infty e^{-rt} F(t; k_0) dt - c = 0,$$

where

$$F(t; k_0) = \frac{1}{N - k_0} \left(\alpha \frac{N}{k_0} + (1 - \alpha) \frac{N}{k_t} - 1\right) A k_t^{1-\beta},$$

and

$$k_t = \frac{N e^{\gamma N t}}{e^{\gamma N t} + \frac{N}{k_0} - 1}.$$

We verify that $\frac{\partial F(t;k_0)}{\partial k_0} < 0$, so $\frac{\partial G}{\partial k_0} < 0$. Following that, we can prove

$$\frac{\partial k_0^{\mathbf{I}}}{\partial \alpha} = -\frac{\partial G/\partial \alpha}{\partial G/\partial k_0^{\mathbf{I}}} > 0; \quad \frac{\partial k_0^{\mathbf{I}}}{\partial A} = -\frac{\partial G/\partial A}{\partial G/\partial k_0^{\mathbf{I}}} > 0 \ ;$$

$$\frac{\partial k_0^{\mathbf{I}}}{\partial c} = -\frac{\partial G/\partial c}{\partial G/\partial k_0^{\mathbf{I}}} < 0; \frac{\partial k_0^{\mathbf{I}}}{\partial r} = -\frac{\partial G/\partial r}{\partial G/\partial k_0^{\mathbf{I}}} < 0.$$

Similarly, in Eq. (A.36), we can prove that $k_0^{\mathbf{II}}$ increases with α and A, but decreases with c and r.

(B) First, it is straightforward to verify Eqs. (A.22) and (A.36) are identical when $\alpha \in \{0, 1\}$, so $k_0^{\mathbf{I}} = k_0^{\mathbf{II}}$.

Second, for any $\alpha \in (0, 1)$ and $t > 0$, we can apply the mean-value theorem to derive

$$\left(\frac{N}{k_0}\right)^{\alpha} \left(\frac{N}{k_t}\right)^{1-\alpha} = \left(\frac{N}{k_t}\right) \left(\frac{k_t}{k_0}\right)^{\alpha} = \left(\frac{N}{k_t}\right) \left(1 + \alpha \left(\frac{k'}{k_0}\right)^{\alpha-1} \frac{(k_t - k_0)}{k_0}\right)$$

where $k_t > k' > k_0$. Therefore,

$$\left(\frac{N}{k_0}\right)^{\alpha} \left(\frac{N}{k_t}\right)^{1-\alpha} < \left(\frac{N}{k_t}\right) \left(1 + \alpha \frac{(k_t - k_0)}{k_0}\right) = \alpha \frac{N}{k_0} + (1-\alpha) \frac{N}{k_t}. \tag{A.37}$$

Given that $k_0^{\mathbf{I}}$ and $k_t^{\mathbf{I}}$ satisfy Eq. (A.22) so that

$$\int_0^{\infty} e^{-rt} \frac{1}{N - k_0^{\mathbf{I}}} \left(\alpha \frac{N}{k_0^{\mathbf{I}}} + (1-\alpha) \frac{N}{k_t^{\mathbf{I}}} - 1\right) A (k_t^{\mathbf{I}})^{1-\beta} dt = c,$$

the same $k_0^{\mathbf{I}}$ and $k_t^{\mathbf{I}}$ would not satisfy Eq. (A.36). Instead,

$$\int_0^{\infty} e^{-rt} \frac{1}{N - k_0^{\mathbf{I}}} \left(\left(\frac{N}{k_0^{\mathbf{I}}}\right)^{\alpha} \left(\frac{N}{k_t^{\mathbf{I}}}\right)^{1-\alpha} - 1\right) A (k_t^{\mathbf{I}})^{1-\beta} dt < c, \tag{A.38}$$

given the inequality (A.37).

The left-hand side of Eq. (A.36) can be written as

$$LHS = \int_0^{\infty} e^{-rt} F(t; k_0) dt$$

where

$$F(t; k_0) = \frac{1}{N - k_0} \left(\left(\frac{N}{k_0}\right)^{\alpha} \left(\frac{N}{k_t}\right)^{1-\alpha} - 1\right) A k_t^{1-\beta},$$

and

$$k_t = \frac{N e^{\gamma N t}}{e^{\gamma N t} + \frac{N}{k_0} - 1}.$$

We verify that $\frac{\partial F(t;k_0)}{\partial k_0} < 0$, so $\frac{\partial LHS}{\partial k_0} < 0$. Therefore, the solution $k_0^{\mathbf{II}}$ that satisfies Eq. (A.36) has to satisfy $k_0^{\mathbf{II}} < k_0^{\mathbf{I}}$. ■

IA-A5. Proof of Proposition 4

Proof. We prove the following results:

(A) For inelastic or unit elastic demand (i.e., $\beta \geq 1$),

$$k_0^{\mathbf{I}} \text{ and } k_0^{\mathbf{II}} \begin{cases} \text{decrease with } \gamma \text{ when } \beta \geq 1 > \alpha, \\ \text{do not vary with } \gamma \text{ when } \beta = \alpha = 1. \end{cases}$$

(B) For elastic demand (i.e., $\beta < 1$),

$$k_0^{\mathbf{I}} \begin{cases} \text{decreases with } \gamma \text{ if } 0 \leq \alpha < \frac{1}{\beta + \frac{N}{k_0^{\mathbf{I}}}(1-\beta)} < 1, \\ \text{increases with } \gamma \text{ if } 1 \geq \alpha > \beta + \frac{k_0^{\mathbf{I}}}{N}(1-\beta) > 0, \end{cases}$$

and

$$k_0^{\mathbf{II}} \begin{cases} \text{decreases with } \gamma \text{ if } 0 \leq \alpha < \beta + \frac{k_0^{\mathbf{II}}}{N}(1-\beta) < 1, \\ \text{increases with } \gamma \text{ if } 1 \geq \alpha > \beta + \left(\frac{k_0^{\mathbf{II}}}{N}\right)^{\alpha}(1-\beta). \end{cases}$$

We first prove the results for Regime 1. Rewrite Eq. (A.22) as

$$G = \int_0^{\infty} e^{-rt} F(t;\gamma) dt - \frac{c}{AN^{1-\beta}} = 0,$$

where

$$F(t;\gamma) = \frac{1}{(N-k_0)} \left(\alpha \frac{N}{k_0} - 1 + (1-\alpha)\left(1 + (\frac{N}{k_0} - 1)e^{-\gamma N t}\right)\right)\left(1 + (\frac{N}{k_0} - 1)e^{-\gamma N t}\right)^{\beta - 1}.$$

Note that

$$\frac{F(t;\gamma)}{\partial \gamma} \propto \left\{ \begin{array}{c} -(1-\alpha)\left(1 + (\frac{N}{k_0} - 1)e^{-\gamma N t}\right) \\ -\left(\alpha \frac{N}{k_0} - 1 + (1-\alpha)\left(1 + (\frac{N}{k_0} - 1)e^{-\gamma N t}\right)\right)(\beta - 1) \end{array} \right\}.$$

Therefore, for inelastic or unit elastic demand (i.e., $\beta \geq 1$), we have $\partial G/\partial \gamma < 0$ (except $\partial G/\partial \gamma = 0$ when $\beta = \alpha = 1$). Recall that $\partial G/\partial k_0 < 0$ from the proof of Proposition 3. We then derive

$$\frac{\partial k_0^{\mathbf{I}}}{\partial \gamma} = -\frac{\partial G/\partial \gamma}{\partial G/\partial k_0^{\mathbf{I}}} < 0 \text{ (except } \frac{\partial k_0^{\mathbf{I}}}{\partial \gamma} = 0 \text{ when } \beta = \alpha = 1).$$

We now consider the case of elastic demand (i.e., $\beta < 1$). Note that

$$\frac{F(t;\gamma)}{\partial \gamma} < 0 \Longleftrightarrow (1-\alpha)\beta\left(1 + (\frac{N}{k_0} - 1)e^{-\gamma N t}\right) > (\alpha \frac{N}{k_0} - 1)(1-\beta),$$

which holds for any $t \geq 0$ if

$$(1-\alpha)\,\beta > (\alpha\frac{N}{k_0} - 1)(1-\beta) \Longleftrightarrow \alpha < \frac{1}{\frac{N}{k_0}(1-\beta)+\beta}.$$

Therefore, we have

$$\frac{\partial k_0^{\mathbf{I}}}{\partial \gamma} = -\frac{\partial G/\partial \gamma}{\partial G/\partial k_0^{\mathbf{I}}} < 0 \quad \text{if} \quad \alpha < \frac{1}{\frac{N}{k_0^{\mathbf{I}}}(1-\beta)+\beta}.$$

Similarly,

$$\frac{F(t;\gamma)}{\partial \gamma} > 0 \iff (1-\alpha)\left(1 + (\frac{N}{k_0} - 1)e^{-\gamma N t}\right)\beta < (\alpha\frac{N}{k_0} - 1)(1-\beta),$$

which holds for any $t \geq 0$ if

$$(1-\alpha)\left(\frac{N}{k_0}\right)\beta < (\alpha\frac{N}{k_0} - 1)(1-\beta) \Longleftrightarrow \alpha > \beta + \frac{k_0}{N}(1-\beta).$$

Therefore, we have

$$\frac{\partial k_0^{\mathbf{I}}}{\partial \gamma} = -\frac{\partial G/\partial \gamma}{\partial G/\partial k_0^{\mathbf{I}}} > 0 \quad \text{if} \quad \alpha > \beta + \frac{k_0^{\mathbf{I}}}{N}(1-\beta).$$

Similarly, with Eq. (A.36), we can prove the results for Regime 2. ■

IA-A6. Proof of Proposition 6

Proof. We first consider the case $\beta < 1$. For any date $\tau \geq 0$, if no further innovators enter, the number of firms at any date $t \geq \tau$ is

$$k_t = \frac{N e^{\gamma N (t-\tau)}}{e^{\gamma N (t-\tau)} + \frac{N}{k_\tau} - 1} \qquad \text{for } t \geq \tau. \tag{A.39}$$

As of date τ, the social return to innovation is

$$SR_\tau = \int_\tau^\infty e^{-r(t-\tau)} \frac{A}{1-\beta} k_t^{1-\beta} dt. \tag{A.40}$$

The current cost of innovation is c per unit, and its marginal social return (even if no further innovations are made) is

$$\frac{\partial SR_\tau}{\partial k_\tau} = \int_\tau^\infty e^{-r(t-\tau)} A k_t^{-\beta} \frac{\partial k_t}{\partial k_\tau} dt, \tag{A.41}$$

where

$$\frac{\partial k_t}{\partial k_\tau} = \frac{N^2 e^{\gamma N(t-\tau)}}{\left(N + \left(e^{\gamma N(t-\tau)} - 1\right) k_\tau\right)^2}, \tag{A.42}$$

which is strictly decreasing in k_τ. And since k_t is increasing in t, $e^{-r(t-\tau)} k_t^{-\beta} \frac{\partial k_t}{\partial k_\tau}$ is also decreasing in k_τ. Therefore, $\frac{\partial SR_\tau}{\partial k_\tau}$ is strictly decreasing in k_τ and so if at date 0 k_0 is chosen so that $\frac{\partial SR_0}{\partial k_0} = c$, thereafter $\frac{\partial SR_\tau}{\partial k_\tau} < c$. Similarly, we can prove the result holds for $\beta \geq 1$. Hence, it is socially optimal to innovate only at date 0.

The planner should then simply choose the scalar k_0^* to maximize social welfare:

$$\max_{k_0} \left\{ \int_0^\infty e^{-rt} U(k_t)\, dt - c k_0 \right\}, \tag{A.43}$$

subject to Eq. (A.1). The objective function is strictly concave in k_0, and so the socially optimal mass of innovators k_0^* solves the first-order condition

$$\int_0^\infty e^{-(r+\gamma N)t} \left(\frac{k_t}{k_0}\right)^2 A k_t^{-\beta} dt = c. \tag{A.44}$$

for any $\beta > 0$.

Interior solution (i.e., $k_0^* < N$).—Given that the social welfare function (A.43) is strictly concave in k_0, for $k_0^* < N$ to hold, one needs

$$\frac{d\left\{\int_0^\infty e^{-rt} U(k_t)\, dt - c k_0\right\}}{dk_0}\Big|_{k_0 = N} < 0,$$

which yields condition (A.45):

$$c > \frac{A N^{-\beta}}{r + \gamma N}. \tag{A.45}$$

Comparative statics.—Rewriting Eq. (A.44), we define

$$G = \int_0^\infty e^{-(r+\gamma N)t} \frac{A}{k_0^{*2}} \left(\frac{N e^{\gamma N t}}{e^{\gamma N t} + (\frac{N}{k_0^*} - 1)}\right)^{2-\beta} dt - c = 0.$$

It follows that

$$\frac{\partial k_0^*}{\partial A} = -\frac{\partial G/\partial A}{\partial G/\partial k_0^*} > 0.$$

Similarly, we can prove $\partial k_0^*/\partial c < 0$ and $\partial k_0^*/\partial r < 0$.

The sign of $\partial k_0^*/\partial \gamma$ depends on $\partial G/\partial \gamma$ and requires some discussions.

$$\frac{\partial G}{\partial \gamma} \propto \int_0^\infty e^{-rt} \left\{ \begin{array}{c} (1-\beta)\left(1 + (\frac{N}{k_0^*} - 1)e^{-\gamma N t}\right)^{\beta-2} \left(e^{\gamma N t} + (\frac{N}{k_0^*} - 1)\right)^{-1} (\frac{N}{k_0^*} - 1)e^{-\gamma N t} N t \\ -\left(1 + (\frac{N}{k_0^*} - 1)e^{-\gamma N t}\right)^{\beta-1} \left(e^{\gamma N t} + (\frac{N}{k_0^*} - 1)\right)^{-2} e^{\gamma N t} N t \end{array} \right\} dt.$$

This implies $\frac{\partial G}{\partial \gamma} < 0$ if $\beta \geq 1$. When $\beta < 1$, the sign of $\frac{\partial G}{\partial \gamma} < 0$ if $(1-\beta)(\frac{N}{k_0^*}-1) < e^{\gamma N t}$. A sufficient condition is that

$$(1-\beta)(\frac{N}{k_0^*}-1) < 1,$$

$\Longleftrightarrow$

$$\beta > 1 - \frac{k_0^*}{N - k_0^*}.$$

The social planner's problem (A.43) requires $\frac{\partial W_0^*}{\partial k_0} = 0$. Applying the envelope theorem, we have

$$\frac{dW_0^*}{d\gamma} = \frac{\partial W_0^*}{\partial k_0}\frac{\partial k_0}{\partial \gamma} + \frac{\partial W_0^*}{\partial \gamma} = \int_0^\infty e^{-rt}\frac{\partial U(k_t)}{\partial \gamma}dt > 0$$

for any $\beta > 0$. Similarly, we can prove that $dW_0^*/dA > 0$, $dW_0^*/dc < 0$, and $dW_0^*/dr < 0$. ■

IA-A7. Socially optimal subsidy or tax

Whenever $\alpha \neq \alpha^*$ in each regime, the planner can use a subsidy s or tax ($s < 0$) to achieve the social optimum. Denote the socially optimal subsidy for Regimes 1 and 2 by $s^{\mathbf{I}*}$ and $s^{\mathbf{II}*}$, respectively. We obtain the following result:

Proposition IA-1: *Social optimum implies* $s^{\mathbf{I}*} < s^{\mathbf{II}*}$ *for* $\alpha \in (0,1)$, $s^{\mathbf{I}*} = s^{\mathbf{II}*} > 0$ *for* $\alpha = 0$, *and* $s^{\mathbf{I}*} = s^{\mathbf{II}*} < 0$ *for* $\alpha = 1$.

Proof. In Regime 1, for a given value of α, Eq. (A.22) yields the equilibrium entry of innovators $k_0^{\mathbf{I}}$. Proposition 7 suggests that whenever $\alpha \neq \alpha^{\mathbf{I}*}$, the number of innovators $k_0^{\mathbf{I}}$ from Eq. (A.22) differs from the social optimum k_0^*, in which case offering an innovation subsidy (or tax) to adjust the innovation cost c would help restore the social optimum. This implies that k_0^* can be achieved by a subsidy or tax $s^{\mathbf{I}*}$ as follows:

$$\frac{1}{N-k_0^*}\int_0^\infty e^{-rt}\left(\alpha\frac{N}{k_0^*} + (1-\alpha)\frac{N}{k_t^*} - 1\right)Ak_t^{*1-\beta}dt \;=\; c - s^{\mathbf{I}*}.$$

The same logic applies to Regime 2 that

$$\frac{1}{N-k_0^*}\int_0^\infty e^{-rt}\left(\left(\frac{N}{k_0^*}\right)^\alpha\left(\frac{N}{k_t^*}\right)^{1-\alpha} - 1\right)Ak_t^{*1-\beta}dt \;=\; c - s^{\mathbf{II}*}.$$

Recall that when $\alpha \in \{0,1\}$, Regimes 1 and 2 coincide. When $\alpha = 0$, both regimes would need more entry of innovators, so a positive subsidy is needed to achieve that, and when $\alpha = 1$, a negative subsidy (tax) is needed. Moreover, for $\alpha \in (0,1)$, according to Proposition 3(B), if a given pair of α and $(c - s^{\mathbf{I}*})$ lead to the social optimum k_0^* in Regime 1, the same parameter values would result in $k_0^{\mathbf{II}} < k_0^*$ in Regime 2. Therefore, a higher subsidy (or a smaller tax) $s^{\mathbf{II}*}$ is needed for adjusting c to achieve k_0^* in Regime 2 given that $k_0^{\mathbf{II}}$ decreases with c as shown by Proposition 3(A). ■

IA-A8. Proof of Proposition 7

Proof. Given that condition (A.45) holds, we have $k_0^* < N$. Proposition 6 shows that the socially optimal innovation k_0^* satisfies Eq. (A.44) that

$$\int_0^\infty e^{-(r+\gamma N)t} \left(\frac{k_t}{k_0}\right)^2 A k_t^{-\beta} dt = c.$$

When $\alpha = 0$, condition (A.45) is equivalent to condition (A.23). Accordingly, we have $k_0^{\mathbf{I}} = k_0^{\mathbf{II}} < N$ and they satisfy Eq. (A.22) or Eq. (A.36) evaluated at $\alpha = 0$ so that

$$\int_0^\infty e^{-rt} \frac{1}{N - k_0} \left(\frac{N}{k_t} - 1\right) A k_t^{1-\beta} dt = c$$

$\Longleftrightarrow$

$$\int_0^\infty e^{-(r+\gamma N)t} \left(\frac{k_0}{k_t}\right) \left(\frac{k_t}{k_0}\right)^2 A k_t^{-\beta} dt = c. \tag{A.46}$$

Note that the left hand side of Eq. (A.46) is smaller than the left hand side of Eq. (A.44) given that $\frac{k_0}{k_t} < 1$ for $t > 0$. Therefore, the solution $k_0^{\mathbf{I}} = k_0^{\mathbf{II}}$ to Eq. (A.46) would cause the left hand side of Eq. (A.44) to exceed c. Because the left hand side of Eq. (A.44) decreases with k_0, this implies $k_0^{\mathbf{I}} = k_0^{\mathbf{II}} < k_0^* < N$ for $\alpha = 0$.

Given that the right-hand side of condition (A.23) increases with α, if there existed an $\alpha' \in (0, 1]$ such that the inequality (A.23) becomes an equality (i.e., $c = \frac{r+\alpha'\gamma N}{r(r+\gamma N)} A N^{-\beta}$), we would then have $k^* < k_0^{\mathbf{I}} = k_0^{\mathbf{II}} = N$ for $\alpha = \alpha'$. In this case, the planner could choose optimal shares $\alpha^{\mathbf{I}*}$ and $\alpha^{\mathbf{II}*}$ such that $0 < \alpha^{\mathbf{I}*} < \alpha^{\mathbf{II}*} < \alpha' \leq 1$ to achieve the socially optimal k^* in Regimes 1 and 2, respectively. Note that $\alpha^{\mathbf{I}*} < \alpha^{\mathbf{II}*}$ results from Proposition 3: If a value of α led to $k_0^{\mathbf{I}} = k^*$ in Regime 1, the same α would lead to $k_0^{\mathbf{II}} < k^*$ in Regime 2, so a larger value of α would be needed to achieve k^*in Regime 2.

Alternatively, if the inequality (A.23) holds for any $\alpha \leq 1$, we have $k_0^{\mathbf{I}} = k_0^{\mathbf{II}} < N$ for $\alpha = 1$ and they satisfy Eq. (A.22) or Eq. (A.36) evaluated at $\alpha = 1$ so that

$$\int_0^\infty e^{-(r+\gamma N)t} \left(\frac{k_t}{k_0}\right)^2 A k_t^{-\beta} dt = c$$

$\Longleftrightarrow$

$$\int_0^\infty e^{-(r+\gamma N)t} \left(\frac{k_0}{N} e^{\gamma N t} - \frac{k_0}{N} + 1\right) \left(\frac{k_t}{k_0}\right)^2 A k_t^{-\beta} dt = c. \tag{A.47}$$

Note that the left hand side of Eq. (A.47) is greater than the left hand side of Eq. (A.44) given that $\frac{k_0}{N}e^{\gamma N t} - \frac{k_0}{N} + 1 > 1$ for $t > 0$. Therefore, the solution $k_0^{\mathbf{I}} = k_0^{\mathbf{II}}$ to Eq. (A.47) would cause the left hand side of Eq. (A.44) to be smaller than c. Because the left hand side of Eq. (A.44) decreases with k_0, this implies $k_0^* < k_0^{\mathbf{I}} = k_0^{\mathbf{II}} < N$ for $\alpha = 1$. In this case, the social planner could choose optimal shares $\alpha^{\mathbf{I}*}$ and $\alpha^{\mathbf{II}*}$ such that $0 < \alpha^{\mathbf{I}*} < \alpha^{\mathbf{II}*} < 1$ to achieve the socially optimal k^* in Regimes 1 and 2, respectively. ■

IA-A9. Industry licensing revenue in Regimes 1 and 2

In Regime 1, Eq. (A.12) shows that an innovator' value at date 0 is

$$v_0^{\mathbf{I}} = \int_0^\infty e^{-rt}\left(\frac{1}{k_t} + \frac{\alpha}{k_0}(\frac{k_t - k_0}{k_t})\right) A k_t^{1-\beta} dt.$$

Excluding the revenues from selling goods, the present value of licensing revenue is

$$l_0^{\mathbf{I}} = v_0^{\mathbf{I}} - \int_0^\infty e^{-rt} A k_t^{-\beta} dt = \int_0^\infty e^{-rt}\left(\alpha(\frac{k_t}{k_0} - 1)\right) A k_t^{-\beta} dt.$$

Therefore, the present value of total licensing revenue paid to innovators in the industry is

$$L_0^{\mathbf{I}} = k_0 l_0^{\mathbf{I}} = \int_0^\infty e^{-rt}\alpha(k_t - k_0) A k_t^{-\beta} dt. \tag{A.48}$$

In Regime 2, Eq. (A.29) shows that an innovator's value at date 0 is

$$v_0^{\mathbf{II}} = \int_0^\infty e^{-rt}\left(\frac{k_t}{k_0}\right)^\alpha A k_t^{-\beta} dt.$$

Excluding the revenue from selling goods, the present value of licensing revenue is

$$l_0^{\mathbf{II}} = v_0^{\mathbf{II}} - \int_0^\infty e^{-rt} A k_t^{-\beta} dt = \int_0^\infty e^{-rt}\left(\left(\frac{k_t}{k_0}\right)^\alpha - 1\right) A k_t^{-\beta} dt.$$

Therefore, the present value of total licensing revenue paid to innovators in the industry is

$$L_0^{\mathbf{II}} = k_0 l_0^{\mathbf{II}} = \int_0^\infty e^{-rt}\left(\left(\frac{k_t}{k_0}\right)^\alpha k_0 - k_0\right) A k_t^{-\beta} dt. \tag{A.49}$$

IA-A10. Socially optimal diffusion

Assume condition (A.23) holds so that not all agents enter at date 0 (i.e., $k_0 < N$). We now prove the welfare results for the unit demand elasticity case (i.e., $\beta = 1$).

Proposition IA-2. *(A) For $\beta = 1$ and all $\alpha \in [0,1]$, social welfare always increases with the diffusion rate γ in Regime 1. (B) For $\beta = 1$ and $\alpha \in \{0,1\}$, Regimes 1 and 2 coincide and social welfare always increases with the diffusion rate γ.*

Proof. (A) With $\beta = 1$ under Regime 1, Eq. (A.22) simplifies to

$$k_0^{\mathbf{I}} = \frac{A(\alpha\gamma N + r)}{cr(r + \gamma N)}. \tag{A.50}$$

Given Eq. (A.50) and $\beta = 1$, social surplus is

$$W_0 = -\frac{A(1-\alpha)}{(r+\gamma N)} + A\int_0^\infty e^{-rt}\left[\gamma N t - \ln\left(e^{\gamma N t} + \frac{Ncr(r+\gamma N)}{A(\alpha\gamma N + r)} - 1\right)\right] dt + \text{constant}.$$

This suggests that

$$\frac{dW_0}{d\gamma} = \frac{A(1-\alpha)N}{(r+\gamma N)^2} + A\int_0^\infty e^{-rt} N t dt - A\int_0^\infty e^{-rt}\left[\frac{Nte^{\gamma N t} + \frac{crN^2}{A(\alpha\gamma N + r)} - \frac{\alpha cr(r+\gamma N)N^2}{A(\alpha\gamma N+r)^2}}{e^{\gamma N t} + \frac{Ncr(r+\gamma N)}{A(\alpha\gamma N + r)} - 1}\right] dt.$$

We then verify that

$$\begin{aligned}
& d\left[\frac{Nte^{\gamma N t} + \frac{crN^2}{A(\alpha\gamma N + r)} - \frac{\alpha cr(r+\gamma N)N^2}{A(\alpha\gamma N+r)^2}}{e^{\gamma N t} + \frac{Ncr(r+\gamma N)}{A(\alpha\gamma N + r)} - 1}\right] / dc \\
\propto \quad & \left(\frac{(1-\alpha)r}{(\alpha\gamma N + r)}\right)\left(e^{\gamma N t} + \frac{Ncr(r+\gamma N)}{A(\alpha\gamma N + r)} - 1\right) \\
& - \left(te^{\gamma N t} + \frac{crN}{A(\alpha\gamma N + r)} - \frac{\alpha cr(r+\gamma N)N}{A(\alpha\gamma N + r)^2}\right)(r+\gamma N) \\
< \quad & 0
\end{aligned}$$

for any $t > 0$. Equation (A.50) implies that $c > \frac{A(\alpha\gamma N + r)}{Nr(r+\gamma N)}$ is needed for $k_0^{\mathbf{I}}$ to be interior solution (i.e., $k_0^{\mathbf{I}} < N$). We then have

$$A\int_0^\infty e^{-rt}\left[\frac{Nte^{\gamma N t} + \frac{crN^2}{A(\alpha\gamma N + r)} - \frac{\alpha cr(r+\gamma N)N^2}{A(\alpha\gamma N+r)^2}}{e^{\gamma N t} + \frac{Ncr(r+\gamma N)}{A(\alpha\gamma N + r)} - 1}\right] dt < A\int_0^\infty e^{-rt}\left[\frac{Nte^{\gamma N t} + \frac{N}{r+\gamma N} - \frac{\alpha N}{(\alpha\gamma N + r)}}{e^{\gamma N t}}\right] dt.$$

Therefore,

$$\begin{aligned}
\frac{dW_0}{d\gamma} \quad > \quad & \frac{A(1-\alpha)N}{(r+\gamma N)^2} + A\int_0^\infty e^{-rt}Ntdt - A\int_0^\infty e^{-rt}\left[\frac{Nte^{\gamma N t} + \frac{N}{r+\gamma N} - \frac{\alpha N}{(\alpha\gamma N + r)}}{e^{\gamma N t}}\right] dt \\
= \quad & \frac{A(1-\alpha)N}{(r+\gamma N)^2} - \frac{A}{(r+\gamma N)}\left(\frac{N}{r+\gamma N} - \frac{\alpha N}{(\alpha\gamma N + r)}\right) \\
= \quad & \frac{A\alpha N}{(r+\gamma N)}\left(\frac{1}{r+\alpha\gamma N} - \frac{1}{r+\gamma N}\right) \geq 0 \text{ for any } \alpha \in [0,1].
\end{aligned}$$

(B) Proposition IA-2(B) is an application of Proposition IA-2(A) by taking $\alpha = 0$ or $\alpha = 1$, and Regimes 1 and 2 coincide in those cases (cf. Proposition 3). ■

IA-A11. Solutions to the extended model in Section 5.4.3 where $c^m > 0$

We assume an entry cost c^m for imitators, where $0 < c^m < c$. Following similar proofs of Propositions 1, 2, and 6, we can verify that innovators enter only at date 0. In the following, we show the solutions that determine the mass of innovators $k_0 \in (0, N)$ in each scenario:

$$\text{No IP protection} \ : \ \frac{1}{N-k_0}\int_0^\infty e^{-rt}\left(N-k_t\right)\left(Ak_t^{-\beta}+c^m\gamma k_t\right)dt = \ c; \tag{A.51}$$

$$\text{Regime 1} \ : \ \frac{\int_0^\infty e^{-rt}\left\{\begin{array}{c}\left(\frac{\alpha N}{k_0}+(1-\alpha)\frac{N}{k_t}-1\right)Ak_t^{1-\beta}\\ +\left(1-\frac{N}{k_0}\alpha\right)c^m\gamma k_t\left(N-k_t\right)\end{array}\right\}dt}{(N-k_0)} = c; \tag{A.52}$$

$$\text{Regime 2} \ : \ \frac{\int_0^\infty e^{-rt}\left\{\begin{array}{c}\left(\left(\frac{N}{k_0}\right)^{\alpha}\left(\frac{N}{k_t}\right)^{1-\alpha}-1\right)Ak_t^{1-\beta}\\ +c^m\gamma\left(N-k_t\right)\left(k_t-\alpha N\left(\frac{k_t}{k_0}\right)^{\alpha}\right)\end{array}\right\}dt}{N-k_0} = c; \tag{A.53}$$

$$\text{Social optimum} \ : \ \int_0^\infty e^{-(r+\gamma N)t}\left(\frac{k_t}{k_0}\right)^2\left(Ak_t^{-\beta}-c^m\gamma\left(N-2k_t\right)\right)dt = \ c. \tag{A.54}$$

Note that Regimes 1 and 2 coincide with the no-IP-protection case if copying ideas is free, so Eqs. (A.52) and (A.53) coincide with Eq. (A.51) when $\alpha = 0$. Moreover, Eqs. (A.51)-(A.54) coincide with their counterparts in Sections 2-4 in the paper when $c^m = 0$.

No IP protection.—Consider the case where imitators can copy ideas for free but they need to pay a cost $c^m < c$ to enter the industry. The value of an innovator at date 0, denoted by v_0, is

$$v_0 = \int_0^\infty e^{-rt}Ak_t^{-\beta}dt, \tag{A.55}$$

where k_t follows the logistic process given by Eq. (A.1). The total value of outsiders $u_0(N-k_0)$ at date 0 equals the imitators' share of the total discounted industry revenues minus their discounted entry costs. Therefore, we have

$$u_0(N-k_0) = \int_0^\infty e^{-rt}\left[\left(k_t-k_0\right)Ak_t^{-\beta}-c^m\gamma k_t\left(N-k_t\right)\right]dt. \tag{A.56}$$

With Eqs. (A.55) and (A.56), the free entry condition $v_0 - u_0 = c$ then yields Eq. (A.51) that pins down the mass of innovators k_0 at date 0.

Regimes 1.—In Regime 1, the value of an innovator who enters at date 0 is

$$v_0 = \int_0^\infty e^{-rt} \left\{ \left(\frac{1}{k_t} + \frac{\alpha}{k_0} (\frac{k_t - k_0}{k_t}) \right) A k_t^{1-\beta} - \frac{\alpha \gamma k_t (N - k_t) c^m}{k_0} \right\} dt. \tag{A.57}$$

At date 0, the total industry discounted revenue, $\int_0^\infty e^{-rt} \left[A k_t^{1-\beta} - c^m \gamma k_t (N - k_t) \right] dt$, is shared by the two groups – the initial entrants k_0 and the outsiders $N - k_0$. With the free entry condition $v_0 - c = u_0$, we have

$$\int_0^\infty e^{-rt} \left[A k_t^{1-\beta} - c^m \gamma k_t (N - k_t) \right] dt = v_0 k_0 + u_0 (N - k_0) = v_0 N - c (N - k_0). \tag{A.58}$$

Plugging Eq. (A.57) into Eq. (A.58) yields Eq. (A.52) which pins down the mass of innovators $k_0^{\mathbf{I}}$ at date 0.

Regime 2.—In Regime 2, any producer (innovator or imitator) that sells an idea can keep the proceeds. Then all producers now have the same value $v_t = \omega_t$. Imitators need to pay an imitation cost $c^m < c$ to enter the industry and also pay $\alpha(v_t - c^m)$ to purchase the idea. The revenue from a single idea sale is $\alpha(v_t - c^m)$, and total revenue from idea sales, $\gamma k_t (N - k_t) \alpha(v_t - c^m)$, is now shared by all the k_t producers. Therefore, v_t now satisfies

$$r v_t = p_t + \gamma (N - k_t) \alpha(v_t - c^m) + \frac{d v_t}{dt}, \tag{A.59}$$

which has the solution

$$v_t = \int_0^\infty e^{-rj} \left(1 - \frac{1 - e^{-\gamma N j}}{\frac{k_0}{N - k_0} e^{\gamma N t} + 1} \right)^{-\alpha} \left[p_{t+j} - \gamma (N - k_{t+j}) \alpha c^m \right] dj. \tag{A.60}$$

Eq. (A.60) implies that

$$v_0 = \int_0^\infty e^{-rt} \left(\frac{k_t}{k_0} \right)^\alpha \left[A k_t^{-\beta} - \gamma (N - k_t) \alpha c^m \right] dt \tag{A.61}$$

At date 0, the total industry discounted revenue, $\int_0^\infty e^{-rt} \left[A k_t^{1-\beta} - c^m \gamma k_t (N - k_t) \right] dt$, is shared by the two groups – the initial entrants k_0 and the outsiders $N - k_0$. With the free entry condition $v_0 - c = u_0$, we have

$$\int_0^\infty e^{-rt} \left[A k_t^{1-\beta} - c^m \gamma k_t (N - k_t) \right] dt = v_0 k_0 + u_0 (N - k_0) = v_0 N - c (N - k_0). \tag{A.62}$$

Plugging Eq. (A.61) into Eq. (A.62) yields Eq. (A.53) which pins down the mass of innovators $k_0^{\mathbf{II}}$ at date 0.

Social optimum.—Given imitators' entry cost $c^m > 0$, the planner would maximize social welfare W_0 given by

$$W_0 = \int_0^\infty e^{-rt} \left[U(k_t) - c^m \gamma k_t (N - k_t) \right] dt - ck_0, \tag{A.63}$$

where k_t is given by Eq. (A.1). The first order condition then yields Eq. (A.54) which pins down the mass of innovators k_0^* at date 0.

IA-A12. Time-varying demand and firm productivity

We now extend our model to consider an industry with time-varying demand and firm productivity. Specifically, we assume each firm's production capacity q_t rises over time, and the total industry output at date t is $q_t k_t$. We also assume that the market size parameter A_t grows with time, and the product price at date t is

$$p_t = A_t (q_t k_t)^{-\beta}.$$

Our findings in Proposition 2 in the paper can then be extended as follows.

Proposition IA-3. *With time-varying A_t and q_t, as long as the profit per firm $p_t q_t = A_t q_t^{1-\beta} k_t^{-\beta}$ does not increase in t, market equilibrium yields:*
(A) In Regime 1, innovators enter only at date 0 and $k_0^{\mathbf{I}} \in (0, N)$ solves

$$\underbrace{\frac{1}{N-k_0} \int_0^\infty e^{-rt} \left(\alpha \frac{N}{k_0} + (1-\alpha) \frac{N}{k_t} - 1 \right) A_t (q_t k_t)^{1-\beta} dt}_{v_0 - u_0} \;=\; c; \tag{A.64}$$

(B) In Regime 2, innovators enter only at date 0 and $k_0^{\mathbf{II}} \in (0, N)$ solves

$$\underbrace{\frac{1}{N-k_0} \int_0^\infty e^{-rt} \left(\left(\frac{N}{k_0} \right)^\alpha \left(\frac{N}{k_t} \right)^{1-\alpha} - 1 \right) A_t (q_t k_t)^{1-\beta} dt}_{v_0 - u_0} \;=\; c; \tag{A.65}$$

where, in each regime, $0 \le \alpha \le 1$ and k_t follows logistic diffusion (A.1).

Proof. (A) In Regime 1, with time-varying A_t and q_t, we first consider a measure-zero innovator who deviates from the equilibrium and enters at date $\tau > 0$. Following a proof similar to that of Proposition 2(A), the value of such a firm at his entry date τ would be

$$v_\tau^\tau = \int_\tau^\infty e^{-r(t-\tau)} \left(\frac{1}{k_t} + \frac{\alpha}{k_\tau} \left(\frac{k_t - k_\tau}{k_t} \right) \right) A_t (q_t k_t)^{1-\beta} dt. \tag{A.66}$$

Defining $s = t - \tau$, Eq. (A.66) becomes

$$v_\tau^\tau = \int_0^\infty e^{-rs} \left(1 - \alpha + \alpha \frac{k_{\tau+s}}{k_\tau} \right) A_{\tau+s} q_{\tau+s}^{1-\beta} k_{\tau+s}^{-\beta} ds. \tag{A.67}$$

Similarly, like in the proof of Proposition 2(A), we can derive the value of an outsider at τ to be

$$u_\tau = \int_0^\infty e^{-rs} \left(\frac{(1-\alpha)\left(e^{\gamma N s} - 1\right)}{N e^{\gamma N s}} \right) A_{\tau+s} \left(q_{\tau+s} k_{\tau+s}\right)^{1-\beta} ds \tag{A.68}$$

Combining Eqs. (A.67) and (A.68), we have

$$v_\tau^\tau - u_\tau = \int_0^\infty e^{-rs} \left(1 - \alpha + \alpha \frac{k_{\tau+s}}{k_\tau} - \frac{(1-\alpha)\left(e^{\gamma N s} - 1\right)}{N e^{\gamma N s}} k_{\tau+s} \right) A_{\tau+s} q_{\tau+s}^{1-\beta} k_{\tau+s}^{-\beta} ds. \tag{A.69}$$

Note the term $\left(1 - \alpha + \alpha \frac{k_{\tau+s}}{k_\tau} - \frac{(1-\alpha)\left(e^{\gamma N s}-1\right)}{N e^{\gamma N s}} k_{\tau+s}\right)$ decreases in τ. Then, as long as $p_{\tau+s} q_{\tau+s} = A_{\tau+s} q_{\tau+s}^{1-\beta} k_{\tau+s}^{-\beta}$ does not increase in τ, $v_\tau^\tau - u_\tau$ decreases with τ. Therefore, given the free entry condition $v_0^0 - u_0 = c$, we have $v_\tau^\tau - u_\tau < c$ for any $\tau > 0$, and so no innovator would enter the industry after date 0.

Let $v_0 \equiv v_0^0$. Equation (A.69) implies that

$$v_0 - u_0 = \frac{1}{N - k_0} \int_0^\infty e^{-rt} \left(\alpha \frac{N}{k_0} + (1-\alpha) \frac{N}{k_t} - 1 \right) A_t \left(q_t k_t\right)^{1-\beta} dt. \tag{A.70}$$

The free entry condition $v_0 - u_0 = c$ then pins down the entry of innovators k_0 at date 0, as shown by Eq. (A.64):

$$\frac{\int_0^\infty e^{-rt} \left(\alpha \frac{N}{k_0} + (1-\alpha) \frac{N}{k_t} - 1 \right) A_t \left(q_t k_t\right)^{1-\beta} dt}{N - k_0} = c.$$

(B) In Regime 2, with time-varying A_t and q_t, we have

$$r v_t = p_t q_t + \gamma \left(N - k_t\right) \alpha v_t + \frac{d v_t}{dt}, \tag{A.71}$$

and

$$r u_t = \gamma k_t \left((1-\alpha) v_t - u_t\right) + \frac{d u_t}{dt}. \tag{A.72}$$

Following a proof similar to that of Proposition 2(B), we can solve Eq. (A.71) to get

$$v_t = \int_0^\infty e^{-rj} \left(1 - \frac{1 - e^{-\gamma N j}}{\frac{k_0}{N - k_0} e^{\gamma N t} + 1} \right)^{-\alpha} A_{t+j} q_{t+j}^{1-\beta} k_{t+j}^{-\beta} dj. \tag{A.73}$$

In the integral, the term $\left(1 - \frac{1-e^{-\gamma N j}}{\frac{k_0}{N-k_0} e^{\gamma N t} + 1}\right)^{-\alpha}$ decreases in t. Therefore, as long as $p_{t+j} q_{t+j} = A_{t+j} q_{t+j}^{1-\beta} k_{t+j}^{-\beta}$ does not increase in t, v_t decreases with t.

Moreover, Eqs. (A.71) and (A.72) imply that

$$\frac{d(v_t - u_t)}{dt} = (r + \gamma k_t)(v_t - u_t) - (p_t q_t + \gamma N \alpha v_t). \tag{A.74}$$

Let $\psi_t \equiv v_t - u_t$. Following a proof similar to that of Proposition 2(B) , we solve Eq. (A.74) to derive

$$\psi_t = \int_0^\infty e^{-rj} \left(1 - \frac{e^{\gamma N j} - 1}{(\frac{N-k_0}{k_0})e^{-\gamma N t} + e^{\gamma N j}}\right)(p_{t+j}q_{t+j} + \gamma N \alpha v_{t+j})dj.$$

Note that in the integral, the terms $\left(1 - \frac{e^{\gamma N j}-1}{(\frac{N-k_0}{k_0})e^{-\gamma N t}+e^{\gamma N j}}\right)$ and $\gamma N \alpha v_{t+j}$ decrease in t. As long as $p_{t+j}q_{t+j} = A_{t+j}q_{t+j}^{1-\beta}k_{t+j}^{-\beta}$ does not increase in t, $\psi_t = v_t - u_t$ strictly decreases with t. Given the free entry condition that $v_0 - u_0 = c$ at date 0, we get $v_t - u_t < c$ at all $t > 0$, so no agent would enter as an innovator after date 0.

Eq. (A.73) implies that

$$v_0 = \int_0^\infty e^{-rt} \left(\frac{k_t}{k_0}\right)^\alpha A_t q_t^{1-\beta} k_t^{-\beta} dt. \tag{A.75}$$

At date 0, the total industry discounted revenue, $\int_0^\infty e^{-rt} A_t q_t^{1-\beta} k_t^{1-\beta} dt$, is shared by the two groups – the initial entrants k_0 and the outsiders $N - k_0$. With the free entry condition $v_0 - c = u_0$,

$$\int_0^\infty e^{-rt} A_t q_t^{1-\beta} k_t^{1-\beta} dt = v_0 k_0 + u_0 (N - k_0) = v_0 N - c(N - k_0). \tag{A.76}$$

Plugging Eq. (A.75) into Eq. (A.76) yields Eq. (A.65):

$$\frac{\int_0^\infty e^{-rt} \left(\left(\frac{N}{k_0}\right)^\alpha \left(\frac{N}{k_t}\right)^{1-\alpha} - 1\right) A_t q_t^{1-\beta} k_t^{1-\beta} dt}{N - k_0} = c.$$

■

IA-A13. Proofs of the claims in Section 6.4

Diffusion process as $N \to \infty$.—The logistic process (A.1) implies that for a given k_t,

$$\frac{dk_t/dt}{k_t} = \gamma(N - k_t) = \gamma N \left(1 - \frac{e^{\lambda t}}{e^{\lambda t} + \frac{N}{k_0} - 1}\right). \tag{A.77}$$

Given the inverse demand function $p_t = A k_t^{-\beta}$, k_0 has to be finite as $N \to \infty$; otherwise $p_0 \to 0$, and no innovator would enter at date 0. Therefore, Eq. (A.77) implies that

$$\left.\frac{dk_t/dt}{k_t}\right|_{N\to\infty} \to \gamma N \to \lambda. \tag{A.78}$$

The logistic diffusion process then converges to $\frac{dk_t}{dt} = \lambda k_t$, and Eq. (A.1) becomes

$$k_t = k_0 e^{\lambda t}. \tag{A.79}$$

Regime 1.—We conjecture that no agent would enter as an innovator after date 0, so k_t is given by Eq. (A.79). Given that an imitator cannot resell the idea, his only revenue comes from selling the good, and his value ω_t satisfies the ordinary differential equation (ODE):

$$r\omega_t = p_t + \frac{d\omega_t}{dt} = A(k_0 e^{\lambda t})^{-\beta} + \frac{d\omega_t}{dt}. \tag{A.80}$$

The ODE has the unique bounded solution

$$\omega_t = \frac{A k_0^{-\beta}}{r + \beta\lambda} e^{-\beta\lambda t}. \tag{A.81}$$

An innovator receives revenue from selling both the good and the idea. The number of ideas sold at t is λk_t and the total date-t revenue from these sales, $\lambda k_t \alpha \omega_t$ is divided among the k_0 innovators. Thus v_t, the value of being an innovator at date t, follows the ODE

$$r v_t = p_t + \frac{\lambda k_t}{k_0} \alpha \omega_t + \frac{dv_t}{dt} = A(k_0 e^{\lambda t})^{-\beta} + \frac{\alpha\lambda A k_0^{-\beta} e^{(1-\beta)\lambda t}}{r + \beta\lambda} + \frac{dv_t}{dt}. \tag{A.82}$$

Unless $\alpha = 0$, innovators receive a fraction of revenues from idea sales, and we shall need to restrict the elasticity of demand to be below unity which means $\beta \geq 1$. Imposing the boundary condition $v_{t\to\infty} < \infty$ yields the unique solution to Eq. (A.82):

$$v_t = \frac{A k_0^{-\beta} e^{-\beta\lambda t}}{r + \beta\lambda} + \frac{\alpha\lambda A k_0^{-\beta}}{(r + \beta\lambda)(r + (\beta - 1)\lambda)} e^{-(\beta-1)\lambda t}. \tag{A.83}$$

Recall that u_t denotes the option value of becoming a future imitator. At $t = 0$, the free entry condition is $v_0 - u_0 = c$. Given that the pool of outsiders is infinite, an outsider's chance of matching with an incumbent is zero so that $u_t = 0$ for all t, implying that $v_0 = c$. Since v_t decreases over time, we verify the conjecture that no one would pay c to become an innovator at any date $t > 0$. Note that if an agent deviates from the equilibrium and enters at date $t > 0$, he would have a lower valuation than an innovator who entered at date 0 (i.e., $v_t^t < v_t^0$) because the latter would have a larger family of imitators to disseminate his idea and collect idea-sale revenues. Therefore, the finding that $v_t - u_t$ declines in t implies that $v_t^t - u_t < c$ at any date $t > 0$.

Combining $v_0 = c$ with Eq. (A.83) yields

$$v_0 = \frac{A k_0^{-\beta}}{r + \beta\lambda} + \frac{\alpha\lambda A k_0^{-\beta}}{(r + \beta\lambda)(r + (\beta - 1)\lambda)} = c. \tag{A.84}$$

Equation (A.84) then determines the entry of innovators at date 0 to be

$$k_0^{\mathbf{I}} = \left(\frac{A(r + (\beta - 1)\lambda + \alpha\lambda)}{c\,(r + \beta\lambda)\,(r + (\beta - 1)\lambda)} \right)^{\frac{1}{\beta}}. \tag{A.85}$$

Regime 2.—Assume that condition $\lambda < r$ holds so that social welfare derived from the innovation is bounded. We conjecture that no agent would enter as an innovator after date 0. Given that imitators can resell the innovation, all the incumbents (be they innovators or imitators) share the same value v_t. The revenue from an idea sale is αv_t and the total date-t revenue from these sales, $\lambda k_t \alpha v_t$, is shared equally among all the incumbents. Then v_t follows the ODE

$$r v_t = p_t + \lambda\alpha v_t + \frac{dv_t}{dt} = A(k_0 e^{\lambda t})^{-\beta} + \lambda\alpha v_t + \frac{dv_t}{dt}. \tag{A.86}$$

The general solution of Eq. (A.86) is

$$v_t = \frac{A k_0^{-\beta} e^{-\beta\lambda t}}{r + \beta\lambda - \lambda\alpha} + A k_0^{-\beta} C e^{(r - \lambda\alpha)t},$$

where C is the constant of integration. Given that $\lambda < r$, the boundary condition $v_{t\to\infty} < \infty$ requires $C = 0$ and yields

$$v_t = \frac{A k_0^{-\beta} e^{-\beta\lambda t}}{r + (\beta - \alpha)\lambda}, \tag{A.87}$$

which decreases with t. Again, given $N \to \infty$, an outsider's chance of matching with an incumbent is zero so that $u_t = 0$. Therefore, $v_t - u_t$ decreases with t and no innovator enters after date 0. Since the free entry condition requires $v_0 = c$, Eq. (A.87) evaluated at $t = 0$ yields

$$k_0^{\mathbf{II}} = \left(\frac{A}{c(r + (\beta - \alpha)\lambda)} \right)^{\frac{1}{\beta}}. \tag{A.88}$$

Socially optimal licensing rate.—The planner maximizes

$$W_0 = \int_0^{\infty} e^{-rt} U(k_t)\, dt - c k_0,$$

where

$$U(k_t) = \begin{cases} \frac{A}{1-\beta} k_t^{1-\beta} & \text{if } \beta \in (0,1), \\ A(\ln k_t + 1 - \ln\varepsilon) & \text{if } \beta = 1, \\ \frac{A}{1-\beta} k_t^{1-\beta} + \frac{\beta A}{\beta - 1}\varepsilon^{1-\beta} & \text{if } \beta > 1. \end{cases}$$

and $k_t = k_0 e^{\lambda t}$. We maintain condition $\lambda < r$ so that social welfare derived from the innovation is bounded.

Consider the case $\beta < 1$ first. For any date $\tau \geq 0$, if no further innovators enter, the number of firms at dates $t \geq \tau$ is

$$k_t = k_\tau \; e^{\lambda(t-\tau)} \quad \text{for } t \geq \tau.$$

As of date τ, the social return to innovation is

$$SR_\tau = \int_\tau^\infty e^{-r(t-\tau)} \frac{A}{1-\beta} k_t^{1-\beta} dt.$$

The current cost of innovation is c per unit, and its marginal social return (even if no further innovations are made) is

$$\frac{\partial SR_\tau}{\partial k_\tau} = \int_\tau^\infty e^{-r(t-\tau)} A k_t^{-\beta} \frac{\partial k_t}{\partial k_\tau} dt = \int_\tau^\infty e^{-(r-\lambda)(t-\tau)} A k_t^{-\beta} dt. \tag{A.89}$$

Let $s \equiv t - \tau$, and then Eq. (A.89) becomes

$$\frac{\partial SR_\tau}{\partial k_\tau} = \int_0^\infty e^{-(r-\lambda)s} A k_{\tau+s}^{-\beta} ds = \int_0^\infty e^{-(r-\lambda)s} A \left(k_\tau e^{\lambda s}\right)^{-\beta} ds,$$

which is strictly decreasing in k_τ. Therefore, if at date 0 k_0 is chosen so that $\frac{\partial SR_0}{\partial k_0} = c$, thereafter we would have $\frac{\partial SR_\tau}{\partial k_\tau} < c$. Similarly, we can prove that the result holds for $\beta \geq 1$. Hence, it is socially optimal to innovate only at date 0.

Accordingly, the social planner chooses k_0 to maximize social welfare:

$$W_0 = \int_0^\infty e^{-rt} U\left(k_t\right) dt - c k_0.$$

We verify that the social welfare function is strictly concave in k_0, and the first-order condition is

$$k_0^* = \left(\frac{A}{(r+(\beta-1)\lambda)c}\right)^{\frac{1}{\beta}}. \tag{A.90}$$

Comparing Eq. (A.90) to Eqs. (A.85) and (A.88) shows that $k_0^* = k_0^{\mathbf{I}} = k_0^{\mathbf{II}}$ iff $\alpha = 1$. I.e., $\alpha^* = 1$ for both Regimes 1 and 2.

Socially optimal diffusion rate.—Recall that the solutions for $k_0^{\mathbf{I}}$ and $k_0^{\mathbf{II}}$ are given by Eqs. (A.85), (A.88):

$$k_0^{\mathbf{I}} = \left(\frac{A(r+(\beta-1)\lambda+\alpha\lambda)}{c\left(r+\beta\lambda\right)\left(r+(\beta-1)\lambda\right)}\right)^{\frac{1}{\beta}}; \quad k_0^{\mathbf{II}} = \left(\frac{A}{c(r+(\beta-\alpha)\lambda)}\right)^{\frac{1}{\beta}}.$$

In each regime, the number of firms grows at a constant rate λ (i.e., $k_t^{\mathbf{I}} = k_0^{\mathbf{I}} e^{\lambda t}$ and $k_t^{\mathbf{II}} = k_0^{\mathbf{II}} e^{\lambda t}$). The planner maximizes social welfare

$$W_0 = \int_0^\infty e^{-rt} U\left(k_t\right) dt - c k_0,$$

where

$$U(k_t) = \begin{cases} \frac{A}{1-\beta} k_t^{1-\beta} & \text{if } \beta \in (0,1), \\ A(\ln k_t + 1 - \ln \varepsilon) & \text{if } \beta = 1, \\ \frac{A}{1-\beta} k_t^{1-\beta} + \frac{\beta A}{\beta - 1} \varepsilon^{1-\beta} & \text{if } \beta > 1. \end{cases}$$

We again assume that $\lambda < r$, and denote by $W_0^{\mathbf{I}}$ and $W_0^{\mathbf{II}}$ the social welfare under Regimes 1 and 2, respectively. With free imitation ($\alpha = 0$), we have $W_0^{\mathbf{I}} = W_0^{\mathbf{II}} = W_0$, where

$$W_0 = \begin{cases} A^{\frac{1}{\beta}} c^{1-\frac{1}{\beta}} \left(\frac{(r+\beta\lambda)^{1-\frac{1}{\beta}}}{(1-\beta)(r-\lambda(1-\beta))} - (r+\beta\lambda)^{-\frac{1}{\beta}} \right) & \text{if } \beta \in (0,1), \\ \frac{A}{r} \ln \frac{A}{c(r+\lambda)} + \frac{A\lambda}{r^2} - \frac{A}{(r+\lambda)} + \frac{A(1-\ln\varepsilon)}{r} & \text{if } \beta = 1, \\ A^{\frac{1}{\beta}} c^{1-\frac{1}{\beta}} \left(\frac{(r+\beta\lambda)^{1-\frac{1}{\beta}}}{(1-\beta)(r-\lambda(1-\beta))} - (r+\beta\lambda)^{-\frac{1}{\beta}} \right) + \frac{\beta A}{r(\beta-1)} \varepsilon^{1-\beta} & \text{if } \beta > 1. \end{cases} \tag{A.91}$$

It is straightforward to show that for any $\beta > 0$, $\frac{\partial W_0}{\partial \lambda} > 0$.

This finding extends to any $\alpha \in (0,1]$, for which we have

$$W_0^{\mathbf{I}} = \begin{cases} \text{N/A} & \text{if } \beta \in (0,1), \\ \frac{A}{r} \ln \left(\frac{A(r+\alpha\lambda)}{c(r+\lambda)r} \right) + \frac{A\lambda}{r^2} - \frac{A(r+\alpha\lambda)}{(r+\lambda)r} + \frac{A(1-\ln\varepsilon)}{r} & \text{if } \beta = 1, \\ A^{\frac{1}{\beta}} c^{1-\frac{1}{\beta}} \left(\begin{array}{c} \frac{1}{(1-\beta)} \left(\frac{(r+(\beta-1)\lambda+\alpha\lambda)}{(r+\beta\lambda)} \right)^{\frac{1-\beta}{\beta}} (r+(\beta-1)\lambda)^{-\frac{1}{\beta}} \\ - \left(\frac{r+(\beta-1)\lambda+\alpha\lambda}{(r+\beta\lambda)(r+(\beta-1)\lambda)} \right)^{\frac{1}{\beta}} \end{array} \right) & \text{if } \beta > 1. \\ + \frac{\beta A}{r(\beta-1)} \varepsilon^{1-\beta} & \end{cases} \tag{A.92}$$

$$W_0^{\mathbf{II}} = \begin{cases} A^{\frac{1}{\beta}} c^{1-\frac{1}{\beta}} \left(\frac{(r+(\beta-\alpha)\lambda)^{1-\frac{1}{\beta}}}{(1-\beta)(r-\lambda(1-\beta))} - (r+(\beta-\alpha)\lambda)^{-\frac{1}{\beta}} \right) & \text{if } \beta \in (0,1), \\ \frac{A}{r} \ln \frac{A}{c(r+(1-\alpha)\lambda)} + \frac{A\lambda}{r^2} - \frac{A}{r+(1-\alpha)\lambda} + \frac{A}{r}(1-\ln\varepsilon) & \text{if } \beta = 1, \\ A^{\frac{1}{\beta}} c^{1-\frac{1}{\beta}} \left(\frac{(r+(\beta-\alpha)\lambda)^{1-\frac{1}{\beta}}}{(1-\beta)(r-\lambda(1-\beta))} - (r+(\beta-\alpha)\lambda)^{-\frac{1}{\beta}} \right) & \text{if } \beta > 1. \\ + \frac{\beta A \varepsilon^{1-\beta}}{r(\beta-1)} & \end{cases} \tag{A.93}$$

We then confirm from Eqs. (A.92) and (A.93) that $\frac{\partial W_0^{\mathbf{I}}}{\partial \lambda} > 0$ for $\beta \geq 1$ and $\frac{\partial W_0^{\mathbf{II}}}{\partial \lambda} > 0$ for $\beta > 0$.

IA-B. Data and robustness checks

IA-B1. Cross-industry licensing rates

Matching the reported royalty rates with licensees' operating profit margins in a sample of 347 companies in 15 industries between 1990-2000, Goldscheider *et al.* (2002) show the empirical relevance of the 25% rule: Across all 15 industries, the median royalty rate as a percentage of average licensee operating profit margins was 26.7%, and the majority of industries had ratios of royalty rates to licensee profit margins of 21-40%, as shown in Fig. (IA-1).

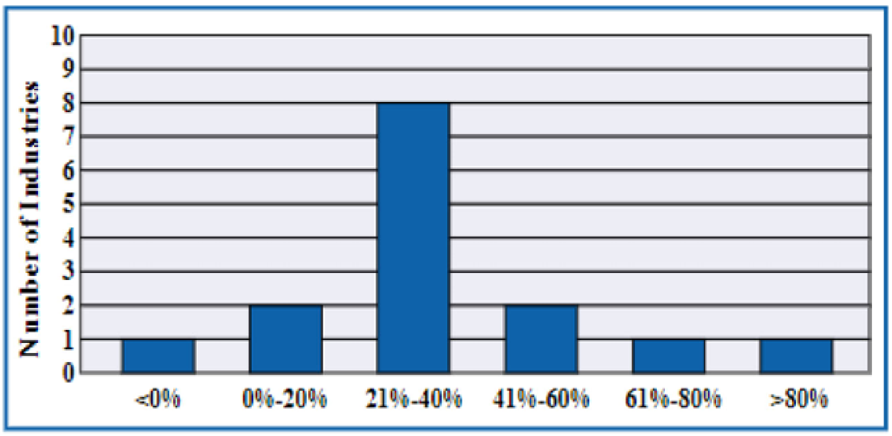

Fig. IA-1. Licensing Rates acoss Industries

Using more recent data, Kemmerer and Lu (2012) show such result holds broadly in a sample of 3,887 companies in 14 industries based on 3,015 patent licensing transactions collected over a 21-year period prior to 2007. They find that the reported royalty rates across industries tend to fall between 25 percent of gross margins and 25 percent of operating margins.

IA-B2. Auto diffusion estimation: Robustness checks

For robustness checks, we re-ran the diffusion regression in the paper for the subsample period 1900-1910 (i.e., removing the first five years of observations). The result shows that

$$\ln \frac{k_t}{N - k_t} = \underset{(0.40)^{**}}{-1.30} + \underset{(0.07)^{***}}{0.52}\ t.$$

Standard errors are reported in parentheses, with three stars and two stars representing statistical significance at 1% and 5%, respectively. The adjusted $R^2 = 0.85$. The estimated $\gamma N = 0.52$ is very similar to the estimate $\gamma N = 0.54$ in the paper (and the constant term is smaller in absolute value because the number of firms in 1900 was higher than that in 1895).

For additional robustness checks, we also estimate the matching function directly by rewriting $\frac{dk_t/dt}{N-k_t} = \gamma k_t$ into a discrete-time version:

$$\frac{k_t - k_{t-1}}{N - k_{t-1}} = \gamma k_{t-1}. \tag{A.94}$$

Note that the left-hand side of Eq. (A.94) is the hazard rate of adopting the new product. We again set $N = 210$ and run the regression model (A.94) using auto-firm numbers data from 1895-1908. The result was

$$\frac{k_t - k_{t-1}}{N - k_{t-1}} = \underset{(0.0004)^{***}}{0.0029}\, k_{t-1},$$

and the standard error is reported in the parentheses. The estimate of γ is statistically significant at 1% and the adjusted $R^2 = 0.77$. The estimate $\gamma = 0.0029$ implies that $\gamma N = 0.61$.

We also redo the exercise by estimating an extended version of Eq. (A.94)

$$\frac{k_t - k_{t-1}}{N - k_{t-1}} = \eta + \gamma k_{t-1}, \tag{A.95}$$

that was proposed by Bass (1969). The Bass model allows the hazard rate of adoption to depend on both the coefficient of innovation η and the coefficient of imitation γ. In our context, η captures the hazard rate of entry by innovators independent of incumbents while γk_{t-1} captures the hazard rate of entry by imitators. The regression result is

$$\frac{k_t - k_{t-1}}{N - k_{t-1}} = \underset{(0.072)}{-0.0339} + \underset{(.0007)^{***}}{.0032}\, k_{t-1},$$

and the standard errors are reported in parentheses. The estimate $\gamma = 0.0032$ (which implies $\gamma N = 0.67$) is statistically significant at 1%, but the estimate of η is not statistically significant, which is consistent with our theoretical prediction that innovators enter only at the beginning of the industry.

IA-B3. Auto demand estimation: Robustness checks

To check robustness, we estimate the auto industry demand function by controlling for changes of population and per capita income over time. In doing so, we use annual data on auto prices p_t and output Q_t from 1900–1929 to estimate a per capita demand function:

$$\ln\left(\frac{Q_t}{pop_t}\right) = a_t - \phi \ln(p_t).$$

where pop_t is U.S. population at year t. The dependent variable is auto demand per capita, and we control for log U.S. GDP per capita (as a proxy for income) in the demand intercept a_t. Both auto price and GDP per capita are in real terms.

As before, to address potential endogeneity of the price variable, we use output per firm lagged by a year as an instrument to estimate the demand elasticity parameter ϕ in a two-stage least-squares regression.

The first-stage regression result (adj. $R^2 = 0.89$) is

$$\ln(p_t) = \underset{(1.08)^{***}}{8.63} + \underset{(0.63)^{**}}{1.62} \times \ln(\frac{GDP_t}{pop_t}) - \underset{(0.03)^{***}}{0.29} \times \ln(\mathit{output\ per\ firm})_{t-1},$$

and the second-stage regression result ($R^2 = 0.83$) is

$$\ln(\frac{Q_t}{pop_t}) = \underset{(7.15)^{***}}{32.36} + \underset{(2.10)}{0.28} \times \ln(\frac{GDP_t}{pop_t}) - \underset{(0.38)^{***}}{3.33} \times \ln(p_t).$$

Standard errors are in parentheses, with three stars and two stars representing statistical significance at 1% and 5%, respectively. The estimate $\phi = 3.33$ is highly statistically significant and the implied inverse demand elasticity $\beta = \frac{1}{\phi} = 0.3$ is similar to our estimate in the paper.

IA-B4. PC diffusion estimation: Robustness checks

To check robustness, we re-ran the diffusion regression in the paper for the subsample period 1980-1992 (i.e., removing the first five years of observations). The result was

$$\ln \frac{k_t}{N - k_t} = \underset{(0.35)^{***}}{-2.83} + \underset{(0.05)^{***}}{0.62}\ t,$$

where the standard errors are in parentheses and the adjusted $R^2 = 0.93$. The estimated $\gamma N = 0.62$ is very similar to the estimate $\gamma N = 0.58$ in the paper (and the constant term is smaller in absolute value because the number of firms in 1980 is higher than that in 1975).

For additional robustness checks, we estimate the matching function (A.94) for the PC industry:

$$\frac{k_t - k_{t-1}}{N - k_{t-1}} = \gamma k_{t-1}.$$

We again set $N = 435$ and run the regression using PC firm number data from 1975-1991. The result is

$$\frac{k_t - k_{t-1}}{N - k_{t-1}} = \underset{(0.00025)^{***}}{0.00092}\ k_{t-1},$$

and the standard error is in the parentheses. The estimate of γ is statistically significant at 1% and the adjusted $R^2 = 0.44$. The estimate $\gamma = 0.00092$ implies that $\gamma N = 0.40$.

We also redo the exercise by estimating a more general version (cf. Eq. (A.95)) that

$$\frac{k_t - k_{t-1}}{N - k_{t-1}} = \eta + \gamma k_{t-1},$$

proposed by Bass (1969). In our context, η captures the hazard rate of entry by innovators independent of incumbents, while γk_{t-1} captures the hazard rate of entry by imitators. The regression result shows that

$$\frac{k_t - k_{t-1}}{N - k_{t-1}} = \underset{(0.08398)}{0.04721} + \underset{(0.00036)^{*}}{0.00078} k_{t-1},$$

and the standard errors are reported in parentheses. The estimate $\gamma = 0.00078$ (which implies $\gamma N = 0.34$) is statistically significant at 5.1%, but the estimate of η is not statistically significant, which is consistent with our theoretical prediction that innovators only enter at the beginning of the industry.

IA-B5. PC demand estimation: Robustness checks

To check robustness, we estimate the PC industry demand function by controlling for changes of population and per capita income over time. In doing so, we use annual data of PC prices p_t and output Q_t from 1975–1992 to estimate the per capita demand function:

$$\ln\left(\frac{Q_t}{pop_t}\right) = a_t - \phi \ln(p_t).$$

The dependent variable is PC demand per capita (where pop_t is U.S. population at year t), and we control for log U.S. GDP per capita (as a proxy for income) in the demand intercept a_t. PC price and GDP per capita are both in real terms.

As before, to address potential endogeneity of the price variable, we use output per firm lagged by a year as an instrument to estimate the demand elasticity parameter ϕ in a two-stage least-squares regression.

The first-stage regression result (adj. $R^2 = 0.95$) is

$$\ln(p_t) = \underset{(0.23)^{***}}{12.44} - \underset{(0.06)^{***}}{0.95} \times \ln\left(\frac{GDP_t}{pop_t}\right) - \underset{(0.01)^{***}}{0.07} \times \ln(\textit{output per firm})_{t-1},$$

and the second-stage regression result ($R^2 = 0.95$) is

$$\ln\left(\frac{Q_t}{pop_t}\right) = \underset{(29.00)^{***}}{143.18} - \underset{(2.63)}{2.90} \times \ln\left(\frac{GDP_t}{pop_t}\right) - \underset{(2.40)^{***}}{15.57} \times \ln(p_t).$$

Standard errors are in parentheses, with three stars indicating statistical significance at 1% level. The estimate $\phi = 15.57$ is highly statistically significant and the implied inverse demand elasticity $\beta = \frac{1}{\phi} = 0.06$ is similar to our estimate in the paper.