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Idea Diffusion and Property Rights

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Abstract

We develop a model of industry evolution that links innovation, diffusion, and intellectual property (IP) protection. Motivated by evidence that most innovations are neither patented nor effectively protected by patents, we analyze diffusion under free imitation and derive the socially optimal licensing rate that balances learning and matching externalities. When IP protection is ineffective, innovators rely on lead time to recover investment, but extending this lead time by slowing down diffusion reduces welfare. Calibrating the model to 18 U.S. industries, including automobiles and personal computers, we quantify the theoretical predictions and show how industry dynamics shape optimal policy design.

Keywords: Innovation, Industry Dynamics, Intellectual Property Rights

JEL Classification: L0, O3

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1 Introduction

Innovation and diffusion are twin engines of technological progress. While innovation drives creation, diffusion determines its impact on productivity and welfare. Yet these two forces are inherently in tension: Rapid diffusion lowers prices and weakens incentives to innovate.

Patent protection has traditionally been viewed as a mechanism to balance this trade-off. However, despite its central role in innovation policy, the patent system fails to effectively protect most innovations. On the one hand, surveys consistently show that only a small fraction of innovations are patented, few are ever litigated, and even fewer yield enforceable protection. This widespread ineffectiveness implies that in many industries, innovation proceeds under *de facto* weak or nonexistent intellectual property (IP) protection. On the other hand, where patent licensing does occur, rates are typically determined through bilateral negotiation between an IP owner and a licensee as a share of the latter's profit—"25 percent"—being a common rule of thumb (see Razgaitis, 1999, Goldscheider, 2011). Whether such private bargaining outcomes align with the social optimum remains unclear.

Our analysis addresses these issues directly. We model a competitive industry with a downward-sloping demand curve for a homogeneous product and a continuum of potential producers. An innovation—or "idea"—enables its holder to produce at zero cost, subject to a capacity constraint. At the outset, agents decide whether to incur a sunk cost to innovate: some do so immediately, while others delay or wait to imitate.

Imitation then occurs through random matching between idea holders and non-holders. Without IP protection, imitation is free; with protection, imitators pay a share of their profits to the idea holders. We consider two IP regimes. In Regime 1, imitators cannot resell ideas—copying from them still requires payment to the original innovator, as under licensing arrangements that forbid sublicensing. In Regime 2, imitators may resell ideas and keep the proceeds, allowing sublicensing.

Solving the model, we derive the industry's equilibrium evolution. Under either free imitation or either IP regime, innovators enter only at the beginning, and imitators follow an S-shaped logistic diffusion path over time. More innovators enter under Regime 1 than under Regime 2 or when innovators' bargaining share is higher, leading to faster industry growth. Calibrating the model to 18 U.S. industries—including automobiles and personal computers—we find that it replicates the key features of their early expansion and yields several policy-relevant insights.

First, when IP protection is absent, innovators must rely on lead-time advantage to recoup their investment. The outcome is not socially optimal because innovators fail to internalize knowledge spillovers, leading to under-investment in innovation and slower industry development.

Second, when IP protection exists, full compensation to innovators is not socially optimal either, since innovators ignore the negative matching externality in diffusion. We derive the socially optimal licensing rate that balances these opposing externalities and maximizes welfare, showing how it depends on demand elasticity, innovation cost, diffusion speed, and on whether sublicensing is permitted. This finding implies that neither the 25 percent rule nor bilateral bargaining is likely to yield the socially optimal outcome, as individual agents neglect industry-level externalities.

Third, when IP protection fails, governments may seek to extend innovators' lead time—for instance, by slowing diffusion—to strengthen the incentive to innovate. Our analysis shows that while such policies do raise innovators' private returns, they also delay capacity expansion and output growth, ultimately reducing social welfare.

Fitting the model to 18 U.S. industries which all exhibit S-shaped producer growth prior to the shakeout—we contribute to the industry life-cycle literature, e.g., Gort and Klepper (1982), Jovanovic and MacDonald (1994), Klepper (1996), Filson (2001), and Hayashi *et al.* (2017). While prior studies emphasize industry shakeouts, we focus on the preceding expansion phase. Our framework shows how, with minimal patent licensing, industries nonetheless sustained innovation and rapid diffusion. Counterfactual analysis reveals the cross-industry distribution of socially optimal licensing rates and contrasts them with observed practice. The results also highlight that while lead-time advantage can support private incentive to innovate, a policy that extends this lead time by slowing diffusion would reduce output growth and overall welfare.

Examining the automobile and personal computer (PC) industries in detail, we find that highly elastic demand in both industries reduced the downward pressure on price caused by imitator entry and thus stimulated innovation, but that it also amplified matching externalities. As a result, the socially optimal licensing rate should be low for both industries, and lower for the more price-elastic PC industry.

In our model, random matching between idea holders and non-holders gives rise to a logistic diffusion process, consistent with the classic technology-diffusion literature (e.g., Griliches 1957, Mansfield 1961, Bass 1969, 2004, Young 2009) and epidemic studies (e.g., Atkeson 2020, Garibaldi, Moen, and Pissarides 2020, among many applications of the SIR model to the spread of the COVID-19 disease).[1] And the quadratic matching function underlying logistic diffusion was recently studied by Lauermann, Nöldeke, and Tröger (2020).

We also contribute to several other strands of research. First, the literature on competitive innovation, such as Boldrin and Levine (2008), documents that innovation often occurs even without enforceable IP rights. They argue that capacity constraints sustain prices above marginal costs for extended periods, generating competitive rents that motivate innovation. Their framework has a single innovator's entry decision and the number of imitators growing at a constant rate, whereas our model endogenizes the number of innovators and generates S-shaped imitation dynamics. The resulting matching externality yields distinct policy implications.

Second, our results on licensing apply out of steady state. Benhabib, Perla, and Tonetti (2021) show that the growth-maximizing licensing rate depends on how imitation affects incentives to innovate and thus affects the movement in the distribution of idea quality, and Hopenhayn and Shi (2020) show that this rate also depends on the parameters in the matching function. Those papers study steady-state aggregate growth, whereas we focus on the transitional dynamics of specific industries. Our model thus connects directly to early industry life-cycle patterns of price, output, and firm entry, implying industry-specific optimal licensing policies. We also find that the optimal licensing rate depends on the licensing structure—the rate should be higher when imitators can resell ideas.

Finally, in our setting, the flow value of an idea depends on how many others use it. Manea (2021) also models bilateral bargaining over ideas, but in his framework, the value of an idea to its user does not depend on its diffusion.

The paper proceeds as follows. Section 2 provides motivating facts. Section 3 develops and solves the baseline model without IP protection. Section 4 introduces two IP regimes and characterizes the resulting equilibrium. Section 5 presents the welfare and policy analyses, and Section 6 applies the model to U.S. automobile and PC industries as well as to 16 other industries studied by Gort and Klepper (1982). Section 7 provides additional discussion, and Section 8 concludes.

[1] Our model focuses on the diffusion process driven by information spillovers. There are also models where diffusion is driven by falling prices of inputs; e.g., David (1969) and Manuelli and Seshadri (2014).

2 Motivating facts

Our study is motivated by several salient features of industry evolution. New industries typically follow a well-documented life-cycle pattern: the number of producers first rises and later falls as the industry evolves from birth to maturity, while industry output continues to grow and prices decline. This pattern was first documented by Gort and Klepper (1982) for 46 new industries in the U.S. and subsequently confirmed in studies of other emerging industries—e.g., see Jovanovic and MacDonald (1994), Klepper (1996), Filson (2001), and Hayashi *et al.* (2017). Jovanovic and MacDonald (1994) show that the later "shakeout" phase can be triggered by the arrival of a mass-production technology: a few incumbents adopt the new production technology, collapse prices, and drive less efficient firms to exit. Before this shakeout, however, industry growth is driven mainly by the entry of small producers whose expansion follows an S-shaped diffusion curve, as emphasized in the classic literature on technology diffusion (e.g., Griliches 1957, Mansfield 1961, Bass 1969, 2004, Young 2009).

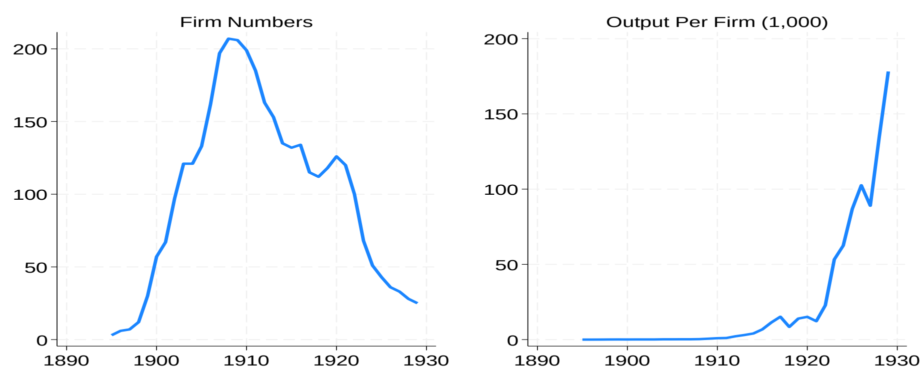

Fig. 1. Auto Firm Numbers and Output Per Firm

Note: Figure 1 plots the time path of firm numbers and output per firm in the U.S. automobile industry from 1895-1929. See Section 6.1.1 for the data sources and industry definition.

The automobile and personal computer industries illustrate this life-cycle pattern vividly. The U.S. automobile industry began in the 1890s and transformed from a nascent sector into a major industry within a few decades. Starting with three firms in 1895, the number of producers exceeded 200 by 1910, before a sharp shakeout following the introduction of the assembly line. The number of firms then fell precipitously, while industry output expanded dramatically. Figure 1 plots firm numbers and output per firm in the U.S. automobile industry from 1895–1929: prior to the shakeout, firm entry followed an S-shaped curve and output per firm remained flat, consistent with capacity constraints.

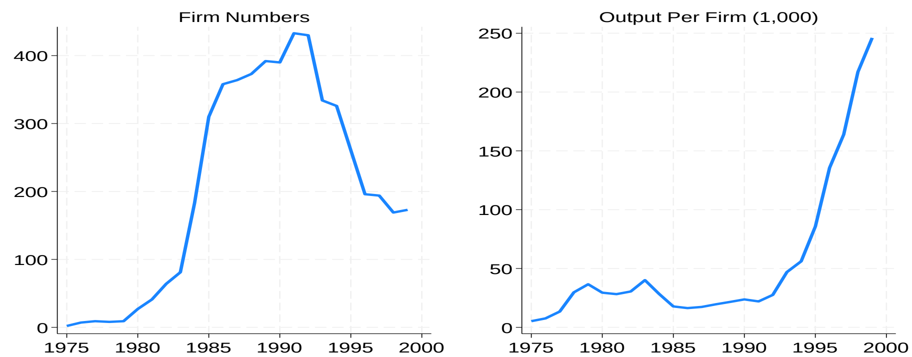

Fig. 2. PC Firm Numbers and Output Per Firm

Note: Figure 2 plots the time path of firm numbers and output per firm in the U.S. PC industry from 1975-1999. See Section 6.1.1 for the data sources and industry definition.

A similar pattern characterized the PC industry eight decades later. Beginning with two firms in 1975, the number of PC producers exceeded 430 by 1992, then declined sharply while total output continued to rise. Figure 2 plots the corresponding data from 1975–1999: as in autos, firm numbers followed an S-shaped diffusion path and average output per firm was roughly constant before the shakeout.

In this paper we focus on the pre-shakeout phase and model producers' entry decisions as innovators or imitators.[2] We derive S-shaped diffusion as an outcome of random matching between idea holders and non-holders. The choice between innovating and imitating depends on relative payoffs, which in turn reflect how innovators are compensated for transferring their ideas.

Our baseline analysis assumes free imitation. This assumption reflects extensive evidence showing that most innovations are not effectively protected by patents. Surveys indicate that only about 18 percent of R&D-performing U.S. firms seek any patent protection, and that merely 23 percent of new consumer products are introduced by firms holding relevant patents. Even among granted patents, fewer than

[2] We assume that shakeout is unexpected in the model. We then extend the model to incorporate anticipated shakeout as a robustness check in Section 6.4.

1.5 percent are litigated and only 0.1 percent reach trial. High litigation costs and the strategic use of patents by assertion entities further limit their protective function (see Internet Appendix IA-0 for an extensive review). Thus, in many industries, innovation proceeds under *de facto* weak or nonexistent IP protection.

The automobile and PC industries provide historical examples. When possible, imitators freely copied or circumvented existing patents. Licensing played little role in their development—creating legal disputes and expenses but not constraining imitation.[3] Accordingly, we calibrate the model under free imitation for these two industries and for 16 other industries studied by Gort and Klepper (1982).[4] We show how industries with minimal patent licensing nonetheless sustained innovation and rapid diffusion. Using counterfactual analysis, we derive the cross-industry distribution of socially optimal licensing rates and quantify welfare gains from alternative policy interventions.

3 Baseline model: free imitation

Consider a competitive market in continuous time. There is a measure N of potential producers. At date 0, a measure k_0 who we call "innovators," invest an amount c each in an innovation that results in the ability to produce one unit of a new good each period at zero cost. They start producing immediately. After that, the innovation spreads to others. At any date $t \geq 0$ the measure of producers (as well as market output) is k_t, and the remaining $N - k_t$ agents are "outsiders." We normalize outsiders'

[3] The patent system did not function properly in the early life cycle of the two industries and led to much controversy. The Selden patent is one of the most famous examples in the early automobile industry. It created major litigation among firms but did not really affect the industry development, and the patent was later overturned by the court (see e.g., Welsh 1948, Howells and Katznelson 2024). The PC industry had a similar experience. Famous examples include that the original computer invention failed to be patented (Kalat, 2021), Texas Instruments created patent wars (Pollack, 1990), and Compaq's and other clone computers skirted IBM's IP (Mitchell, 2017).

[4] Most innovations are neither patented nor effectively protected by existing patent systems. In cases where patent licensing does occur, the "25 percent of licensee profits" rule is a common benchmark used in both private negotiations and litigation. For robustness, we also recalibrate the model under the 25 percent rule in Sections 6.4 and 6.5.

earnings to zero and denote by u_t an outsider's date-t option value of entering the industry in the future.

The total output of the homogeneous good is k_t, and the product price is

$$p_t = Ak_t^{-\beta}, \tag{1}$$

where A is a market size parameter and $\beta > 0$ the inverse demand elasticity.

Two types of producers.—All producers have an idea and all are equally productive,[5] but some are "innovators" while the others are "imitators." An innovator has invested a cost c to invent an idea for production, and an imitator has copied a producer's idea. In the baseline model, we assume no IP protection so copying an idea from others is free, and imitators incur no other cost to enter the industry.[6] We shall then introduce IP protection and explore socially optimal policies.

3.1 Diffusion process

Ideas spread via a classic logistic process, in which the conditional probability for an outsider to imitate an idea is a linear function of k_t:

$$\frac{dk_t/dt}{N - k_t} = \gamma k_t. \tag{2}$$

The parameter $\gamma > 0$ captures the positive influence of existing idea holders on future imitators, which could occur either directly (e.g., through meetings between incumbents and outsiders) or indirectly (e.g., outsiders may reverse engineer ideas through observing existing outputs in the market).[7]

An outsider can also enter as an innovator after date 0, but Proposition 1 will show that no one will choose to do so. Thus, for $t > 0$ imitation is the only way that agents become producers, and Eq. (2) then implies that the mass of producers evolves as

$$\frac{dk_t}{dt} = \gamma k_t \left(N - k_t\right). \tag{3}$$

[5]Section 7.2 extends the analysis to allow firms to have heterogeneous production capacity.

[6]Section 6.4 extends the model to allow for a positive entry cost $c^m < c$ for imitators. Our analysis and findings continue to hold, but the solutions become more complicated.

[7]Based on survey results from 130 lines of business, Levin *et al.* (1987) show that reverse engineering, conversations with employees of innovating firms, learning from publications, technical meetings or patent disclosure, and licensing technology are main alternative methods of learning and imitating new processes and products.

The solution to (3) is

$$k_t = \frac{Ne^{\gamma Nt}}{e^{\gamma Nt} + \frac{N}{k_0} - 1}. \tag{4}$$

Figure 3 shows that the estimated logistic diffusion model (cf. Eq. (4)) matches the time paths of firm numbers well for the U.S. automobile and PC industries.

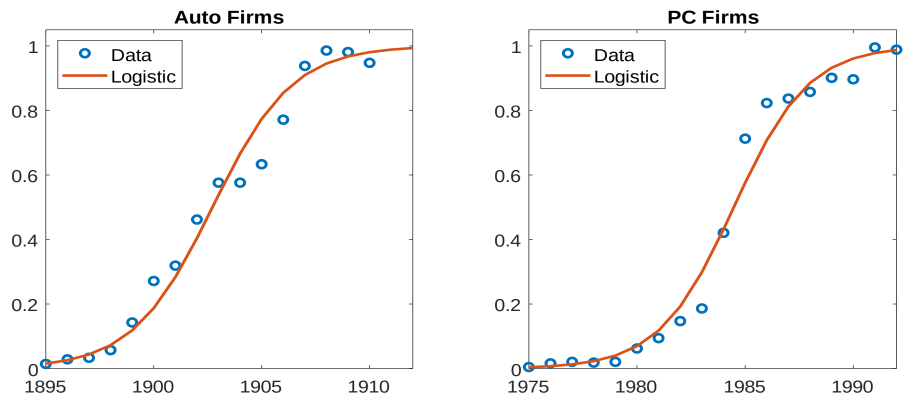

Fig. 3. Logistic Diffusion: Fitting Time Paths of Firm Numbers

Note: Figure 3 fits a logistic diffusion model to the time paths of firm numbers in the U.S. automobile and PC industries prior to the shakeout. In the figure, firm counts are expressed as a fraction of total potential entrants, proxied by the peak number of firms observed in each industry.

3.2 Value functions

A producer sells only goods and not ideas, so his flow revenue is p_t at date t and his present value v_t satisfies

$$rv_t = p_t + \frac{dv_t}{dt}, \tag{5}$$

where r is the interest rate.

An outsider's hazard rate for learning an idea to become a producer is γk_t. Accordingly, his value u_t at date t satisfies

$$ru_t = \gamma k_t (v_t - u_t) + \frac{du_t}{dt}. \tag{6}$$

3.3 Equilibrium outcome

At equilibrium, k_0, the mass of innovators at date 0, is endogenously determined. The free entry condition requires that

$$v_0 - u_0 = c \quad \text{if} \quad k_0 < N, \tag{7}$$

or $v_0 - u_0 \geq c$ if $k_0 = N$.

In what follows, we shall assume that innovation is costly enough to prevent the N agents from all innovating at date 0:

$$c > \frac{AN^{-\beta}}{r + \gamma N}. \tag{8}$$

Proposition 1 *Given condition (8), innovators enter only at date 0 and* $k_0 \in (0, N)$ *solves*

$$\underbrace{\frac{1}{N - k_0} \int_0^{\infty} e^{-rt} \left(\frac{N}{k_t} - 1\right) A k_t^{1-\beta} dt}_{v_0 - u_0} = c, \tag{9}$$

where k_t *is given by Eq. (4).*

Proof. See Internet Appendix IA-A1. ■

In Proposition 1, condition (8) guarantees the existence of interior solution for k_0: Isoelastic demand in Eq. (1) implies that $k_0 > 0$, otherwise p_0 would be infinite, and condition (8) ensures that $k_0 < N$.[8] The equilibrium price of the good declines over time (as more producers enter), and so does the value of the idea. As a result, $v_t - u_t < c$ for all $t > 0$ so that innovators enter only at date 0. Equation (9) yields the solution for k_0 which rises with A but falls with c, r and γ.

4 Intellectual property rights and licensing

We now introduce IP protection and assume idea sellers negotiate with imitators and collect a licensing fee $\alpha\omega_t$, where ω_t is an imitator's present value at t. The parameter $\alpha \in [0, 1]$ is an idea seller's compensation share or licensing rate, which nests our baseline model (i.e., $\alpha = 0$).[9] We shall analyze two regimes that differ in how much revenue innovators get from idea sales.

[8]Note that if all agents were to innovate at date 0 (i.e., $k_0 = N$), product price would be fixed at $AN^{-\beta}$ and each firm's present value would be $v_0 = \frac{AN^{-\beta}}{r}$. However, $c > v_0 = \frac{AN^{-\beta}}{r}$ would be sufficient but not necessary for preventing the N agents from all innovating at date 0. In fact, Eq. (6) implies that when $k_0 = N$, a marginal outsider has the option value $u_0 = \frac{\gamma N}{r(r+\gamma N)} AN^{-\beta} > 0$ to become an imitator. Condition (8) hence eliminates the equilibrium in which $k_0 = N$ by requiring $c > v_0 - u_0$ for $k_0 = N$. See Internet Appendix IA-A1 for the formal proof.

[9]This bargaining protocol differs from Nash bargaining in which the innovator and imitator would split the joint surplus $\omega_t - u_t$ from the idea transfer. This alternative bargaining is easier to enforce

4.1 Regime 1: imitators cannot resell ideas

In Regime 1, the original innovators receive all of their ideas' sales revenue. An imitator may learn the idea from any incumbent producer or by reverse engineering, but he then has to pay the idea's *original* innovator.

Since an imitator cannot resell the innovation, his only revenue comes from selling the good, and his value ω_t satisfies

$$r\omega_t = p_t + \frac{d\omega_t}{dt}. \tag{10}$$

An innovator receives revenue from selling both the good and the idea. The number of ideas sold at t is $\gamma k_t (N - k_t)$, and the total date-t revenue from these sales is $\gamma k_t (N - k_t) \alpha \omega_t$. Proposition 2 will show that at equilibrium, innovators enter only at date 0. In that case, the total date-t revenue from idea sales is divided equally among the k_0 innovators. Therefore, the date-t value v_t of an innovator satisfies

$$rv_t = p_t + \frac{\gamma k_t (N - k_t)}{k_0} \alpha \omega_t + \frac{dv_t}{dt}. \tag{11}$$

An outsider's hazard rate for meeting a producer is γk_t. Therefore, his value at date t, u_t, satisfies

$$ru_t = \gamma k_t [(1 - \alpha)\omega_t - u_t] + \frac{du_t}{dt}, \tag{12}$$

with k_0 again satisfying (7).

4.2 Regime 2: imitators can resell ideas

In Regime 2, imitators can keep the proceeds from the ideas that they resell. An incoming idea buyer pays the agent from whom he copies the idea.

All producers now have the same value, i.e., $v_t = \omega_t$. Proposition 2 will show that innovators will again enter only at date 0. The revenue from a single idea sale is αv_t, and the total revenue from idea sales, $\gamma k_t (N - k_t) \alpha v_t$, is now shared by all the k_t producers. Therefore, v_t now satisfies

$$rv_t = p_t + \gamma (N - k_t) \alpha v_t + \frac{dv_t}{dt}. \tag{13}$$

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than a Nash bargaining because the courts need to know only ω_t and not the imitators' outside options. Section 7.4 shows that α does coincide with the Nash bargaining share in a limiting version of the model in which $N \to \infty$.

The value of an outsider, u_t, now satisfies

$$ru_t = \gamma k_t \left((1-\alpha)v_t - u_t\right) + \frac{du_t}{dt}. \quad (14)$$

Motivation for the two regimes.—Both regimes reflect practice. Regime 2 removes the burden of enforcing the no-reselling constraint, but imitators need to pay for a licensing fee that incorporates their future revenues from reselling, which may be infeasible if they are financially constrained. By contrast, Regime 1 requires a smaller licensing payment, but the no-reselling constraint on idea buyers may be hard to enforce.

4.3 Equilibrium outcomes

Proposition 2 will show that in both regimes innovators enter only at $t = 0$, so that the time path of firm numbers is given by Eq. (4). Thus the diffusion process is the same in both regimes, but k_0 will generally differ because in Eq. (7) both v_0 and u_0 depend on the regime.

Intuitively, in Regime 1, where imitators cannot resell ideas, each new imitator must trace and compensate the original innovator of the idea acquired through matching with an incumbent firm. Therefore, an innovator's value depends on his entry date. If a measure-0 outsider were to deviate from the equilibrium and enter as an innovator at date $\tau > 0$, his value at date $t \geq \tau$, denoted by v_t^τ, would satisfy

$$rv_t^\tau = p_t + \frac{\gamma k_t \left(N - k_t\right)}{k_\tau} \alpha \omega_t + \frac{dv_t^\tau}{dt}. \quad (15)$$

This differs from v_t in Eq. (11) because this deviator does not have the same $1/k_0$ chance to share the industry's licensing revenues as the date-0 innovators. His and his offspring's chances of matching with new imitators depend on the number of incumbents k_τ at his entry date τ. Each period, this deviator would have only a $1/k_\tau$ chance, instead of $1/k_0$, to disseminate his innovation and share the industry's licensing revenues. The later an innovator enters the market, the smaller the chance he has to disseminate his idea and collect licensing revenues. Since the number of firms continues to rise, it does not pay for an innovator to enter after date 0.

By contrast, in Regime 2, where imitators can resell ideas, an innovator's value at any date t does not depend on his entry date. If any innovator were to enter the industry after date 0, he would share the same value v_t as any incumbent, be it an innovator or an imitator. The free entry condition requires that $v_0 - u_0 = c$ and we can formally verify that at equilibrium $v_t - u_t < c$ for any $t > 0$ so that even in Regime 2, innovators enter only at date 0.

In what follows, we shall assume that innovation is costly enough to prevent N agents from all innovating at date 0. As Internet Appendixes IA-A2 and IA-A3 show, this requires that

$$c > \frac{r + \alpha\gamma N}{r(r + \gamma N)} A N^{-\beta}. \tag{16}$$

To distinguish Regimes 1 and 2, we use the superscripts **I** and **II**, and so the masses of date-0 innovators will be denoted by $k_0^{\mathbf{I}}$ and $k_0^{\mathbf{II}}$. Proposition 2 states the conditions that determine $k_0^{\mathbf{I}}$ and $k_0^{\mathbf{II}}$, respectively.

Proposition 2 *Given condition (16), market equilibrium yields:*

(A) In Regime 1, innovators enter only at date 0 and $k_0^{\mathbf{I}} \in (0, N)$ *solves*

$$\underbrace{\frac{1}{N - k_0} \int_0^{\infty} e^{-rt} \left(\alpha \frac{N}{k_0} + (1 - \alpha) \frac{N}{k_t} - 1 \right) A k_t^{1-\beta} dt}_{v_0 - u_0} \quad = \quad c; \tag{17}$$

(B) In Regime 2, innovators enter only at date 0 and $k_0^{\mathbf{II}} \in (0, N)$ *solves*

$$\underbrace{\frac{1}{N - k_0} \int_0^{\infty} e^{-rt} \left(\left(\frac{N}{k_0} \right)^{\alpha} \left(\frac{N}{k_t} \right)^{1-\alpha} - 1 \right) A k_t^{1-\beta} dt}_{v_0 - u_0} \quad = \quad c; \tag{18}$$

where, in each regime, k_t *is given by Eq. (4).*

Proof. See Internet Appendixes IA-A2 and IA-A3. ■

4.4 Properties of equilibrium

Equations (17) and (18) pin down the solutions for $k_0^{\mathbf{I}}$ and $k_0^{\mathbf{II}}$, respectively. These implicit functions can be simplified under unit price elasticity ($\beta = 1$) in the following two examples: $\alpha = 0$ and $\alpha = 1$. In these two cases, Eqs. (17) and (18) are equivalent:

$$k_0^{\mathbf{I}} = k_0^{\mathbf{II}} = \frac{A}{c(r + \gamma N)} \qquad \text{(when } \alpha = 0\text{)}, \tag{19}$$

$$k_0^{\mathbf{I}} = k_0^{\mathbf{II}} = \frac{A}{rc} \qquad \text{(when } \alpha = 1\text{)}, \tag{20}$$

i.e., Regimes 1 and 2 generate the same mass of innovators in each case. This follows because, when idea sellers are not compensated at all ($\alpha = 0$) or when they are fully compensated ($\alpha = 1$), innovators' payoffs do not depend on whether they are paid for their ideas directly (in Regime 1) or indirectly (in Regime 2). Moreover, Eqs. (19) and (20) suggest that a larger market size A encourages innovation, while a bigger innovation cost c or a higher interest rate r does the opposite. Across these two cases, k_0 is larger when idea sellers are fully compensated, which suggests α has a positive effect on innovation. The effect of the diffusion rate γ is more involved. In Eq. (19), γ shows a negative effect on k_0, but it plays no role in Eq. (20), which suggests the effect may depend on the values of α and β.[10]

The intuition from the special cases generalizes. Assuming condition (16) holds, comparing Eqs. (17) and (18) yields the findings in Propositions 3 and 4:

Proposition 3

$$(A) \quad k_0^{\mathbf{I}} \;\; \textit{and} \;\; k_0^{\mathbf{II}} \begin{cases} \textit{rise with } \alpha \textit{ and } A, \\ \textit{fall with } c \textit{ and } r. \end{cases} \tag{21}$$

(B) All parameters being equal across the two regimes,

$$\frac{k_0^{\mathbf{I}}}{k_0^{\mathbf{II}}} = \begin{cases} 1 & \textit{for } \alpha \in \{0, 1\} \\ > 1 & \textit{for } \alpha \in (0, 1) \end{cases} . \tag{22}$$

Proof. See Internet Appendix IA-A4. ■

Proposition 3 shows that the intuition from the special examples holds generally. It also shows that for $\alpha \in (0, 1)$, all else equal, fewer innovators enter in Regime 2 than in Regime 1. This is because innovators' revenues get discounted when they collect the payoff of ideas indirectly in Regime 2. Since the two regimes share the same diffusion process, this implies that industry output is higher for all t under Regime 1 due to its larger entry of innovators at date 0.

Next, we ask how the diffusion rate γ affects innovation. The effect of γ on $k_0^{\mathbf{I}}$ and $k_0^{\mathbf{II}}$ hinges on the values of α and β as follows:

[10] Note when $\beta = 1$, industry revenue becomes constant over time, so the diffusion rate γ would not affect the present value of industry revenue which is the value of innovators when $\alpha = 1$.

Proposition 4 *(A) For inelastic or unit elastic demand (i.e., $\beta \geq 1$),*

$$k_0^{\mathbf{I}} \text{ and } k_0^{\mathbf{II}} \begin{cases} \text{fall with } \gamma \text{ when } \beta \geq 1 > \alpha, \\ \text{do not vary with } \gamma \text{ when } \beta = \alpha = 1. \end{cases}$$

(B) For elastic demand (i.e., $\beta < 1$),

$$k_0^{\mathbf{I}} \text{ and } k_0^{\mathbf{II}} \begin{cases} \text{fall with } \gamma \text{ if } \alpha \text{ is sufficiently low,} \\ \text{rise with } \gamma \text{ if } \alpha \text{ is sufficiently high.} \end{cases}$$

Proof. See Internet Appendix IA-A5. ■

The intuition for these results is as follows: There are two channels through which diffusion affects innovation. One is the negative price effect, captured by β—the faster the diffusion, the lower the revenue from selling the good. The other is the positive idea-selling effect, captured by α—the faster the diffusion, the larger the revenue from selling the idea. When demand is inelastic ($\beta > 1$), a faster inflow of imitators reduces the industry revenue stream, so the price effect dominates and the innovators' value at date 0 drops even with the highest compensation share ($\alpha = 1$). This is also true for the unit demand elasticity case when $\alpha < \beta = 1$. When demand is elastic ($\beta < 1$), a faster inflow of imitators raises the industry revenue stream. If α is sufficiently high compared with β, the idea-selling effect dominates, which raises the incentive to innovate. Otherwise, the price effect dominates, which dampens innovation.

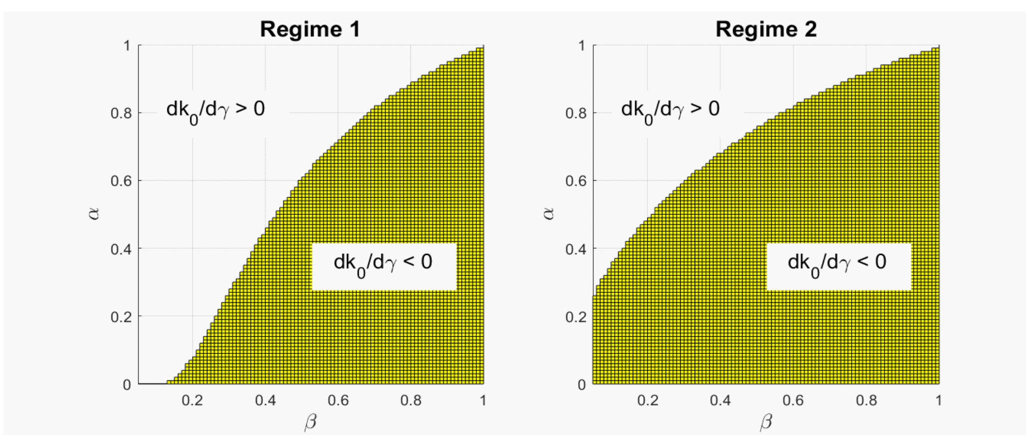

Fig. 4. Effect of γ on Innovation under elastic Demand

Note: Figure 4 numerically solves $dk_0/d\gamma$ for the full parameter space of $\alpha \in [0,1]$ when $\beta < 1$. The simulation assumes A=1, N=1, r=0.05, c=30, and γ= 0.55.

Figure 4 illustrates Proposition 4(B): under elastic demand (i.e., $\beta < 1$), a higher diffusion rate γ reduces innovation k_0 (i.e., $\partial k_0/\partial\gamma < 0$) if β is sufficiently high or α is sufficiently low in Regimes 1 and 2. Comparing regimes, a higher γ is more likely to reduce innovation in Regime 2 for a given value of α because innovators are paid indirectly.

The effect of the diffusion rate γ on innovation has direct implications for industry growth patterns. If a rise in γ raises k_0, output would rise at all dates. Proposition 4(B) shows this can happen for a subset of parameter values when demand is elastic (i.e., $\beta < 1$) and idea sellers' bargaining share α is high, as illustrated by the unshaded areas in Fig. 4. For the remaining parameter values (as shown by the shaded areas in Fig. 4) or when demand is inelastic or unit elastic (i.e., $\beta \geq 1$), a rise in γ would reduce k_0. Proposition 5 proves that the higher-γ trajectory would then overtake the lower-γ one:

Proposition 5 *(Industry overtaking) In both Regimes 1 and 2, whenever a larger diffusion rate $\gamma^i (> \gamma^j)$ leads to a smaller $k_0^i (< k_0^j)$, there exists a date t' where*

$$t' = \frac{1}{(\gamma^i - \gamma^j)N} \ln\left(\frac{k_0^j\left(N - k_0^i\right)}{k_0^i\left(N - k_0^j\right)}\right) \tag{23}$$

such that $k_t^i > k_t^j$ for all $t > t'$.

Proof. Eq. (4) states that $k_t^l = \frac{Ne^{\gamma^l Nt}}{e^{\gamma^l Nt} + \frac{N}{k_0^l} - 1}$ for $l \in \{i, j\}$. Therefore, $k_t^i > k_t^j \iff \frac{k_t^i}{N - k_t^i} > \frac{k_t^j}{N - k^j} \iff \frac{e^{\gamma^i Nt}}{\frac{N}{k_0^i} - 1} > \frac{e^{\gamma^j Nt}}{\frac{N}{k_0^j} - 1} \iff t > t'$, where t' is given by Eq. (23). ■

Thus the model generates overtaking when two sectors or two locations face the same environment (i.e., same A, α, c and N) except that γ is higher in one than in the other. For example, a high-tech sector may use technology based on ideas that spread faster than they do in other sectors. As a result, high-tech sectors would have a higher γ and would start smaller but grow faster.

There could also be legal or regulatory reasons why γ differs over regions. For example, California bans non-compete contracts and therefore indirectly encourages labor turnover and spin-offs,[11] whereas Massachusetts enforces those contracts. As Saxenian (1994), Gilson (1999), and Franco and Mitchell (2008) argue, this may help

[11] Spin-offs are firms founded by former employees of incumbent firms to conduct businesses in

explain why Silicon Valley has overtaken Massachusetts' Route 128 in developing high-tech industry. Our model generates such a pattern if γ is higher in California than in Massachusetts. Under conditions in Proposition 4, Route 128 would then have a higher initial entry rate of firms (i.e., a higher k_0) than Silicon Valley. Thus our model would predict the type of overtaking portrayed in Fig. 5.[12]

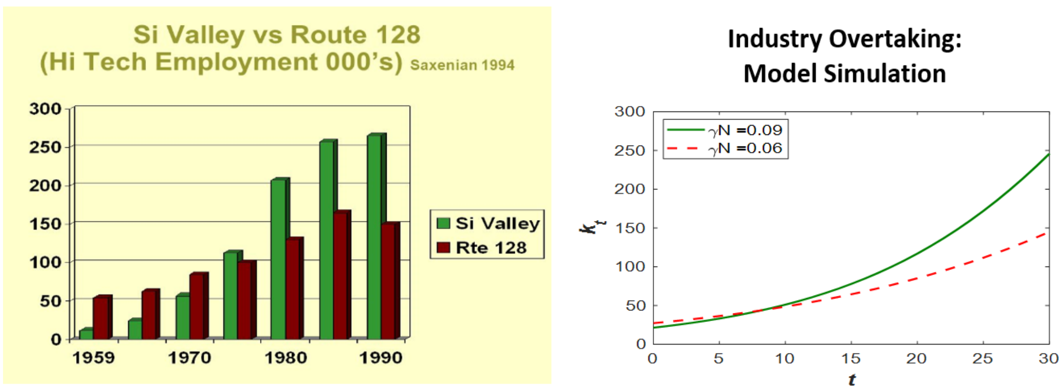

Fig. 5. Industry Overtaking: Data and Model

Note: Figure 5 illustrates that our model (right) can generate the overtaking pattern in Saxenian (1994)'s data (left). The simulation assumes $\alpha = 0$, $\beta = 1$, $A/c = 3$, $N = 1000$, $r = 0.05$, and plots k_t in two locations with high *vs.* low diffusion rate (γN=0.09 vs. γN=0.06).

5 Welfare and policy analysis

We now study the welfare implications of the model. Consumers' utility from consuming output k is the integral under the demand curve. For $\beta \in (0,1)$, aggregate utility at output k is

$$U(k) = \int_0^k As^{-\beta} ds = \frac{A}{1-\beta} k^{1-\beta}. \tag{24}$$

For $\beta \geq 1$ the above integral is infinite; to ensure consumer surplus is finite, we put a maximum, $A\varepsilon^{-\beta}$, on the willingness to pay. Let $D(s) = \min\left(A\varepsilon^{-\beta}, As^{-\beta}\right)$

the same industry. While our model does not include labor in production, employees could work in the same company but not be involved in directly producing the new product. And then they could learn about the idea internally.

[12]Figure 5 is not a quantitative analysis, only an illustrative example. We conduct a quantitative analysis on noncompetes and industry overtaking in the context of the early automobile industry in Section 6.3.2.

and define aggregate utility as $U(k) = \int_0^k D(s)\, ds = A\left(\int_0^\varepsilon \varepsilon^{-\beta} ds + \int_\varepsilon^k s^{-\beta} ds\right)$ where $\varepsilon \ll k$. Accordingly, for $\beta = 1$ we have

$$U(k) = A(\ln k + 1 - \ln \varepsilon), \tag{25}$$

and for $\beta > 1$, we have

$$U(k) = \frac{\beta A}{\beta - 1}\varepsilon^{1-\beta} + \frac{A}{1-\beta}k^{1-\beta}. \tag{26}$$

5.1 Planner's problem

The planner tries to maximize social welfare W_0 given by

$$W_0 = \int_0^\infty e^{-rt} U(k_t)\, dt - ck_0, \tag{27}$$

where k_t follows Eq. (4).

Given the logistic diffusion process, a matching externality arises when N is finite because an innovator's matching rate $\frac{dk_t/dt}{k_t} = \gamma(N - k_t)$ falls with k_t while an imitator's matching rate $\frac{dk_t/dt}{N-k_t} = \gamma k_t$ rises with k_t. Section 7.4 shows that if N grows but γ shrinks so that $\gamma N \to \lambda > 0$ as $N \to \infty$, Eq. (3) converges to $\frac{dk_t}{dt} = \lambda k_t$ so that $k_t = k_0 e^{\lambda t}$, which coincides with exponential diffusion. The externality then disappears because neither matching rate depends on k_t since $\frac{dk_t/dt}{k_t} \to \lambda$ for innovators and $\frac{dk_t/dt}{N-k_t} \to 0$ for imitators. However, exponential diffusion does not match general industry growth patterns and the policy implications also differ.

In what follows we assume that condition (8) holds so that innovation is costly enough to keep the socially optimal entry mass of innovators k_0^* to be an interior solution (i.e., $k_0^* < N$) at date 0. We then have the following proposition:

Proposition 6 *Given condition (8), social optimum requires that innovators enter only at date 0 and k_0^* solves*

$$\underbrace{\int_0^\infty e^{-(r+\gamma N)t}\left(\frac{k_t}{k_0}\right)^2 A k_t^{-\beta} dt}_{\text{marginal social return to } k_0} \;=\; c. \tag{28}$$

Proof. See Internet Appendix IA-A6. ■

The claim of Proposition 6 is intuitive. As of date $\tau \geq 0$, the social return to innovation is

$$SR_\tau = \int_\tau^\infty e^{-r(t-\tau)} U(k_t)\, dt$$

and one can verify that the marginal social return $\partial SR_\tau / \partial k_\tau$ is strictly decreasing in k_τ. So if k_0^* is chosen so that $\partial SR_0 / \partial k_0 = c$ at date 0, thereafter $\partial SR_\tau / \partial k_\tau < c$ for any $\tau > 0$. Hence, it is socially optimal to innovate only at date 0. And the condition $\partial SR_0 / \partial k_0 = c$ yields Eq. (28). Finally, welfare given by Eq. (27) is strictly concave in k_0, so for $k_0^* < N$ to hold, one needs $\frac{dW_0}{dk_0}|_{k_0=N} < 0$, which yields condition (8). Henceforth, we assume that condition (8) always holds.

Denote the socially optimal welfare by W_0^*. We have the following comparative-static findings: both k_0^* and W_0^* increase in market size A but decrease in innovation cost c and interest rate r. Moreover, while k_0^* falls with the diffusion rate γ if the demand is not too elastic, W_0^* always rises with γ (see Internet Appendix IA-A6 for the proofs).

Next, we discuss policy implications of the model. We show that the planner can achieve k_0^* by enforcing a socially optimal licensing rate α^*,[13] and that policy interventions that raise the diffusion speed γ are socially desirable.

5.2 Socially optimal licensing

For any $\alpha \in [0, 1]$, the interior solution condition (16) faced by market participants relates to the interior solution condition (8) faced by the social planner by the inequality

$$\underbrace{\frac{r + \alpha\gamma N}{r(r + \gamma N)} A N^{-\beta}}_{\text{RHS of (16)}} \geq \underbrace{\frac{A N^{-\beta}}{r + \gamma N}}_{\text{RHS of (8)}}.$$

The two are identical if $\alpha = 0$.

Denote the socially optimal compensation shares for idea sellers in Regimes 1 and 2 by $\alpha^{\mathbf{I}*}$ and $\alpha^{\mathbf{II}*}$, respectively. Assume that condition (8) always holds. If there exists an $\alpha' \in (0, 1]$ such that the inequality (16) becomes an equality (i.e.,

[13] The planner can also achieve k_0^* by providing an innovation subsidy (or tax) s^* (see Internet Appendix IA-A7 for the analysis).

$c = \frac{r+\alpha'\gamma N}{r(r+\gamma N)} AN^{-\beta}$), one can prove that the planner could choose the optimal shares $\alpha^{\mathbf{I}*}$ and $\alpha^{\mathbf{II}*}$ such that $0 < \alpha^{\mathbf{I}*} < \alpha^{\mathbf{II}*} < \alpha'$ to achieve the socially optimal k^* in Regimes 1 and 2, respectively. This is illustrated by Fig. 6(A). Alternatively, if the inequality (16) holds for any $\alpha \leq 1$, one can prove that $0 < \alpha^{\mathbf{I}*} < \alpha^{\mathbf{II}*} < 1$, as illustrated by Fig. 6(B). Note that $\alpha^{\mathbf{I}*} < \alpha^{\mathbf{II}*}$ results from Proposition 3: if there is a value of α that leads to $k_0^{\mathbf{I}} = k^*$ in Regime 1, the same α would lead to $k_0^{\mathbf{II}} < k^*$ in Regime 2, so a larger value of α is needed to achieve k^* in Regime 2.

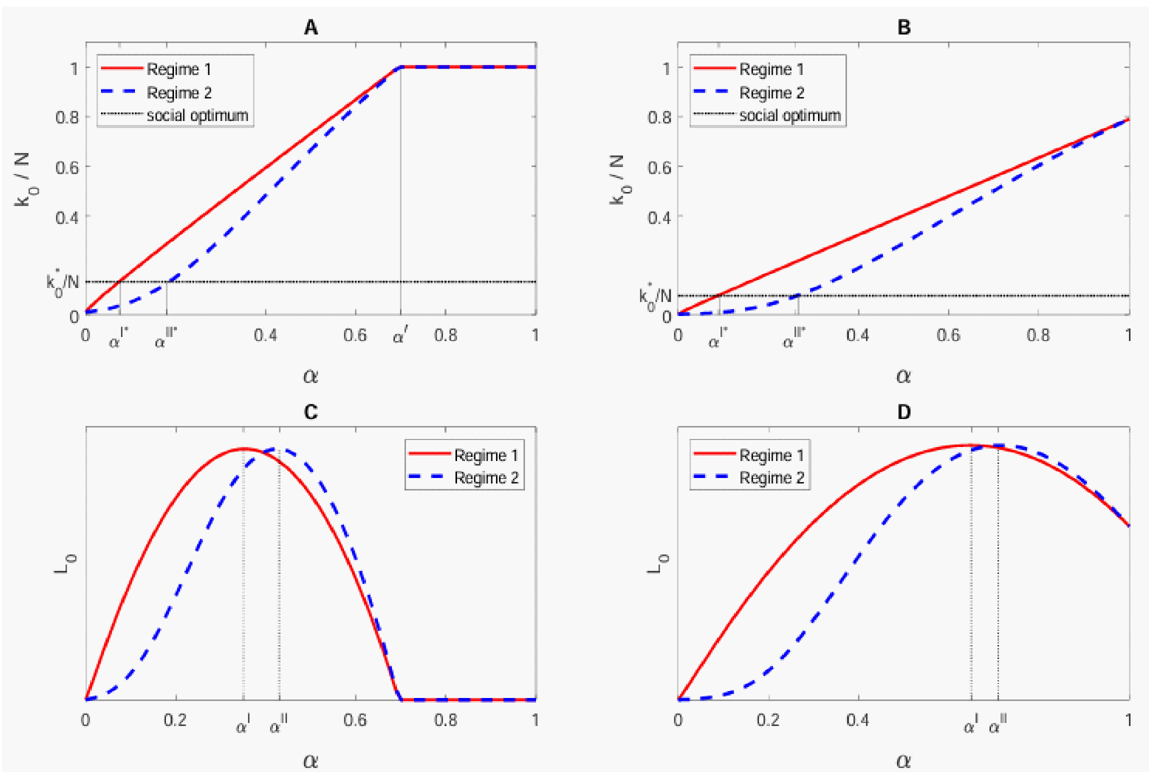

Fig. 6. Welfare-Maximizing vs. Licensing-Revenue-Maximizing α

Note: Figure 6 compares the welfare-maximizing α and the licensing-revenue-maximizing α across Regimes 1 and 2, as well as their effects on the entry share of innovators (k_0/N) and the present value of industry licensing revenues at date 0 (L_0). Figures 6(A) and 6(C) illustrate a scenario in which all firms would enter as innovators for some $\alpha' \leq 1$, whereas Figures 6(B) and 6(D) depict a case where not all firms would enter as innovators even when $\alpha = 1$.

The above findings are stated formally in Proposition 7.

Proposition 7

$$0 < \alpha^{\mathbf{I}*} < \alpha^{\mathbf{II}*} < 1. \tag{29}$$

Proof. See Internet Appendix IA-A8. ■

Proposition 7 holds because if $\alpha^{\mathbf{I}} = \alpha^{\mathbf{II}} = 0$, no innovator would internalize knowledge spillovers they create for imitators, so fewer innovators enter than is socially optimal. On the other hand, if $\alpha^{\mathbf{I}} = \alpha^{\mathbf{II}} = 1$, innovators would not fully internalize the negative matching externality they impose on one another, so more innovators would enter than is socially optimal.

Licensing-revenue-maximizing α.—In general, the licensing rate that maximizes licensing revenue does not maximize social welfare. To see this, we plot the licensing revenues in Fig. 6(C) and 6(D), corresponding to Fig. 6(A) and 6(B) respectively. The present value of total licensing revenue paid to innovators is

$$L_0^{\mathbf{I}} = \int_0^\infty e^{-rt}\alpha(k_t - k_0)Ak_t^{-\beta}dt \tag{30}$$

in Regime 1, and

$$L_0^{\mathbf{II}} = \int_0^\infty e^{-rt}\left(\left(\frac{k_t}{k_0}\right)^\alpha k_0 - k_0\right)Ak_t^{-\beta}dt \tag{31}$$

in Regime 2 (See Internet Appendix IA-A9 for the proof).

Figure 6 compares the socially optimal α and the licensing-revenue-maximizing α for the two regimes. Figures 6(A) and 6(C) which are plotted using the same parameter values, show the case in which all the agents would innovate at date 0 when α is sufficiently large. As a function of α, licensing revenue has an inverted-U shape: When $\alpha = 0$, no imitator pays the licensing fee, so licensing revenue is zero; when α is sufficiently large, no one enters as an imitator, so licensing revenue is again zero. Regimes 1 and 2 each have a unique α that maximizes licensing revenue, and each is larger than the socially optimal α. In fact, because the innovation cost c is ignored in the licensing revenue maximization, the value of α that maximizes licensing revenue would induce too many agents to innovate at date 0 and in so doing, overspend the innovation cost c. Figures 6(B) and 6(D) (that also use the same parameter values) show a similar finding in the case where not all the agents would innovate at date 0 for any α. The licensing revenue again displays an inverted-U curve with regard to α and remains positive at $\alpha = 1$.[14]

[14] Benhabib, Perla, and Tonetti (2021) also derive an inverted-U curve of licensing revenue with regard to the licensing rate. When the licensing rate is low, increasing it would not deter much imitation and hence would raise licensing revenue. However, if licensing rate is high, increasing it further would deter too much imitation and would ultimately generate less licensing revenue.

Policy implications.—Our analysis offers new insights into licensing policies. In practice, licensing rates in each industry are usually determined by negotiation between an IP owner and a licensee for a split of the latter's profit. In an IP violation, the court evaluates and grants the IP owner damages also based on a hypothetical negotiation between the two parties (see e.g., Razgaitis, 1999, Goldscheider, 2011). Historically, the "25 percent rule" has commonly been used in licensing practice (see Internet Appendix IA-B1),[15] which prescribes that the licensor should receive 25 percent of the licensee's profit because the latter takes on more responsibility of producing and marketing the good and should keep a larger share. The rule was also upheld by courts until 2011, after which courts have relied on more comprehensive factors—specifically, the Georgia-Pacific factors, named after a landmark 1970 court case—to reconstruct hypothetical negotiations and determine reasonable licensing rates.

Our analysis, however, suggests that neither the fixed 25 percent rule nor bilateral bargaining is likely to yield the socially optimal licensing rate. Rather, the optimal rate should address industry-level learning and matching externalities that lie beyond the profit considerations of individual market participants, and it should reflect market factors (e.g., demand elasticity, diffusion speed, and innovation cost) and the type of licensing (e.g., sublicensing permitted or not).

5.3 Socially optimal diffusion

Consider the baseline model in which incumbents are not compensated by imitators (i.e., $\alpha = 0$). In this case, imitation causes the maximal disincentive for innovation. From the social welfare viewpoint, should the planner reduce the diffusion speed γ (e.g., by restricting entry of imitators) to enhance incentives for innovation?

Note that when $\alpha = 0$, Proposition 4 shows that the mass of innovators decreases in γ. Therefore, a policy that reduces the diffusion rate γ would boost the entry of innovators. Such a policy, however, would not necessarily raise welfare. Internet Appendix IA-A10 shows that for the unit demand elasticity case (i.e., $\beta = 1$), social

[15] Using a sample of 347 firms across 15 industries from 1990 to 2000, Goldscheider *et al.* (2002) find most royalty rates conform to the 25 percent rule. Using more recent data, Kemmerer and Lu (2012) show such a result holds broadly in a sample of 3,887 companies across 14 industries based on 3,015 patent licensing transactions collected over a 21-year period prior to 2007.

welfare always increases in the diffusion rate γ. This suggests that in the numerical example of Fig. 5, a higher value of γ not only helps Silicon Valley overtake Route 128 in industry size but also yields higher social welfare.[16] In Section 6, we empirically analyze the U.S. automobile and PC industries as well as 16 other industries studied by Gort and Klepper (1982). In all the cases we find that welfare rises with γ.

6 Empirical applications

In this section, we use the model to quantitatively replicate industry evolution patterns and to assess the associated welfare and policy implications. We first examine two industries in detail—automobiles and personal computers—and then extend the analysis to 16 additional industries originally studied by Gort and Klepper (1982).

6.1 Auto and PC: parameter estimation

We estimate key model parameters for the U.S. auto and PC industries during their pre-shakeout phases. We first estimate the diffusion parameters from the time path of firm counts and then the demand parameters from price and output data together with instrumental variables. The estimates are used for calibration in Section 6.2.

6.1.1 Data

The data come from the following sources:

Auto.—Firm counts are derived from Smith (1970), which documents every U.S. car make from 1895 to 1969. Annual price and output data for 1900–1929 are taken from Thomas (1977). Information on spin-off firms is drawn from Cabral *et al.* (2018).

PC.—Firm counts are compiled from Stavins (1995) and the *Thomas Register of American Manufacturers*, covering desktop and portable computers. Annual price and output data come from the *Information Technology Industry Data Book*, and import values for computers and accessories are from the U.S. Census Bureau.

In addition, annual data of U.S. population, real GDP, and the GDP deflator are from $Williamson$(2020).

[16] A comprehensive numerical analysis shows that welfare always increases in γ for different values of $\alpha \in [0, 1]$ and $\beta > 0$ for both Regimes 1 and 2.

6.1.2 Estimation procedure

Diffusion estimation We estimate three parameters: the total pool of potential entrants N, the diffusion rate γN, and the initial innovator mass k_0. Taking N as the observed peak firm count, we estimate a logistic diffusion equation derived from Eq. (4):

$$\ln \frac{k_t}{N - k_t} = z + \lambda t, \tag{32}$$

where k_t is firm count in year t.[17] Ordinary least squares (OLS) yields the intercept z and slope λ. We then recover k_0 and γN from these estimates given that $z = \log(k_0/(N - k_0))$ and $\lambda = \gamma N$.

For robustness checks, we re-estimate the diffusion process by using a subsample and by using the matching function (3) that allows differencing the data. The results are virtually identical (see Internet Appendixes IA-B2 and IA-B4).

Demand estimation We estimate two demand parameters, β and $\tilde{A}$, for the isoelastic demand $p_t = \tilde{A} Q_t^{-\beta}$.[18] In so doing, we estimate a log-log demand function

$$\ln Q_t = a - \phi \ln p_t,$$

where Q_t and p_t denote industry output and real price (in 2012 prices) in year t, respectively. To address price endogeneity, we apply two-stage least squares (2SLS), using lagged output per firm as an instrument. We then recover β and $\tilde{A}$ from the second-stage estimates given that $\beta = 1/\phi$ and $\tilde{A} = e^{a/\phi}$.

Output per firm, while assumed fixed in our theory, grew over the long term due to technological progress. If unobserved demand shocks are not serially correlated, lagged output per firm can serve as a valid supply shifter to trace out the demand curve. To check robustness, we use the *spin-off share* (for automobiles) and *real imports* (for PCs) as alternative instruments,[19] and we also re-run the 2SLS regressions

[17] Note that Eq. (4) implies $\frac{k_t}{N - k_t} = \frac{e^{\gamma N t}}{\frac{N}{k_0} - 1}$, which leads to Eq. (32).

[18] In the model, we normalize a firm's output to 1, so $Q_t = k_t$ and the inverse demand function is $p_t = A k_t^{-\beta}$. In the empirical analysis, we denote a firm's output by q, so $Q_t = q k_t$ and the corresponding inverse demand function becomes $p_t = \tilde{A} Q_t^{-\beta}$.

[19] Spin-off firms tend to outperform *de novo* entrants (Klepper, 2010). Accordingly, the share of spin-off firms in the automobile industry affects supply and can serve as an alternative instrument

by controlling changes of population and per capita income over time. The results remain unchanged (see Internet Appendixes IA-B3 and IA-B5).

6.1.3 Estimation results

Table 1 summarizes the parameter estimates for both industries. All coefficient estimates are highly significant (see Internet Appendix IA-B for more details). The logistic regressions fit remarkably well (adjusted $R^2 > 0.95$). The PC industry exhibits a higher price elasticity ($\phi = 14.58$, $\beta = 0.07$) than the automobile industry ($\phi = 3.61$, $\beta = 0.28$).

Table 1. Parameter Estimation Results for Auto and PC

Parameter	**Automobile**	**Personal Computer**
Diffusion Estimation	Sample: 1895–1910 (16 obs)	Sample: 1975–1992 (18 obs)
Intercept (z)	-4.17***(0.28)	-5.49***(0.29)
Slope ($\lambda = \gamma N$)	0.54***(0.03)	0.58***(0.03)
N (peak firm count)	210	435
Set $k_0 = N/(1+e^{-z})$	3.20	1.78
Demand Estimation	Sample: 1900–1929 (30 obs)	Sample: 1975–1992 (18 obs)
Intercept (a)	47.04***(12.52)	137.15***(12.52)
Price coefficient ($-\phi$)	-3.61***(0.48)	-14.58***(1.49)
Set $\beta = 1/\phi$	0.28	0.07
Set $\tilde{A} = e^{a/\phi}$	456,102	12,170

Notes: *** significant at 1%; standard errors in parentheses. Demand estimation uses 2SLS with lagged output per firm as the instrument, which is highly significant in the first stage.

to identify demand (Cabral *et al.*, 2018). Similarly, import growth shifts the supply curve in the emerging PC industry and thus provides another valid instrument for demand estimation. In our analysis, we use economically grounded instruments and clearly state the identifying assumptions. Nevertheless, demand estimation may still face complications such as substitution or complementarity across products. Berry and Haile (2021) offer a comprehensive discussion of these challenges and possible solutions.

6.2 Auto and PC: model calibration

To calibrate the model, we choose values of N, γN, k_0, β and $\tilde{A}$ from the diffusion and demand estimation above. In the model, a firm's output is normalized to 1 per period. While this does not affect the theoretical analysis, we account for a firm's production size in the empirical applications. In doing so, we denote by q a firm's output and by Q the industry output, so $Q_t = qk_t$ at date t. Accordingly, we revise Eqs. (17), (18) and (28) by replacing A with $\tilde{A}q^{1-\beta}$ (where $\tilde{A}$ and β are from the demand estimation above) as follows:

$$\text{Regime 1:} \quad \frac{1}{N-k_0}\int_0^\infty e^{-rt}\left(\alpha\frac{N}{k_0}+(1-\alpha)\frac{N}{k_t}-1\right)\tilde{A}q^{1-\beta}k_t^{1-\beta}dt = c; \tag{33}$$

$$\text{Regime 2:} \quad \frac{1}{N-k_0}\int_0^\infty e^{-rt}\left(\left(\frac{N}{k_0}\right)^{\alpha}\left(\frac{N}{k_t}\right)^{1-\alpha}-1\right)\tilde{A}q^{1-\beta}k_t^{1-\beta}dt = c; \tag{34}$$

$$\text{Social optimum:} \quad \int_0^\infty e^{-(r+\gamma N)t}\left(\frac{k_t}{k_0}\right)^2 \tilde{A}q^{1-\beta}k_t^{-\beta}dt = \ c. \tag{35}$$

In the automobile case, a firm on average produced less than 1,000 cars a year up to 1910, and we calibrate $q = 900$ based on output per firm in 1910 and $\tilde{A}q^{1-\beta} = 61.11$ (million). In the PC case, using output per firm in 1992, we calibrate $q = 27,500$ and $\tilde{A}q^{1-\beta} = 163.63$ (million). We also set $r = 0.05$.

Licensing played little role in the creation of automobile and PC firms in the early life cycle of the two industries. Thus, we set $\alpha = 0$ for both industries in the calibration. When $\alpha = 0$, Regimes 1 and 2 both reduce to the free-imitation baseline given by Eq. (9). We then use

$$\text{Free imitation:} \quad \frac{1}{N-k_0}\int_0^\infty e^{-rt}\left(\frac{N}{k_t}-1\right)\tilde{A}q^{1-\beta}k_t^{1-\beta}dt = c \tag{36}$$

to solve for c.

Table 2 summarizes the benchmark parameter values calibrated for the automobile and PC industries, which we shall use to study the socially optimal licensing rate $\alpha^* > 0$ for a counterfactually successful IP protection system.[20]

[20] We consider alternative model parameter values in Section 6.4 for robustness checks.

Table 2. Parameter Values for the Model Calibration

	α	r	N	γN	k_0	β	$\tilde{A}q^{1-\beta}$	implied c
Auto	0	0.05	210	0.54	3.20	0.28	61.11	172.70
PC	0	0.05	435	0.58	1.78	0.07	163.63	986.87

Figure 7 plots the calibrated model dynamics for the automobile industry. The number of firms k_t grows along a logistic curve. Meanwhile, v_t falls while u_t rises over time. The initial difference $v_0 - u_0$ equals the model-implied innovation cost $c_{\text{Auto}} = \$172.70$ million (in 2012 prices). By 1910, the value of a producer v_t falls to \$274 million and the value of a future imitator rises to \$250 million. Because almost all the potential entrants N have entered the industry by then, the total value of firms $v_{1910}k_{1910}$ is very close to the present value of the industry revenue $p_{1910}Q_{1910}/r$.

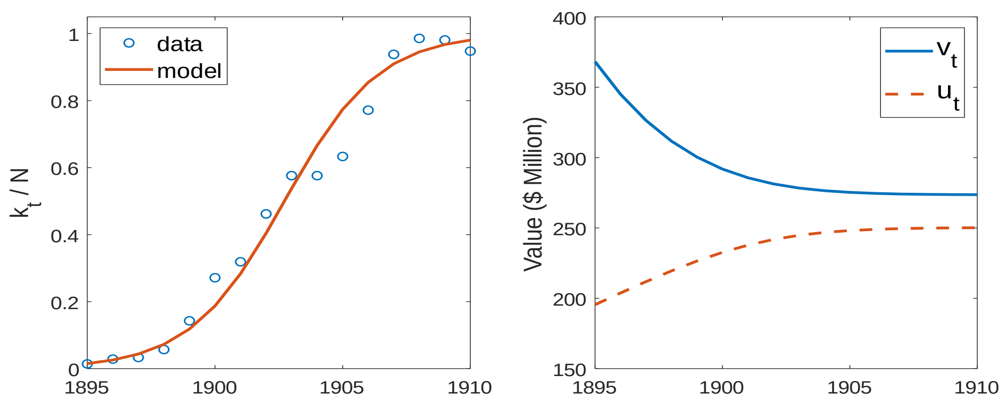

Fig. 7. Model Calibration: Auto

Note: Figure 7 plots the model-calibrated automobile firm numbers as a share of total potential entrants (left panel) and the model-implied value of an incumbent firm versus that of an outsider (right panel), 1895-1910.

Figure 8 plots the calibration results for the PC industry. Again, the number of firms k_t grows along a logistic curve, and v_t falls while u_t rises over time. The initial difference $v_0 - u_0$ equals the model-implied innovation cost $c_{\text{PC}} = \$\ 986.87$ million (in 2012 prices). By 1992, the value of a producer v_t falls to \$2.14 billion and the value of a future imitator rises to \$1.97 billion. Because almost all the potential entrants have entered by then, the total value of firms $v_{1992}k_{1992}$ is very close to the present value of the industry revenue $p_{1992}Q_{1992}/r$.

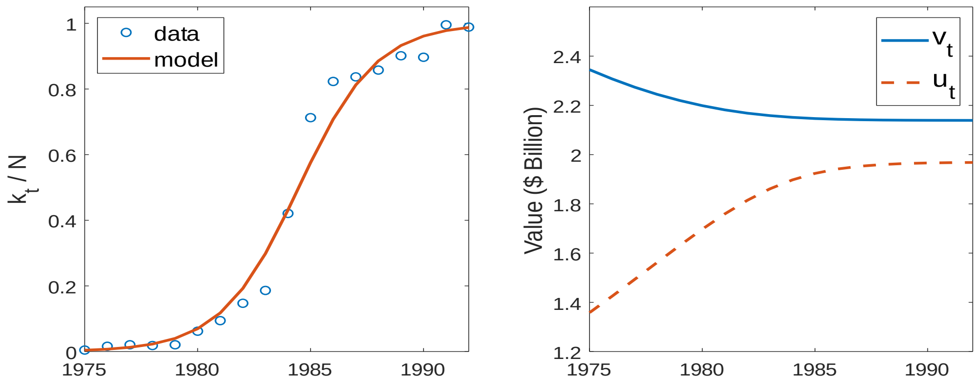

Fig. 8. Model Calibration: PC

Note: Figure 8 plots the model-calibrated PC firm numbers as a share of total potential entrants (left panel) and the model-implied value of an incumbent firm versus that of an outsider (right panel), 1975-1992.

6.3 Auto and PC: counterfactual analyses

Using the calibrated model, we conduct counterfactual analyses to evaluate welfare and policy outcomes.

6.3.1 Socially optimal licensing rate

We begin by evaluating the effect of the compensation share α in Regimes 1 and 2, starting with the auto industry. Given the innovation cost c_{Auto} derived from the model calibration, we solve for the equilibrium industry dynamics under each counterfactual value of $\alpha \in (0, 1]$. In particular, Eqs. (33) and (34) determine the counterfactual mass of entering innovators k_0 at date 0. Figure 9 shows that k_0 rises monotonically with α in both Regimes 1 and 2 when $0 < \alpha < 0.60$, and that Regime 1 yields a higher k_0 than Regime 2. When $\alpha \geq 0.60$, k_0 in both regimes reaches the corner solution $k_0 = N$. Equation (35) identifies the socially optimal entry share of innovators, $k_0^*/N = 14.88\%$, which is substantially higher than 1.52% under $\alpha = 0$. The social optimum can be attained by setting $\alpha^{\mathbf{I}*}_{\text{Auto}} = 0.069$ in Regime 1 or $\alpha^{\mathbf{II}*}_{\text{Auto}} = 0.166$ in Regime 2, yielding a total social surplus of $W^*_{0,\text{Auto}} = \$64.46$ billion (in 2012 prices), or 12.63% higher than under the free-imitation benchmark.

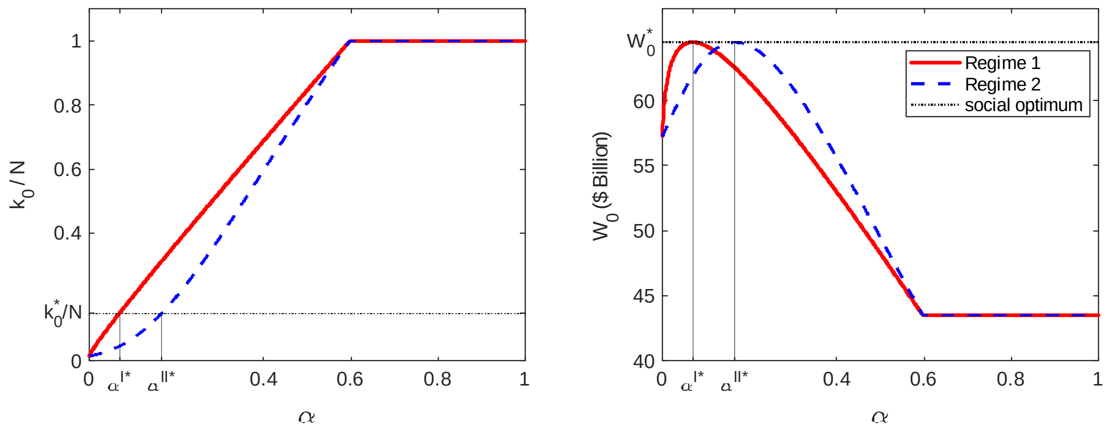

Fig. 9. Effect of α : Auto

Note: Figure 9 compares Regime 1, Regime 2 and social optimum across different values of α in terms of the entry share of innovators (left panel) and welfare (right panel) in the auto industry.

We can similarly evaluate the effect of α for the PC industry.

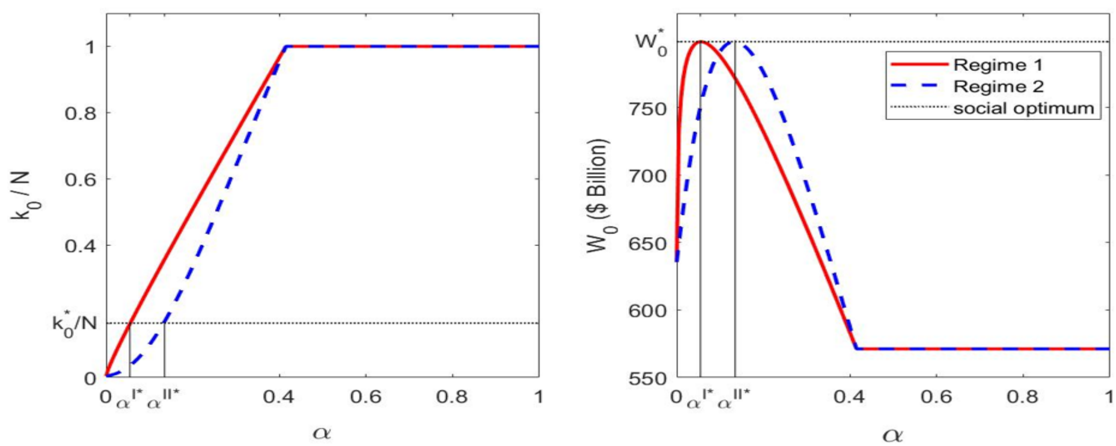

Fig. 10. Effect of α : PC

Note: Figure 10 compares Regime 1, Regime 2 and social optimum across different values of α in terms of the entry share of innovators (left panel) and welfare (right panel) in the PC industry.

Given the innovation cost c_{PC} derived from the model calibration, Eqs. (33) and (34) pin down the entry mass of innovators k_0 for each counterfactual value of $\alpha \in (0, 1]$. Figure 10 shows that k_0 rises monotonically with α for both Regimes 1 and 2 when $0 < \alpha < 0.42$, and that Regime 1 yields a higher k_0 than Regime 2.

When $\alpha \geq 0.42$, k_0 in both regimes reaches the corner solution $k_0 = N$. The socially optimal entry share of innovators, $k_0^*/N = 16.41\%$, is markedly higher than 0.41% under $\alpha = 0$. The social optimum can be achieved by setting $\alpha_{\text{PC}}^{\mathbf{I}*} = 0.056$ in Regime 1 or $\alpha_{\text{PC}}^{\mathbf{II}*} = 0.133$ in Regime 2, yielding a total social surplus of $W_{0,\text{PC}}^* = \$798.9$ billion (in 2012 prices), or 25.72% higher than under the free-imitation benchmark.

Comparative statics for α^*.—Figure 11 plots comparative statics for the socially optimal compensation share α^* under Regimes 1 and 2 based on the auto calibration.

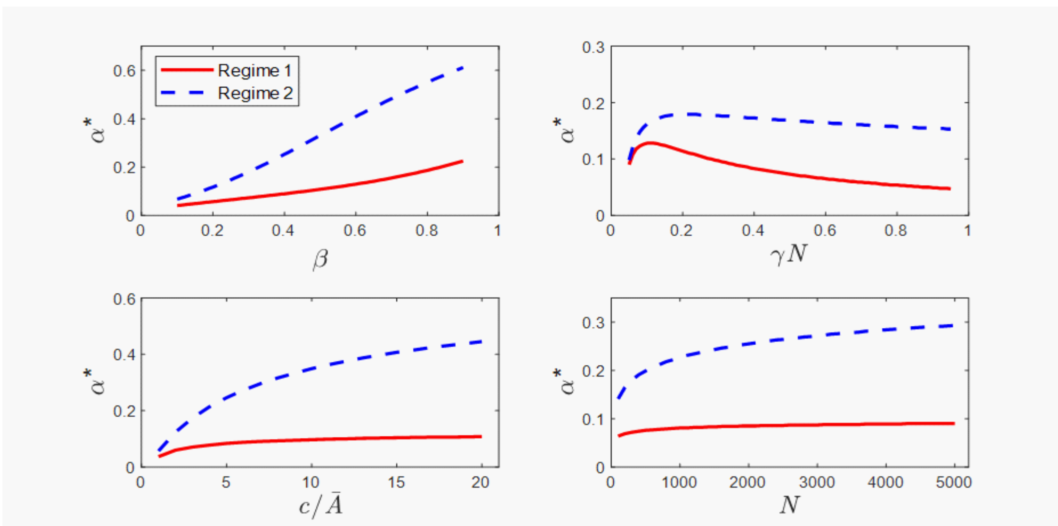

Fig. 11. Comparative Statics for α^* under Regimes 1 and 2

Note: Figure 11 plots how the socially optimal compensation share α^* reacts to changes of model parameter values based on the calibration to the automobile industry.

The results show the following:

- α^* *rises with* β.—A higher β means a lower price elasticity, which leads price to decline faster which discourages k_0. This makes the matching externality less of a concern, so α^* rises.

- α^* *falls with* γ *(holding* N *fixed, when* γ *is sufficiently large)*.—A higher γ implies a better imitation technology, so the planner needs less innovation when γ is sufficiently large and so α^* falls.

- α^* *rises with* c *but falls with* $\bar{A}$ $(\equiv \tilde{A}q^{1-\beta})$.—A higher c or a lower $\bar{A}$ discourages k_0. This makes the matching externality less of a concern, so α^* rises.

- *α^* rises with N (holding $\gamma N = \lambda$ fixed).*—A higher N leads to a faster price decline which discourages k_0. This, together with a larger pool of potential adopters N, makes the matching externality less of a concern, so α^* rises.

- *Comparison of Regimes 1 and 2.*—α^* is higher under Regime 2 than under Regime 1, and the difference rises with β, γ, $c/\bar{A}$, and N.

The comparative statics help explain the difference in α^* between the auto and the PC. Compared with the auto, the PC industry has a smaller β and a larger γN, and these two dominate the offsetting forces of the larger $c/\bar{A}$ and larger N and hence $\alpha^*_{\text{PC}} < \alpha^*_{\text{Auto}}$ under each regime. Quantitatively, by comparing counterfactuals that let one industry take on the other industry's parameter values, we find that the smaller β (i.e., the higher price elasticity) accounts for most for the smaller α^*_{PC}.

6.3.2 Socially optimal diffusion rate

We now evaluate the effects of varying the diffusion rate γ (holding N fixed). In the baseline case of free imitation ($\alpha = 0$), should the planner slow down diffusion to stimulate innovation?

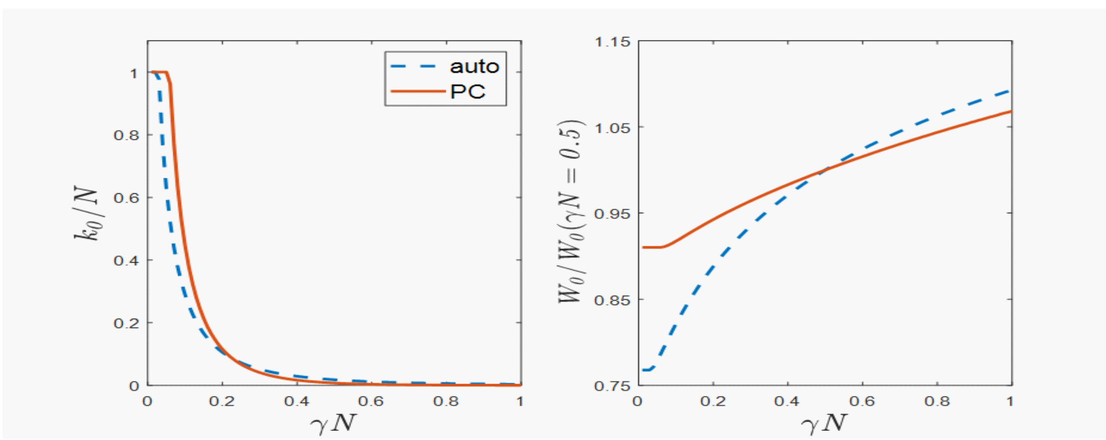

Fig. 12. Effect of the Diffusion Rate γ

Note: Figure 12 plots how the diffusion rate γ affects the entry share of innovators (left panel) and welfare (right panel) in both automobile and PC industries.

Figure 12 shows that for both the auto and PC industries, k_0 falls with γ while W_0 rises with γ. Therefore, if the planner were to reduce γ, the entry of innovators

k_0 would rise but social welfare would fall. The intuition is that while slowing down diffusion could encourage entry of innovators, it would forego too much free imitation and the welfare effect of the latter dominates.

To further study how policy affects diffusion and welfare, we split the auto data by region according to regulatory regimes. Specifically, we compare the diffusion of auto producers in Michigan to that in the rest of the country from 1895-1910. The result shows that for Michigan, the diffusion estimates are

$$\ln \frac{k_t}{N - k_t} = \underset{(0.89)^{***}}{-6.03} + \underset{(0.09)^{***}}{0.68}\, t,$$

and the adjusted $R^2 = 0.86$. For the rest of country, the diffusion estimates are

$$\ln \frac{k_t}{N - k_t} = \underset{(0.23)^{***}}{-3.72} + \underset{(0.03)^{***}}{0.46}\, t,$$

and the adjusted $R^2 = 0.95$. Compared to the estimate $\gamma N = 0.54$ based on the national sample (cf. Table 1), the estimated diffusion rate for Michigan $\gamma N^{\text{Michigan}} = 0.68$ is higher, while that for the rest of the country $\gamma N^{\text{rest}} = 0.46$ is lower.

A key reason for the higher diffusion rate in Michigan was its policy on noncompetes. In the early years of the auto industry, Michigan had a legal and political culture that prioritized worker freedom over employer protection, which culminated in its 1905 Anti-Trust Act that banned non-compete agreements outright. This set Michigan apart from most other states where noncompetes were widely enforced. Auto firms in Michigan had a much higher spin-off rate than other places (Cabral *et al.*, 2018), as the pattern shown in Fig. 5 would suggest. Counterfactually, suppose that all other states were to adopt policies similar to Michigan's and achieve the higher diffusion rate $\gamma N = 0.68$ nationwide. Our model can then assess the welfare impact: as shown in Fig. 12, raising the diffusion rate from 0.54 to 0.68 nationwide would have led to a lower initial mass of innovators (falling from $k_0 = 3.20$ to $k_0 = 1.87$) but a 3.1% higher social welfare (rising from $W_0 = \$57.23$ billion to $W_0 = \$58.98$ billion).[21]

[21] Related to our analysis, Marx *et al.* (2009) explore an exogenous reversal of the ban on employee noncompete agreements in Michigan in 1985 as a natural experiment. Using micro data and a differences-in-differences approach, and controlling for changes in the auto industry central to Michigan's economy, they find that the enforcement of noncompetes attenuates mobility, most sharply for inventors with firm-specific skills, and for those who specialize in narrow technical fields.

6.4 Auto and PC: robustness checks

To assess robustness, we extend the analysis under alternative modeling assumptions regarding anticipated shakeouts, imitators' entry costs, and effective IP protection. In all cases, the results remain consistent with our main findings—Restricting diffusion speed γ always lowers social welfare in both industries, and the socially optimal licensing rate α^* adjusts systematically as predicted by the model's comparative statics.

Anticipated shakeout.—Our analysis treats the shakeout as an unexpected event. Alternatively, the model can be extended to allow for *anticipated* shakeouts. Specifically, we may assume that firms expect a disruptive innovation to arrive at a Poisson rate ρ, which would render existing technologies obsolete and drive firm values to zero. The functional forms of our original model remain valid, except that the discount rate now becomes $r + \rho$. As shown in Internet Appendix IA-C1, α^* rises in both the automobile and PC industries under this extension due to the higher effective discount rate, despite the lower calibrated innovation cost c.

Imitators' entry costs.—Our analysis assumes that imitators incur no cost other than the licensing fee, giving them the strongest competitive advantage and simplifying the model solution. In Internet Appendix IA-C2, we extend the model to require imitators to pay a positive imitation cost $c^m < c$—for instance, to absorb new technology and establish production. All else equal, introducing $c^m > 0$ induces more agents to innovate, yielding a higher calibrated innovation cost c to match observed innovator entry. Consequently, α^* rises in both the automobile and PC industries under this extension, but the change is small.

Effective IP protection.—Our baseline analysis assumes no effective IP protection in the automobile and PC industries ($\alpha = 0$). For robustness, we consider an alternative, extreme scenario in which both industries fully enforced IP protection under the 25 percent rule ($\alpha = 0.25$). In Internet Appendix IA-C3, we recalibrate the model for both industries. Relative to the baseline, a higher α incentivizes more agents to innovate, leading to a higher innovation cost c calibrated to match observed innovator entry. Consequently, α^* rises in both industries compared with the baseline, as the model predicts. Moreover, α^* remains smaller for the PC industry than for the automobile industry in Regime 1.

6.5 Cross-industry analysis beyond auto and PC

We now extend the empirical analysis to a broad set of industries originally studied by Gort and Klepper (1982). Their classic work tracks the number of producers in 46 industries from each industry's initial year of production through 1973, documenting a common life-cycle pattern across most industries: the number of producers first rises and later falls. For 23 of these industries, they also collect annual data on price and output, showing that industry output continues to expand while prices decline over the life cycle.

Table 3. Model Application to 18 Industries

Product	Year of initial production	k^*_0/N	α^{*I}	α^{*II}
Auto	1895	15%	0.07	0.17
PC	1975	16%	0.06	0.13
Computers, Pre-PC	1935	21%	0.13	0.23
Electric blankets	1911	15%	0.37	0.53
Electric shavers	1930	14%	0.06	0.15
Freezers, Home and farm	1929	19%	0.15	0.28
Lasers	1960	48%	0.04	0.06
Nylon	1939	14%	0.16	0.32
Penicillin	1943	12%	0.15	0.32
Pens, Ballpoint	1945	16%	0.25	0.41
Records, Phonograph	1887	9%	0.37	0.58
Streptomycin	1945	10%	0.13	0.30
Styrene	1935	7%	0.33	0.57
Tapes, Recording	1947	8%	0.20	0.43
Television	1929	11%	0.12	0.30
Tires, Automobile	1896	22%	0.09	0.17
Transistors	1948	30%	0.06	0.10
Zippers	1904	7%	0.31	0.55

We use the original Gort and Klepper (1982) dataset and apply our model to the 23 industries with available annual data on firm numbers, output, and prices. Using the same procedure as in Section 6.1, we estimate the diffusion and demand parameters for each industry. We then exclude seven industries that fail either the diffusion or demand estimation.[22] This leaves 16 industries for which we can successfully calibrate

[22] Of the 23 industries, one fails the diffusion estimation as the shakeout begins almost immediately after the industry's emergence. Six others fail the demand estimation, yielding either negative or statistically insignificant estimates of ϕ—suggesting that in those cases demand may be time-varying or that lagged output per firm is an invalid instrument.

the model and conduct counterfactual analysis. For each industry, we derive the socially optimal licensing rates $\alpha^{\mathbf{I}^*}$ and $\alpha^{\mathbf{II}^*}$ under Regimes 1 and 2, and verify that reducing diffusion rate γ always lowers social welfare. Table 3 lists these industries, together with automobile and PC, along with their year of initial production, and reports the model-implied socially optimal innovator entry rate k_0^*/N, and the socially optimal licensing rates $\alpha^{\mathbf{I}^*}$ for Regime 1 and $\alpha^{\mathbf{II}^*}$ for Regime 2.

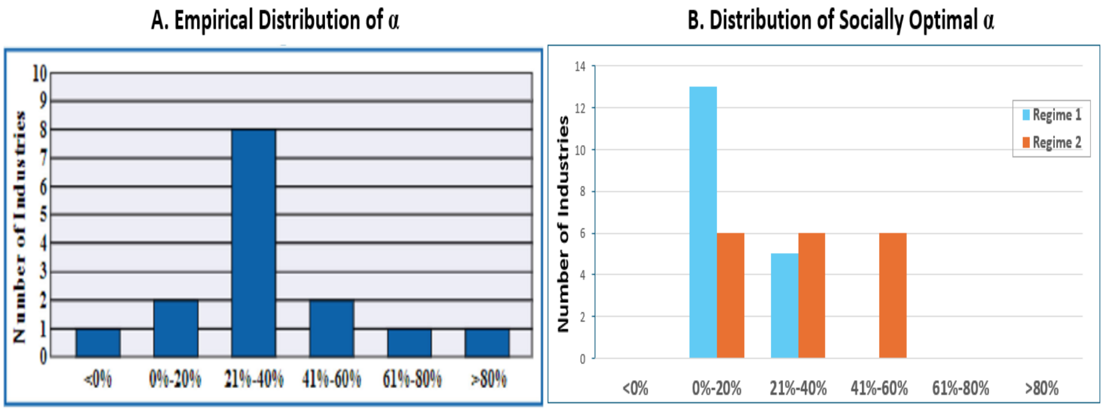

Fig. 13. Distribution of α: Empirical vs. Socially Optimal

Note: Figure 13 contrasts the empirical distribution of α across industries (Panel A) with the socially optimal distribution implied by the model (Panel B).

Figure 13 compares the cross-industry distribution of observed α to the socially optimal rates implied by our model. The left-hand panel, Fig. 13(A), is taken from Goldscheider *et al.* (2002), which reports licensing fees paid as a share of licensee profits across 347 firms in 15 industries between 1990 and 2000. The empirical distribution confirms the prevalence of the "25 percent rule" in practice—most industries cluster around a licensing rate of approximately 25 percent, consistent with the standard bilateral bargaining benchmark between licensors and licensees.

In contrast, when applied to 18 industries, our model yields markedly different cross-industry distributions of socially optimal licensing rates, as Fig. 13(B) shows. These differences arise from market-level determinants such as demand elasticity, innovation cost, diffusion speed, and whether sublicensing is permitted. The results underscore that industry-level licensing outcomes driven by bilateral bargaining need not align with welfare-maximizing rates shaped by aggregate diffusion dynamics.

We assume free imitation ($\alpha = 0$) in the baseline analysis, consistent with evidence that most innovations are neither patented nor effectively protected by patents. For robustness, we consider an alternative, extreme scenario in which all industries in the Gort and Klepper (1982) dataset fully enforced IP protection under the 25 percent rule ($\alpha = 0.25$). We then recalibrate the model for each industry and derive the corresponding socially optimal licensing rate α^*. A higher α incentivizes more agents to innovate, leading to a higher calibrated innovation cost c to match observed innovator entry. Consequently, α^* rises across all industries in both Regimes 1 and 2, shifting the cross-industry distributions of α^* to the right relative to Fig. 13(B), while remaining markedly different from the empirical distribution of α shown in Fig. 13(A). Moreover, the cross-industry ranking of α^* obtained under the free-imitation baseline persists under the alternative scenario in Regime 1, though not necessarily in Regime 2. Detailed results are provided in Internet Appendix IA-C3.

7 Additional discussion

7.1 Matching function specification

In our model, the matching function (3) features increasing returns to scale. However, the assumption on returns to scale is inessential for our analysis. To see why, let us generalize Eq. (3) to

$$\frac{dk_t}{dt} = \hat{\gamma} k_t \left(N - k_t\right) \qquad \text{where} \qquad \hat{\gamma} = \frac{\gamma}{N^{\psi}}.$$

The solution for k_t then becomes

$$k_t = \frac{N e^{\hat{\gamma} N t}}{e^{\hat{\gamma} N t} + \frac{N}{k_0} - 1} = \frac{N e^{\gamma N^{1-\psi} t}}{e^{\gamma N^{1-\psi} t} + \frac{N}{k_0} - 1}.$$

By rescaling the diffusion parameter γ by the constant $\frac{1}{N^{\psi}}$, the matching function features increasing returns to scale if $\psi < 1$, constant returns if $\psi = 1$, and decreasing returns if $\psi > 1$. In the model, we assumed that $\psi = 0$, but our analysis and findings would hold for any ψ because a time series study takes N and $\frac{\gamma}{N^{\psi}}$ as given; the value of ψ plays no role except in a counterfactual that would involve changing the value of N.

Also, the labor search literature often assumes a Cobb-Douglas matching function:

$$\frac{dk_t}{dt} = \gamma k_t^{\theta} \left(N - k_t\right)^{1-\theta},$$

where $0 < \theta < 1$. However, the Cobb-Douglas formulation does not appear to fit data better, and more importantly, it does not have a closed-form solution for the time path of k_t. Therefore the logistic formulation we use has analytical advantages.

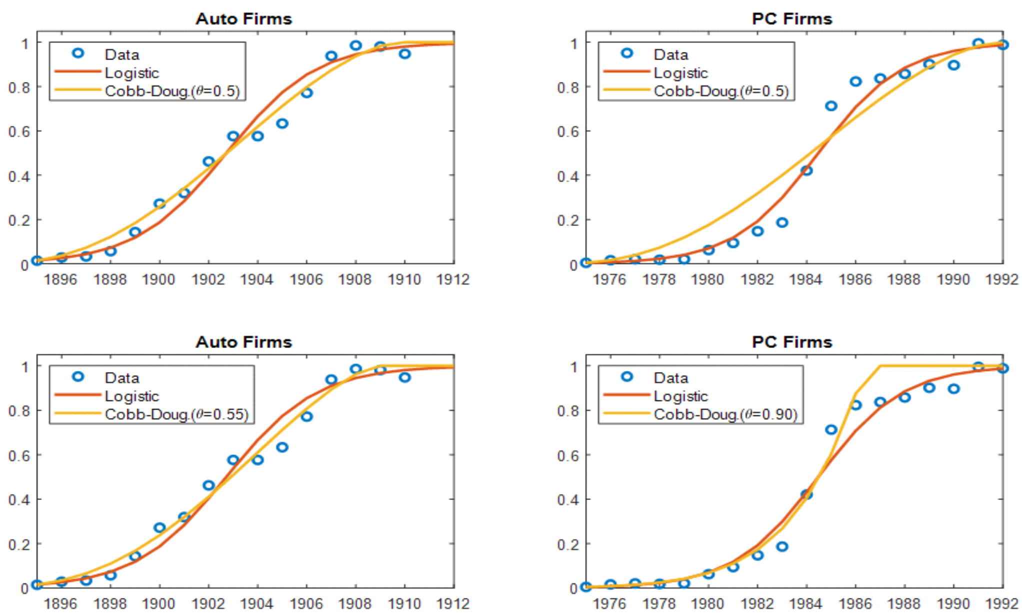

Fig. 14. Diffusion Models: Fitting Time Paths of Firm Numbers

Note: Figure 14 compares data fitting of logistic diffusion model with that of symmetric Cobb-Douglas matching model (top panels) and with that of best fitting Cobb-Douglas matching model (bottom panels) in the automobile and PC industries.

Figure 14 shows that the estimated logistic diffusion model (cf. Eq. (4)) matches the time paths of firm numbers well for U.S. automobile and PC industries. Compared to the symmetric Cobb-Douglas counterpart (i.e., $\theta = 0.5$), logistic diffusion shows a more pronounced inflection point and fits better for the PC industry, as shown in the top panels. In the bottom panels, we compare logistic diffusion with the best fitting Cobb-Douglas formulation for each industry without restricting θ. The former still fits better for the PC industry.[23]

[23] By maximizing the fit with a Cobb-Douglas function without restricting θ, we estimate that

7.2 Heterogeneous production capacity

The model assumes that all firms have the same constant production capacity. The constancy of output per firm prior to the shakeout shown in Figs. 1 and 2 suggests that this is not unreasonable in the pre-shakeout period. Still, the model can accommodate some output heterogeneity with the following minor adjustment: suppose the production capacity of innovator i is s_i, a random variable with CDF $H(s_i)$ with support $(0, \overline{s})$ and a mean equal to unity.

Among innovators we assume that the variables s_i are random, independent over i, but are realized after the innovator has paid the cost c. We also assume that matching is undirected so that H pertains to the imitators, too. Industry output would be

$$Q_t = k_t \int_0^{\overline{s}} s dH(s) = k_t.$$

Then $Q_t = k_t$, and p_t would remain the same as before. Ex ante expected revenue of firm i would be $p_t E(s_i) = p_t$. The rest of the analysis would stay unchanged.

A subset of the values of s in the support of H could then develop into "dominant designs" in the terminology of Utterback and Abernathy (1975) and survive the shakeout.

7.3 Industry growth factors

Our model can also be extended to incorporate growing market demand and firm productivity. In doing so, one may extend the model setup by allowing the market demand parameter A_t and each firm's production capacity q_t to rise over time. As a result, at date $t > 0$ industry output is $Q_t = q_t k_t$, and the product price is

$$p_t = A_t Q_t^{-\beta} = A_t (q_t k_t)^{-\beta}.$$

As long as profit per firm (i.e., $p_t q_t = A_t q_t^{1-\beta} k_t^{-\beta}$) does not increase in t, we find that at equilibrium, innovators still enter the industry only at date 0. In Regime 1,

$\theta^{\text{auto}} = 0.55$ and $\theta^{\text{PC}} = 0.90$. Figure 14 shows that Cobb-Douglas curves fit slightly better in the auto case ($R^2 = 0.986$ when $\theta = 0.5$ and $R^2 = 0.987$ when $\theta = 0.55$) than the logistic curve ($R^2 = 0.974$), but the logistic curve fits better in the PC case ($R^2 = 0.981$) than the Cobb-Douglas curves ($R^2 = 0.936$ when $\theta = 0.5$ and $R^2 = 0.969$ when $\theta = 0.90$).

the number of innovators $k_0^{\mathbf{I}} \in (0, N)$ solves

$$\underbrace{\frac{1}{N-k_0}\int_0^{\infty} e^{-rt}\left(\alpha\frac{N}{k_0}+(1-\alpha)\frac{N}{k_t}-1\right)A_t\left(q_t k_t\right)^{1-\beta}dt}_{v_0-u_0} \quad = \quad c,$$

and in Regime 2, the number of innovators $k_0^{\mathbf{II}} \in (0, N)$ solves

$$\underbrace{\frac{1}{N-k_0}\int_0^{\infty} e^{-rt}\left(\left(\frac{N}{k_0}\right)^{\alpha}\left(\frac{N}{k_t}\right)^{1-\alpha}-1\right)A_t\left(q_t k_t\right)^{1-\beta}dt}_{v_0-u_0} \quad = \quad c,$$

where, in each regime, k_t follows the logistic diffusion process in Eq. (4) (see Internet Appendix IA-A11 for the proof).

The intuition is that despite the growth in market demand and firm productivity, the entry of imitators may continue to bring down the competitive rents enjoyed by earlier entrants. As a result, innovators enter only at date 0, and the private and social trade-offs that we have studied between innovation and imitation continue to exist. To apply the model extension to data, one needs to specify the laws of motion for A_t and q_t explicitly. Given that our model with time-invariant A_t and q_t fits the U.S. auto and PC industry data well, we leave the empirical application of this extension for future research.[24]

7.4 The $N \to \infty$ limit

The special case where $N \to \infty$ does not fit the industry data well, but the solutions are simpler and may help illustrate the findings of our model. In this subsection, we study a limiting version of our model with $N \to \infty$. All proofs are provided in Internet Appendix IA-A12.

[24] There may be exceptional cases where market demand or firm capacity expands so much that profit per firm rises over time. If so, some agents could enter as innovators after date 0. That scenario might be relevant for some products or industries, but from a modeling point of view, it blurs the tradeoff between innovation and imitation. We run Bass model regressions using both auto and PC industry data by allowing some firms to enter the industry as independent innovators after date 0, but the effect is statistically insignificant (see Internet Appendixes IA-B2 and IA-B4).

Let N get large but at the same time reduce γ so that $\gamma N \to \lambda > 0$, a constant. The logistic diffusion process (3) then converges to $\frac{dk_t}{dt} = \lambda k_t$, and then (4) becomes $k_t = k_0 e^{\lambda t}$. This is essentially the exponential diffusion process considered by the models of competitive innovation (i.e., Boldrin and Levine, 2002, 2008, and Quah, 2002). Those studies embed such a diffusion process in a growth model, and we now incorporate it into our industry dynamic model.

Assuming that $\lambda < r$ so that welfare is bounded, we prove that, at equilibrium, innovators only enter at date 0, and the number of innovators is

$$k_0^{\mathbf{I}} = \left(\frac{A(r + (\beta - 1)\lambda + \alpha\lambda)}{c\,(r + \beta\lambda)\,(r + (\beta - 1)\lambda)} \right)^{\frac{1}{\beta}}, \tag{37}$$

$$k_0^{\mathbf{II}} = \left(\frac{A}{c(r + (\beta - \alpha)\lambda)} \right)^{\frac{1}{\beta}} \tag{38}$$

in Regime 1 and Regime 2, respectively.

Equations (37) and (38) imply that $k_0^{\mathbf{I}} = k_0^{\mathbf{II}} > 0$ for $\alpha \in \{0, 1\}$ and $k_0^{\mathbf{I}} > k_0^{\mathbf{II}} > 0$ for $\alpha \in (0, 1)$. Hence, innovation occurs even when $\alpha = 0$, and Regime 1 yields more innovation unless $\alpha \in \{0, 1\}$.

Socially optimal licensing rate.—While $\alpha = 0$ is compatible with positive innovation, it is not optimal. We prove that it is socially optimal to innovate only at date 0 and that the corresponding number of innovators is

$$k_0^* = \left(\frac{A}{(r + (\beta - 1)\lambda)c} \right)^{\frac{1}{\beta}}. \tag{39}$$

Comparing Eq. (39) to Eqs. (37) and (38) shows that $k_0^* = k_0^{\mathbf{I}} = k_0^{\mathbf{II}}$ iff $\alpha = 1$. I.e., $\alpha^* = 1$ for both Regimes 1 and 2.

Why does the limiting model yield an optimal licensing rate different from that when N is finite? The key is that when N is finite, there is a matching externality that an innovator creates and ignores, which reduces other agents' innovation payoff. In the limiting model, however, there is no such externality —an innovator's matching rate is fixed at $\frac{dk_t/dt}{k_t} = \lambda$ while an imitator's matching rate is fixed at 0 given that $N \to \infty$. Therefore, the finite-N version not only fits the industry evolution pattern better but also incorporates the important matching externality.

Socially optimal diffusion rate.—Parallel to the logistic diffusion case where the planner does not want to reduce γ, here the planner does not want to reduce λ under the exponential diffusion:

$$\frac{\partial W_0}{\partial \lambda} > 0 \quad \text{for } \alpha \in [0,1], \text{ and } \beta > 0 \tag{40}$$

for both Regimes 1 and 2.

8 Conclusion

This paper developed a unified model linking innovation, diffusion, and intellectual property (IP) protection in industry evolution. By endogenizing innovators' entry and by modeling idea diffusion through random matching, the framework reproduces the S-shaped producer growth observed in the early life cycle of U.S. industries, including automobiles and personal computers.

Our analysis yields two main findings. First, optimal licensing must balance the positive learning externality with the negative matching externality arising in the diffusion of new technologies. The resulting socially optimal licensing rate α^* varies systematically with demand elasticity, innovation cost, diffusion speed, and whether sublicensing is permitted—departing sharply from the uniform 25 percent rule commonly used in practice.

Second, when IP protection is weak, innovators rely on lead-time advantage to recover investment. However, policies that attempt to strengthen innovation incentives by slowing down diffusion inevitably reduce welfare, as the losses from delayed imitation and output expansion outweigh the private gains to innovators.

Taken together, these results highlight that diffusion policy and innovation policy are inherently intertwined. Ensuring rapid diffusion while providing sufficient—but not excessive—rewards to innovators maximizes welfare and promotes dynamic, competitive industry growth. Future research may extend this framework to endogenous quality improvements, network diffusion, and modern innovation ecosystems such as software and artificial intelligence.

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Internet Appendix to "Idea Diffusion and Property Rights"

Boyan Jovanovic and Zhu Wang

IA-0. Evidence on limited patent protection

Despite the centrality of intellectual property (IP) rights in innovation policy, the real-world effectiveness of patents in protecting innovations is very limited. Evidence suggests that for the majority of innovators, patents offer weak or non-existent protection; Innovators rely primarily on first-mover advantage and their lead time over imitators to recoup their investment in innovation.

1) A majority of innovations are never patented

- Mezzanotti and Simcoe (2023) report on the Business R&D and Innovation survey, which was conducted between 2008 and 2015 by the US Census Bureau and the National Science Foundation. This survey asked more than 40,000 US firms, from a nationally representative sample, about their use of intellectual property. Only 18% of firms that perform any R&D seek patent protection. Liu (2025) further finds from a 2022 survey that for U.S. manufacturing firms with multinational operation, only 17.6% of innovations considered for patent protection eventually filed applications.

- Cohen, Nelson, and Walsh (2000) describes the results of a 1994 survey administered to about 1500 R&D labs in the US manufacturing sector. In this survey, labs were asked to estimate the percent of innovations that were patented. Even in this sample where the innovators have the highest tendency to patent, less than half of their inventions were patented: across all sectors, on average labs reported they patented 49% of their product innovations and 31% of their process innovations.

- Fontana *et al.* (2013): Since 1963, the magazine Research and Development has run the annual "R&D 100 award" competition to identify the 100 most technologically significant products available for sale/licensing in the previous year. Fontana and coauthors gather data on 2802 award-winning inventions over 1977-2004 and find that only 9.1% of them have an associated patent.

- Argente *et al.* (2023) use the Nielsen Retail Measurement Services data to get information on how often new consumer products are patented. Their dataset covers more than one million products sold over 2006-2015. They find that only 23% of new products are introduced by firms who have patents related to that product category.

2) Few patents are litigated, and many are weaponized

- Among the patents that are filed, only a small fraction are ever litigated. Of approximately 200,000 patents issued annually in the U.S., fewer than 1.5% face litigation, and a mere 0.1% proceed to trial (Wikipedia, "Economics of Patents"). Even when patents are litigated, enforcement outcomes are inconsistent, and many patents are broad, low-quality filings used as leverage by patent assertion entities (PAEs), commonly known as patent trolls.
- In 2012, PAEs accounted for 61% of U.S. patent lawsuits, and by 2014 this share had risen to 67%. These entities often target small firms that lack resources to engage in prolonged legal battles, forcing settlements regardless of the patent's substantive merit (Scherer, 2015). Consequently, the enforcement landscape is skewed in favor of entities that weaponize patent portfolios rather than genuinely protect innovative output.

3) Patent litigation is costly and enforcement is uneven

- For most innovators, pursuing patent litigation is financially nonviable. The cost of defending a patent lawsuit can easily exceed $2.5 million for cases with stakes over $25 million, and even for smaller disputes, legal costs remain prohibitively high (American Intellectual Property Law Association, 2019). In fact, for certain sectors like software, the average cost of litigation doubles the expected value of the patent, making enforcement a negative net-return activity for many small innovators (Wikipedia, "Criticism of Patents").
- This uneven enforcement capacity creates a systemic disadvantage for startups and SMEs, which often lack the resources to navigate complex litigation or even defend their patents against infringement by larger competitors.

4) Alternative approaches

The profits from innovation erode sharply when imitation occurs within a few years, which is often the case in fast-moving industries like consumer electronics and software. In these markets, leading innovators rely more on speed-to-market strategies and brand differentiation than on legal protections (Scherer, 2015).

Given the limitations of patents, many innovators turn to alternative strategies to protect and monetize their creations:

- Trade secrets: Many firms opt to keep critical innovations as trade secrets to avoid disclosure through patenting, though the approach is more commonly used in sectors where reverse engineering is difficult.
- First-mover and lead-time advantages: Rapid market entry and establishing brand loyalty are often more effective than legal barriers to imitation.

- Open innovation and collaboration: In fields like software and biotech, open-source models and collaborative IP-sharing arrangements provide mutual protection and foster innovation ecosystems (e.g., Linux, CRISPR consortia).

Motivated by the above evidence, we analyze innovation and diffusion under free imitation, and derive the socially optimal licensing rate that balances learning and matching externalities in the industry evolution. We also show when intellectual property protection is ineffective, innovators rely on lead time to recover investment, but extending this lead time by a policy that slows diffusion would reduce social welfare.

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IA-A. Proofs

IA-A1. Proof of Proposition 1

Proof. We first conjecture that no agent enters as an innovator after date 0 and $k_0 < N$, so the time path of firm numbers follows logistic diffusion:

$$k_t = \frac{Ne^{\gamma Nt}}{e^{\gamma Nt} + \frac{N}{k_0} - 1}. \tag{A.1}$$

The value of a producer v_t is determined by

$$rv_t = p_t + \frac{dv_t}{dt}. \tag{A.2}$$

Solving the differential equation (A.2) yields

$$v_t = \int_t^\infty e^{-r(s-t)} p_s ds = \int_t^\infty e^{-r(s-t)} A k_s^{-\beta} ds. \tag{A.3}$$

Because k_s rises with s, v_t falls with t.

Next, we show that $v_t - u_t$ falls with t. Recall that the value of an outsider, u_t, is determined by

$$ru_t = \gamma k_t \left(v_t - u_t\right) + \frac{du_t}{dt}, \tag{A.4}$$

which, together with Eq. (A.2), implies that

$$\frac{d\left(v_t - u_t\right)}{dt} = (r + \gamma k_t)(v_t - u_t) - p_t. \tag{A.5}$$

Let $\psi_t \equiv v_t - u_t$. Then Eq. (A.5) reads

$$\frac{d\psi_t}{dt} - (r + \gamma k_t)\psi_t = -p_t.$$

Define $z_t = \exp \int -(r + \gamma k_t)dt$. We then have

$$\frac{d(z_t \psi_t)}{dt} = -z_t p_t,$$

which yields the general solution

$$\psi_t = z_t^{-1} \int -z_t p_t dt + z_t^{-1} C,$$

where C is the constant of integration.

Given that $k_t = \frac{Ne^{\gamma Nt}}{e^{\gamma Nt} + \frac{N}{k_0} - 1}$, we can solve for z_t:

$$z_t = e^{-rt} \frac{k_0}{N - k_0 + k_0 e^{\gamma Nt}}.$$

Since ψ_t needs to be bounded as $t \to \infty$, we have $C = 0$. We then have

$$\begin{aligned}\psi_t &= z_t^{-1} \int -z_t p_t dt \\ &= e^{rt} \frac{N - k_0 + k_0 e^{\gamma N t}}{k_0} \int -e^{-rt} \frac{k_0}{N - k_0 + k_0 e^{\gamma N t}} p_t dt \\ &= e^{rt} \frac{N - k_0 + k_0 e^{\gamma N t}}{k_0} \int_t^\infty e^{-rs} \frac{k_0}{N - k_0 + k_0 e^{\gamma N s}} p_s ds \\ &= \int_t^\infty e^{-r(s-t)} \left(1 - \frac{e^{\gamma N(s-t)} - 1}{(\frac{N-k_0}{k_0}) e^{-\gamma N t} + e^{\gamma N(s-t)}}\right) p_s ds. \end{aligned} \tag{A.6}$$

Define $j = s - t$. We can rewrite Eq. (A.6) as

$$\psi_t = \int_0^\infty e^{-rj} \left(1 - \frac{e^{\gamma N j} - 1}{(\frac{N-k_0}{k_0}) e^{-\gamma N t} + e^{\gamma N j}}\right) p_{t+j} dj.$$

In the integral, the terms $\left(1 - \frac{e^{\gamma N j} - 1}{(\frac{N-k_0}{k_0}) e^{-\gamma N t} + e^{\gamma N j}}\right)$ and p_{t+j} both fall with t, hence $\psi_t = v_t - u_t$ strictly falls with t. Given the free entry condition $v_0 - u_0 = c$, we have $v_t - u_t < c$ for any $t > 0$, and so no agent would want to enter as an innovator after date 0.

Eq. (A.3) implies that

$$v_0 = \int_0^\infty e^{-rt} A k_t^{-\beta} dt. \tag{A.7}$$

At date 0, the total industry discounted revenue, $\int_0^\infty e^{-rt} A k_t^{1-\beta} dt$, is shared by the two groups – the initial entrants k_0 and the outsiders $N - k_0$. With the free entry condition $v_0 - c = u_0$, we have

$$\int_0^\infty e^{-rt} A k_t^{1-\beta} dt = v_0 k_0 + u_0 (N - k_0) = v_0 N - c (N - k_0). \tag{A.8}$$

Plugging Eq. (A.7) into Eq. (A.8) yields Eq. (A.9):

$$\frac{\int_0^\infty e^{-rt} \left(\frac{N}{k_t} - 1\right) A k_t^{1-\beta} dt}{N - k_0} = c. \tag{A.9}$$

Interior solution (i.e., $k_0 < N$).—We have shown that k_0 is determined by Eq. (A.9). Note that as $k_0 \to N$, we have $k_t \to N$. Hence,the numerator and the denominator on the left hand side of Eq. (A.9) both converge to 0 as $k_0 \to N$. Applying L'Hôpital's rule, the left hand side of Eq. (A.9) converges to $\frac{AN^{-\beta}}{r+\gamma N}$ as

$k_0 \to N$. We can further prove that Eq. (A.9) yields $dk_0/dc < 0$, and so the baseline model has an interior solution $k_0 < N$ iff condition (A.10) holds, i.e., iff

$$c > \frac{AN^{-\beta}}{r + \gamma N}. \tag{A.10}$$

■

IA-A2. Proof of Proposition 2(A)

Proof. In Regime 1, potential adopters can copy an idea from an imitator but the fee goes to the idea's original innovator. We first assume that, at equilibrium, innovators enter only at date 0 and $k_0 < N$, so that the time path of firm numbers is determined by Eq. (A.1). We then ask if any agent would want to deviate by entering as an innovator at a date $\tau > 0$.

The entry of a measure-zero innovator at $\tau > 0$ would not change the industry quantity and price through Eq. (A.1). Upon entry, the value of this innovator comes from two sources: One is that he will get a fraction $1/k_t$ of the total industry revenue $Ak_t^{1-\beta}$ at each date $t \geq \tau$ by selling goods; the other is that he will get a chance $1/k_\tau$ to collect idea-sale revenues from new imitators at each date $t \geq \tau$ (note that k_τ is the number of all incumbent firms at his entry date τ, so $1/k_\tau$ is the probability for new imitators at each date $t \geq \tau$ to trace him as the original innovator of the idea they copy). At each date $t \geq \tau$, a fraction $\frac{k_t - k_\tau}{k_t}$ of firms in the industry are imitators who enter between date τ and date t, so this new innovator at his entry date τ expects to have $1/k_\tau$ chance to receive the discounted sum of the fraction $\alpha\left(k_t - k_\tau\right)/k_t$ of the total industry revenue $Ak_t^{1-\beta}$ as his idea-sale revenue starting from date τ.

Therefore, the value of this new innovator at his entry date τ, denoted by v_τ^τ, is

$$v_\tau^\tau = \int_\tau^\infty e^{-r(t-\tau)} \left(\frac{1}{k_t} + \frac{\alpha}{k_\tau} (\frac{k_t - k_\tau}{k_t}) \right) Ak_t^{1-\beta} dt \tag{A.11}$$

Note that v_τ^τ varies by entry date τ because the number of existing firms k_τ rises with τ. By contrast, the value of an innovator who entered at date 0 would have the date-τ value

$$v_\tau^0 = \int_\tau^\infty e^{-r(t-\tau)} \left(\frac{1}{k_t} + \frac{\alpha}{k_0} (\frac{k_t - k_\tau}{k_t}) \right) Ak_t^{1-\beta} dt, \tag{A.12}$$

so $v_\tau^\tau < v_\tau^0$ for any $\tau > 0$, and $v_\tau^\tau = v_\tau^0$ for $\tau = 0$.

Equation (A.1) implies that for any date $t \geq \tau$,

$$k_t = \frac{Ne^{\gamma N(t-\tau)}}{e^{\gamma N(t-\tau)} + (\frac{N}{k_\tau} - 1)}, \tag{A.13}$$

and that

$$\frac{k_t}{k_\tau} = \frac{Ne^{\gamma N(t-\tau)}}{N + (e^{\gamma N(t-\tau)} - 1)k_\tau}. \tag{A.14}$$

We can rewrite Eq. (A.11) as

$$v_\tau^\tau = \int_\tau^\infty e^{-r(t-\tau)} \left(1 - \alpha + \alpha \frac{k_t}{k_\tau}\right) A k_t^{-\beta} dt. \tag{A.15}$$

Defining $s = t - \tau$, Eq. (A.15) becomes

$$v_\tau^\tau = \int_0^\infty e^{-rs} \left(1 - \alpha + \alpha \frac{k_{\tau+s}}{k_\tau}\right) A k_{\tau+s}^{-\beta} ds. \tag{A.16}$$

Note that Eqs. (A.13) and (A.14) imply that

$$\frac{k_{\tau+s}}{k_\tau} = \frac{N e^{\gamma N s}}{N + (e^{\gamma N s} - 1) k_\tau}, \quad k_{\tau+s}^{-\beta} = \left(\frac{N e^{\gamma N s}}{e^{\gamma N s} + (\frac{N}{k_\tau} - 1)}\right)^{-\beta},$$

which both decrease in k_τ. In Eq. (A.16), because k_τ increases in τ, $\frac{k_{\tau+s}}{k_\tau}$ and $k_{\tau+s}^{-\beta}$ decrease in τ, and hence v_τ^τ decreases in τ.

Similarly, because an imitator can keep $(1-\alpha)$ share of his output, the total value of outsiders $u_\tau(N - k_\tau)$ at date τ equals the imitators' share of the total discounted industry revenues from date τ and onward. Therefore, we have

$$u_\tau(N - k_\tau) = \int_\tau^\infty e^{-r(t-\tau)} \left(\frac{(1-\alpha)(k_t - k_\tau)}{k_t}\right) A k_t^{1-\beta} dt,$$

which implies

$$u_\tau = \int_\tau^\infty e^{-r(t-\tau)} \left(\frac{(1-\alpha)(k_t - k_\tau)}{k_t(N - k_\tau)})\right) A k_t^{1-\beta} dt. \tag{A.17}$$

Inserting Eq. (A.14) into Eq. (A.17), we derive

$$u_\tau = \int_\tau^\infty e^{-r(t-\tau)} \left(\frac{(1-\alpha)\left(e^{\gamma N(t-\tau)} - 1\right)}{N e^{\gamma N(t-\tau)}}\right) A k_t^{1-\beta} dt. \tag{A.18}$$

Again, defining $s = t - \tau$, Eq. (A.18) becomes

$$u_\tau = \int_0^\infty e^{-rs} \left(\frac{(1-\alpha)\left(e^{\gamma N s} - 1\right)}{N e^{\gamma N s}}\right) A k_{\tau+s}^{1-\beta} ds. \tag{A.19}$$

Equation (A.19) implies that if $\beta = 1$, u_τ is a constant that does not vary with τ; if $\beta > 1$, $k_{\tau+s}^{1-\beta}$ falls with k_τ so u_τ falls with τ; and if $\beta < 1$, $k_{\tau+s}^{1-\beta}$ rises with k_τ so u_τ rises with τ. Moreover, combining Eqs. (A.16) and (A.19), we have

$$v_\tau^\tau - u_\tau = \int_0^\infty e^{-rs} \left(1 - \alpha + \alpha \frac{k_{\tau+s}}{k_\tau} - \frac{(1-\alpha)\left(e^{\gamma N s} - 1\right)}{N e^{\gamma N s}} k_{\tau+s}\right) A k_{\tau+s}^{-\beta} ds. \tag{A.20}$$

Within the integral of Eq. (A.20), both terms $\left(1-\alpha+\alpha\frac{k_{\tau+s}}{k_\tau}-\frac{(1-\alpha)\left(e^{\gamma Ns}-1\right)}{Ne^{\gamma Ns}}k_{\tau+s}\right)$ and $k_{\tau+s}^{-\beta}$ decrease in τ, so $v_\tau^\tau-u_\tau$ falls with τ. Therefore, given the free entry condition $v_0^0-u_0=c$, we have $v_\tau^\tau-u_\tau<c$ for any $\tau>0$, so no innovator would enter the industry after date 0.

Let $v_0=v_0^0$. Equations (A.12) and (A.17) yield

$$v_0-u_0=\frac{1}{N-k_0}\int_0^\infty e^{-rt}\left(\alpha\frac{N}{k_0}+(1-\alpha)\frac{N}{k_t}-1\right)Ak_t^{1-\beta}dt. \tag{A.21}$$

The free entry condition $v_0-u_0=c$ then pins down the mass of innovators k_0 at date 0, as shown by Eq. (A.22):

$$\frac{\int_0^\infty e^{-rt}\left(\alpha\frac{N}{k_0}+(1-\alpha)\frac{N}{k_t}-1\right)Ak_t^{1-\beta}dt}{N-k_0}=c. \tag{A.22}$$

Interior solution (i.e., $k_0^{\mathbf{I}}<N$).—We have shown above that $k_0^{\mathbf{I}}$ is determined by Eq. (A.22). Note that as $k_0\to N$, we have $k_t\to N$. Hence, the numerator and the denominator of the left hand side of Eq. (A.22) both go to 0 as $k_0\to N$. Applying L'Hôpital's rule, the left hand side of Eq. (A.22) converges to $\frac{r+\alpha\gamma N}{r(r+\gamma N)}AN^{-\beta}$ as $k_0\to N$. Proposition 3(A) shows that $dk_0^{\mathbf{I}}/dc<0$. Therefore, the model has an interior solution $k_0^{\mathbf{I}}<N$ in Regime 1 iff condition (A.23) holds, i.e., iff

$$c>\frac{r+\alpha\gamma N}{r(r+\gamma N)}AN^{-\beta}. \tag{A.23}$$

Additional properties of the dynamic path.—The proof above confirms that innovators only enter at date 0. The time path of firm numbers is given by Eq. (A.1), and the time path of u_t is solved by Eq. (A.19). Following that, the dynamic paths of ω_t, v_t^τ and v_t for any $t\geq 0$ can also be derived. Recall that

$$r\omega_t=p_t+\frac{d\omega_t}{dt},$$

which yields

$$\omega_t=\int_t^\infty e^{-r(s-t)}p_sds=\int_t^\infty e^{-r(s-t)}Ak_s^{-\beta}ds.$$

Because k_s increases in s, ω_t declines in t.

Counterfactually, suppose a marginal innovator did enter at date $\tau>0$. For any $t\geq\tau$, he would collect $Ak_s^{-\beta}$ in each period $s\geq t$ by selling goods, and collect a fraction $\frac{\alpha(k_s-k_t)}{k_sk_\tau}$ of the total industry revenues $Ak_s^{1-\beta}$ from new entrants after date t by selling ideas. Therefore, his value at date t would be determined by

$$v_t^\tau=A\int_t^\infty e^{-r(s-t)}k_s^{-\beta}ds+\frac{A}{k_\tau}\int_t^\infty e^{-r(s-t)}\alpha(1-\frac{k_t}{k_s})k_s^{1-\beta}ds. \tag{A.24}$$

Because k_τ increases in τ, v_t^τ declines in τ.

Finally, Eq. (A.24) suggests that an innovator who had entered at date 0 would have a value v_t for any $t \geq 0$:

$$v_t = v_t^0 = A \int_t^\infty e^{-r(s-t)} k_s^{-\beta} ds + \frac{A}{k_0} \int_t^\infty e^{-r(s-t)} \alpha (1 - \frac{k_t}{k_s}) k_s^{1-\beta} ds. \tag{A.25}$$

Equation (A.25) suggests that the time path of v_t depends on parameter values. For example, v_t may fall with t when $\alpha = 0$ or when $\beta \geq 1$, or v_t may initially rise and later fall with t if α is close to 1, β is close to zero, and c is sufficiently large. ■

IA-A3. Proof of Proposition 2(B)

Proof. In Regime 2, all firms at date-t share the same value v_t regardless of their entry date or type. We first conjecture that no agent would enter as an innovator after date 0 and $k_0 < N$, so that the time path of firm numbers is determined by Eq. (A.1).

Recall in Regime 2, v_t is determined by

$$r v_t = p_t + \gamma (N - k_t) \alpha v_t + \frac{dv_t}{dt} \tag{A.26}$$

$\Rightarrow$

$$\frac{dv_t}{dt} - [r - \gamma (N - k_t) \alpha] v_t = -p_t. \tag{A.27}$$

Defining $z_t = \exp\left(\int -[r - \gamma (N - k_t) \alpha] dt\right)$, we can rewrite Eq. (A.27) as

$$\frac{d}{dt}(z_t v_t) = -z_t p_t,$$

which yields the general solution

$$v_t = z_t^{-1} \int -z_t p_t dt + z_t^{-1} C,$$

where C is the constant of integration.

Given that $k_t = \frac{N e^{\gamma N t}}{e^{\gamma N t} + \frac{N}{k_0} - 1}$, we can solve for z_t:

$$z_t = \exp\left(\int -[r - \gamma (N - k_t) \alpha] dt\right) = e^{-rt} \left(\frac{N - k_0}{k_0} e^{-\gamma N t} + 1\right)^{-\alpha}.$$

Accordingly,

$$\begin{aligned} v_t &= e^{rt} \left(\frac{N - k_0}{k_0} e^{-\gamma N t} + 1\right)^{\alpha} \int -e^{-rt} \left(\frac{N - k_0}{k_0} e^{-\gamma N t} + 1\right)^{-\alpha} p_t dt \\ &\quad + e^{rt} \left(\frac{N - k_0}{k_0} e^{-\gamma N t} + 1\right)^{\alpha} C, \end{aligned}$$

which requires that $C = 0$ because v_t needs to be bounded as $t \to \infty$. We then solve for v_t as follows:

$$\begin{aligned} v_t &= e^{rt}\left(\frac{N-k_0}{k_0}e^{-\gamma Nt}+1\right)^{\alpha}\int -e^{-rt}\left(\frac{N-k_0}{k_0}e^{-\gamma Nt}+1\right)^{-\alpha}p_t dt \\ &= e^{rt}\left(\frac{N-k_0}{k_0}e^{-\gamma Nt}+1\right)^{\alpha}\int_t^{\infty} e^{-rs}\left(\frac{N-k_0}{k_0}e^{-\gamma Ns}+1\right)^{-\alpha}p_s ds \\ &= \int_t^{\infty} e^{-r(s-t)}\left(\frac{\frac{N-k_0}{k_0}e^{-\gamma Ns}+1}{\frac{N-k_0}{k_0}e^{-\gamma Nt}+1}\right)^{-\alpha}p_s ds \\ &= \int_t^{\infty} e^{-r(s-t)}\left(1-\frac{1-e^{-\gamma N(s-t)}}{\frac{k_0}{N-k_0}e^{\gamma Nt}+1}\right)^{-\alpha}p_s ds. \end{aligned} \tag{A.28}$$

Defining $j = s - t$, we can rewrite Eq. (A.28) as

$$v_t = \int_0^{\infty} e^{-rj}\left(1-\frac{1-e^{-\gamma Nj}}{\frac{k_0}{N-k_0}e^{\gamma Nt}+1}\right)^{-\alpha}p_{t+j}dj. \tag{A.29}$$

In the integral, both terms $\left(1-\frac{1-e^{-\gamma Nj}}{\frac{k_0}{N-k_0}e^{\gamma Nt}+1}\right)^{-\alpha}$ and $p_{t+j} = Ak_{t+j}^{-\beta}$ decrease in t. Therefore, v_t falls with t.

Next, we show $v_t - u_t$ falls with t. Recall in Regime 2, u_t is determined by

$$ru_t = \gamma k_t\left((1-\alpha)v_t - u_t\right) + \frac{du_t}{dt}, \tag{A.30}$$

which, together with Eq. (A.26), implies that

$$\frac{d\left(v_t - u_t\right)}{dt} = (r+\gamma k_t)(v_t-u_t) - (p_t + \gamma N\alpha v_t). \tag{A.31}$$

Defining $\psi_t \equiv v_t - u_t$, we can rewrite Eq. (A.31) as

$$\frac{d\psi_t}{dt} - (r+\gamma k_t)\psi_t = -(p_t+\gamma N\alpha v_t).$$

Define $z_t = \exp\int -(r+\gamma k_t)dt$. We then have

$$\frac{d(z_t\psi_t)}{dt} = -z_t(p_t+\gamma N\alpha v_t),$$

which yields the general solution

$$\psi_t = z_t^{-1}\int -z_t(p_t+\gamma N\alpha v_t)dt + z_t^{-1}C,$$

where C is the constant of integration.

Given that $k_t = \frac{Ne^{\gamma N t}}{e^{\gamma N t}+\frac{N}{k_0}-1}$, we can solve z_t :

$$z_t = e^{-rt}\frac{k_0}{N-k_0+k_0e^{\gamma N t}}.$$

Again, ψ_t needs to be bounded as $t\to\infty$, so $C=0$. We then have

$$\begin{aligned}
\psi_t &= z_t^{-1}\int -z_t(p_t+\gamma N\alpha v_t)dt \\
&= e^{rt}\frac{N-k_0+k_0e^{\gamma N t}}{k_0}\int -e^{-rt}\frac{k_0}{N-k_0+k_0e^{\gamma N t}}(p_t+\gamma N\alpha v_t)dt \\
&= e^{rt}\frac{N-k_0+k_0e^{\gamma N t}}{k_0}\int_t^\infty e^{-rs}\frac{k_0}{N-k_0+k_0e^{\gamma N s}}(p_s+\gamma N\alpha v_s)ds \\
&= \int_t^\infty e^{-r(s-t)}\left(1-\frac{e^{\gamma N(s-t)}-1}{(\frac{N-k_0}{k_0})e^{-\gamma N t}+e^{\gamma N(s-t)}}\right)(p_s+\gamma N\alpha v_s)ds. \qquad \text{(A.32)}
\end{aligned}$$

Define $j=s-t$. Then Eq. (A.32) reads

$$\psi_t = \int_0^\infty e^{-rj}\left(1-\frac{e^{\gamma N j}-1}{(\frac{N-k_0}{k_0})e^{-\gamma N t}+e^{\gamma N j}}\right)(p_{t+j}+\gamma N\alpha v_{t+j})dj. \qquad \text{(A.33)}$$

Note that in the integral, the terms $\left(1-\frac{e^{\gamma N j}-1}{(\frac{N-k_0}{k_0})e^{-\gamma N t}+e^{\gamma N j}}\right)$ and $(p_{t+j}+\gamma N\alpha v_{t+j})$ both fall with t, hence $\psi_t = v_t-u_t$ strictly decreases with t. Given the free entry condition that $v_0-u_0=c$ at date 0, we know $v_t-u_t<c$ at any date $t>0$, so no agent would enter as an innovator after date 0.

Note that Eq. (A.28) implies that

$$v_t = \int_t^\infty e^{-r(s-t)}\left(\frac{k_s}{k_t}\right)^\alpha p_s ds,$$

so that

$$v_0 = \int_0^\infty e^{-rt}\left(\frac{k_t}{k_0}\right)^\alpha Ak_t^{-\beta}dt. \qquad \text{(A.34)}$$

At date 0, the total industry discounted revenue, $\int_0^\infty e^{-rt}Ak_t^{1-\beta}dt$, is shared by the two groups – the initial incumbents k_0 and the outsiders $N-k_0$. With the free entry condition $v_0-c=u_0$ we have

$$\int_0^\infty e^{-rt}Ak_t^{1-\beta}dt = v_0k_0+u_0(N-k_0) = v_0N-c(N-k_0). \qquad \text{(A.35)}$$

Plugging Eq. (A.34) into Eq. (A.35) yields Eq. (A.36):

$$\frac{\int_0^\infty e^{-rt}\left(\left(\frac{N}{k_0}\right)^\alpha\left(\frac{N}{k_t}\right)^{1-\alpha}-1\right)Ak_t^{1-\beta}dt}{N-k_0} = c. \qquad \text{(A.36)}$$

Interior solution (i.e., $k_0^{\mathbf{II}} < N$).—We have shown that $k_0^{\mathbf{II}}$ is determined by Eq. (A.36). Note that as $k_0 \to N$, we have $k_t \to N$. Hence, the numerator and the denominator of the left hand side of Eq. (A.36) both go to 0 as $k_0 \to N$. Applying L'Hôpital's rule, the left hand side of Eq. (A.36) converges to $\frac{r+\alpha\gamma N}{r(r+\gamma N)}AN^{-\beta}$ as $k_0 \to N$. Proposition 3(A) shows that $dk_0^{\mathbf{II}}/dc < 0$. Therefore, the model has an interior solution $k_0^{\mathbf{II}} < N$ in Regime 2 iff condition (A.23) holds, i.e., iff

$$c > \frac{r+\alpha\gamma N}{r(r+\gamma N)}AN^{-\beta}.$$

■

IA-A4. Proof of Proposition 3

Proof. (A) We first prove that $k_0^{\mathbf{I}}$ rises with α and A, but falls with c and r. Rewrite Eq. (A.22) as

$$G = \int_0^\infty e^{-rt}F(t;k_0)dt - c = 0,$$

where

$$F(t;k_0) = \frac{1}{N-k_0}\left(\alpha\frac{N}{k_0} + (1-\alpha)\frac{N}{k_t} - 1\right)Ak_t^{1-\beta},$$

and

$$k_t = \frac{Ne^{\gamma Nt}}{e^{\gamma Nt} + \frac{N}{k_0} - 1}.$$

We verify that $\frac{\partial F(t;k_0)}{\partial k_0} < 0$, so $\frac{\partial G}{\partial k_0} < 0$. Following that, we can prove

$$\frac{\partial k_0^{\mathbf{I}}}{\partial \alpha} = -\frac{\partial G/\partial \alpha}{\partial G/\partial k_0^{\mathbf{I}}} > 0; \quad \frac{\partial k_0^{\mathbf{I}}}{\partial A} = -\frac{\partial G/\partial A}{\partial G/\partial k_0^{\mathbf{I}}} > 0\ ;$$

$$\frac{\partial k_0^{\mathbf{I}}}{\partial c} = -\frac{\partial G/\partial c}{\partial G/\partial k_0^{\mathbf{I}}} < 0; \frac{\partial k_0^{\mathbf{I}}}{\partial r} = -\frac{\partial G/\partial r}{\partial G/\partial k_0^{\mathbf{I}}} < 0.$$

Similarly, in Eq. (A.36), we can prove that $k_0^{\mathbf{II}}$ rises with α and A, but falls with c and r.

(B) First, it is straightforward to verify Eqs. (A.22) and (A.36) are identical when $\alpha \in \{0,1\}$, so $k_0^{\mathbf{I}} = k_0^{\mathbf{II}}$.

Second, for any $\alpha \in (0,1)$ and $t > 0$, we can apply the mean-value theorem to derive

$$\left(\frac{N}{k_0}\right)^\alpha \left(\frac{N}{k_t}\right)^{1-\alpha} = \left(\frac{N}{k_t}\right)\left(\frac{k_t}{k_0}\right)^\alpha = \left(\frac{N}{k_t}\right)\left(1 + \alpha\left(\frac{k'}{k_0}\right)^{\alpha-1}\frac{(k_t-k_0)}{k_0}\right)$$

where $k_t > k' > k_0$. Therefore,

$$\left(\frac{N}{k_0}\right)^{\alpha}\left(\frac{N}{k_t}\right)^{1-\alpha} < \left(\frac{N}{k_t}\right)\left(1+\alpha\frac{(k_t-k_0)}{k_0}\right) = \alpha\frac{N}{k_0} + (1-\alpha)\frac{N}{k_t}. \tag{A.37}$$

Given that $k_0^{\mathbf{I}}$ and $k_t^{\mathbf{I}}$ satisfy Eq. (A.22) so that

$$\int_0^{\infty} e^{-rt}\frac{1}{N-k_0^{\mathbf{I}}}\left(\alpha\frac{N}{k_0^{\mathbf{I}}} + (1-\alpha)\frac{N}{k_t^{\mathbf{I}}} - 1\right)A(k_t^{\mathbf{I}})^{1-\beta}dt = c,$$

the same $k_0^{\mathbf{I}}$ and $k_t^{\mathbf{I}}$ would not satisfy Eq. (A.36). Instead,

$$\int_0^{\infty} e^{-rt}\frac{1}{N-k_0^{\mathbf{I}}}\left(\left(\frac{N}{k_0^{\mathbf{I}}}\right)^{\alpha}\left(\frac{N}{k_t^{\mathbf{I}}}\right)^{1-\alpha} - 1\right)A(k_t^{\mathbf{I}})^{1-\beta}dt < c, \tag{A.38}$$

given the inequality (A.37).

The left-hand side of Eq. (A.36) can be written as

$$LHS = \int_0^{\infty} e^{-rt}F(t;k_0)dt$$

where

$$F(t;k_0) = \frac{1}{N-k_0}\left(\left(\frac{N}{k_0}\right)^{\alpha}\left(\frac{N}{k_t}\right)^{1-\alpha} - 1\right)Ak_t^{1-\beta},$$

and

$$k_t = \frac{Ne^{\gamma N t}}{e^{\gamma N t} + \frac{N}{k_0} - 1}.$$

We verify that $\frac{\partial F(t;k_0)}{\partial k_0} < 0$, so $\frac{\partial LHS}{\partial k_0} < 0$. Therefore, the solution $k_0^{\mathbf{II}}$ that satisfies Eq. (A.36) has to satisfy $k_0^{\mathbf{II}} < k_0^{\mathbf{I}}$. ■

IA-A5. Proof of Proposition 4

Proof. We prove the following results:

(A) For inelastic or unit elastic demand (i.e., $\beta \geq 1$**),**

$$k_0^{\mathbf{I}} \text{ and } k_0^{\mathbf{II}} \begin{cases} \text{fall with } \gamma \text{ when } \beta \geq 1 > \alpha, \\ \text{do not vary with } \gamma \text{ when } \beta = \alpha = 1. \end{cases}$$

(B) For elastic demand (i.e., $\beta < 1$),

$$k_0^{\mathbf{I}} \begin{cases} \text{falls with } \gamma \text{ if } 0 \leq \alpha < \frac{1}{\beta + \frac{N}{k_0^{\mathbf{I}}}(1-\beta)} < 1, \\ \text{rises with } \gamma \text{ if } 1 \geq \alpha > \beta + \frac{k_0^{\mathbf{I}}}{N}(1-\beta) > 0, \end{cases}$$

and

$$k_0^{\mathbf{II}} \begin{cases} \text{falls with } \gamma \text{ if } 0 \leq \alpha < \beta + \frac{k_0^{\mathbf{II}}}{N}(1-\beta) < 1, \\ \text{rises with } \gamma \text{ if } 1 \geq \alpha > \beta + \left(\frac{k_0^{\mathbf{II}}}{N}\right)^{\alpha}(1-\beta). \end{cases}$$

We first prove the results for Regime 1. Rewrite Eq. (A.22) as

$$G = \int_0^\infty e^{-rt} F(t;\gamma) dt - \frac{c}{AN^{1-\beta}} = 0,$$

where

$$F(t;\gamma) = \frac{1}{(N-k_0)}\left(\alpha\frac{N}{k_0} - 1 + (1-\alpha)\left(1 + (\frac{N}{k_0}-1)e^{-\gamma N t}\right)\right)\left(1 + (\frac{N}{k_0}-1)e^{-\gamma N t}\right)^{\beta-1}.$$

Note that

$$\frac{F(t;\gamma)}{\partial\gamma} \propto \left\{ \begin{array}{c} -(1-\alpha)\left(1+(\frac{N}{k_0}-1)e^{-\gamma N t}\right) \\ -\left(\alpha\frac{N}{k_0} - 1 + (1-\alpha)\left(1+(\frac{N}{k_0}-1)e^{-\gamma N t}\right)\right)(\beta-1) \end{array} \right\}.$$

Therefore, for inelastic or unit elastic demand (i.e., $\beta \geq 1$), we have $\partial G/\partial\gamma < 0$ (except $\partial G/\partial\gamma = 0$ when $\beta = \alpha = 1$). Recall that $\partial G/\partial k_0 < 0$ from the proof of Proposition 3. We then derive

$$\frac{\partial k_0^{\mathbf{I}}}{\partial\gamma} = -\frac{\partial G/\partial\gamma}{\partial G/\partial k_0^{\mathbf{I}}} < 0 \text{ (except } \frac{\partial k_0^{\mathbf{I}}}{\partial\gamma} = 0 \text{ when } \beta = \alpha = 1).$$

We now consider the case of elastic demand (i.e., $\beta < 1$). Note that

$$\frac{F(t;\gamma)}{\partial\gamma} < 0 \Longleftrightarrow (1-\alpha)\,\beta\left(1+(\frac{N}{k_0}-1)e^{-\gamma N t}\right) > (\alpha\frac{N}{k_0}-1)(1-\beta),$$

which holds for any $t \geq 0$ if

$$(1-\alpha)\,\beta > (\alpha\frac{N}{k_0}-1)(1-\beta) \Longleftrightarrow \alpha < \frac{1}{\frac{N}{k_0}(1-\beta)+\beta}.$$

Therefore, we have

$$\frac{\partial k_0^{\mathbf{I}}}{\partial\gamma} = -\frac{\partial G/\partial\gamma}{\partial G/\partial k_0^{\mathbf{I}}} < 0 \quad \text{if} \quad \alpha < \frac{1}{\frac{N}{k_0^{\mathbf{I}}}(1-\beta)+\beta}.$$

Similarly,

$$\frac{F(t;\gamma)}{\partial\gamma} > 0 \iff (1-\alpha)\left(1+(\frac{N}{k_0}-1)e^{-\gamma N t}\right)\beta < (\alpha\frac{N}{k_0}-1)(1-\beta),$$

which holds for any $t \geq 0$ if

$$(1-\alpha)\left(\frac{N}{k_0}\right)\beta < (\alpha\frac{N}{k_0}-1)(1-\beta) \Longleftrightarrow \alpha > \beta + \frac{k_0}{N}(1-\beta).$$

Therefore, we have

$$\frac{\partial k_0^{\mathbf{I}}}{\partial \gamma} = -\frac{\partial G/\partial \gamma}{\partial G/\partial k_0^{\mathbf{I}}} > 0 \text{ if } \alpha > \beta + \frac{k_0^{\mathbf{I}}}{N}(1-\beta).$$

Similarly, with Eq. (A.36), we can prove the results for Regime 2. ■

IA-A6. Proof of Proposition 6

Proof. We first consider the case $\beta < 1$. For any date $\tau \geq 0$, if no further innovators enter, the number of firms at any date $t \geq \tau$ is

$$k_t = \frac{Ne^{\gamma N(t-\tau)}}{e^{\gamma N(t-\tau)} + \frac{N}{k_\tau} - 1} \qquad \text{for } t \geq \tau. \tag{A.39}$$

As of date τ, the social return to innovation is

$$SR_\tau = \int_\tau^\infty e^{-r(t-\tau)} \frac{A}{1-\beta} k_t^{1-\beta} dt. \tag{A.40}$$

The current cost of innovation is c per unit, and its marginal social return (even if no further innovations are made) is

$$\frac{\partial SR_\tau}{\partial k_\tau} = \int_\tau^\infty e^{-r(t-\tau)} A k_t^{-\beta} \frac{\partial k_t}{\partial k_\tau} dt, \tag{A.41}$$

where

$$\frac{\partial k_t}{\partial k_\tau} = \frac{N^2 e^{\gamma N(t-\tau)}}{\left(N + \left(e^{\gamma N(t-\tau)} - 1\right) k_\tau\right)^2}, \tag{A.42}$$

which is strictly decreasing in k_τ. And since k_t is increasing in t, $e^{-r(t-\tau)} k_t^{-\beta} \frac{\partial k_t}{\partial k_\tau}$ is also decreasing in k_τ. Therefore, $\frac{\partial SR_\tau}{\partial k_\tau}$ is strictly decreasing in k_τ and so if at date 0 k_0 is chosen so that $\frac{\partial SR_0}{\partial k_0} = c$, thereafter $\frac{\partial SR_\tau}{\partial k_\tau} < c$. Similarly, we can prove the result holds for $\beta \geq 1$. Hence, it is socially optimal to innovate only at date 0.

The planner should then simply choose the scalar k_0^* to maximize social welfare:

$$\max_{k_0} \left\{ \int_0^\infty e^{-rt} U\left(k_t\right) dt - ck_0 \right\}, \tag{A.43}$$

subject to Eq. (A.1). The objective function is strictly concave in k_0, and so the socially optimal mass of innovators k_0^* solves the first-order condition

$$\int_0^\infty e^{-(r+\gamma N)t} \left(\frac{k_t}{k_0}\right)^2 A k_t^{-\beta} dt = c. \tag{A.44}$$

for any $\beta > 0$.

Interior solution (i.e., $k_0^* < N$).—Given that the social welfare function (A.43) is strictly concave in k_0, for $k_0^* < N$ to hold, one needs

$$\frac{d\left\{\int_0^\infty e^{-rt} U(k_t)\, dt - ck_0\right\}}{dk_0}|_{k_0=N} < 0,$$

which yields condition (A.45):

$$c > \frac{AN^{-\beta}}{r+\gamma N}. \tag{A.45}$$

Comparative statics.—Rewriting Eq. (A.44), we define

$$G = \int_0^\infty e^{-(r+\gamma N)t} \frac{A}{k_0^{*2}} \left(\frac{Ne^{\gamma Nt}}{e^{\gamma Nt} + (\frac{N}{k_0^*} - 1)}\right)^{2-\beta} dt - c = 0.$$

It follows that

$$\frac{\partial k_0^*}{\partial A} = -\frac{\partial G/\partial A}{\partial G/\partial k_0^*} > 0.$$

Similarly, we can prove $\partial k_0^*/\partial c < 0$ and $\partial k_0^*/\partial r < 0$.

The sign of $\partial k_0^*/\partial\gamma$ depends on $\partial G/\partial\gamma$ and requires some discussions.

$$\frac{\partial G}{\partial \gamma} \propto \int_0^\infty e^{-rt} \left\{ \begin{array}{c} (1-\beta)\left(1+(\frac{N}{k_0^*}-1)e^{-\gamma Nt}\right)^{\beta-2}\left(e^{\gamma Nt}+(\frac{N}{k_0^*}-1)\right)^{-1}(\frac{N}{k_0^*}-1)e^{-\gamma Nt}Nt \\ -\left(1+(\frac{N}{k_0^*}-1)e^{-\gamma Nt}\right)^{\beta-1}\left(e^{\gamma Nt}+(\frac{N}{k_0^*}-1)\right)^{-2}e^{\gamma Nt}Nt \end{array} \right\} dt.$$

This implies $\frac{\partial G}{\partial \gamma} < 0$ if $\beta \geq 1$. When $\beta < 1$, the sign of $\frac{\partial G}{\partial \gamma} < 0$ if $(1-\beta)(\frac{N}{k_0^*} - 1) < e^{\gamma Nt}$. A sufficient condition is that

$$(1-\beta)(\frac{N}{k_0^*} - 1) < 1,$$

$\Longleftrightarrow$

$$\beta > 1 - \frac{k_0^*}{N - k_0^*}.$$

The social planner's problem (A.43) requires $\frac{\partial W_0^*}{\partial k_0} = 0$. Applying the envelope theorem, we have

$$\frac{dW_0^*}{d\gamma} = \frac{\partial W_0^*}{\partial k_0}\frac{\partial k_0}{\partial \gamma} + \frac{\partial W_0^*}{\partial \gamma} = \int_0^\infty e^{-rt}\frac{\partial U(k_t)}{\partial \gamma} dt > 0$$

for any $\beta > 0$. Similarly, we can prove that $dW_0^*/dA > 0$, $dW_0^*/dc < 0$, and $dW_0^*/dr < 0$. ■

IA-A7. Socially optimal subsidy or tax

Whenever $\alpha \neq \alpha^*$ in each regime, the planner can use a subsidy s or tax ($s < 0$) to achieve the social optimum. Denote the socially optimal subsidy for Regimes 1 and 2 by $s^{\mathbf{I}*}$ and $s^{\mathbf{II}*}$, respectively. We obtain the following result:

Proposition IA-1: *Social optimum implies* $s^{\mathbf{I}*} < s^{\mathbf{II}*}$ *for* $\alpha \in (0,1)$, $s^{\mathbf{I}*} = s^{\mathbf{II}*} > 0$ *for* $\alpha = 0$, *and* $s^{\mathbf{I}*} = s^{\mathbf{II}*} < 0$ *for* $\alpha = 1$.

Proof. In Regime 1, for a given value of α, Eq. (A.22) yields the equilibrium entry of innovators $k_0^{\mathbf{I}}$. Proposition 7 suggests that whenever $\alpha \neq \alpha^{\mathbf{I}*}$, the number of innovators $k_0^{\mathbf{I}}$ from Eq. (A.22) differs from the social optimum k_0^*, in which case offering an innovation subsidy (or tax) to adjust the innovation cost c would help restore the social optimum. This implies that k_0^* can be achieved by a subsidy or tax $s^{\mathbf{I}*}$ as follows:

$$\frac{1}{N - k_0^*} \int_0^\infty e^{-rt} \left(\alpha \frac{N}{k_0^*} + (1-\alpha) \frac{N}{k_t^*} - 1 \right) A k_t^{*1-\beta} dt \;=\; c - s^{\mathbf{I}*}.$$

The same logic applies to Regime 2 that

$$\frac{1}{N - k_0^*} \int_0^\infty e^{-rt} \left(\left(\frac{N}{k_0^*} \right)^{\alpha} \left(\frac{N}{k_t^*} \right)^{1-\alpha} - 1 \right) A k_t^{*1-\beta} dt \;=\; c - s^{\mathbf{II}*}.$$

Recall that when $\alpha \in \{0,1\}$, Regimes 1 and 2 coincide. When $\alpha = 0$, both regimes would need more entry of innovators, so a positive subsidy is needed to achieve that, and when $\alpha = 1$, a negative subsidy (tax) is needed. Moreover, for $\alpha \in (0,1)$, according to Proposition 3(B), if a given pair of α and $(c - s^{\mathbf{I}*})$ lead to the social optimum k_0^* in Regime 1, the same parameter values would result in $k_0^{\mathbf{II}} < k_0^*$ in Regime 2. Therefore, a higher subsidy (or a smaller tax) $s^{\mathbf{II}*}$ is needed for adjusting c to achieve k_0^* in Regime 2 given that $k_0^{\mathbf{II}}$ falls with c as shown by Proposition 3(A). ■

IA-A8. Proof of Proposition 7

Proof. Given that condition (A.45) holds, we have $k_0^* < N$. Proposition 6 shows that the socially optimal innovation k_0^* satisfies Eq. (A.44) that

$$\int_0^\infty e^{-(r+\gamma N)t} \left(\frac{k_t}{k_0} \right)^2 A k_t^{-\beta} dt = c.$$

When $\alpha = 0$, condition (A.45) is equivalent to condition (A.23). Accordingly, we have $k_0^{\mathbf{I}} = k_0^{\mathbf{II}} < N$ and they satisfy Eq. (A.22) or Eq. (A.36) evaluated at $\alpha = 0$ so that

$$\int_0^\infty e^{-rt} \frac{1}{N - k_0} \left(\frac{N}{k_t} - 1 \right) A k_t^{1-\beta} dt = c$$

$\Longleftrightarrow$

$$\int_0^\infty e^{-(r+\gamma N)t} \left(\frac{k_0}{k_t} \right) \left(\frac{k_t}{k_0} \right)^2 A k_t^{-\beta} dt = c. \tag{A.46}$$

Note that the left hand side of Eq. (A.46) is smaller than the left hand side of Eq. (A.44) given that $\frac{k_0}{k_t} < 1$ for $t > 0$. Therefore, the solution $k_0^{\mathbf{I}} = k_0^{\mathbf{II}}$ to Eq. (A.46) would cause the left hand side of Eq. (A.44) to exceed c. Because the left hand side of Eq. (A.44) decreases with k_0, this implies $k_0^{\mathbf{I}} = k_0^{\mathbf{II}} < k_0^* < N$ for $\alpha = 0$.

Given that the right-hand side of condition (A.23) rises with α, if there existed an $\alpha' \in (0, 1]$ such that the inequality (A.23) becomes an equality (i.e., $c = \frac{r+\alpha'\gamma N}{r(r+\gamma N)} A N^{-\beta}$), we would then have $k^* < k_0^{\mathbf{I}} = k_0^{\mathbf{II}} = N$ for $\alpha = \alpha'$. In this case, the planner could choose optimal shares $\alpha^{\mathbf{I}*}$ and $\alpha^{\mathbf{II}*}$ such that $0 < \alpha^{\mathbf{I}*} < \alpha^{\mathbf{II}*} < \alpha'$ ≤ 1 to achieve the socially optimal k^* in Regimes 1 and 2, respectively. Note that $\alpha^{\mathbf{I}*} < \alpha^{\mathbf{II}*}$ results from Proposition 3: If a value of α led to $k_0^{\mathbf{I}} = k^*$ in Regime 1, the same α would lead to $k_0^{\mathbf{II}} < k^*$ in Regime 2, so a larger value of α would be needed to achieve k^*in Regime 2.

Alternatively, if the inequality (A.23) holds for any $\alpha \leq 1$, we have $k_0^{\mathbf{I}} = k_0^{\mathbf{II}} < N$ for $\alpha = 1$ and they satisfy Eq. (A.22) or Eq. (A.36) evaluated at $\alpha = 1$ so that

$$\int_0^\infty e^{-(r+\gamma N)t} \left(\frac{k_t}{k_0} \right)^2 A k_t^{-\beta} dt = c$$

$\Longleftrightarrow$

$$\int_0^\infty e^{-(r+\gamma N)t} \left(\frac{k_0}{N} e^{\gamma N t} - \frac{k_0}{N} + 1 \right) \left(\frac{k_t}{k_0} \right)^2 A k_t^{-\beta} dt = c. \tag{A.47}$$

Note that the left hand side of Eq. (A.47) is greater than the left hand side of Eq. (A.44) given that $\frac{k_0}{N} e^{\gamma N t} - \frac{k_0}{N} + 1 > 1$ for $t > 0$. Therefore, the solution $k_0^{\mathbf{I}} = k_0^{\mathbf{II}}$ to Eq. (A.47) would cause the left hand side of Eq. (A.44) to be smaller than c. Because the left hand side of Eq. (A.44) falls with k_0, this implies $k_0^* < k_0^{\mathbf{I}} = k_0^{\mathbf{II}} < N$ for $\alpha = 1$. In this case, the social planner could choose optimal shares $\alpha^{\mathbf{I}*}$ and $\alpha^{\mathbf{II}*}$ such that $0 < \alpha^{\mathbf{I}*} < \alpha^{\mathbf{II}*} < 1$ to achieve the socially optimal k^* in Regimes 1 and 2, respectively. ■

IA-A9. Industry licensing revenue in Regimes 1 and 2

In Regime 1, Eq. (A.12) shows that an innovator' value at date 0 is

$$v_0^{\mathbf{I}} = \int_0^\infty e^{-rt} \left(\frac{1}{k_t} + \frac{\alpha}{k_0} \left(\frac{k_t - k_0}{k_t} \right) \right) A k_t^{1-\beta} dt.$$

Excluding the revenues from selling goods, the present value of licensing revenue is

$$l_0^{\mathbf{I}} = v_0^{\mathbf{I}} - \int_0^\infty e^{-rt} A k_t^{-\beta} dt = \int_0^\infty e^{-rt} \left(\alpha (\frac{k_t}{k_0} - 1) \right) A k_t^{-\beta} dt.$$

Therefore, the present value of total licensing revenue paid to innovators in the industry is

$$L_0^{\mathbf{I}} = k_0 l_0^{\mathbf{I}} = \int_0^\infty e^{-rt} \alpha (k_t - k_0) A k_t^{-\beta} dt. \tag{A.48}$$

In Regime 2, Eq. (A.29) shows that an innovator's value at date 0 is

$$v_0^{\mathbf{II}} = \int_0^\infty e^{-rt} \left(\frac{k_t}{k_0} \right)^\alpha A k_t^{-\beta} dt.$$

Excluding the revenue from selling goods, the present value of licensing revenue is

$$l_0^{\mathbf{II}} = v_0^{\mathbf{II}} - \int_0^\infty e^{-rt} A k_t^{-\beta} dt = \int_0^\infty e^{-rt} \left(\left(\frac{k_t}{k_0} \right)^\alpha - 1 \right) A k_t^{-\beta} dt.$$

Therefore, the present value of total licensing revenue paid to innovators in the industry is

$$L_0^{\mathbf{II}} = k_0 l_0^{\mathbf{II}} = \int_0^\infty e^{-rt} \left(\left(\frac{k_t}{k_0} \right)^\alpha k_0 - k_0 \right) A k_t^{-\beta} dt. \tag{A.49}$$

IA-A10. Socially optimal diffusion

Assume condition (A.23) holds so that not all agents enter at date 0 (i.e., $k_0 < N$). We now prove the welfare results for the unit demand elasticity case (i.e., $\beta = 1$).

Proposition IA-2. *(A) For $\beta = 1$ and all $\alpha \in [0, 1]$, social welfare always rises with the diffusion rate γ in Regime 1. (B) For $\beta = 1$ and $\alpha \in \{0, 1\}$, Regimes 1 and 2 coincide and social welfare always rises with the diffusion rate γ.*

Proof. (A) With $\beta = 1$ under Regime 1, Eq. (A.22) simplifies to

$$k_0^{\mathbf{I}} = \frac{A(\alpha\gamma N + r)}{cr(r + \gamma N)}. \tag{A.50}$$

Given Eq. (A.50) and $\beta = 1$, social surplus is

$$W_0 = -\frac{A(1-\alpha)}{(r+\gamma N)} + A \int_0^\infty e^{-rt} \left[\gamma N t - \ln \left(e^{\gamma N t} + \frac{N cr(r+\gamma N)}{A(\alpha\gamma N + r)} - 1 \right) \right] dt + \text{constant}.$$

This suggests that

$$\frac{dW_0}{d\gamma} = \frac{A(1-\alpha)N}{(r+\gamma N)^2} + A\int_0^\infty e^{-rt}Ntdt - A\int_0^\infty e^{-rt}\left[\frac{Nte^{\gamma Nt} + \frac{crN^2}{A(\alpha\gamma N+r)} - \frac{\alpha cr(r+\gamma N)N^2}{A(\alpha\gamma N+r)^2}}{e^{\gamma Nt} + \frac{Ncr(r+\gamma N)}{A(\alpha\gamma N+r)} - 1}\right]dt.$$

We then verify that

$$\begin{aligned} & d\left[\frac{Nte^{\gamma Nt} + \frac{crN^2}{A(\alpha\gamma N+r)} - \frac{\alpha cr(r+\gamma N)N^2}{A(\alpha\gamma N+r)^2}}{e^{\gamma Nt} + \frac{Ncr(r+\gamma N)}{A(\alpha\gamma N+r)} - 1}\right]/dc \\ \propto & \left(\frac{(1-\alpha)r}{(\alpha\gamma N+r)}\right)\left(e^{\gamma Nt} + \frac{Ncr(r+\gamma N)}{A(\alpha\gamma N+r)} - 1\right) \\ & -\left(te^{\gamma Nt} + \frac{crN}{A(\alpha\gamma N+r)} - \frac{\alpha cr(r+\gamma N)N}{A(\alpha\gamma N+r)^2}\right)(r+\gamma N) \\ < & \ 0 \end{aligned}$$

for any $t > 0$. Equation (A.50) implies that $c > \frac{A(\alpha\gamma N+r)}{Nr(r+\gamma N)}$ is needed for $k_0^{\mathbf{I}}$ to be interior solution (i.e., $k_0^{\mathbf{I}} < N$). We then have

$$A\int_0^\infty e^{-rt}\left[\frac{Nte^{\gamma Nt} + \frac{crN^2}{A(\alpha\gamma N+r)} - \frac{\alpha cr(r+\gamma N)N^2}{A(\alpha\gamma N+r)^2}}{e^{\gamma Nt} + \frac{Ncr(r+\gamma N)}{A(\alpha\gamma N+r)} - 1}\right]dt < A\int_0^\infty e^{-rt}\left[\frac{Nte^{\gamma Nt} + \frac{N}{r+\gamma N} - \frac{\alpha N}{(\alpha\gamma N+r)}}{e^{\gamma Nt}}\right]dt.$$

Therefore,

$$\begin{aligned} \frac{dW_0}{d\gamma} &> \frac{A(1-\alpha)N}{(r+\gamma N)^2} + A\int_0^\infty e^{-rt}Ntdt - A\int_0^\infty e^{-rt}\left[\frac{Nte^{\gamma Nt} + \frac{N}{r+\gamma N} - \frac{\alpha N}{(\alpha\gamma N+r)}}{e^{\gamma Nt}}\right]dt \\ &= \frac{A(1-\alpha)N}{(r+\gamma N)^2} - \frac{A}{(r+\gamma N)}\left(\frac{N}{r+\gamma N} - \frac{\alpha N}{(\alpha\gamma N+r)}\right) \\ &= \frac{A\alpha N}{(r+\gamma N)}\left(\frac{1}{r+\alpha\gamma N} - \frac{1}{r+\gamma N}\right) \geq 0 \text{ for any } \alpha\in[0,1]. \end{aligned}$$

(B) Proposition IA-2(B) is an application of Proposition IA-2(A) by taking $\alpha = 0$ or $\alpha = 1$, and Regimes 1 and 2 coincide in those cases (cf. Proposition 3). ■

IA-A11. Time-varying demand and firm productivity

We now extend our model to consider an industry with time-varying demand and firm productivity. Specifically, we assume each firm's production capacity q_t rises over time, and the total industry output at date t is $q_t k_t$. We also assume that the market size parameter A_t grows with time, and the product price at date t is

$$p_t = A_t\left(q_t k_t\right)^{-\beta}.$$

Our findings in Proposition 2 in the paper can then be extended as follows.

Proposition IA-3. *With time-varying A_t and q_t, as long as the profit per firm $p_t q_t = A_t q_t^{1-\beta} k_t^{-\beta}$ does not increase in t, market equilibrium yields:*
(A) In Regime 1, innovators enter only at date 0 and $k_0^{\mathbf{I}} \in (0, N)$ solves

$$\underbrace{\frac{1}{N-k_0}\int_0^\infty e^{-rt}\left(\alpha\frac{N}{k_0}+(1-\alpha)\frac{N}{k_t}-1\right)A_t\left(q_t k_t\right)^{1-\beta}dt}_{v_0-u_0} \;=\; c; \tag{A.51}$$

(B) In Regime 2, innovators enter only at date 0 and $k_0^{\mathbf{II}} \in (0, N)$ solves

$$\underbrace{\frac{1}{N-k_0}\int_0^\infty e^{-rt}\left(\left(\frac{N}{k_0}\right)^{\alpha}\left(\frac{N}{k_t}\right)^{1-\alpha}-1\right)A_t\left(q_t k_t\right)^{1-\beta}dt}_{v_0-u_0} \;=\; c; \tag{A.52}$$

where, in each regime, $0 \le \alpha \le 1$ and k_t follows logistic diffusion (A.1).

Proof. (A) In Regime 1, with time-varying A_t and q_t, we first consider a measure-zero innovator who deviates from the equilibrium and enters at date $\tau > 0$. Following a proof similar to that of Proposition 2(A), the value of such a firm at his entry date τ would be

$$v_\tau^\tau = \int_\tau^\infty e^{-r(t-\tau)}\left(\frac{1}{k_t}+\frac{\alpha}{k_\tau}\left(\frac{k_t-k_\tau}{k_t}\right)\right)A_t\left(q_t k_t\right)^{1-\beta}dt. \tag{A.53}$$

Defining $s = t-\tau$, Eq. (A.53) becomes

$$v_\tau^\tau = \int_0^\infty e^{-rs}\left(1-\alpha+\alpha\frac{k_{\tau+s}}{k_\tau}\right)A_{\tau+s}q_{\tau+s}^{1-\beta}k_{\tau+s}^{-\beta}ds. \tag{A.54}$$

Similarly, like in the proof of Proposition 2(A), we can derive the value of an outsider at τ to be

$$u_\tau = \int_0^\infty e^{-rs}\left(\frac{(1-\alpha)\left(e^{\gamma Ns}-1\right)}{Ne^{\gamma Ns}}\right)A_{\tau+s}\left(q_{\tau+s}k_{\tau+s}\right)^{1-\beta}ds \tag{A.55}$$

Combining Eqs. (A.54) and (A.55), we have

$$v_\tau^\tau - u_\tau = \int_0^\infty e^{-rs}\left(1-\alpha+\alpha\frac{k_{\tau+s}}{k_\tau}-\frac{(1-\alpha)\left(e^{\gamma Ns}-1\right)}{Ne^{\gamma Ns}}k_{\tau+s}\right)A_{\tau+s}q_{\tau+s}^{1-\beta}k_{\tau+s}^{-\beta}ds. \tag{A.56}$$

Note the term $\left(1-\alpha+\alpha\frac{k_{\tau+s}}{k_\tau}-\frac{(1-\alpha)\left(e^{\gamma Ns}-1\right)}{Ne^{\gamma Ns}}k_{\tau+s}\right)$ falls with τ. Then, as long as $p_{\tau+s}q_{\tau+s} = A_{\tau+s}q_{\tau+s}^{1-\beta}k_{\tau+s}^{-\beta}$ does not increase in τ, $v_\tau^\tau - u_\tau$ rises with τ. Therefore, given the free entry condition $v_0^0 - u_0 = c$, we have $v_\tau^\tau - u_\tau < c$ for any $\tau > 0$, and so no innovator would enter the industry after date 0.

Let $v_0 \equiv v_0^0$. Equation (A.56) implies that

$$v_0 - u_0 = \frac{1}{N - k_0} \int_0^\infty e^{-rt} \left(\alpha \frac{N}{k_0} + (1-\alpha) \frac{N}{k_t} - 1 \right) A_t \left(q_t k_t \right)^{1-\beta} dt. \tag{A.57}$$

The free entry condition $v_0 - u_0 = c$ then pins down the entry of innovators k_0 at date 0, as shown by Eq. (A.51):

$$\frac{\int_0^\infty e^{-rt} \left(\alpha \frac{N}{k_0} + (1-\alpha) \frac{N}{k_t} - 1 \right) A_t \left(q_t k_t \right)^{1-\beta} dt}{N - k_0} = c.$$

(B) In Regime 2, with time-varying A_t and q_t, we have

$$r v_t = p_t q_t + \gamma \left(N - k_t \right) \alpha v_t + \frac{d v_t}{dt}, \tag{A.58}$$

and

$$r u_t = \gamma k_t \left((1-\alpha) v_t - u_t \right) + \frac{d u_t}{dt}. \tag{A.59}$$

Following a proof similar to that of Proposition 2(B), we can solve Eq. (A.58) to get

$$v_t = \int_0^\infty e^{-rj} \left(1 - \frac{1 - e^{-\gamma N j}}{\frac{k_0}{N-k_0} e^{\gamma N t} + 1} \right)^{-\alpha} A_{t+j} q_{t+j}^{1-\beta} k_{t+j}^{-\beta} dj. \tag{A.60}$$

In the integral, the term $\left(1 - \frac{1 - e^{-\gamma N j}}{\frac{k_0}{N-k_0} e^{\gamma N t} + 1} \right)^{-\alpha}$ falls in t. Therefore, as long as $p_{t+j} q_{t+j} = A_{t+j} q_{t+j}^{1-\beta} k_{t+j}^{-\beta}$ does not increase in t, v_t falls with t.

Moreover, Eqs. (A.58) and (A.59) imply that

$$\frac{d \left(v_t - u_t \right)}{dt} = (r + \gamma k_t)(v_t - u_t) - (p_t q_t + \gamma N \alpha v_t). \tag{A.61}$$

Let $\psi_t \equiv v_t - u_t$. Following a proof similar to that of Proposition 2(B) , we solve Eq. (A.61) to derive

$$\psi_t = \int_0^\infty e^{-rj} \left(1 - \frac{e^{\gamma N j} - 1}{(\frac{N-k_0}{k_0}) e^{-\gamma N t} + e^{\gamma N j}} \right) (p_{t+j} q_{t+j} + \gamma N \alpha v_{t+j}) dj.$$

Note that in the integral, the terms $\left(1 - \frac{e^{\gamma N j} - 1}{(\frac{N-k_0}{k_0}) e^{-\gamma N t} + e^{\gamma N j}} \right)$ and $\gamma N \alpha v_{t+j}$ decrease in t. As long as $p_{t+j} q_{t+j} = A_{t+j} q_{t+j}^{1-\beta} k_{t+j}^{-\beta}$ does not increase in t, $\psi_t = v_t - u_t$ strictly decreases with t. Given the free entry condition that $v_0 - u_0 = c$ at date 0, we get $v_t - u_t < c$ at all $t > 0$, so no agent would enter as an innovator after date 0.

Eq. (A.60) implies that

$$v_0 = \int_0^\infty e^{-rt} \left(\frac{k_t}{k_0}\right)^\alpha A_t q_t^{1-\beta} k_t^{-\beta} dt. \tag{A.62}$$

At date 0, the total industry discounted revenue, $\int_0^\infty e^{-rt} A_t q_t^{1-\beta} k_t^{1-\beta} dt$, is shared by the two groups – the initial entrants k_0 and the outsiders $N - k_0$. With the free entry condition $v_0 - c = u_0$,

$$\int_0^\infty e^{-rt} A_t q_t^{1-\beta} k_t^{1-\beta} dt = v_0 k_0 + u_0 (N - k_0) = v_0 N - c (N - k_0). \tag{A.63}$$

Plugging Eq. (A.62) into Eq. (A.63) yields Eq. (A.52):

$$\frac{\int_0^\infty e^{-rt} \left(\left(\frac{N}{k_0}\right)^\alpha \left(\frac{N}{k_t}\right)^{1-\alpha} - 1 \right) A_t q_t^{1-\beta} k_t^{1-\beta} dt}{N - k_0} = c.$$

■

IA-A12. Proofs of the claims in Section 6.4

Diffusion process as $N \to \infty$.—The logistic process (A.1) implies that for a given k_t,

$$\frac{dk_t/dt}{k_t} = \gamma (N - k_t) = \gamma N (1 - \frac{e^{\lambda t}}{e^{\lambda t} + \frac{N}{k_0} - 1}). \tag{A.64}$$

Given the inverse demand function $p_t = A k_t^{-\beta}$, k_0 has to be finite as $N \to \infty$; otherwise $p_0 \to 0$, and no innovator would enter at date 0. Therefore, Eq. (A.64) implies that

$$\left.\frac{dk_t/dt}{k_t}\right|_{N \to \infty} \to \gamma N \to \lambda. \tag{A.65}$$

The logistic diffusion process then converges to $\frac{dk_t}{dt} = \lambda k_t$, and Eq. (A.1) becomes

$$k_t = k_0 e^{\lambda t}. \tag{A.66}$$

Regime 1.—We conjecture that no agent would enter as an innovator after date 0, so k_t is given by Eq. (A.66). Given that an imitator cannot resell the idea, his only revenue comes from selling the good, and his value ω_t satisfies the ordinary differential equation (ODE):

$$r\omega_t = p_t + \frac{d\omega_t}{dt} = A(k_0 e^{\lambda t})^{-\beta} + \frac{d\omega_t}{dt}. \tag{A.67}$$

The ODE has the unique bounded solution

$$\omega_t = \frac{Ak_0^{-\beta}}{r+\beta\lambda} e^{-\beta\lambda t}. \tag{A.68}$$

An innovator receives revenue from selling both the good and the idea. The number of ideas sold at t is λk_t and the total date-t revenue from these sales, $\lambda k_t \alpha \omega_t$ is divided among the k_0 innovators. Thus v_t, the value of being an innovator at date t, follows the ODE

$$rv_t = p_t + \frac{\lambda k_t}{k_0}\alpha\omega_t + \frac{dv_t}{dt} = A(k_0 e^{\lambda t})^{-\beta} + \frac{\alpha\lambda A k_0^{-\beta} e^{(1-\beta)\lambda t}}{r+\beta\lambda} + \frac{dv_t}{dt}. \tag{A.69}$$

Unless $\alpha = 0$, innovators receive a fraction of revenues from idea sales, and we shall need to restrict the elasticity of demand to be below unity which means $\beta \geq 1$. Imposing the boundary condition $v_{t\to\infty} < \infty$ yields the unique solution to Eq. (A.69):

$$v_t = \frac{Ak_0^{-\beta} e^{-\beta\lambda t}}{r+\beta\lambda} + \frac{\alpha\lambda A k_0^{-\beta}}{(r+\beta\lambda)(r+(\beta-1)\lambda)} e^{-(\beta-1)\lambda t}. \tag{A.70}$$

Recall that u_t denotes the option value of becoming a future imitator. At $t = 0$, the free entry condition is $v_0 - u_0 = c$. Given that the pool of outsiders is infinite, an outsider's chance of matching with an incumbent is zero so that $u_t = 0$ for all t, implying that $v_0 = c$. Since v_t falls over time, we verify the conjecture that no one would pay c to become an innovator at any date $t > 0$. Note that if an agent deviates from the equilibrium and enters at date $t > 0$, he would have a lower valuation than an innovator who entered at date 0 (i.e., $v_t^t < v_t^0$) because the latter would have a larger family of imitators to disseminate his idea and collect idea-sale revenues. Therefore, the finding that $v_t - u_t$ declines in t implies that $v_t^t - u_t < c$ at any date $t > 0$.

Combining $v_0 = c$ with Eq. (A.70) yields

$$v_0 = \frac{Ak_0^{-\beta}}{r+\beta\lambda} + \frac{\alpha\lambda A k_0^{-\beta}}{(r+\beta\lambda)(r+(\beta-1)\lambda)} = c. \tag{A.71}$$

Equation (A.71) then determines the entry of innovators at date 0 to be

$$k_0^{\mathbf{I}} = \left(\frac{A(r+(\beta-1)\lambda+\alpha\lambda)}{c\,(r+\beta\lambda)(r+(\beta-1)\lambda)} \right)^{\frac{1}{\beta}}. \tag{A.72}$$

Regime 2.—Assume that condition $\lambda < r$ holds so that social welfare derived from the innovation is bounded. We conjecture that no agent would enter as an innovator after date 0. Given that imitators can resell the innovation, all the incumbents (be they innovators or imitators) share the same value v_t. The revenue from an idea sale is αv_t and the total date-t revenue from these sales, $\lambda k_t \alpha v_t$, is shared equally among all the incumbents. Then v_t follows the ODE

$$rv_t = p_t + \lambda\alpha v_t + \frac{dv_t}{dt} = A(k_0 e^{\lambda t})^{-\beta} + \lambda\alpha v_t + \frac{dv_t}{dt}. \tag{A.73}$$

The general solution of Eq. (A.73) is

$$v_t = \frac{Ak_0^{-\beta} e^{-\beta\lambda t}}{r + \beta\lambda - \lambda\alpha} + Ak_0^{-\beta} C e^{(r-\lambda\alpha)t},$$

where C is the constant of integration. Given that $\lambda < r$, the boundary condition $v_{t\to\infty} < \infty$ requires $C = 0$ and yields

$$v_t = \frac{Ak_0^{-\beta} e^{-\beta\lambda t}}{r + (\beta - \alpha)\lambda}, \tag{A.74}$$

which falls with t. Again, given $N \to \infty$, an outsider's chance of matching with an incumbent is zero so that $u_t = 0$. Therefore, $v_t - u_t$ falls with t and no innovator enters after date 0. Since the free entry condition requires $v_0 = c$, Eq. (A.74) evaluated at $t = 0$ yields

$$k_0^{\mathbf{II}} = \left(\frac{A}{c(r + (\beta - \alpha)\lambda)} \right)^{\frac{1}{\beta}}. \tag{A.75}$$

Socially optimal licensing rate.—The planner maximizes

$$W_0 = \int_0^\infty e^{-rt} U(k_t)\, dt - ck_0,$$

where

$$U(k_t) = \begin{cases} \frac{A}{1-\beta} k_t^{1-\beta} & \text{if } \beta \in (0,1), \\ A(\ln k_t + 1 - \ln \varepsilon) & \text{if } \beta = 1, \\ \frac{A}{1-\beta} k_t^{1-\beta} + \frac{\beta A}{\beta - 1} \varepsilon^{1-\beta} & \text{if } \beta > 1. \end{cases}$$

and $k_t = k_0 e^{\lambda t}$. We maintain condition $\lambda < r$ so that social welfare derived from the innovation is bounded.

Consider the case $\beta < 1$ first. For any date $\tau \geq 0$, if no further innovators enter, the number of firms at dates $t \geq \tau$ is

$$k_t = k_\tau\ e^{\lambda(t-\tau)} \quad \text{for } t \geq \tau.$$

As of date τ, the social return to innovation is

$$SR_\tau = \int_\tau^\infty e^{-r(t-\tau)} \frac{A}{1-\beta} k_t^{1-\beta} dt.$$

The current cost of innovation is c per unit, and its marginal social return (even if no further innovations are made) is

$$\frac{\partial SR_\tau}{\partial k_\tau} = \int_\tau^\infty e^{-r(t-\tau)} Ak_t^{-\beta} \frac{\partial k_t}{\partial k_\tau} dt = \int_\tau^\infty e^{-(r-\lambda)(t-\tau)} Ak_t^{-\beta} dt. \tag{A.76}$$

Let $s \equiv t - \tau$, and then Eq. (A.76) becomes

$$\frac{\partial SR_\tau}{\partial k_\tau} = \int_0^\infty e^{-(r-\lambda)s} A k_{\tau+s}^{-\beta} ds = \int_0^\infty e^{-(r-\lambda)s} A \left(k_\tau e^{\lambda s}\right)^{-\beta} ds,$$

which is strictly decreasing in k_τ. Therefore, if at date 0 k_0 is chosen so that $\frac{\partial SR_0}{\partial k_0} = c$, thereafter we would have $\frac{\partial SR_\tau}{\partial k_\tau} < c$. Similarly, we can prove that the result holds for $\beta \geq 1$. Hence, it is socially optimal to innovate only at date 0.

Accordingly, the social planner chooses k_0 to maximize social welfare:

$$W_0 = \int_0^\infty e^{-rt} U(k_t)\, dt - ck_0.$$

We verify that the social welfare function is strictly concave in k_0, and the first-order condition is

$$k_0^* = \left(\frac{A}{(r + (\beta - 1)\lambda)c}\right)^{\frac{1}{\beta}}. \tag{A.77}$$

Comparing Eq. (A.77) to Eqs. (A.72) and (A.75) shows that $k_0^* = k_0^{\mathbf{I}} = k_0^{\mathbf{II}}$ iff $\alpha = 1$. I.e., $\alpha^* = 1$ for both Regimes 1 and 2.

Socially optimal diffusion rate.—Recall that the solutions for $k_0^{\mathbf{I}}$ and $k_0^{\mathbf{II}}$ are given by Eqs. (A.72), (A.75):

$$k_0^{\mathbf{I}} = \left(\frac{A(r + (\beta - 1)\lambda + \alpha\lambda)}{c\,(r + \beta\lambda)\,(r + (\beta - 1)\lambda)}\right)^{\frac{1}{\beta}}; \quad k_0^{\mathbf{II}} = \left(\frac{A}{c(r + (\beta - \alpha)\lambda)}\right)^{\frac{1}{\beta}}.$$

In each regime, the number of firms grows at a constant rate λ (i.e., $k_t^{\mathbf{I}} = k_0^{\mathbf{I}} e^{\lambda t}$ and $k_t^{\mathbf{II}} = k_0^{\mathbf{II}} e^{\lambda t}$). The planner maximizes social welfare

$$W_0 = \int_0^\infty e^{-rt} U(k_t)\, dt - ck_0,$$

where

$$U(k_t) = \begin{cases} \frac{A}{1-\beta} k_t^{1-\beta} & \text{if } \beta \in (0,1), \\ A(\ln k_t + 1 - \ln\varepsilon) & \text{if } \beta = 1, \\ \frac{A}{1-\beta} k_t^{1-\beta} + \frac{\beta A}{\beta-1}\varepsilon^{1-\beta} & \text{if } \beta > 1. \end{cases}$$

We again assume that $\lambda < r$, and denote by $W_0^{\mathbf{I}}$ and $W_0^{\mathbf{II}}$ the social welfare under Regimes 1 and 2, respectively. With free imitation ($\alpha = 0$), we have $W_0^{\mathbf{I}} = W_0^{\mathbf{II}} = W_0$, where

$$W_0 = \begin{cases} A^{\frac{1}{\beta}} c^{1-\frac{1}{\beta}} \left(\frac{(r+\beta\lambda)^{1-\frac{1}{\beta}}}{(1-\beta)(r-\lambda(1-\beta))} - (r+\beta\lambda)^{-\frac{1}{\beta}}\right) & \text{if } \beta \in (0,1), \\ \frac{A}{r} \ln \frac{A}{c(r+\lambda)} + \frac{A\lambda}{r^2} - \frac{A}{(r+\lambda)} + \frac{A(1-\ln\varepsilon)}{r} & \text{if } \beta = 1, \\ A^{\frac{1}{\beta}} c^{1-\frac{1}{\beta}} \left(\frac{(r+\beta\lambda)^{1-\frac{1}{\beta}}}{(1-\beta)(r-\lambda(1-\beta))} - (r+\beta\lambda)^{-\frac{1}{\beta}}\right) + \frac{\beta A}{r(\beta-1)}\varepsilon^{1-\beta} & \text{if } \beta > 1. \end{cases} \tag{A.78}$$

It is straightforward to show that for any $\beta > 0$, $\frac{\partial W_0}{\partial \lambda} > 0$.

This finding extends to any $\alpha \in (0, 1]$, for which we have

$$W_0^{\mathbf{I}} = \begin{cases} \text{N/A} & \text{if } \beta \in (0,1)\,, \\ \frac{A}{r} \ln\left(\frac{A(r+\alpha\lambda)}{c(r+\lambda)r}\right) + \frac{A\lambda}{r^2} - \frac{A(r+\alpha\lambda)}{(r+\lambda)r} + \frac{A(1-\ln\varepsilon)}{r} & \text{if } \beta = 1, \\ A^{\frac{1}{\beta}} c^{1-\frac{1}{\beta}} \left(\begin{array}{c} \frac{1}{(1-\beta)} \left(\frac{(r+(\beta-1)\lambda+\alpha\lambda)}{(r+\beta\lambda)}\right)^{\frac{1-\beta}{\beta}} (r+(\beta-1)\lambda)^{-\frac{1}{\beta}} \\ -\left(\frac{r+(\beta-1)\lambda+\alpha\lambda}{(r+\beta\lambda)(r+(\beta-1)\lambda)}\right)^{\frac{1}{\beta}} \end{array} \right) & \text{if } \beta > 1. \\ +\frac{\beta A}{r(\beta-1)}\varepsilon^{1-\beta} & \end{cases} \tag{A.79}$$

$$W_0^{\mathbf{II}} = \begin{cases} A^{\frac{1}{\beta}} c^{1-\frac{1}{\beta}} \left(\frac{(r+(\beta-\alpha)\lambda)^{1-\frac{1}{\beta}}}{(1-\beta)(r-\lambda(1-\beta))} - (r+(\beta-\alpha)\lambda)^{-\frac{1}{\beta}} \right) & \text{if } \beta \in (0,1)\,, \\ \frac{A}{r} \ln \frac{A}{c(r+(1-\alpha)\lambda)} + \frac{A\lambda}{r^2} - \frac{A}{r+(1-\alpha)\lambda} + \frac{A}{r}(1-\ln\varepsilon) & \text{if } \beta = 1, \\ A^{\frac{1}{\beta}} c^{1-\frac{1}{\beta}} \left(\frac{(r+(\beta-\alpha)\lambda)^{1-\frac{1}{\beta}}}{(1-\beta)(r-\lambda(1-\beta))} - (r+(\beta-\alpha)\lambda)^{-\frac{1}{\beta}} \right) & \text{if } \beta > 1. \\ +\frac{\beta A \varepsilon^{1-\beta}}{r(\beta-1)} & \end{cases} \tag{A.80}$$

We then confirm from Eqs. (A.79) and (A.80) that $\frac{\partial W_0^{\mathbf{I}}}{\partial \lambda} > 0$ for $\beta \geq 1$ and $\frac{\partial W_0^{\mathbf{II}}}{\partial \lambda} > 0$ for $\beta > 0$.

IA-B. Data and robustness of regression analysis

IA-B1. Cross-industry licensing rates

Using a sample of 347 firms across 15 industries from 1990 to 2000, Goldscheider *et al.* (2002) show the empirical relevance of the 25 percent rule: Across all 15 industries, the median royalty rate as a percentage of average licensee operating profit margins was 26.7%, and the majority of industries had ratios of 21-40%, as shown in Fig. (IA-1).

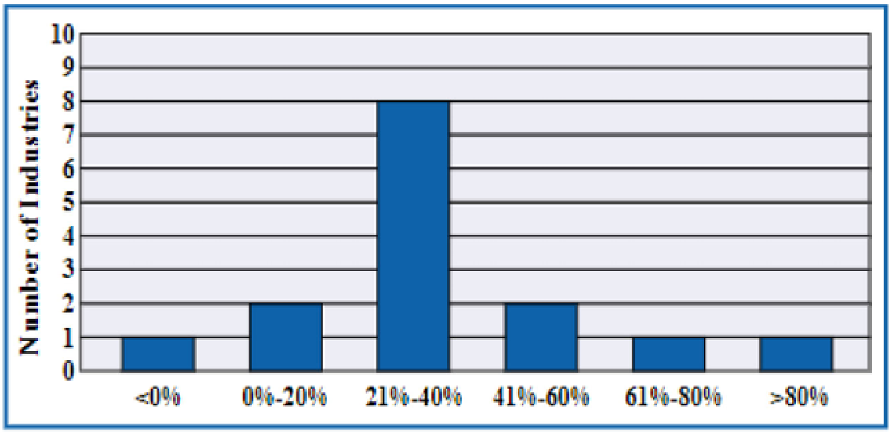

Fig. IA-1. Licensing Rates acoss Industries

Using more recent data, Kemmerer and Lu (2012) show such result holds broadly in a sample of 3,887 companies in 14 industries based on 3,015 patent licensing transactions collected over a 21-year period prior to 2007. They find that the reported royalty rates across industries tend to fall between 25 percent of gross margins and 25 percent of operating margins.

IA-B2. Auto diffusion estimation

We use the data of firm numbers in the pre-shakeout period, 1895-1910, to estimate the diffusion parameters. We estimate the diffusion process of k_t as follows:

$$\ln \frac{k_t}{N - k_t} = z + \lambda t,$$

where $z = \ln \frac{k_0}{N - k_0}$, and $\lambda = \gamma N$.

We assume that the shakeout started after almost all the potential firms had entered the industry. Accordingly, we set $N = 210$ and run the regression model. The result shows that

$$\ln \frac{k_t}{N - k_t} = \underset{(0.28)^{***}}{-4.17} + \underset{(0.03)^{***}}{0.54}\ t,$$

and the standard errors are reported in parentheses. The estimates of z and λ are both statistically significant at 1% (denoted by three stars) and the adjusted $R^2 = 0.95$. Based on the estimates of diffusion parameters, we calibrate $\gamma N = 0.54$ and $k_0 = 3.20$ (i.e., $\ln \frac{k_0}{N-k_0} = -4.17$).

For robustness checks, we re-ran the diffusion regression in the paper for the subsample period 1900-1910 (i.e., removing the first five years of observations). The result shows that

$$\ln \frac{k_t}{N-k_t} = \underset{(0.40)^{**}}{-1.30} + \underset{(0.07)^{***}}{0.52}\, t.$$

Standard errors are reported in parentheses, with three stars and two stars representing statistical significance at 1% and 5%, respectively. The adjusted $R^2 = 0.85$. The estimated $\gamma N = 0.52$ is very similar to the estimate $\gamma N = 0.54$ in the paper (and the constant term is smaller in absolute value because the number of firms in 1900 was higher than that in 1895).

For additional robustness checks, we also estimate the matching function directly by rewriting $\frac{dk_t/dt}{N-k_t} = \gamma k_t$ into a discrete-time version:

$$\frac{k_t - k_{t-1}}{N - k_{t-1}} = \gamma k_{t-1}. \tag{B.1}$$

Note that the left-hand side of Eq. (B.1) is the hazard rate of adopting the new product. We again set $N = 210$ and run the regression model (B.1) using auto-firm numbers data from 1895-1908. The result was

$$\frac{k_t - k_{t-1}}{N - k_{t-1}} = \underset{(0.0004)^{***}}{0.0029}\, k_{t-1},$$

and the standard error is reported in the parentheses. The estimate of γ is statistically significant at 1% and the adjusted $R^2 = 0.77$. The estimate $\gamma = 0.0029$ implies that $\gamma N = 0.61$.

We also redo the exercise by estimating an extended version of Eq. (B.1)

$$\frac{k_t - k_{t-1}}{N - k_{t-1}} = \eta + \gamma k_{t-1}, \tag{B.2}$$

that was proposed by Bass (1969). The Bass model allows the hazard rate of adoption to depend on both the coefficient of innovation η and the coefficient of imitation γ. In our context, η captures the hazard rate of entry by innovators independent of incumbents while γk_{t-1} captures the hazard rate of entry by imitators. The regression result is

$$\frac{k_t - k_{t-1}}{N - k_{t-1}} = \underset{(0.072)}{-0.0339} + \underset{(.0007)^{***}}{.0032}\, k_{t-1},$$

and the standard errors are reported in parentheses. The estimate $\gamma = 0.0032$ (which implies $\gamma N = 0.67$) is statistically significant at 1%, but the estimate of η is not statistically significant, which is consistent with our theoretical prediction that innovators enter only at the beginning of the industry.

IA-B3. Auto demand estimation

We estimate the auto demand function using annual data on real auto price p_t (in 2012 prices) and industry output Q_t from 1900–1929. Our model implies a simple log-log demand function

$$\ln(Q_t) = a - \phi \ln(p_t).$$

To address potential endogeneity of the price variable, we use output per firm lagged by a year as an instrumental variable to estimate the demand elasticity parameter ϕ in a two-stage least-squares (2SLS) regression. Output per firm, while assumed fixed in our theory, did grow over the long term due to technological progress. If unobserved demand shocks are not serially correlated, lagged output per firm can serve as a valid supply shifter to trace out the demand curve.

The first-stage regression result (adj. $R^2 = 0.87$) is

$$\ln(p_t) = \underset{(0.14)^{***}}{11.37} - \underset{(0.02)^{***}}{0.24} \times \ln(\textit{output per firm})_{t-1},$$

and the second-stage regression result ($R^2 = 0.82$) is

$$\ln(Q_t) = \underset{(2.74)^{***}}{47.04} - \underset{(0.29)^{***}}{3.61} \times \ln(p_t).$$

All the estimates are statistically significant at 1%.

The IV estimation gives $\phi = 3.61$ and $a = 47.04$. Because our model specifies an inverse demand function that implies

$$\ln Q_t = \frac{1}{\beta} \ln \tilde{A} - \frac{1}{\beta} \ln p_t,$$

this yields that $\beta = 0.28$ (i.e., $\frac{1}{\beta} = \phi = 3.61$) and $\tilde{A} = 456,102$ (i.e., $\frac{1}{\beta} \ln \tilde{A} = 47.04$).

To check if lagged output per firm is a valid instrument, we follow Cabral *et al.* (2018) and use an alternative instrumental variable, the share of spin-off firms in the auto industry. Using this alternative instrument and with the same sample range (1900-1929), the first-stage regression result (adj. $R^2 = 0.84$) is

$$\ln(p_t) = \underset{(0.09)^{***}}{10.52} - \underset{(0.38)^{***}}{4.77} \times (\textit{share of spin-off firms})_{t-1},$$

and the second-stage regression result ($R^2 = 0.82$) is

$$\ln(Q_t) = \underset{(2.79)^{***}}{47.28} - \underset{(0.29)^{***}}{3.63} \times \ln(p_t).$$

The estimates are all statistically significant at 1% and very close to our estimates above.

For a further robustness check, we estimate the auto industry demand function by controlling for changes of population and per capita income over time. In doing so, we use annual data on auto prices p_t and output Q_t from 1900–1929 to estimate a per capita demand function:

$$\ln(\frac{Q_t}{pop_t}) = a_t - \phi \ln(p_t).$$

where pop_t is U.S. population at year t. The dependent variable is auto demand per capita, and we control for log U.S. GDP per capita (as a proxy for income) in the demand intercept a_t. Both auto price and GDP per capita are in real terms.

As before, to address potential endogeneity of the price variable, we use output per firm lagged by a year as an instrument to estimate the demand elasticity parameter ϕ in a two-stage least-squares regression.

The first-stage regression result (adj. $R^2 = 0.89$) is

$$\ln(p_t) = \underset{(1.08)^{***}}{8.63} + \underset{(0.63)^{**}}{1.62} \times \ln(\frac{GDP_t}{pop_t}) - \underset{(0.03)^{***}}{0.29} \times \ln(\textit{output per firm})_{t-1},$$

and the second-stage regression result ($R^2 = 0.83$) is

$$\ln(\frac{Q_t}{pop_t}) = \underset{(7.15)^{***}}{32.36} + \underset{(2.10)}{0.28} \times \ln(\frac{GDP_t}{pop_t}) - \underset{(0.38)^{***}}{3.33} \times \ln(p_t).$$

Standard errors are in parentheses, with three stars and two stars representing statistical significance at 1% and 5%, respectively. The estimate $\phi = 3.33$ is highly statistically significant and the implied inverse demand elasticity $\beta = \frac{1}{\phi} = 0.3$ is similar to our estimate in the paper.

IA-B4. PC diffusion estimation

We use the data of firm numbers in the pre-shakeout period, 1975-1992, to estimate the diffusion parameters. We assume the shakeout started after almost all the potential PC firms had entered the industry. Accordingly, we set $N = 435$ and run the following regression model. The result shows that

$$\ln \frac{k_t}{N - k_t} = \underset{(0.29)^{***}}{-5.49} + \underset{(0.03)^{***}}{0.58}\, t,$$

with the standard errors reported in parentheses. The coefficient estimates are statistically significant at 1% and the adjusted $R^2 = 0.96$. Based on the estimates of diffusion parameters, we calibrate $\gamma N = 0.58$, and $k_0 = 1.78$ (i.e., $\ln \frac{k_0}{N-k_0} = -5.49$).

To check robustness, we re-ran the diffusion regression in the paper for the subsample period 1980-1992 (i.e., removing the first five years of observations). The result was

$$\ln \frac{k_t}{N - k_t} = \underset{(0.35)^{***}}{-2.83} + \underset{(0.05)^{***}}{0.62}\, t,$$

where the standard errors are in parentheses and the adjusted $R^2 = 0.93$. The estimated $\gamma N = 0.62$ is very similar to the estimate $\gamma N = 0.58$ in the paper (and the constant term is smaller in absolute value because the number of firms in 1980 is higher than that in 1975).

For additional robustness checks, we estimate the matching function (B.1) for the PC industry:

$$\frac{k_t - k_{t-1}}{N - k_{t-1}} = \gamma k_{t-1}.$$

We again set $N = 435$ and run the regression using PC firm number data from 1975-1991. The result is

$$\frac{k_t - k_{t-1}}{N - k_{t-1}} = \underset{(0.00025)^{***}}{0.00092}\, k_{t-1},$$

and the standard error is in the parentheses. The estimate of γ is statistically significant at 1% and the adjusted $R^2 = 0.44$. The estimate $\gamma = 0.00092$ implies that $\gamma N = 0.40$.

We also redo the exercise by estimating a more general version (cf. Eq. (B.2)) that

$$\frac{k_t - k_{t-1}}{N - k_{t-1}} = \eta + \gamma k_{t-1},$$

proposed by Bass (1969). In our context, η captures the hazard rate of entry by innovators independent of incumbents, while γk_{t-1} captures the hazard rate of entry by imitators. The regression result shows that

$$\frac{k_t - k_{t-1}}{N - k_{t-1}} = \underset{(0.08398)}{0.04721} + \underset{(0.00036)^{*}}{0.00078} k_{t-1},$$

and the standard errors are reported in parentheses. The estimate $\gamma = 0.00078$ (which implies $\gamma N = 0.34$) is statistically significant at 5.1%, but the estimate of η is not statistically significant, which is consistent with our theoretical prediction that innovators only enter at the beginning of the industry.

IA-B5. PC demand estimation

We estimate the PC demand function using annual data on real PC price p_t (in 2012 prices) and industry output Q_t from 1975-1992. As before, in order to address potential endogeneity of the price variable, we use output per firm lagged by a year as an instrument to estimate the demand elasticity ϕ.

The first-stage regression result (adj. $R^2 = 0.23$) is

$$\ln(p_t) = \underset{(0.50)^{***}}{9.62} - \underset{(0.05)^{**}}{0.12} \times \ln(\textit{output per firm})_{t-1},$$

and the second-stage regression result ($R^2 = 0.94$) is

$$\ln(Q_t) = \underset{(12.52)^{***}}{137.15} - \underset{(1.49)^{***}}{14.58} \times \ln(p_t).$$

Standard errors are in parentheses, with two and three stars indicating statistical significance at 5% and 1%, respectively. The IV estimation gives $\phi = 14.58$ and $a = 137.15$. This yields $\beta = 0.07$ (i.e., $\frac{1}{\beta} = \phi = 14.58$) and $\tilde{A} = 12,170$ (i.e., $\frac{1}{\beta} \ln \tilde{A} = a = 137.15$).

To check if lagged output per firm is a valid instrument for the PC industry, we use the real import value of computers and accessories (in 2012 prices) as an alternative instrument. Using this alternative instrument and with the same sample range (1975-1992), the first-stage regression result (adj. $R^2 = 0.91$) is

$$\ln(p_t) = \underset{(0.02)^{***}}{8.68} - \underset{(0.01)^{***}}{0.11} \times \ln(\textit{value of imports})_t,$$

and the second-stage regression result ($R^2 = 0.96$) is

$$\ln(Q_t) = \underset{(6.54)^{***}}{137.58} - \underset{(0.78)^{***}}{14.64} \times \ln(p_t).$$

The estimates are all statistically significant at 1% and very close to our estimates above.

For a further robustness check, we estimate the PC industry demand function by controlling for changes of population and per capita income over time. In doing so, we use annual data of PC prices p_t and output Q_t from 1975–1992 to estimate the per capita demand function

$$\ln(\frac{Q_t}{pop_t}) = a_t - \phi \ln(p_t).$$

The dependent variable is PC demand per capita (where pop_t is U.S. population at year t), and we control for log U.S. GDP per capita (as a proxy for income) in the demand intercept a_t. PC price and GDP per capita are both in real terms.

As before, to address potential endogeneity of the price variable, we use output per firm lagged by a year as an instrument to estimate the demand elasticity parameter ϕ in a two-stage least-squares regression.

The first-stage regression result (adj. $R^2 = 0.95$) is

$$\ln(p_t) = \underset{(0.23)^{***}}{12.44} - \underset{(0.06)^{***}}{0.95} \times \ln(\frac{GDP_t}{pop_t}) - \underset{(0.01)^{***}}{0.07} \times \ln(\textit{output per firm})_{t-1},$$

and the second-stage regression result ($R^2 = 0.95$) is

$$\ln(\frac{Q_t}{pop_t}) = \underset{(29.00)^{***}}{143.18} - \underset{(2.63)}{2.90} \times \ln(\frac{GDP_t}{pop_t}) - \underset{(2.40)^{***}}{15.57} \times \ln(p_t).$$

Standard errors are in parentheses, with three stars indicating statistical significance at 1% level. The estimate $\phi = 15.57$ is highly statistically significant and the implied inverse demand elasticity $\beta = \frac{1}{\phi} = 0.06$ is similar to our estimate in the paper.

IA-C. Alternative model assumptions

IA-C1. Anticipated shakeout

Our model can be extended to allow the shakeout to be anticipated. Specifically, we could assume that the industry expects a disruptive innovation to arrive at the Poisson rate ρ. This innovation would make obsolete existing technologies and drive firm values to zero.[25]

Accordingly, the value of an incumbent firm under Regime 2 satisfies

$$rv_t = p_t + \gamma\left(N - k_t\right)\alpha v_t - \rho v_t + \frac{dv_t}{dt},$$

i.e.,

$$\left(r + \rho\right)v_t = p_t + \gamma\left(N - k_t\right)\alpha v_t + \frac{dv_t}{dt}. \tag{C.1}$$

Including $\rho > 0$ in Eq. (C.1) is equivalent to raising r to $r + \rho$ in Eq. (13). Similarly, we can revise the value function conditions for outsiders as well as for Regime 1 and for the social planner's problem. The original functional forms of our model hold, except that r becomes $r + \rho$.

Considering that the shakeout occurred in the 16th year for the auto industry and in the 18th year for the PC industry, we take the average and calibrate $\rho = 1/17 = 0.06$. Accordingly, we set $r + \rho = 0.05 + 0.06 = 0.11$ and redo the model calibration and counterfactual analysis.

Regarding the socially optimal licensing rate, we now find in the auto case, $\alpha^{\mathbf{I}*}_{\text{Auto}} = 0.145$ under Regime 1 or $\alpha^{\mathbf{II}*}_{\text{Auto}} = 0.295$ under Regime 2, while in the PC case, $\alpha^{\mathbf{I}*}_{\text{PC}} = 0.115$ or $\alpha^{\mathbf{II}*}_{\text{PC}} = 0.235$. The values of α^* are larger than the benchmark analysis due to the higher discount $r + \rho$ in spite of the lower c implied by the re-calibrated model.

IA-C2. Imitators' entry cost

In our analysis, we have assumed that except for paying the licensing fee, imitators incur no entry cost. This extreme case offers imitators the greatest advantage in competing against innovators and also simplifies the solution.

We now extend the model to include a positive entry cost c^m for imitators, where $c^m < c$. One interpretation is that imitators may have to pay such a cost to absorb the new technology and set up production. With $c^m > 0$, the net present value for an outsider to become an imitator at date t changes from ω_t to $\omega_t - c^m$.

[25] For example, an industry may expect a disruptive innovation (e.g., the assembly line in the auto case) to arrive at the Poisson rate ρ. This innovation would require an incumbent firm to incur a big capital investment to produce a newly designed product at a massive scale. When that innovation did arrive, the new (and lower) equilibrium price could support the capital investment made by only a few firms and the rest would have to exit. As a result, the present value of an investing firm (net of its investment costs) would be zero, and the value of an exiting firm would also be zero.

As a result, the value functions for imitators, innovators and outsiders in the free-imitation baseline model become

$$\omega_t = v_t, \qquad rv_t = p_t + \frac{dv_t}{dt}, \tag{C.2}$$

$$\text{and} \quad ru_t = \gamma k_t \left(v_t - c^m - u_t\right) + \frac{du_t}{dt}. \tag{C.3}$$

The counterparts of (C.2) and (C.3) in Regime 1 become

$$r\omega_t = p_t + \frac{d\omega_t}{dt}, \qquad rv_t = p_t + \frac{\gamma k_t \left(N - k_t\right)}{k_0}\alpha(\omega_t - c^m) + \frac{dv_t}{dt}, \tag{C.4}$$

$$\text{and} \quad ru_t = \gamma k_t \left[(1-\alpha)(\omega_t - c^m) - u_t\right] + \frac{du_t}{dt}; \tag{C.5}$$

and those in Regime 2 become

$$\omega_t = v_t, \qquad rv_t = p_t + \gamma \left(N - k_t\right) \alpha(v_t - c^m) + \frac{dv_t}{dt}, \tag{C.6}$$

$$\text{and} \quad ru_t = \gamma k_t \left((1-\alpha)(v_t - c^m) - u_t\right) + \frac{du_t}{dt}. \tag{C.7}$$

The planner also takes c^m into account when maximizing social welfare

$$W_0 = \int_0^\infty e^{-rt} \left[U\left(k_t\right) - \gamma k_t \left(N - k_t\right) c^m\right] dt - ck_0,$$

where $\gamma k_t \left(N - k_t\right) c^m$ is the entry cost incurred by the inflow of imitators.

Following similar proofs of Propositions 1, 2, and 6, we can verify that innovators enter only at date 0. In the following, we show the conditions that determine the mass of innovators $k_0 \in (0, N)$ in each scenario:

$$\text{No IP protection} : \quad \frac{1}{N - k_0} \int_0^\infty e^{-rt} \left(N - k_t\right) \left(Ak_t^{-\beta} + c^m \gamma k_t\right) dt = c; \tag{C.8}$$

$$\text{Regime 1} : \quad \frac{\int_0^\infty e^{-rt} \left\{ \begin{array}{c} \left(\frac{\alpha N}{k_0} + (1-\alpha)\frac{N}{k_t} - 1\right) Ak_t^{1-\beta} \\ + \left(1 - \frac{N}{k_0}\alpha\right) c^m \gamma k_t \left(N - k_t\right) \end{array} \right\} dt}{\left(N - k_0\right)} = c; \tag{C.9}$$

$$\text{Regime 2} : \quad \frac{\int_0^\infty e^{-rt} \left\{ \begin{array}{c} \left(\left(\frac{N}{k_0}\right)^\alpha \left(\frac{N}{k_t}\right)^{1-\alpha} - 1\right) Ak_t^{1-\beta} \\ + c^m \gamma \left(N - k_t\right) \left(k_t - \alpha N \left(\frac{k_t}{k_0}\right)^\alpha\right) \end{array} \right\} dt}{N - k_0} = c; \tag{C.10}$$

$$\text{Social optimum} : \quad \int_0^\infty e^{-(r+\gamma N)t} \left(\frac{k_t}{k_0}\right)^2 \left(Ak_t^{-\beta} - c^m \gamma \left(N - 2k_t\right)\right) dt = c. \tag{C.11}$$

We then calibrate the extended model to the U.S. auto and PC industries by assuming that c^m is a fraction of c. All else equal, adding an entry cost for imitators incentivizes more agents to innovate. Then, for the extended model to match the observed number of innovators in the data, the calibrated innovation cost c needs to be higher. As a result, the socially optimal licensing rate α^* becomes higher. Comparing across industries, since the PC is more price elastic than the auto, the former continues to warrant a smaller socially optimal licensing rate than the latter.

Table IA-1. The Effect of Imitators' Entry Cost

	$\alpha^{\mathbf{I}^*}_{\text{Auto}}$	$\alpha^{\mathbf{II}^*}_{\text{Auto}}$	$\alpha^{\mathbf{I}^*}_{\text{PC}}$	$\alpha^{\mathbf{II}^*}_{\text{Pc}}$
$c^m = 0$	0.069	0.166	0.056	0.133
$c^m = 0.25c$	0.073	0.179	0.057	0.136
$c^m = 0.5c$	0.084	0.212	0.058	0.143

Table IA-1 compares the model findings for $c^m = 0$, $c^m = 0.25c$, and $c^m = 0.5c$. The results confirm that the higher c^m is as a fraction of c, the higher is the value of α^* in Regimes 1 and 2 for both industries, and that $\alpha^{\mathbf{I}^*}_{\text{PC}} < \alpha^{\mathbf{I}^*}_{\text{Auto}}$ and $\alpha^{\mathbf{II}^*}_{\text{Pc}} < \alpha^{\mathbf{II}^*}_{\text{Auto}}$ continue to hold. Moreover, comparing with our benchmark analysis where $c^m = 0$, the rise in the model-implied socially optimal licensing rate α^* is quantitatively small when $c^m = 0.25c$ or $c^m = 0.5c$.

IA-C3. Effective IP protection

We assume free imitation ($\alpha = 0$) in the baseline analysis, consistent with evidence that most innovations are neither patented nor effectively protected by patents. For robustness, we consider an alternative, extreme scenario in which all industries fully enforced IP protection under the 25 percent rule ($\alpha = 0.25$). We then recalibrate the model for automobile, PC and 16 additional industries in the Gort and Klepper (1982) dataset and derive the corresponding socially optimal licensing rate α^*. A higher α incentivizes more agents to innovate, leading to a higher calibrated innovation cost c to match observed innovator entry. Consequently, α^* rises across all industries in both Regimes 1 and 2, shifting the cross-industry distributions of α^* to the right relative to Fig. 13(B), while remaining markedly different from the empirical distribution of α shown in Fig. 13(A). Moreover, the cross-industry ranking of α^* obtained under the free-imitation baseline persists under the alternative scenario in Regime 1, though not necessarily in Regime 2.

Table IA-2 reports the results under the assumption that industry data are generated by $\alpha = 0.25$ under Regime 1. The corresponding cross-industry distributions of α^* under Regimes 1 and 2 are plotted in Fig. IA-2.

Table IA-2. Model Application to 18 Industries
(assuming data are generated by $\alpha = 0.25$ in Regime 1)

Product	k^*_0/N	α^{*I}	α^{*II}
Auto	0.01	0.12	0.56
PC	0.00	0.09	0.63
Computers, Pre-PC	0.01	0.22	0.69
Electric blankets	0.09	0.46	0.65
Electric shavers	0.00	0.10	0.59
Freezers, Home and farm	0.15	0.25	0.42
Lasers	0.08	0.10	0.29
Nylon	0.01	0.26	0.67
Penicillin	0.07	0.21	0.45
Pens, Ballpoint	0.06	0.35	0.60
Records, Phonograph	0.03	0.48	0.74
Streptomycin	0.05	0.19	0.48
Styrene	0.03	0.42	0.69
Tapes, Recording	0.02	0.28	0.64
Television	0.00	0.19	0.72
Tires, Automobile	0.03	0.16	0.51
Transistors	0.07	0.12	0.33
Zippers	0.01	0.43	0.77

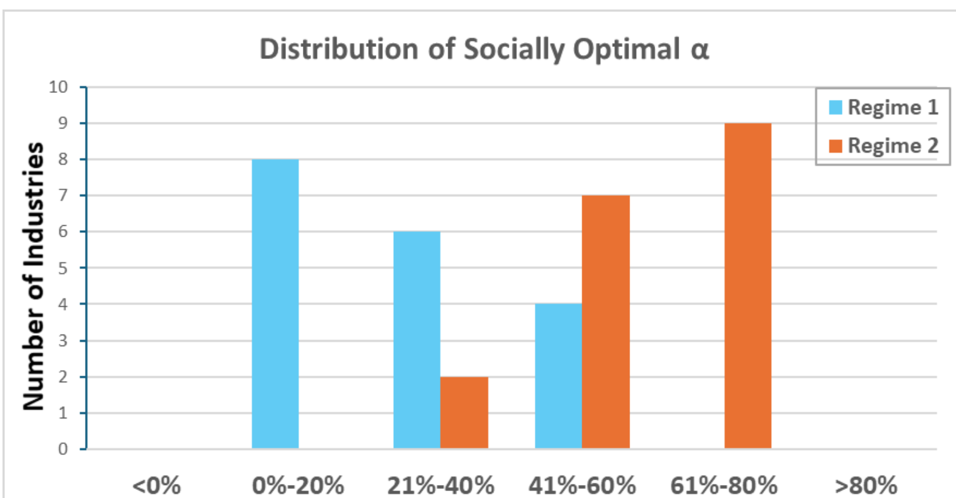

Fig. IA-2. Cross-Industry Distribution of α^* (assuming data are generated by α=0.25 in Regime 1)

Table IA-3 reports the results under the assumption that industry data are generated by $\alpha = 0.25$ under Regime 2. The corresponding cross-industry distributions of α^* under Regimes 1 and 2 are plotted in Fig. IA-3.

Table IA-3. Model Application to 18 Industries
(assuming data are generated by $\alpha = 0.25$ in Regime 2)

Product	k^*_0/N	α^{*I}	α^{*II}
Auto	0.04	0.10	0.35
PC	0.03	0.08	0.37
Computers, Pre-PC	0.06	0.18	0.44
Electric blankets	0.12	0.42	0.60
Electric shavers	0.03	0.08	0.35
Freezers, Home and farm	0.16	0.19	0.34
Lasers	0.14	0.09	0.21
Nylon	0.06	0.21	0.47
Penicillin	0.10	0.18	0.38
Pens, Ballpoint	0.11	0.30	0.50
Records, Phonograph	0.06	0.41	0.65
Streptomycin	0.07	0.16	0.38
Styrene	0.05	0.37	0.63
Tapes, Recording	0.05	0.23	0.52
Television	0.03	0.16	0.47
Tires, Automobile	0.08	0.13	0.34
Transistors	0.12	0.10	0.24
Zippers	0.04	0.35	0.64

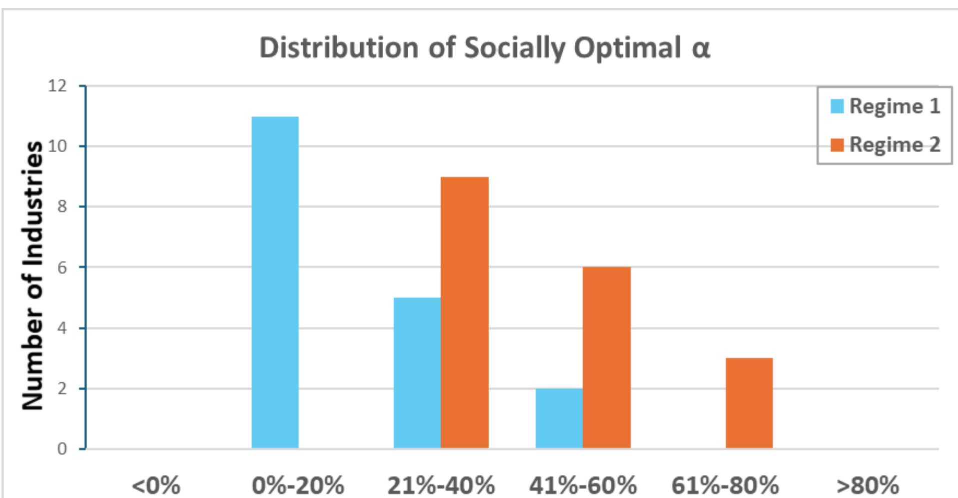

Fig. IA-3. Cross-Industry Distribution of α^* (assuming data are generated by α=0.25 in Regime 2)