How Do Sectoral Shocks Shape Future GDP?
Technical Appendix*

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*The views expressed in this paper are those of the authors and do not necessarily represent the views of the Federal Reserve Bank of New York, the Federal Reserve Bank of Richmond, or the Federal Reserve System.

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1 Capital Accumulation and the Aggregate Effects of Sectoral Shocks

In the economic environment below, we denote real variables by capital letters. We then denote the nominal value of any real variable, X, by $\widetilde{X}$. We further denote matrices and vectors by bold letters, $\boldsymbol{X}$, and the log of variables by lowercase letters unless explicitly stated, x or $\boldsymbol{x}$.

1.1 The Production Environment

Consider an economy with N production sectors indexed by $i \in \{1, ..., N\}$ (or j). In each period, t, and sector, i, firms produce gross output using the constant-returns-to scale (CRS) technology,

$$\boxed{Q_{it} = Z_{it}\, \mathcal{F}_i\left(L_{it}, K_{it}, \boldsymbol{M}_{it}\right),}$$

where L_{it} and K_{it} denote respectively labor and capital inputs used by sector i, and $\boldsymbol{M}_{it} \in \mathbb{R}_+^N$ is a vector of intermediate inputs, $\boldsymbol{M}_{it} = (M_{1it}, ..., M_{Nit})$, where M_{jit} denotes the quantity of materials produced in sector j used in sector i. Aggregate labor supply, L_t, is given exogenously.

New capital goods in sector i, K_{it+1}, are produced using existing capital, K_{it} and investment, $\boldsymbol{X}_t$, with the CRS technology,

$$\boxed{K_{it+1} = \mathcal{K}_i\left(K_{it}, \boldsymbol{X}_{it}\right),}$$

where $\boldsymbol{X}_{it} \in \mathbb{R}_+^N$ is a vector of investment inputs, $\boldsymbol{X}_{it} = (X_{1it}, ..., X_{Nit})$, where X_{jit} denotes investment produced in sector j used in sector i.

We assume the following market structure. At the beginning of period t, firms producing gross output in sector i purchase capital, K_{it}, at price, P_{it}^K, and hire workers at wage rate, W_t. They then produce output, Q_{it}, which they sell at price, P_{it}. Once production is complete, these firms sell their capital at price, $P_{it}^K - R_{it}$, where R_{it} is the rental price of capital in sector i at date t. Thus, we can think of $P_{it}^K - R_{it}$ as the scrap value of capital used to produce gross output. Firms producing new capital goods then purchase investment inputs, $\boldsymbol{X}_{it}$, along with existing capital to produce the stock of new capital goods, K_{it+1}, which they sell at price P_{it+1}^K. This market structure implies the following relationship,

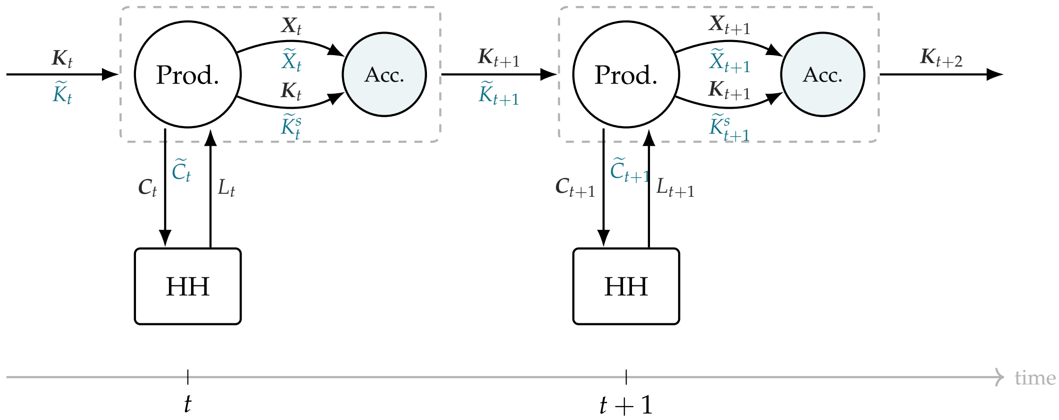
$$\boxed{P_{it+1}^K = (1 + r_t) P_{it+1}^K.} \tag{1}$$

where r_t is the nominal rate of interest on a risk free bond between periods t and $t + 1$.

Output-producing firms maximize,

$$\boxed{P_{it} Z_{it}\, \mathcal{F}_i\left(L_{it}, K_{it}, \boldsymbol{M}_{it}\right) - R_{it} K_{it} - \sum_j P_{jt} M_{jit} - W_t L_{it},}$$

Figure 1: **Production and capital accumulation**



while firms producing new capital goods maximize,

$$\boxed{P^K_{it+1}\mathcal{K}_i\left(K_{it},\boldsymbol{X}_{it}\right)-\left(P^K_{it}-R_{it}\right)K_{it}-\sum_j P_{jt}X_{jit}.}$$

The corresponding first-order conditions (FOCs) are:

$$P_{it}Z_{it}\frac{\partial\mathcal{F}_i}{\partial M_{jit}}=P_{jt}, \tag{2}$$

$$P_{it}Z_{it}\frac{\partial\mathcal{F}_i}{\partial K_{it}}=R_{it}, \tag{3}$$

$$P_{it}Z_{it}\frac{\partial\mathcal{F}_i}{\partial L_{it}}=W_t, \tag{4}$$

$$P^K_{it+1}\frac{\partial\mathcal{K}_i}{\partial K_{it}}=P^K_{it}-R_{it}, \tag{5}$$

$$P^K_{it+1}\frac{\partial\mathcal{K}_i}{\partial X_{jit}}=P_{jt}. \tag{6}$$

We define real net output, or real value added, in sector i at date t as,

$$\boxed{Y_{it}\equiv Q_{it}-\sum_j M_{jit}.}$$

Value Added

Let $\widetilde{Y}_t\equiv\sum_i P_{it}Y_{it}$ denote the nominal value of aggregate GDP at date t. Then,

$$\boxed{\widetilde{Y}_t=\widetilde{C}_t+\widetilde{X}_t\equiv\sum_i P_{it}C_{it}+\sum_{j,i}P_{jt}X_{jit}.}$$

Using the CRS property of $\mathcal{K}_i$, we can write nominal aggregate value added as

$$\begin{aligned}
\widetilde{Y}_t &= \sum_i P_{it} C_{it} + \sum_{j,i} P_{jt} X_{jit}, \\
&= \sum_i P_{it} C_{it} + \sum_i P^K_{it+1} \sum_j \frac{\partial \mathcal{K}_i}{\partial X_{jit}} X_{jit}, \text{ by equation (6)}, \\
&= \sum_i P_{it} C_{it} + \sum_i P^K_{it+1} \left(K_{it+1} - \frac{\partial \mathcal{K}_i}{\partial K_{it}} K_{it} \right), \text{ by the CRS property of } \mathcal{K}_i, \\
&= \sum_i P_{it} C_{it} + \sum_i P^K_{it+1} K_{it+1} - \sum_i \left(P^K_{it} - R_{it} \right) K_{it}, \\
&= \sum_i P_{it} C_{it} + \sum_i R_{it} K_{it} + \sum_i P^K_{it+1} K_{it+1} - \sum_i P^K_{it} K_{it}, \\
&= \sum_i P_{it} C_{it} + \sum_i R_{it} K_{it} + \sum_i P^K_{it+1} K_{it+1} - (1 + r_{t-1}) \sum_i P^K_{it} K_{it}.
\end{aligned}$$

In other words, the nominal value of undertaking investment comprises two components. One is the service flow from capital associated with current production, $\sum_i R_{it} K_{it}$. The other is the resulting change in the value of the capital stock associated with future production, $\sum_i P^K_{it+1} K_{it+1} - (1 + r_{t-1}) \sum_i P^K_{it} K_{it}$.

On the income side, we have that

$$\widetilde{Y}_t = \sum_i P_{it} Q_{it} - \sum_{j,i} P_{jt} M_{jit}.$$

Using the CRS property of $\mathcal{F}_i$, we obtain,

$$\begin{aligned}
\widetilde{Y}_t &= \sum_i P_{it} Q_{it} - \sum_{j,i} P_{jt} M_{jit} \\
&= \sum_i P_{it} Q_{it} - \sum_i P_{it} Z_{it} \sum_j \frac{\partial \mathcal{F}_i}{\partial M_{jit}} M_{jit} \\
&= \sum_i P_{it} Q_{it} - \sum_i P_{it} Z_{it} \left(\mathcal{F}_i - \frac{\partial \mathcal{F}_i}{\partial K_{it}} K_{it} - \frac{\partial \mathcal{F}_i}{\partial L_{it}} L_{it} \right) \\
&= W_t L_t + \widetilde{K}^s_t,
\end{aligned}$$

where $\widetilde{K}^s_t \equiv \sum_i R_{it} K_{it}$ is the value of current capital services.

Finally, these equations imply that

$$\boxed{\widetilde{X}_t = \widetilde{K}^s_t + \widetilde{K}_{t+1} - (1 + r_{t-1}) \widetilde{K}_t,}$$

where $\widetilde{K}_{t+1} \equiv \sum_i P^K_{it+1} K_{it+1}$ reflects the nominal value of the aggregate capital stock at the end of period t. It then also follows that the nominal aggregate wage bill satisfies $W_t L_t = \widetilde{C}_t + \widetilde{K}_{t+1} - (1 + r_{t-1}) \widetilde{K}_t$.

1.1.1 Real GDP, Consumption, and Capital

So far, we have defined value added in nominal terms. We next define the (percentage) change in real GDP, independently of prices.

Definition 1 (Change in Real GDP)**.** *Consider a change in the economic environment, and define the resulting change in* real *GDP as,*

$$\boxed{d\log Y_t \equiv \sum_i S_{it}\, d\log Y_{it},} \tag{7}$$

where $S_{it} \equiv (P_{it}Y_{it})/\widetilde{Y}_t$ is sector i's nominal share in aggregate GDP (its aggregate value added share). Similarly, we can define the change in real consumption and investment as,

$$d\log C_t \equiv \sum_i \tfrac{P_{it}C_{it}}{\widetilde{C}_t}\, d\log C_{it}, \tag{8}$$

$$d\log X_t \equiv \sum_{j,i} \tfrac{P_{jt}X_{jit}}{\widetilde{X}_t}\, d\log X_{jit}, \tag{9}$$

such that

$$\boxed{d\log Y_t = S_{ct}\, d\log C_t + S_{xt}\, d\log X_t,} \tag{10}$$

where $S_{ct} \equiv \widetilde{C}_t/\widetilde{Y}_t$ and $S_{xt} \equiv \widetilde{X}_t/\widetilde{Y}_t$ denote respectively the nominal shares of aggregate consumption and aggregate investment in GDP.

Put another way,

$$\begin{aligned} d\log Y_t &= \sum_i S_{it}\, d\log Y_{it}, \\ &= \sum_i S_{it} \left(\frac{P_{it}C_{it}}{P_{it}Y_{it}}\, d\log C_{it} + \sum_j \frac{P_{it}X_{ijt}}{P_{it}Y_{it}}\, d\log X_{ijt} \right), \\ &= \sum_i \frac{P_{it}C_{it}}{\widetilde{Y}_t}\, d\log C_{it} + \sum_{j,i} \frac{P_{it}X_{ijt}}{\widetilde{Y}_t}\, d\log X_{ijt}, \end{aligned}$$

which gives equation (10).

Finally,

$$\begin{aligned}
d\log Y_t &= S_{ct}\, d\log C_t + S_{xt}\, d\log X_t, \\
&= S_{ct}\, d\log C_t + \sum_{j,i} \tfrac{P_{jt}X_{jit}}{\widetilde{Y}_t}\, d\log X_{jit} \\
&= S_{ct}\, d\log C_t + \sum_i \frac{P^K_{it+1}}{\widetilde{Y}_t} \sum_j \frac{\partial \mathcal{K}_i}{\partial X_{jit}} X_{jit}\, d\log X_{jit} \\
&= S_{ct}\, d\log C_t + \sum_i \frac{P^K_{it+1}K_{it+1}}{\widetilde{Y}_t} \sum_j \frac{\partial \log \mathcal{K}_i}{\partial \log X_{jit}}\, d\log X_{jit} \\
&= S_{ct}\, d\log C_t + \sum_i \frac{P^K_{it+1}K_{it+1}}{\widetilde{Y}_t} \left(d\log K_{it+1} - \frac{\partial \log \mathcal{K}_i}{\partial \log K_{it}} d\log K_{it} \right) \\
&= S_{ct}\, d\log C_t + \sum_i \frac{P^K_{it+1}K_{it+1}}{\widetilde{Y}_t} \left(d\log K_{it+1} - \frac{(P^K_{it} - R_{it})K_{it}}{P^K_{it+1}K_{it+1}} d\log K_{it} \right) \\
&= S_{ct}\, d\log C_t + \\
&\sum_i \frac{P^K_{it+1}K_{it+1}}{\widetilde{Y}_t} d\log K_{it+1} + \sum_i \frac{R_{it}K_{it}}{\widetilde{Y}_t} d\log K_{it} - (1 + r_{t-1}) \sum_i \frac{P^K_{it}K_{it}}{\widetilde{Y}_t} d\log K_{it}. \qquad (11)
\end{aligned}$$

We then define the change in the real value of capital as follows.

Definition 2 (Change in Real Aggregate Capital)**.** *Consider a change in the economic environment. We define the resulting* real *change in the the service flow from capital, $d\log K^s_t$, and the value of the capital stock, $d\log K_t$, as*

$$d\log K^s_t \equiv \sum_i \tfrac{R_{it}K_{it}}{\widetilde{K}^s_t}\, d\log K_{it}, \qquad (12)$$

$$d\log K_t \equiv \sum_i \tfrac{P^K_{it}K_{it}}{\widetilde{K}_t}\, d\log K_{it}. \qquad (13)$$

The change in the service flow from capital, $d\log K^s_t$, reflects the change in the productive capacity of capital across sectors with respect to current production. In contrast, the change in the stock of capital, $d\log K_t$, captures the continuation value of capital with respect to production in all future periods. We can think of the latter as the change in the real value of wealth embodied in the stocks of capital across all sectors. It follows that we can now write equation (11) as

$$d\log Y_t = S_{ct}\, d\log C_t + S_{kt}\, d\log K^s_{jt} + S^+_{kt}\, d\log K_{t+1} - (1 + r_{t-1}) \left(\frac{\widetilde{Y}_{t-1}}{\widetilde{Y}_t} \right) S^+_{kt-1}\, d\log K_t, \qquad (14)$$

where $S_{kt} \equiv (\sum_i R_{it}K_{it})/\widetilde{Y}_t$ denotes the income share of capital services, and $S^+_{kt} \equiv (\sum_i P^K_{it+1}K_{it+1})/\widetilde{Y}_t$ denotes the ratio of total assets (capital) to income (GDP) at the end of period t. Put differently, a change in the real value of investment, $S_{xt} d\log X_t$, reflects two underlying changes, i) a change in the service flow from capital corresponding to a change in current production, $S_{kt}\, d\log K^s_{jt}$, and ii) a change in the continuation value of the capital stock associated with changes in future production, $S^{+}_{kt}\, d\log K_{t+1} - \frac{\widetilde{Y}_{t-1}(1+r_{t-1})}{\widetilde{Y}_t}\, S^{+}_{kt-1}\, d\log K_t$.

1.1.2 Relationship to Tobin's Q

We can re-write the problem faced by investment firms by first defining $\mathcal{C}^I_{it}$ as the cost of producing new capital goods, K_{it+1}

$$\mathcal{C}^I_{it}\left(K_{it+1}\right) \equiv \min_{\boldsymbol{X}_{jit}} \left(P^K_{it} - R_{it}\right) K_{it} + \sum_j P_{jt} X_{jit} \text{ subject to } K_{it+1} = \mathcal{K}_i\left(K_{it}, \boldsymbol{X}_{it}\right),$$

and then maximizing profits, $P^K_{it+1} K_{it+1} - \mathcal{C}^I_{it}\left(K_{it+1}\right)$.

Written in this way, Tobin's Q is defined as

$$\mathcal{Q}_{it} \equiv \frac{P^K_{it+1}}{\partial \mathcal{C}^I_{it}\left(K_{it+1}\right) / \partial K_{it+1}}.$$

Under our baseline assumptions, where $\mathcal{K}_i$ is a CRS aggregator, we can write $\mathcal{C}^I_{it}\left(K_{it+1}\right) = P^I_{it} K_{it+1}$. In this case, equilibrium ensures that $P^K_{it+1} = P^I_{it}$ and, therefore, that Tobin's Q is unity, $\mathcal{Q}_{it} \equiv 1$. In other words, this specification corresponds to a case with *no adjustment costs.* When $\mathcal{K}_i$ is given by a DRS aggregator, Tobin's Q exceeds unity.

1.2 Hulten's Theorem: The Static Aggregate Effects of Sectoral Productivity Changes

In a seminal paper, Hulten (1978) relies on the characterization of production efficiency in a static environment to show that the effect of a unit productivity shock in a sector on real aggregate GDP, to a first-order of approximation, is the ratio of that sector's gross output to GDP (sectoral Domar weights).

In this context, Hulten's theorem considers a change in sector i's productivity in any period, t, $d\log Z_{it}$, that leaves productivity in all other sectors, aggregate labor supply, and sectoral capital unchanged. Then,

$$\boxed{\frac{d\log Y_t}{d\log Z_{it}} = D_{it} \equiv \frac{P_{it} Q_{it}}{\widetilde{Y}_t}}. \tag{15}$$

Proof. We begin by defining a single-period aggregate value added function derived from maximizing aggregate value added given a vector of arbitrary sectoral prices, $\check{\boldsymbol{P}}_t$, the size of the labor force, L_t, and a vector summarizing sectoral capital stocks, $\boldsymbol{K}_t$:

$$\mathcal{T}\left(\check{\boldsymbol{P}}_t; \boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right) = \max_{\{\boldsymbol{L}_{it}, \boldsymbol{M}_t\}} \sum_i \check{P}_{it} \left(Z_{it}\, \mathcal{F}_i\left(L_{it}, K_{it}, \boldsymbol{M}_{it}\right) - \sum_j M_{ijt}\right) \text{ subject to } \sum_j L_{jt} \leq L_t.$$

The corresponding first-order conditions (FOCs) are

$$\check{P}_{it} Z_{it} \frac{\partial \mathcal{F}_{it}}{\partial M_{jit}} = \check{P}_{jt}, \text{ and } \check{P}_{it} Z_{it} \frac{\partial \mathcal{F}_{it}}{\partial L_{it}} = \check{W}_t, \tag{16}$$

where $\check{W}_t$ is the Lagrange multiplier associated with the constraint on sectoral labor input. When evaluated at equilibrium prices, $\boldsymbol{P}_t$ and W_t, these FOCs coincide with those making up the competitive equilibrium in equations (2) and (4).

We then define the production possibilities function, $\mathcal{Z}\left(\boldsymbol{Y}_t; \boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right) \leq 0$, such that[1]

$$\mathcal{Z}\left(\boldsymbol{Y}_t; \boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right) \equiv \sup_{\check{\boldsymbol{P}}_t} \ \check{\boldsymbol{P}}_t \cdot \boldsymbol{Y}_t - \mathcal{T}\left(\check{\boldsymbol{P}}_t; \boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right), \tag{17}$$

where *the production possibilities frontier* is then given by $\mathcal{Z}\left(\boldsymbol{Y}_t; \boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right) = 0$.

The FOC characterizing the supremum in equation (17) is given by

$$Y_{it} = \frac{\partial \mathcal{T}_t}{\partial \check{P}_{it}} = Q_{it} - \sum_j M_{ijt}. \tag{18}$$

In a competitive equilibrium, equations (16) and (18) (market clearing) are satisfied at equilibrium prices, $\boldsymbol{P}_t$ and W_t. Applying the envelope condition to the definition of the production possibilities function in equation (17), we obtain

$$\frac{\partial \mathcal{Z}_t}{\partial Z_{it}} = -\frac{\partial \mathcal{T}_t}{\partial Z_{it}} = -\frac{P_{it} Q_{it}}{Z_{it}}, \tag{19}$$

$$\frac{\partial \mathcal{Z}_t}{\partial Y_{it}} = P_{it}, \tag{20}$$

$$\frac{\partial \mathcal{Z}_t}{\partial K_{it}} = -\frac{\partial \mathcal{T}_t}{\partial K_{it}} = -P_{it} Z_{it} \frac{\partial \mathcal{F}_{it}}{\partial K_{it}} = -R_{it}, \tag{21}$$

where the value of $\boldsymbol{P}_t$ that achieves the supremum in the definition of $\mathcal{Z}$ coincides with market equilibrium prices.

The definition of the production possibilities frontier, $\mathcal{Z}_t = 0$, implies that

$$\frac{\partial \mathcal{Z}_t}{\partial Z_{it}} + \sum_j \frac{\partial \mathcal{Z}_t}{\partial Y_{jt}} \frac{dY_{jt}}{dZ_{it}} = 0.$$

[1] Alternatively, the production possibilities function may be described by the feasible set associated with *net* sectoral production, defined as

$$\mathcal{P}\left(\boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right) \equiv \cap_{\widetilde{\boldsymbol{P}} \in \mathbb{R}_+^I} \left\{ \boldsymbol{Q} : \widetilde{\boldsymbol{P}} \cdot \boldsymbol{Q} - \mathcal{T}\left(\widetilde{\boldsymbol{P}}; \boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right) \leq 0 \right\},$$

characterized by the condition, $\mathcal{Z}\left(\boldsymbol{Y}_t; \boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right) \leq 0$.

Then,

$$\begin{aligned}
\frac{\partial \log Y_t}{\partial \log Z_{it}} &= \sum_j \tfrac{P_{jt}Y_{jt}}{\widetilde{Y}_t} \frac{d \log Y_{jt}}{d \log Z_{it}} \\
&= \frac{Z_{it}}{\widetilde{Y}_t} \sum_j P_{jt} \frac{dY_{jt}}{dZ_{it}} \\
&= \frac{Z_{it}}{\widetilde{Y}_t} \sum_j \frac{\partial \mathcal{Z}_t}{\partial Y_{jt}} \frac{dY_{jt}}{dZ_{it}} \text{ from equation (20)} \\
&= -\frac{Z_{it}}{\widetilde{Y}_t} \frac{\partial \mathcal{Z}_t}{\partial Z_{it}} \\
&= \frac{P_{it}Q_{it}}{\widetilde{Y}_t} = D_{it} \text{ from equation (19).}
\end{aligned}$$

□

1.2.1 An Alternative Proof

In anticipation of our main theorem 1, we present an alternative proof of the static aggregate effects of sectoral shocks highlighted by Hulten's theorem. Within a broader dynamic environment, this alternative proof recasts the theorem as applying to the current period, taking future prices and allocations as given.

Proof. Define a function capturing the total value of both consumption and end-period capital given arbitrary vectors of product prices, $\check{\boldsymbol{P}}_t$, and end-of-period capital prices, $\check{\boldsymbol{P}}^K_{t+1}$,

$$\mathcal{T}^{(1)}\left(\check{\boldsymbol{P}}_t, \check{\boldsymbol{P}}^K_{t+1}; \boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right) = \max_{\{\boldsymbol{Y}_t, \boldsymbol{X}_t\}} \sum_i \check{P}_{it}\left(Y_{it} - \sum_j X_{ijt}\right) + \sum_i \check{P}^K_{it+1} \mathcal{K}_i\left(K_{it}, \boldsymbol{X}_{it}\right), \tag{22}$$

$$\text{subject to } \mathcal{Z}\left(\boldsymbol{Y}_t; \boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right) \leq 0, \tag{23}$$

with corresponding FOCs,

$$\check{P}_{it} = \Lambda_t \frac{\partial \mathcal{Z}_t}{\partial Y_{it}}, \tag{24}$$

$$\check{P}_{it} = \check{P}^K_{jt+1} \frac{\partial \mathcal{K}_j}{\partial X_{ijt}}, \tag{25}$$

where Λ_t is the Lagrange multiplier associated with the constraint in equation (23).

Analogously to the previous section, we can now define a single-period dynamic production possibilities function,[2]

$$\mathcal{Z}^{(1)}\left(\boldsymbol{C}_t, \boldsymbol{K}_{t+1}; \boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right) \equiv \sup_{\check{\boldsymbol{P}}_t, \check{\boldsymbol{P}}^K_{t+1}} \check{\boldsymbol{P}}_t \cdot \boldsymbol{C}_t + \check{\boldsymbol{P}}^K_{t+1} \cdot \boldsymbol{K}_{t+1} - \mathcal{T}^{(1)}\left(\check{\boldsymbol{P}}_t, \check{\boldsymbol{P}}^K_{t+1}; \boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right),$$

with corresponding production possibilities frontier given by $\mathcal{Z}^{(1)}\left(\boldsymbol{C}_t, \boldsymbol{K}_{t+1}; \boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right) = 0$. The corresponding FOCs are,

$$C_{it} = \frac{\partial \mathcal{T}_t^{(1)}}{\partial \check{P}_{it}} = Y_{it} - \sum_j X_{ijt}, \tag{26}$$

$$K_{it+1} = \frac{\partial \mathcal{T}_t^{(1)}}{\partial \check{P}^K_{it+1}} = \mathcal{K}_i\left(K_{it}, \boldsymbol{X}_{it}\right). \tag{27}$$

Absent distortions, these equations characterizing production efficiency hold in competitive equilibrium. Then, comparing equation (6) with equation (25), and substituting equation (20) into equation (24), it follows that $\Lambda_t = 1$, and that the values of $\left(\check{\boldsymbol{P}}_t, \check{\boldsymbol{P}}^K_{t+1}\right)$ that achieve the supremum in the definition of $\mathcal{T}^{(1)}$ satisfy $\check{P}_{it} = P_{it}$ and $\check{P}^K_{it+1} = P^K_{it+1}$.

Applying the envelope condition to the single-period dynamic production possibilities frontier, we obtain

$$\begin{aligned}
\frac{\partial \mathcal{Z}_t^{(1)}}{\partial Z_{it}} &= -\frac{\partial \mathcal{T}_t^{(1)}}{\partial Z_{it}} = \Lambda_t \frac{\partial \mathcal{Z}_t}{\partial Z_{it}} = \frac{\partial \mathcal{Z}_t}{\partial Z_{it}}, \\
\frac{\partial \mathcal{Z}_t^{(1)}}{\partial K_{it}} &= -\frac{\partial \mathcal{T}_t^{(1)}}{\partial K_{it}} = \Lambda_t \frac{\partial \mathcal{Z}_t}{\partial K_{it}} - P^K_{it+1} \frac{\partial \mathcal{K}_{it}}{\partial K_{it}} \\
&= -P_{it} Z_{it} \frac{\partial \mathcal{F}_{it}}{\partial K_{it}} - P^K_{it+1} \frac{\partial \mathcal{K}_{it}}{\partial K_{it}} = -R_{it} - P^K_{it} + R_{it} = -P^K_{it}, \\
\frac{\partial \mathcal{Z}_t^{(1)}}{\partial C_{it}} &= P_{it}, \\
\frac{\partial \mathcal{Z}_t^{(1)}}{\partial K_{i,t+1}} &= P^K_{i,t+1}.
\end{aligned}$$

It is now straightforward to re-derive Hulten's theorem. First, $\mathcal{Z}_t^{(1)} = 0$ implies that

$$\frac{\partial \mathcal{Z}_t^{(1)}}{\partial Z_{it}} + \sum_j \frac{\partial \mathcal{Z}_t^{(1)}}{\partial C_{jt}} \frac{dC_{jt}}{dZ_{it}} + \sum_j \frac{\partial \mathcal{Z}_t^{(1)}}{\partial K_{jt+1}} \frac{dK_{jt+1}}{dZ_{it}} = 0.$$

[2] In this case, the feasible set for consumption and capital goods satisfies

$$\mathcal{P}^{(1)}\left(\boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right) \equiv \cap_{\left(\check{P}_t, \tilde{R}_{t+1}\right) \in \mathbb{R}^I_+} \left\{\left(\boldsymbol{C}_t, \boldsymbol{K}_{t+1}\right) : \check{\boldsymbol{P}}_t \cdot \boldsymbol{C}_t + \tilde{\boldsymbol{P}}^K_{t+1} \cdot \boldsymbol{K}_{t+1} - \mathcal{T}^{(1)}\left(\check{\boldsymbol{P}}_t, \tilde{\boldsymbol{R}}_{t+1}; \boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right) \leq 0\right\},$$

characterized by the condition $\mathcal{Z}^{(1)}\left(\boldsymbol{C}_t, \boldsymbol{K}_{t+1}; \boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right) \leq 0$.

Then, from the envelope conditions, we can re-write this expression as

$$
\begin{aligned}
&\frac{P_{it}Q_{it}}{Z_{it}} = \sum_j P_{jt}\frac{dC_{jt}}{dZ_{it}} + \sum_j P^K_{jt+1}\frac{dK_{jt+1}}{dZ_{it}} \\
\Rightarrow & P_{it}Q_{it} = \sum_j P_{jt}C_{jt}\frac{d\log C_{jt}}{d\log Z_{it}} + \sum_j P^K_{jt+1}K_{jt+1}\frac{d\log K_{jt+1}}{d\log Z_{it}} \\
\Rightarrow & D_{it} = \frac{P_{it}Q_{it}}{\widetilde{Y}_t} = \sum_j \frac{P_{jt}C_{jt}}{\widetilde{Y}_t}\frac{d\log C_{jt}}{d\log Z_{it}} + \sum_j \frac{P^K_{jt+1}K_{jt+1}}{\widetilde{Y}_t}\frac{d\log K_{jt+1}}{d\log Z_{it}}
\end{aligned}
$$

Since K_{jt} is given in period t in all sectors, $d\log K_{jt} = 0\ \forall j$ in equation (11) or, alternatively, in equation (14). It then follows that

$$
\frac{d\log Y_t}{d\log Z_{it}} = \frac{P_{it}Q_{it}}{\widetilde{Y}_t} = D_{it}.
$$

□

1.3 Hulten's Theorem in a Dynamic Economy

We now address the aggregate effects of sectoral shocks in a dynamic setting. In this section, we continue to consider general preferences and technologies but assume perfect foresight.

Theorem 1 (Hulten's Theorem in a Dynamic Economy)**.** *Consider a shock to the productivity of sector i, known in period t, that takes place in some period $\tau \geq t$. Then, for any $T \geq 0$, sector i's Domar weight at date τ equates to the following sequence of responses in real aggregate consumption and capital over periods t to $t+T$:*

$$
\boxed{D_{i\tau} = \sum_{t\leq t'\leq t+T}\left(\prod_{t''=t}^{t'-1}\frac{1}{1+r_{t''}}\right)\left(\frac{\widetilde{Y}_{t'}}{\widetilde{Y}_\tau}\right)S_{c,t'}\frac{d\log C_{t'}}{d\log Z_{i\tau}} + \left(\prod_{t'=t}^{t+T-1}\frac{1}{1+r_{t'}}\right)\left(\frac{\widetilde{Y}_{t+T}}{\widetilde{Y}_\tau}\right)S^+_{k,t+T}\frac{d\log K_{t+T+1}}{d\log Z_{i\tau}},}
$$

where changes in real aggregate consumption and the stock of capital are defined as in equations (8) and (13).

Proof. We extend the function characterizing current period total consumption and end-of-period capital to some period, T , given vectors of product prices, $\widehat{\boldsymbol{P}}_t$ to $\widehat{\boldsymbol{P}}_{t+T}$, and capital prices in the final period, $\widehat{\boldsymbol{P}}^K_{t+T+1}$. Here, as shown below, the ′ $\frown$ ′ notation implicitly captures the notion of discounting in the mapping to market equilibrium prices. Thus, we define

$$
\begin{aligned}
&\mathcal{T}^{(T)}\left(\widehat{\boldsymbol{P}}_t,\cdots,\widehat{\boldsymbol{P}}_{t+T},\widehat{\boldsymbol{P}}^K_{t+T+1};\boldsymbol{K}_t,L_t,\cdots,L_{t+T};\boldsymbol{Z}_t,\cdots,\boldsymbol{Z}_{t+T}\right)\\
&=\max_{\{\boldsymbol{Y}_t,\boldsymbol{X}_t\}}\sum_i \widehat{P}_{it}\left(Y_{it}-\sum_j X_{ijt}\right)+\ \cdots\ +\sum_i \widehat{P}_{it+T}\left(Y_{it+T}-\sum_j X_{ijt+T}\right)\\
&+\cdots+\sum_i \widehat{P}_{it+T+1}\mathcal{K}_i\left(K_{it+T},\boldsymbol{X}_{it+T}\right)\\
&\ \text{subject to}\\
&\mathcal{Z}\left(\boldsymbol{Y}_t;\boldsymbol{K}_t,L_t;\boldsymbol{Z}_t\right)\leq 0,\ \cdots,\mathcal{Z}\left(\boldsymbol{Y}_{t+T};\boldsymbol{K}_{t+T},L_{t+T};\boldsymbol{Z}_{t+T}\right)\leq 0,\\
&K_{it+1}=\mathcal{K}_i\left(K_{it},\boldsymbol{X}_{it}\right),\ \cdots,K_{it+T+1}=\mathcal{K}_i\left(K_{it+T},\boldsymbol{X}_{it+T}\right),\ i=1\cdots N,
\end{aligned}
$$

with corresponding FOCs,

$$\widehat{P}_{i\tau}=\Lambda_\tau\frac{\partial\mathcal{Z}_\tau}{\partial Y_{i\tau}},\qquad t\leq\tau\leq t+T, \tag{28}$$

$$\eta_{i\tau-1}-\eta_{i\tau}\frac{\partial\mathcal{K}_i}{\partial K_{i\tau}}=-\Lambda_\tau\frac{\partial\mathcal{Z}_\tau}{\partial K_{i\tau}},\qquad t+1\leq\tau\leq t+T, \tag{29}$$

$$\widehat{P}_{i\tau}=\eta_{j\tau}\frac{\partial\mathcal{K}_j}{\partial X_{ij\tau}},\qquad t\leq\tau\leq t+T-1, \tag{30}$$

$$\widehat{P}_{it+T}=\widehat{P}^K_{jt+T+1}\frac{\partial\mathcal{K}_j}{\partial X_{ijt+T}}, \tag{31}$$

where Λ_τ and $\eta_{i\tau}$ denote the Lagrange multipliers associated with the constraints on $\mathcal{Z}_\tau$ and $K_{i\tau+1}$ respectively.

We can then define the dynamic production possibilities function,

$$\mathcal{Z}^{(T)}\left(\boldsymbol{C}_t,\cdots,\boldsymbol{C}_{t+T},\boldsymbol{K}_{t+T+1};\boldsymbol{K}_t,L_t,\cdots,L_{t+T};\boldsymbol{Z}_t,\cdots,Z_{t+T}\right)\leq 0,$$

such that

$$
\begin{aligned}
&\mathcal{Z}^{(T)}\left(\boldsymbol{C}_t,\cdots,\boldsymbol{C}_{t+T},\boldsymbol{K}_{t+T+1};\boldsymbol{K}_t,L_t,\cdots,L_{t+T};\boldsymbol{Z}_t,\cdots,Z_{t+T}\right)\\
&\equiv\sup_{\widehat{\boldsymbol{P}}_t,\widehat{\boldsymbol{P}}^K_{t+1}}\ \widehat{\boldsymbol{P}}_t\cdot\boldsymbol{C}_t+\cdots+\widehat{\boldsymbol{P}}_{t+T}\cdot\boldsymbol{C}_{t+T}+\widehat{\boldsymbol{P}}^K_{t+T+1}\cdot\boldsymbol{K}_{t+T+1}\\
&\qquad\qquad-\mathcal{T}^{(T)}\left(\widehat{\boldsymbol{P}}_t,\cdots,\widehat{\boldsymbol{P}}_{t+T},\widehat{\boldsymbol{P}}^K_{t+T+1};\boldsymbol{K}_t,L_t,\cdots,L_{t+T};\boldsymbol{Z}_t,\cdots,Z_{t+Tt}\right),
\end{aligned}
$$

with corresponding dynamic production possibilities frontier,

$$\mathcal{Z}^{(T)}\left(\boldsymbol{C}_t,\cdots,\boldsymbol{C}_{t+T},\boldsymbol{K}_{t+T+1};\boldsymbol{K}_t,L_t,\cdots,L_{t+T};\boldsymbol{Z}_t,\cdots,Z_{t+T}\right)=0.$$

The corresponding FOCs are,

$$C_{i\tau} = \frac{\partial \mathcal{T}^{(T)}}{\partial \widehat{P}_{i\tau}} = Y_{i\tau} - \sum_j X_{ij\tau}, \qquad t \leq \tau \leq t+T, \tag{32}$$

$$K_{it+T+1} = \frac{\partial \mathcal{T}^{(T)}}{\partial \widehat{P}_{it+T+1}} = \mathcal{K}_i \left(K_{it+T}, \boldsymbol{X}_{it+T} \right), \tag{33}$$

where the RHS of these last two equations follows from the envelope condition.

Comparing these FOCs with market equilibrium conditions, equations (20) and (28) imply $\widehat{P}_{it} = \Lambda_t P_{it}$. Then, from equations (6) and (30), we find

$$\frac{\widehat{P}_{i\tau}}{\eta_{j\tau}} = \frac{P_{i\tau}}{P^K_{j\tau+1}} \;\Rightarrow\; \eta_{j\tau} = \Lambda_\tau \, P^K_{j\tau+1}. \tag{34}$$

In addition, equations (21) and (29) imply

$$\begin{aligned}
\Lambda_\tau \, R_{i\tau} &= \eta_{i\tau-1} - \eta_{i\tau} \frac{\partial \mathcal{K}_i}{\partial K_{i\tau}}, \\
&= \eta_{i\tau-1} - \eta_{i\tau} \frac{P^K_{i\tau} - R_{i\tau}}{P^K_{i\tau+1}}, \\
&= \Lambda_{\tau-1} \, P^K_{i\tau} - \Lambda_\tau \left(P^K_{i\tau} - R_{i\tau} \right), \\
&= \Lambda_{\tau-1} \, P^K_{i\tau} - \Lambda_\tau \left((1 + r_{\tau-1}) \, P^K_{i\tau} - R_{i\tau} \right), \\
&= \Lambda_\tau \, R_{i\tau} + \Lambda_\tau \, P^K_{i\tau} \left(\frac{\Lambda_{\tau-1}}{\Lambda_\tau} - (1 + r_{\tau-1}) \right).
\end{aligned}$$

where the second equality follows from equation (5), and we have substituted equations (1) and (34) in the third and fourth equalities. It follows that

$$\frac{\Lambda_\tau}{\Lambda_{\tau-1}} = \frac{1}{1 + r_{\tau-1}}, \tag{35}$$

and, letting $\Lambda_t = 1$, we find that

$$\Lambda_\tau = \prod_{t'=t}^{\tau-1} \frac{1}{1 + r_{t'}}, \qquad t+1 \leq \tau \leq t+T. \tag{36}$$

In addition, combing equation (6) in the final period,

$$P^K_{jt+T+1} \frac{\partial \mathcal{K}_j}{\partial X_{ijt+T}} = P_{it+T},$$

with equation (31) gives

$$\begin{aligned}\widehat{P}^K_{jt+T+1} &= \frac{\widehat{P}_{it+T}}{P_{it+T}} P^K_{jt+T+1}\\ &= \Lambda_{t+\tau} P^K_{jt+T+1}\end{aligned}$$

which maps the prices, $\left(\widehat{\boldsymbol{P}}_t, \widehat{\boldsymbol{P}}^K_{t+1}\right)$, in the supremum definition of $\mathcal{Z}^{(T)}$ into market equilibrium prices.

Applying the envelope condition, we now have,

$$\begin{aligned}\frac{\partial \mathcal{Z}^{(T)}}{\partial Z_{i\tau}} &= -\frac{\partial \mathcal{T}^{(T)}}{\partial Z_{i\tau}} = \Lambda_\tau \frac{\partial \mathcal{Z}_\tau}{\partial Z_{i\tau}} = \Lambda_\tau \frac{P_{i\tau} Q_{i\tau}}{Z_{i\tau}},\\ \frac{\partial \mathcal{Z}^{(T)}_\tau}{\partial C_{i\tau}} &= \widehat{P}_{i\tau} = \Lambda_\tau P_{i\tau}, \qquad t \le \tau \le t+T,\\ \frac{\partial \mathcal{Z}^{(T)}_t}{\partial K_{it+T}} &= \widehat{P}^K_{it+T+1} = \Lambda_{t+T}\, P^K_{it+T+1}.\end{aligned}$$

Then, the dynamic production possibilities frontier implies that in a competitive equilibrium,

$$\frac{\partial \mathcal{Z}^{(T)}}{\partial Z_{i\tau}} + \sum_{t \le t' \le T} \sum_j \frac{\partial \mathcal{Z}^{(T)}}{\partial C_{jt'}} \frac{dC_{jt'}}{dZ_{i\tau}} + \sum_j \frac{\partial \mathcal{Z}^{(T)}}{\partial K_{j,t+T+1}} \frac{dK_{j,t+T+1}}{dZ_{i\tau}} = 0,$$

which implies

$$\begin{aligned}\frac{P_{i\tau} Q_{i\tau}}{\widetilde{Y}_\tau} &= Z_{i\tau}\left[\sum_{t \le t' \le T} \widehat{\Lambda}_{t'} \left(\sum_j \frac{P_{jt'}}{\widetilde{Y}_\tau} \frac{dC_{jt'}}{dZ_{i\tau}}\right) + \widehat{\Lambda}_{t+T} \sum_j \frac{P^K_{j,t+T+1}}{\widetilde{Y}_\tau} \frac{dK_{j,t+T+1}}{dZ_{i\tau}}\right]\\ &= \sum_{t \le t' \le t+T} \left(\prod_{t''=t}^{t'-1} \frac{1}{1+r_{t''}}\right) \left(\frac{\widetilde{Y}_{t'}}{\widetilde{Y}_\tau}\right) \left(\sum_j \frac{P_{jt'} C_{jt'}}{\widetilde{Y}_{t'}} \frac{d\log C_{t'}}{d\log Z_{i\tau}}\right)\\ &\qquad \left(\prod_{t'=t}^{t+T-1} \frac{1}{1+r_{t'}}\right) \left(\frac{\widetilde{Y}_{t+T}}{\widetilde{Y}_\tau}\right) \sum_j \frac{P^K_{j,t+T+1} K_{j,t+T+1}}{\widetilde{Y}_{t+T}} \frac{d\log K_{j,t+T+1}}{d\log Z_{i\tau}},\\ &= \sum_{t \le t' \le t+T} \left(\prod_{t''=t}^{t'-1} \frac{1}{1+r_{t''}}\right) \left(\frac{\widetilde{Y}_{t'}}{\widetilde{Y}_\tau}\right) S_{ct'} \frac{d\log C_{t'}}{d\log Z_{i\tau}} +\\ &\qquad \left(\prod_{t'=t}^{t+T-1} \frac{1}{1+r_{t'}}\right) \left(\frac{\widetilde{Y}_{t+T}}{\widetilde{Y}_\tau}\right) S^+_{kt+T} \frac{d\log K_{t+T+1}}{d\log Z_{i\tau}}.\end{aligned} \tag{37}$$

□

More generally, Theorem 1 gives us three key corollaries below,

Corollary 1. *Let $\tau = t$ and $T = 0$, then*

$$\boxed{D_{it} = S_{c,t}\frac{d\log C_t}{d\log Z_{it}} + S^{+}_{k,t}\frac{d\log K_{t+1}}{d\log Z_{it}} = \frac{d\log Y_t}{d\log Z_{it}},}.$$

As in the static economy, Domar weights reflect the effects of sectoral productivity shocks on GDP in the current period. However, in contrast to the static economy, there now exists a distinction between consumption and GDP. Thus, we have,

Corollary 2. *Let $T > 0$ and consider a productivity shock in sector i that takes place at date t. Then,*

$$\begin{aligned} & D_{it} = D^{c}_{it} + D^{x}_{it} \\ & \textit{where } D^{c}_{it} = S_{ct}\frac{d\log C_t}{d\log Z_{it}}, \\ & \qquad D^{x}_{it} = \sum_{t'=t+1}^{t+T}\left(\prod_{t''=t}^{t'-1}\frac{1}{1+r_{t''}}\right)\left(\frac{\widetilde{Y}_{t'}}{\widetilde{Y}_t}\right)S_{ct'}\frac{d\log C_{t'}}{d\log Z_{it}} \\ & \qquad\qquad + \left(\prod_{t'=t}^{t+T-1}\frac{1}{1+r_{t'}}\right)\left(\frac{\widetilde{Y}_{t+T}}{\widetilde{Y}_t}\right)S^{+}_{kt+T}\frac{d\log K_{t+T+1}}{d\log Z_{it}} \end{aligned} \tag{38}$$

It follows from corollary 1 and corollary 2 that

$$\begin{aligned} S_{x,t}\frac{d\log X_t}{d\log Z_{it}} = D^{x}_{it} = & \sum_{t'=t+1}^{t+T}\left(\prod_{t''=t}^{t'-1}\frac{1}{1+r_{t''}}\right)\left(\frac{\widetilde{Y}_{t'}}{\widetilde{Y}_t}\right)S_{ct'}\frac{d\log C_{t'}}{d\log Z_{it}} \\ & + \left(\prod_{t'=t}^{t+T-1}\frac{1}{1+r_{t'}}\right)\left(\frac{\widetilde{Y}_{t+T}}{\widetilde{Y}_t}\right)S^{+}_{kt+T}\frac{d\log K_{t+T+1}}{d\log Z_{it}}. \end{aligned}$$

Corollary 3. *Let $T > 0$ and consider a productivity shock in sector i that takes place at date t. Then,*

$$\boxed{\frac{d\log Y_\tau}{\partial\log Z_{it}} = S_{k,\tau}\frac{d\log K^{s}_{\tau}}{d\log Z_{it}}, \qquad t < \tau.}$$

Proof. Consider equation (37) from the standpoint of periods $T-1$ and T, and equate the right-hand-sides of these equations to obtain, for $\tau < t+T$,

$$S^{+}_{kt+T-1}\frac{d\log K_{t+T}}{d\log Z_{i\tau}} = \frac{\widetilde{Y}_{t+T}}{(1+r_{t+T-1})\widetilde{Y}_{t+T-1}}\left(S_{ct+T}\frac{d\log C_{t+T}}{d\log Z_{i\tau}} + S^{+}_{kt+T}\frac{d\log K_{t+T+1}}{d\log Z_{i\tau}}\right). \tag{39}$$

Now, using equation (14), while replacing $t+T$ with t, gives

$$\begin{aligned}\frac{d\log Y_t}{\partial \log Z_{i\tau}} &= S_{ct}\,\frac{d\log C_t}{d\log Z_{i\tau}} + S_{kt}\,\frac{d\log K^s_{jt}}{d\log Z_{i\tau}} \\ &\quad + S^+_{kt}\,\frac{d\log K_{t+1}}{d\log Z_{i\tau}} - \frac{\widetilde{Y}_{t-1}\left(1+r_{t-1}\right)}{\widetilde{Y}_t}\,S^+_{kt-1}\,d\log K_t, \\ &= S_{kt}\,\frac{d\log K^s_t}{d\log Z_{i\tau}}, \qquad \tau < t.\end{aligned}$$

□

1.4 Hulten's Theorem and Consumption Efficiency

To this point, our findings on the aggregate effects of sectoral shocks over time have relied solely on efficient production. These findings also have a partial interpretation in terms of efficient consumption. Thus, consider the production environment described in Section 1.1 where, in each period t, a representative household maximizes the following present-discounted utility,

$$\mathcal{V}_t \equiv \sum_{\tau=t}^{\infty} \beta^{\tau-t}\, L_\tau\, \mathcal{U}\left(\frac{\boldsymbol{C}_\tau}{L_\tau}\right), \tag{40}$$

given its labor income. The next proposition relates Domar weights to the response in continuation utility, $\mathcal{V}_t$, to sectoral shocks.

Proposition 1. *Let household preferences be given by equation (40) in the economic environment described in Section 1.1. Then, sector i's Domar weight gives the following weighted response in continuation utility to a productivity change in sector i,*

$$\boxed{D_{it} = \varepsilon_t\,\frac{S_{ct}}{L_t\left(\mathcal{U}_t/\mathcal{V}_t\right)}\,\frac{\partial \log \mathcal{V}_t}{\partial \log Z_{it}},} \tag{41}$$

where $L_t\left(\mathcal{U}_t/\mathcal{V}_t\right)$ is the ratio of current utility to lifetime utility, and where $\varepsilon_t \equiv \partial \log \mathcal{E}\left(\mathcal{U}_t;\boldsymbol{P}_t\right)/\partial \log \mathcal{U}_t$ is the utility elasticity of the expenditure function $\mathcal{E}\left(\cdot;\cdot\right)$.

Proof. We can write

$$\begin{gathered}\mathcal{V}_t\left(\boldsymbol{K}_t; L^{(t)}; \boldsymbol{Z}^{(t)}\right) = \max_{\boldsymbol{C}_t,\boldsymbol{K}_{t+1}} L_t\,\mathcal{U}\left(\frac{\boldsymbol{C}_t}{L_t}\right) + \beta\,\mathcal{V}_{t+1}\left(\boldsymbol{K}_{t+1}; L^{(t+1)}; \boldsymbol{Z}^{(t+1)}\right), \\ \text{subject to } \mathcal{Z}^{(1)}\left(\boldsymbol{C}_t, \boldsymbol{K}_{t+1}; \boldsymbol{K}_t, L_t; \boldsymbol{Z}_t\right) \leq 0.\end{gathered}$$

The corresponding FOCs are

$$\partial_i \mathcal{U}_t = \xi_t \frac{\partial \mathcal{Z}_t^{(1)}}{\partial C_{it}} = \xi_t\, P_{it}, \tag{42}$$

$$\beta \frac{\partial \mathcal{V}_{t+1}}{\partial K_{it+1}} = \xi_t \frac{\partial \mathcal{Z}_t^{(1)}}{\partial K_{it+1}} = \xi_t\, P^K_{i,t+1},$$

where $\partial_i \mathcal{U}_t$ denotes the partial derivative of $\mathcal{U}\left(\frac{C_t}{L_t}\right)$ with respect to its i^{th} argument, and where ξ_t is the Lagrange multiplier associated with the constraint.

In the static utility maximization problem, ξ_t is the marginal utility of total per capita nominal consumption expenditure, $\widetilde{C}_t/L_t$, at date t. Thus, let $\mathcal{E}\left(\cdot;\boldsymbol{P}_t\right)$ denote the expenditure function corresponding to the utility function, $\mathcal{U}\left(\cdot\right)$. We then have

$$\frac{1}{\xi_t} = \frac{\partial \mathcal{E}\left(\mathcal{U}_t;\boldsymbol{P}_t\right)}{\partial \mathcal{U}_t} = \frac{\widetilde{C}_t/L_t}{\mathcal{U}_t}\,\varepsilon_t,$$

where $\varepsilon_t \equiv \partial \log \mathcal{E}\left(\mathcal{U}_t;\boldsymbol{P}_t\right)/\partial \log \mathcal{U}_t$. In addition, the envelope condition gives

$$\frac{\partial \mathcal{V}_t}{\partial K_{it}} = \xi_t \frac{\partial \mathcal{Z}_t^{(1)}}{\partial K_{it}},$$

$$\frac{\partial \mathcal{V}_t}{\partial Z_{it}} = \xi_t \frac{\partial \mathcal{Z}_t^{(1)}}{\partial Z_{it}} = \xi_t \frac{P_{it} Q_{it}}{Z_{it}}.$$

The last equation, in particular, implies

$$D_{it} = \frac{Z_{it}}{\xi_t} \frac{1}{\widetilde{Y}_t} \frac{\partial \mathcal{V}_t}{\partial Z_{it}},$$

$$= \varepsilon_t \frac{\widetilde{C}_t}{\widetilde{Y}_t} \frac{\mathcal{V}_t}{L_t \mathcal{U}_t} \frac{\partial \log \mathcal{V}_t}{\partial \log Z_{it}}.$$

□

As an example, consider $\mathcal{U}\left(\boldsymbol{C}\right) \equiv \log \mathcal{C}\left(\boldsymbol{C}\right)$ where $\mathcal{C}\left(\boldsymbol{C}\right)$ is homogenous function of first order. Then $\varepsilon_t \equiv \mathcal{U}_t$ and equation (41) becomes,

$$\boxed{D_{it} = S_{ct} \frac{\partial\left(\mathcal{V}_t/L_t\right)}{\partial \log Z_{it}}.} \tag{43}$$

2 A Structural Model with Production Linkages in Investment and Intermediate Inputs

Preferences are given by

$$\mathcal{V} = \sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathcal{U}(C_\tau) = \sum_{\tau=t}^{\infty} \beta^{\tau-t} \log C_\tau,$$

where C_τ is a consumption bundle defined by $C_\tau \equiv \prod_i C_{i\tau}^{\theta_i}$, $\theta_i > 0\ \forall i$ and $\sum_i \theta_i = 1$, which we take to be the numéraire good.

In each sector, i, value added from capital, K_{it}, and labor, L_{it}, is produced according to,

$$Y_{it} = \left(\frac{K_{it}}{\alpha_i}\right)^{\alpha_i}\left(\frac{L_{it}}{(1-\alpha_i)}\right)^{(1-\alpha_i)}, \tag{44}$$

where $\alpha_i \in (0,1)$. Gross output is then produced by combining the value added from capital and labor with materials according to

$$Q_{it} = Z_{it}\left(\frac{Y_{it}}{\gamma_i}\right)^{\gamma_i}\prod_j\left(\frac{M_{jit}}{(1-\gamma_i)\,\phi_{ji}}\right)^{(1-\gamma_i)\phi_{ji}}, \tag{45}$$

where $\gamma_i, \in (0,1)$, and $\sum_j \phi_{ji} = 1,\ \phi_{ji} > 0\ \forall j, i$, so that production requires intermediate inputs in every sector. In equation (45), Z_{it} represents gross output TFP in sector i and value added TFP, Z_{it}^Y, is then $Z_{it}^Y = (Z_{it})^{\frac{1}{\gamma_i}}$.[3] We let sectoral productivity follows the process,

$$z_{it} = z_{it-1} + g_i^z + \epsilon_{it}, \tag{46}$$

where ϵ_{it} is *i.i.d.* across sectors and over time with $E_{t-1}(\epsilon_{it}) = 0\ \forall i$ and t. Equation (46) implies that, as in the empirical data, real quantities in the economy will be stationary in (log) first differences but not in (log) levels. That said, because we aim to work with a global solution to the household's problem, rather than one linearized around a non-stochastic steady state, our analysis will allow for a seamless transition between levels and growth rates.

Let

$$K_{it+1} = \chi_i K_{it}^{1-\zeta_i}\prod_j\left(\frac{X_{jit}}{\omega_{ji}}\right)^{\zeta_i\omega_{ji}},\ K_{i0} \text{ given } \forall i,$$

where $\sum_j \omega_{ji} = 1,\ \omega_{ji} \geq 0\ \forall j, i$. In each sector, additions to the capital stock reflect a geometric average of investment in new capital and past capital stocks. The parameter, ζ_i, captures adjustment costs in investment. When $\zeta_i = 0$, adjustment costs are infinite and capital becomes a static factor of production. In that case, sectoral shocks cannot propagate to future periods by way of capital accumulation and, therefore, cannot affect future production or consumption.[4]

[3] This implies that $z_{it}^Y = \frac{1}{\gamma_i} z_{it}$, where recall that $z_{it}^Y = \log Z_{it}^Y$ and similarly for z_{it}. All results described throughout this paper may be derived in terms of either gross output TFP or value added TFP. Importantly, while Domar weights are the appropriate weights for aggregating sectoral gross output TFP, sectoral value added TFP is appropriately aggregated using sectoral value added shares in GDP. That is, Hulten's theorem, framed in terms of value added TFP, yields value added shares in GDP rather than Domar weights. Unlike Domar weights, sectoral value added shares in GDP are convenient in that they sum to unity and directly reflect the relative sizes of sectors in the economy.

[4] In the steady state, $K_i = \chi_i^{\frac{1}{\zeta_i}} \prod_j \left(X_{ji}/\omega_{ji}\right)^{\omega_{ji}}$, so that $1/\chi_i^{\frac{1}{\zeta_i}}$ denotes the investment rate needed to maintain a constant capital stock in the long run.

Finally, market clearing in goods requires that in each sector, i,

$$C_{it} + \sum_{j=1}^{N} M_{jit} + \sum_{j=1}^{N} X_{jit} = Q_{it},$$

while, in the labor market,

$$\sum_{i}^{N} L_{it} = L_t.$$

We use the following notation to define the vectors, $\boldsymbol{c}_t \equiv (c_{it})$, $\boldsymbol{k}_t \equiv (k_{it})$, $\boldsymbol{z}_t \equiv (z_{it})$, etc., and the model parameters, $\boldsymbol{\theta} \equiv (\theta_i)$, $\boldsymbol{\alpha}_d \equiv \operatorname{diag}(\alpha_i)$, $\boldsymbol{\zeta}_d \equiv \operatorname{diag}(\zeta_i)$, $\boldsymbol{\gamma}_d \equiv \operatorname{diag}(\gamma_i)$, $\boldsymbol{\Phi} \equiv [\phi_{ij}]$, $\boldsymbol{\Omega} \equiv [\omega_{ij}]$, etc.

2.1 Dynamic Programming Formulation

Recall that for any real variable, X, $x = \log X$. We solve the planner's problem by solving for the recursive version of the Bellman equation,

$$\mathcal{V}(\boldsymbol{k}_t, \boldsymbol{z}_t) = \max_{\boldsymbol{c}_t, \boldsymbol{k}_{t+1}, \boldsymbol{m}_t, \boldsymbol{x}_t} \{\boldsymbol{\theta}' \boldsymbol{c}_t + \beta E_t [\mathcal{V}(\boldsymbol{k}_{t+1}, \boldsymbol{z}_{t+1})]\},$$

and derive the corresponding policy functions and value function, expressed in terms of the log of the state variables, $\boldsymbol{k}_t$ and $\boldsymbol{z}_t$. We conjecture that the value function is linear in the (log of the) state variables,

$$\mathcal{V}(\boldsymbol{k}_t, \boldsymbol{z}_t) = \boldsymbol{B}_k' \boldsymbol{k}_t + \boldsymbol{B}_z' \boldsymbol{z}_t + \boldsymbol{B}, \tag{47}$$

where a 'prime' denotes the transpose of a vector or matrix. Here, the coefficients $\boldsymbol{B}_k$ and $\boldsymbol{B}_z$ are $N \times 1$ vectors and $\boldsymbol{B}$ is a scalar. We include the problem's constraints by way of Lagrange multipliers. Given the process governing sectoral productivity shocks in equation (46), $E_t(\boldsymbol{z}_{t+1}) = \boldsymbol{z}_t + \boldsymbol{g}^z$ and, since next period's capital stock is known once chosen this period, the Bellman equation with constraints becomes

$$\begin{aligned}\mathcal{V}\left(\boldsymbol{k}_t,\boldsymbol{z}_t\right) = & \max_{\boldsymbol{c}_t,\boldsymbol{k}_{t+1},\boldsymbol{m}_t,\boldsymbol{x}_t}\left\{\boldsymbol{\theta}'\boldsymbol{c}_t + \beta\left(\boldsymbol{B}_k'\boldsymbol{k}_{t+1} + \boldsymbol{B}_z'\boldsymbol{z}_t + \boldsymbol{B}_z'\boldsymbol{g}^z + \boldsymbol{B}\right)\right\}, \\ & + \sum_{i=1}^{N}\psi_{it}\left(z_{it} + \gamma_i y_{it} + (1-\gamma_i)\sum_{j=1}^{N}\phi_{ji}m_{jit} - q_{it} - \varpi_i\right) \\ & + \sum_{i=1}^{N}\nu_{it}\left(\alpha_i k_{it} + (1-\alpha_i)\ell_{it} - y_{it} - \varrho_i\right) \\ & + \sum_{i=1}^{N}\mu_{it}\left(\exp\left(q_{it}\right) - \exp\left(c_{it}\right) - \sum_{j=1}^{N}\exp\left(m_{ijt}\right) - \sum_{j=1}^{N}\exp\left(x_{ijt}\right)\right) \\ & + \sum_{i=1}^{N}\eta_{it}\left(\left(1-\zeta_i\right)k_{it} + \zeta_i\sum_{j=1}^{N}\omega_{ji}x_{jit} - k_{it+1} - \zeta_i\sum_{j}\omega_{ji}\log\omega_{ji} + \log\chi_i\right) \\ & + \sum_{i=1}^{N}\xi_t\left[L_t - \sum_i\exp\left(\ell_{it}\right)\right],\end{aligned} \tag{P1}$$

where $\varpi_i = \gamma_i\log\gamma_i + (1-\gamma_i)\sum_j\phi_{ji}\log(1-\gamma_i)\phi_{ji}$ and $\varrho_i = \alpha_i\log\alpha_i + (1-\alpha_i)\log(1-\alpha_i)$. The goal is to obtain the policy functions, $c_{it} = c_i(\boldsymbol{k}_t,\boldsymbol{z}_t)$ and $k_{it+1} = k_i(\boldsymbol{k}_t,\boldsymbol{z}_t)$, by solving for the unknown coefficients, $\boldsymbol{B}_k$, $\boldsymbol{B}_z$, and $\boldsymbol{B}$ in terms of the model parameters, $\boldsymbol{\theta}$, $\boldsymbol{\alpha}_d$, γ_d, $\boldsymbol{\Phi}$, $\boldsymbol{\Omega}$, etc.

2.2 Solving for the Policy Functions

The FOC's with respect to q_{it}, y_{it}, c_{it}, k_{it+1}, m_{jit}, x_{jit}, and ℓ_{it} are

$$\psi_{it} = \mu_{it}Q_{it}, \tag{48}$$

$$\psi_{it}\gamma_i = \nu_{it}, \tag{49}$$

$$\theta_i = \mu_{it}C_{it}, \tag{50}$$

$$\beta\,B_{ki} = \eta_{it}, \tag{51}$$

$$\psi_{it}\ \left(1-\gamma_i\right)\phi_{ji} = \mu_{jt}M_{jit}, \tag{52}$$

$$\eta_{it}\zeta_i\omega_{ji} = \mu_{jt}X_{jit}, \tag{53}$$

$$\nu_{it}\ \left(1-\alpha_i\right) = \xi_t L_{it}. \tag{54}$$

The last FOC with respect to labor input can be expressed alternatively as

$$\gamma_i\psi_{it}\ \left(1-\alpha_i\right) = \xi_t L_{it}.$$

First, let us solve for k_{it+1} from the capital accumulation equation and equation (53),

$$\begin{aligned} k_{it+1} &= (1-\zeta_i)\,k_{it} + \zeta_i \sum_j \omega_{ji} \log \frac{\eta_{it}\zeta_i\omega_{ji}}{\mu_{jt}} - \zeta_i \sum_j \omega_{ji} \log \omega_{ji} + \log \chi_i \\ &= (1-\zeta_i)\,k_{it} + \zeta_i \log \eta_{it} - \zeta_i \sum_j \omega_{ji} \log \mu_{jt} + \zeta_i \log \zeta_i + \log \chi_i, \end{aligned}$$

or, in vector form,

$$\boldsymbol{k}_{t+1} = (I - \boldsymbol{\zeta}_d)\,\boldsymbol{k}_t + \boldsymbol{\zeta}_d \log \boldsymbol{\eta}_t - \boldsymbol{\zeta}_d \boldsymbol{\Omega}' \log \boldsymbol{\mu}_t + \boldsymbol{\zeta}_d \log \boldsymbol{\zeta} + \log \boldsymbol{\chi}. \tag{55}$$

Equation (51) gives us

$$\beta \boldsymbol{B}_k = \boldsymbol{\eta}_t. \tag{56}$$

Substituting the FOC's (48), (50), (52), and (53) into the economy's resource constraint gives

$$\frac{\psi_{it}}{\mu_{it}} = \frac{\theta_i}{\mu_{it}} + \sum_j \frac{\omega_{ij}\zeta_j\eta_{jt}}{\mu_{it}} + \sum_j \frac{(1-\gamma_j)\,\phi_{ij}\,\psi_{jt}}{\mu_{it}},$$

or

$$(\boldsymbol{I} - \boldsymbol{\Phi}\,(\boldsymbol{I} - \gamma_d))\,\boldsymbol{\psi}_t = \boldsymbol{\theta} + \boldsymbol{\Omega}\,\zeta_d \boldsymbol{\eta}_t.$$

It then follows that

$$\begin{aligned} \boldsymbol{\psi}_t &= (\boldsymbol{I} - \boldsymbol{\Phi}\,(\boldsymbol{I} - \gamma_d))^{-1}\,(\boldsymbol{\theta} + \boldsymbol{\Omega}\,\zeta_d \boldsymbol{\eta}_t) \\ &= (\boldsymbol{I} - \boldsymbol{\Phi}\,(\boldsymbol{I} - \gamma_d))^{-1}\,(\boldsymbol{\theta} + \beta \boldsymbol{\Omega} \boldsymbol{\zeta}_d\, \boldsymbol{B}_k)\,. \end{aligned} \tag{57}$$

From equation (54), we have that $\sum_i \psi_{it}\,(1-\alpha_i)\,\gamma_i = \xi_t L_t$ or (assuming WLOG $L_t = 1$),

$$\begin{aligned} \xi_t &= \boldsymbol{1}'\,(\boldsymbol{I} - \boldsymbol{\alpha}_d)\,\gamma_d \boldsymbol{\psi}_t \\ &= \boldsymbol{1}'\,(\boldsymbol{I} - \boldsymbol{\alpha}_d)\,\gamma_d\,(\boldsymbol{I} - \boldsymbol{\Phi}\,(\boldsymbol{I} - \gamma_d))^{-1}\,(\boldsymbol{\theta} + \beta \boldsymbol{\Omega} \delta_d\, \boldsymbol{B}_k)\,, \end{aligned} \tag{58}$$

where $\boldsymbol{1}$ is an $N \times 1$ vector of ones. Since $\boldsymbol{\psi}_t$, $\boldsymbol{\eta}_t$, and ξ_t are constant, hereafter we omit their time subscript.

The production of value added in log terms implies that

$$y_{it} = \alpha_i k_{it} + (1-\alpha_i)\ell_{it} - \varrho_i,$$

and that of gross output,

$$q_{it} = z_{it} + \gamma_i y_{it} + (1-\gamma_i) \sum_j \phi_{ji} m_{jit} - \varpi_i,$$

so that substituting equations (48), (52) and (54) into this last expression, we obtain

$$\log\left(\frac{\psi_{it}}{\mu_{it}}\right) = z_{it} + \alpha_i\gamma_i k_{it} + (1-\alpha_i)\,\gamma_i \log\left(\frac{(1-\alpha_i)\,\gamma_i\,\psi_{it}}{\xi_t}\right) + (1-\gamma_i)\sum_j \phi_{ji}\log\left(\frac{(1-\gamma_i)\,\phi_{ji}\,\psi_{it}}{\mu_{jt}}\right) - \varpi_i - \gamma_i\varrho_i,$$

which gives us the following expression,

$$\left(\boldsymbol{I} - (\boldsymbol{I}-\boldsymbol{\gamma}_d)\,\boldsymbol{\Phi}'\right)\widehat{\boldsymbol{\mu}}_t = \boldsymbol{z}_t - \boldsymbol{\alpha}_d\boldsymbol{\gamma}_d\boldsymbol{k}_t + \left(\boldsymbol{I} - (\boldsymbol{I}-\boldsymbol{\alpha}_d)\,\boldsymbol{\gamma}_d - (\boldsymbol{I}-\boldsymbol{\gamma}_d)\right)\widehat{\boldsymbol{\psi}} + (\boldsymbol{I}-\boldsymbol{\alpha}_d)\,\boldsymbol{\gamma}_d\mathbf{1}\widehat{\xi} + \boldsymbol{d}, \tag{59}$$

where $\widehat{\boldsymbol{\mu}}_t = \log\boldsymbol{\mu}_t$, $\widehat{\boldsymbol{\psi}} = \log\boldsymbol{\psi}$, $\widehat{\boldsymbol{\xi}} = \log\boldsymbol{\xi}$, and, recalling the definition of ϖ_i and ϱ_i,

$$d_i = \alpha_i\gamma_i\log\alpha_i\gamma_i.$$

Define the Leontief inverse,[5]

$$\boxed{\boldsymbol{\mathcal{L}} = \left(\boldsymbol{I} - (\boldsymbol{I}-\boldsymbol{\gamma}_d)\,\boldsymbol{\Phi}'\right)^{-1}.} \tag{60}$$

We can then re-write equation (59) as

$$\widehat{\boldsymbol{\mu}}_t = -\boldsymbol{\mathcal{L}}\boldsymbol{z}_t - \boldsymbol{\mathcal{L}}\boldsymbol{\alpha}_d\boldsymbol{\gamma}_d\boldsymbol{k}_t + \boldsymbol{A}_\mu, \tag{61}$$

where $\boldsymbol{A}_\mu = \boldsymbol{\mathcal{L}}\left[\left(\boldsymbol{I} - (\boldsymbol{I}-\boldsymbol{\alpha}_d)\,\boldsymbol{\gamma}_d - (\boldsymbol{I}-\boldsymbol{\gamma}_d)\right)\widehat{\boldsymbol{\psi}} + (\boldsymbol{I}-\boldsymbol{\alpha}_d)\,\boldsymbol{\gamma}_d\widehat{\boldsymbol{\xi}} + \boldsymbol{d}\right]$ is constant. Equation (50) then gives us the policy function for consumption,

$$\boldsymbol{c}_t\left(\boldsymbol{k}_t, \boldsymbol{z}_t\right) = \boldsymbol{\mathcal{L}}\boldsymbol{z}_t + \boldsymbol{\mathcal{L}}\boldsymbol{\alpha}_d\boldsymbol{\gamma}_d\boldsymbol{k}_t + \log\boldsymbol{\theta} - \boldsymbol{A}_\mu. \tag{62}$$

Similarly, using the capital accumulation equation (55) along with equations (56) and (61) gives

$$\begin{aligned}\boldsymbol{k}_{t+1}\left(\boldsymbol{k}_t, \boldsymbol{z}_t\right) &= (I-\boldsymbol{\zeta}_d)\,\boldsymbol{k}_t + \boldsymbol{\zeta}_d\log\boldsymbol{\eta}_t - \boldsymbol{\zeta}_d\boldsymbol{\Omega}'\log\boldsymbol{\mu}_t + \boldsymbol{\zeta}_d\log\boldsymbol{\zeta} + \log\boldsymbol{\chi}\\ &= (I-\boldsymbol{\zeta}_d)\,\boldsymbol{k}_t + \boldsymbol{\zeta}_d\boldsymbol{\Omega}'\left(\boldsymbol{\mathcal{L}}\boldsymbol{z}_t + \boldsymbol{\mathcal{L}}\boldsymbol{\alpha}_d\boldsymbol{\gamma}_d\boldsymbol{k}_t - \boldsymbol{A}_\mu\right) + \boldsymbol{\zeta}_d\log\beta\boldsymbol{B}_k + \boldsymbol{\zeta}_d\log\boldsymbol{\zeta} + \log\boldsymbol{\chi},\end{aligned}$$

or

$$\boldsymbol{k}_{t+1}\left(\boldsymbol{k}_t, \boldsymbol{z}_t\right) = \boldsymbol{\zeta}_d\boldsymbol{\Omega}'\boldsymbol{\mathcal{L}}\boldsymbol{z}_t + \left(\boldsymbol{I} - \boldsymbol{\zeta}_d + \boldsymbol{\zeta}_d\boldsymbol{\Omega}'\boldsymbol{\mathcal{L}}\boldsymbol{\alpha}_d\boldsymbol{\gamma}_d\right)\boldsymbol{k}_t + \boldsymbol{A}_k, \tag{63}$$

where $\boldsymbol{A}_k = \boldsymbol{\zeta}_d\log\beta\,\boldsymbol{B}_k - \boldsymbol{\zeta}_d\boldsymbol{\Omega}'\boldsymbol{A}_\mu + \boldsymbol{\zeta}_d\log\boldsymbol{\zeta} + \log\boldsymbol{\chi}$ is a constant.

Note that since value added TFP is given by $\boldsymbol{z}_t^Y = \boldsymbol{\gamma}_d^{-1}\boldsymbol{z}_t$, we can alternatively express these

[5] Observe then that in equations (57) and (58), $\left(\boldsymbol{I} - \boldsymbol{\Phi}\,(\boldsymbol{I}-\boldsymbol{\gamma}_d)\right)^{-1} = \boldsymbol{\mathcal{L}}'$.

policy functions as,

$$c_t\left(k_t, z_t^Y\right) = \mathcal{L}\gamma_d z_t^Y + \mathcal{L}\alpha_d\gamma_d k_t + \log\theta - A_\mu, \tag{64}$$

and

$$k_{t+1}\left(k_t, z_t^Y\right) = \zeta_d\Omega'\mathcal{L}\gamma_d z_t^Y + \left(I - \zeta_d + \zeta_d\Omega'\mathcal{L}\alpha_d\gamma_d\right)k_t + A_k. \tag{65}$$

2.3 Solving the Bellman Equation

We now solve for the unknowns, B_k, B_z and B in our conjectured functional form for the value function (47). The policy functions (62) and (63) imply that the Bellman equation takes the form,

$$\begin{aligned} B_k'k_t + B_z'z_t + B = \theta'\left(\mathcal{L}z_t + \mathcal{L}\alpha_d\gamma_d k_t + \log\theta - A_\mu\right) \\ + \beta\left(B_k'\left(\zeta_d\Omega'\mathcal{L}z_t + \left(I - \zeta_d + \zeta_d\Omega'\mathcal{L}\alpha_d\gamma_d\right)k_t + A_k\right) + B_z'z_t + B_z'g^z + B\right). \end{aligned}$$

Collecting terms associated with k_t gives us a solution for B_k,

$$B_k' = \theta'\mathcal{L}\alpha_d\gamma_d + \beta B_k'\left(I - \zeta_d + \zeta_d\Omega'\mathcal{L}\alpha_d\gamma_d\right), \tag{66}$$

or

$$B_k' = \theta'\mathcal{L}\alpha_d\gamma_d\left[I - \beta\left(I - \zeta_d + \zeta_d\Omega'\mathcal{L}\alpha_d\gamma_d\right)\right]^{-1}. \tag{67}$$

In particular, using the policy functions (62) and (63), one can show that[6]

$$B_k' = \sum_{\tau=t+1}\beta^{\tau-(t+1)}\frac{\partial c_\tau}{\partial k_{t+1}}, \tag{68}$$

where recall that c_τ is the log of aggregate consumption at date τ, $\log C_\tau$ (a scalar), and k_{t+1} is the vector of capital stocks (in logs). In other words, given that k_t is fixed in period t, B_k captures the change in households' future consumption induced by changes in investment across sectors.

Proof. From the definition of the log of aggregate consumption, we can express the RHS of

[6] Let $y = (y_1, ..., y_n)'$ and $x = (x_1, ..., x_m)'$. We define matrix derivatives using the numerical layout convention, $\left[\frac{\partial y}{\partial x}\right]_{ij} = \frac{\partial y_i}{\partial x_j}$.

equation (68) as:

$$
\begin{aligned}
\sum_{\tau=t+1} \beta^{\tau-(t+1)} \frac{\partial c_\tau}{\partial \boldsymbol{k}_{t+1}} &= \sum_{\tau=t+1} \beta^{\tau-(t+1)} \boldsymbol{\theta}' \frac{\partial \boldsymbol{c}_\tau}{\partial \boldsymbol{k}_{t+1}} \\
&= \sum_{\tau=t+1} \beta^{\tau-(t+1)} \boldsymbol{\theta}' \frac{\partial \boldsymbol{c}_\tau}{\partial \boldsymbol{k}_\tau} \frac{\partial \boldsymbol{k}_\tau}{\partial \boldsymbol{k}_{t+1}} \\
&= \sum_{\tau=t+1} \beta^{\tau-(t+1)} \boldsymbol{\theta}' \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d \frac{\partial \boldsymbol{k}_\tau}{\partial \boldsymbol{k}_{t+1}} \\
&= \boldsymbol{\theta}' \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d \sum_{\tau=t+1} \beta^{\tau-(t+1)} \frac{\partial \boldsymbol{k}_\tau}{\partial \boldsymbol{k}_{t+1}} \\
&= \boldsymbol{\theta}' \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d \sum_{\tau=t+1} \beta^{\tau-(t+1)} \left(\boldsymbol{I} - \zeta_d + \zeta_d \boldsymbol{\Omega}' \boldsymbol{\alpha}_d \gamma_d \right)^{\tau-(t+1)} \\
&= \boldsymbol{\theta}' \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d \left[\boldsymbol{I} - \beta \left(\boldsymbol{I} - \zeta_d + \zeta_d \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d \right) \right]^{-1} \\
&= \boldsymbol{B}_k'.
\end{aligned}
$$

□

Finally, given $\boldsymbol{B}_k$, it follows that $\boldsymbol{B}_z$ solves

$$\boldsymbol{B}_z' = \boldsymbol{\theta}' \boldsymbol{\mathcal{L}} + \beta \, \boldsymbol{B}_k' \zeta_d \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} + \beta \boldsymbol{B}_z', \tag{69}$$

while $\boldsymbol{B}$ solves

$$\boldsymbol{B} = \boldsymbol{\theta}' \left(\log \boldsymbol{\theta} - \boldsymbol{A}_\mu \right) + \beta \left(\boldsymbol{B}_k' \boldsymbol{A}_k + \boldsymbol{B}_z' \boldsymbol{g}^z + \boldsymbol{B} \right). \tag{70}$$

2.4 The Dynamics of Sectoral Growth Rates

As illustrated in Figure 2, sectoral TFP in practice is stationary in first differences but not in levels. Real quantities such as consumption, value added or gross output inherit these properties both at the sectoral and aggregate level, as they do in the model given the process described in equation (46). It then becomes necessary to recast the analysis of the aggregate implications of sectoral productivity disturbances in terms of growth rates. To that end, we simply take the first difference in the policy functions (62) and (63) to obtain,

$$\boxed{\Delta \boldsymbol{c}_t = \boldsymbol{\mathcal{L}} \Delta \boldsymbol{z}_t + \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d \Delta \boldsymbol{k}_t,} \tag{71}$$

and

$$\boxed{\Delta \boldsymbol{k}_{t+1} = \zeta_d \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} \Delta \boldsymbol{z}_t + \left(\boldsymbol{I} - \zeta_d + \zeta_d \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d \right) \Delta \boldsymbol{k}_t.} \tag{72}$$

In the long run, absent shocks, the economy eventually reaches a steady state in which all real quantities grow at constant rates. In particular, let Δc^* and Δk^* denote the long-run growth rate of consumption and capital in the different sectors. Then, in a non-stochastic steady state,

$\Delta \boldsymbol{z}_t = \boldsymbol{g}^z \; \forall t$ so that

$$\Delta k^* = (I - \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d)^{-1} \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} \boldsymbol{g}^z, \tag{73}$$

and

$$\Delta c^* = \boldsymbol{\mathcal{L}} \left(\boldsymbol{I} + \boldsymbol{\alpha}_d \gamma_d (I - \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d)^{-1} \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} \right) \boldsymbol{g}^z. \tag{74}$$

While sectoral growth rates differ in that steady state, the aggregate economy evolves along a balanced growth path in which all shares are constant (see Foerster et al. (2022)). More generally, given equations (71) and (72), the equilibrium trajectories of all other variables may readily be obtained. For example, $\Delta \boldsymbol{\mu}_t = -\Delta \boldsymbol{c}_t$ and so on.

2.5 Domar Weights and Sectoral Value Added Shares in GDP

We now turn to the behavior of the aggregate economy. To do so, we first need to describe the weights used in aggregation. In particular, aggregate TFP growth, denoted Δz_t, may be described either as the value-added-share-weighted sum of value-added TFP growth, $\sum_i S_{it} \Delta z^Y_{it}$, where $S_{it} = (P_{it} Q_{it} - \sum_j P_{jt} M_{jit}) / \widetilde{Y}_t$, or the Domar-weighted sum of gross output TFP growth, $\sum_i D_{it} \Delta z_{it}$, where $D_{it} = (P_{it} Q_{it}) / \widetilde{Y}_t$.

Nominal value added in sector i is defined as

$$P_{it} Q_{it} - \sum_j P_{jt} M_{jit},$$

where P_{it} is the price of sector i's goods in units of the consumption bundle, C_t (recall that C_t is the numéraire good where $C_t = \sum_i P_{it} C_{it}$). Value added in sector i is that sector's gross production net of material purchases. Nominal GDP is then

$$\begin{aligned}
\sum_i P_{it} Q_{it} - \sum_{i,j} P_{jt} M_{jit} &= \sum_i P_{it} Q_{it} - \sum_{i,j} P_{it} M_{ijt} \\
&= \sum_i P_{it} C_{it} + \sum_{i,j} P_{it} X_{ijt} \\
&= \widetilde{C}_t + \widetilde{X}_t.
\end{aligned}$$

Let Λ_t denote the marginal utility of total consumption, C_t at date t. It follows that the marginal utility of consuming good i at date t is $\mu_{it} = \Lambda_t P_{it}$. From equation (48), we can then write sector i's nominal gross output as

$$P_{it} Q_{it} = \frac{1}{\Lambda_t} \mu_{it} Q_{it} = \frac{\psi_i}{\Lambda_t},$$

Moreover, value added in sector i is

$$\begin{aligned} P_{it}Q_{it} - \sum_j P_{jt}M_{jit} &= \frac{1}{\Lambda_t}\mu_{it}Q_{it} - \sum_j \frac{1}{\Lambda_t}\mu_{jt}M_{jit} \\ &= \frac{1}{\Lambda_t}\psi_i - \frac{1}{\Lambda_t}\sum_j \psi_i(1-\gamma_i)\phi_{ji} \\ &= \frac{1}{\Lambda_t}\psi_i - \frac{1}{\Lambda_t}\psi_i(1-\gamma_i)\sum_j \phi_{ji} \\ &= \frac{1}{\Lambda_t}\gamma_i\psi_i. \end{aligned}$$

Recall from equation (57) that

$$\boldsymbol{\psi} = \boldsymbol{\mathcal{L}}' \left(\boldsymbol{\theta} + \beta\,\boldsymbol{\Omega}\,\boldsymbol{\zeta}_d\,\boldsymbol{B}_k\right),$$

and observe that $\mathbf{1}'\gamma_d\boldsymbol{\mathcal{L}}' = \mathbf{1}'$. Therefore, nominal GDP is

$$\begin{aligned} \widetilde{Y}_t &= \frac{1}{\Lambda_t}\mathbf{1}'\gamma_d\boldsymbol{\mathcal{L}}' \left(\boldsymbol{\theta} + \beta\,\boldsymbol{\Omega}\,\boldsymbol{\zeta}_d\,\boldsymbol{B}_k\right) \\ &= \frac{1}{\Lambda_t}\mathbf{1}' \left(\boldsymbol{\theta} + \beta\,\boldsymbol{\Omega}\,\boldsymbol{\zeta}_d\,\boldsymbol{B}_k\right) \\ &= \frac{1}{\Lambda_t}\left(1 + \beta\,\boldsymbol{\zeta}'\,\boldsymbol{B}_k\right). \end{aligned}$$

Sector i's value added share of GDP is then given by

$$\boldsymbol{S} = \left[\frac{\gamma_d\boldsymbol{\psi}}{1 + \beta\,\boldsymbol{\zeta}'\,\boldsymbol{B}_k}\right]_i = \left[\frac{\gamma_d\boldsymbol{\mathcal{L}}' \left(\boldsymbol{\theta} + \beta\,\boldsymbol{\Omega}\,\boldsymbol{\zeta}_d\,\boldsymbol{B}_k\right)}{1 + \beta\,\boldsymbol{\zeta}'\,\boldsymbol{B}_k}\right]_i,$$

while sector i's Domar weight, D_i, is

$$\begin{aligned} D_i &= \frac{P_{it}Q_{it}}{P_{it}Q_{it} - \sum_j P_{jt}M_{jit}} \times \frac{P_{it}Q_{it} - \sum_j P_{jt}M_{jit}}{\widetilde{Y}_t}, \\ &= \frac{(1/\Lambda_t)\psi_i}{(1/\Lambda_t)\gamma_i\psi_i} \times S_{it}, \\ &= \frac{1}{\gamma_i}S_{it}, \end{aligned}$$

where γ_i is then sector i's value added share in gross output. In vector form,

$$\boxed{\boldsymbol{D} = \gamma_d^{-1}\boldsymbol{S}.} \tag{75}$$

We can express Domar Weights as

$$\begin{aligned}
\boldsymbol{D} &= \boldsymbol{D}^c + \boldsymbol{D}^x \\
&= \frac{\boldsymbol{\mathcal{L}}'\boldsymbol{\theta}}{1+\beta\,\boldsymbol{\zeta}'\,\boldsymbol{B}_k} + \frac{\boldsymbol{\mathcal{L}}'\beta\boldsymbol{\Omega}\boldsymbol{\zeta}_d\,\boldsymbol{B}_k}{1+\beta\,\boldsymbol{\zeta}'\,\boldsymbol{B}_k} \\
&= \frac{\widetilde{C}_t}{\widetilde{Y}_t}\frac{\partial c_t}{\partial \boldsymbol{z}_t} + \frac{\widetilde{C}_t}{\widetilde{Y}_t}\sum_{\tau=t+1}\beta^{\tau-t}\frac{\partial c_\tau}{\partial \boldsymbol{z}_t},
\end{aligned}$$

which gives us corollary 2 in this economy. In particular, observe that, in this case, $S_{ct} = S_c\ \forall t$,

$$\begin{aligned}
\left(\prod_{t''=t}^{t'-1}\frac{1}{1+r_{t''}}\right)\left(\frac{\widetilde{Y}_{t'}}{\widetilde{Y}_t}\right) &= \beta^{t'-t}\frac{\Lambda_{t'}}{\Lambda_t}\frac{\Lambda_t}{\Lambda_{t'}} \\
&= \beta^{t'-t},
\end{aligned}$$

so that the first term in the expression for D_{it}^x in corollary 2 collapses to

$$S_c\sum_{t'=t+1}^{t+T}\beta^{t'-t}\frac{d\log C_{t'}}{d\log Z_{it}},$$

while the last term vanishes by the transversality condition.

Proof. Domar weights are given by

$$\begin{aligned}
\boldsymbol{D} &= \frac{\boldsymbol{\mathcal{L}}'\left(\boldsymbol{\theta}+\beta\boldsymbol{\Omega}\boldsymbol{\zeta}_d\,\boldsymbol{B}_k\right)}{1+\beta\,\boldsymbol{\zeta}'\,\boldsymbol{B}_k} \\
&= \frac{\boldsymbol{\mathcal{L}}'\boldsymbol{\theta}}{1+\beta\,\boldsymbol{\zeta}'\,\boldsymbol{B}_k} + \frac{\boldsymbol{\mathcal{L}}'\beta\boldsymbol{\Omega}\boldsymbol{\zeta}_d\,\boldsymbol{B}_k}{1+\beta\,\boldsymbol{\zeta}'\,\boldsymbol{B}_k}.
\end{aligned}$$

$\widetilde{C}_t = \frac{1}{\Lambda_t}$ while $\widetilde{Y}_t$ is given by $\widetilde{C}_t + \widetilde{X}_t = \frac{1}{\Lambda_t}(1+\beta\boldsymbol{\zeta}'\boldsymbol{B_k})$. Then

$$\begin{aligned}
\boldsymbol{D}^{c\prime} &= \frac{\widetilde{C}_t}{\widetilde{Y}_t}\frac{\partial c_t}{\partial \boldsymbol{z}_t} \\
&= \frac{\frac{1}{\Lambda_t}}{\frac{1}{\Lambda_t}(1+\beta\boldsymbol{\zeta}'\boldsymbol{B_k})}\boldsymbol{\theta}'\frac{\partial \boldsymbol{c}_t}{\partial \boldsymbol{z}_t} \\
&= \frac{1}{1+\beta\boldsymbol{\zeta}'\boldsymbol{B_k}}\boldsymbol{\theta}'\boldsymbol{\mathcal{L}}, \quad \text{by equation (64).}
\end{aligned}$$

In addition, using equations (64) and (65), we have that

$$
\begin{aligned}
\boldsymbol{D}^{x\prime} &= \frac{\widetilde{C}_t}{\widetilde{Y}_t} \sum_{\tau=t+1} \beta^{\tau-t} \frac{\partial c_\tau}{\partial \boldsymbol{z}_t} \\
&= \frac{1}{1+\beta\boldsymbol{\zeta}'\boldsymbol{B}_k} \sum_{\tau=t+1} \beta^{\tau-t} \boldsymbol{\theta}' \frac{\partial \boldsymbol{c}_\tau}{\partial \boldsymbol{z}_t} \\
&= \frac{1}{1+\beta\boldsymbol{\zeta}'\boldsymbol{B}_k} \sum_{\tau=t+1} \beta^{\tau-t} \boldsymbol{\theta}' \frac{\partial \boldsymbol{c}_\tau}{\partial \boldsymbol{k}_\tau} \frac{\partial \boldsymbol{k}_\tau}{\partial \boldsymbol{k}_{t+1}} \frac{\partial \boldsymbol{k}_{t+1}}{\partial \boldsymbol{z}_t} \\
&= \frac{1}{1+\beta\boldsymbol{\zeta}'\boldsymbol{B}_k} \boldsymbol{\theta}' \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d \beta \sum_{\tau=t+1} \beta^{\tau-(t+1)} \frac{\partial \boldsymbol{k}_\tau}{\partial \boldsymbol{k}_{t+1}} \frac{\partial \boldsymbol{k}_{t+1}}{\partial \boldsymbol{z}_t} \\
&= \frac{1}{1+\beta\boldsymbol{\zeta}'\boldsymbol{B}_k} \boldsymbol{\theta}' \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d \beta \left[\sum_{\tau=t+1} \beta^{\tau-(t+1)} \left(\boldsymbol{I} - \boldsymbol{\zeta}_d + \boldsymbol{\zeta}_d \boldsymbol{\Omega}' \boldsymbol{\alpha}_d \gamma_d \right)^{\tau-(t+1)} \right] \boldsymbol{\zeta}_d \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} \\
&= \frac{1}{1+\beta\boldsymbol{\zeta}'\boldsymbol{B}_k} \beta \boldsymbol{\theta}' \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d \left[\boldsymbol{I} - \beta \left(\boldsymbol{I} - \boldsymbol{\zeta}_d + \boldsymbol{\zeta}_d \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d \right) \right]^{-1} \boldsymbol{\zeta}_d \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} \\
&= \frac{\beta \boldsymbol{B}_k' \boldsymbol{\zeta}_d \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}}}{1+\beta\boldsymbol{\zeta}'\boldsymbol{B}_k}.
\end{aligned}
$$

□

Because household welfare is now defined over time, Domar weights explicitly capture both static and forward looking components. The static component, $\boldsymbol{D}^c$, reflects intermediate input linkages, $\mathcal{L} = \left(\boldsymbol{I} - (\boldsymbol{I} - \gamma_d)\, \boldsymbol{\Phi}'\right)^{-1}$, summarized as the *influence vector* in Acemoglu et al. (2012), and captures the effects of sectoral productivity shocks on current utility. The forward-looking component, $\boldsymbol{D}^x$, reflects investment flow linkages, $\boldsymbol{\Omega}'\mathcal{L}$, and captures the effects of sectoral productivity shocks on future utility. In addition, since $\mathcal{V}_t \equiv \sum_{\tau=t}^{\infty} \beta^{\tau-t} L_\tau \mathcal{U}\left(C_\tau / L_\tau\right)$, it also follows that

$$
\begin{aligned}
\boldsymbol{D} &= S_c \sum_{\tau=t} \beta^{\tau-t} \frac{\partial c_\tau}{\partial \boldsymbol{z}_t} \\
&= S_c \frac{\partial \mathcal{V}}{\partial \boldsymbol{z}_t},
\end{aligned}
$$

as in proposition 1 and equation (43) with $L_t = 1$.

3 Dynamic Aggregate Effects of Sectoral Productivity Disturbances

3.1 Real GDP Growth

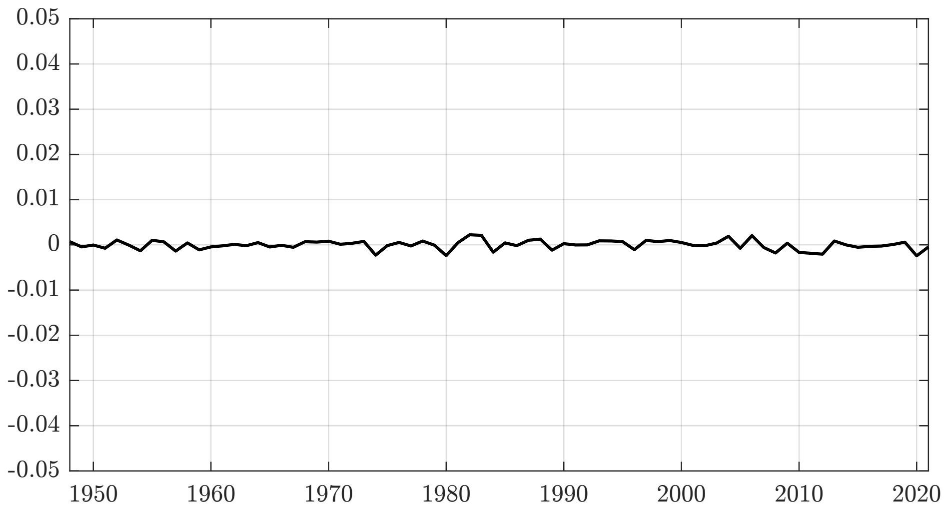
The growth rate of aggregate real GDP is

$$
\boxed{\Delta y_t = \Delta \log\left(\widetilde{C}_t + \widetilde{X}_t\right) - \sum_i S_{it} \Delta p^Y_{i,t},} \tag{76}
$$

where p_{it}^{Y} is the (log) deflator corresponding to nominal value added in sector i, where

$$\begin{aligned} \boldsymbol{p}^{Y} &= \gamma_d^{-1}(\boldsymbol{I} - (\boldsymbol{I} - \gamma_d)\boldsymbol{\Phi}')\boldsymbol{p} \\ &= \gamma_d^{-1}\boldsymbol{\mathcal{L}}^{-1}\boldsymbol{p}. \end{aligned}$$

Figure 2: Model Implied Path of $\Delta p_t^{Y} \equiv \sum_i S_{it}\Delta p_{i,t}^{Y}$



Consider first the term, $\Delta \log\left(\widetilde{C}_t + \widetilde{X}_t\right)$, where

$$\widetilde{C}_t + \widetilde{X}_t = \frac{1}{\Lambda_t}\left(1 + \beta\boldsymbol{\zeta}'\boldsymbol{B}_k\right)$$

so that

$$\Delta \log\left(\widetilde{C}_t + \widetilde{X}_t\right) = -\Delta \log \Lambda_t = -\Delta\lambda_t,$$

where

$$\lambda_t = \log \Lambda_t = -\log C_t = -\sum_{i=1}^{I} \theta_i\, c_{it} = -\boldsymbol{\theta}'\boldsymbol{c}_t. \tag{77}$$

Next, we turn our attention to the term, S_{it}. Nominal value added shares in GDP are

$$S_{it} = \frac{\left[\gamma_d\boldsymbol{\mathcal{L}}'\left(\boldsymbol{\theta} + \beta\boldsymbol{\Omega}\boldsymbol{\zeta}_d\,\boldsymbol{B}_k\right)\right]_i}{1 + \beta\,\boldsymbol{\zeta}'\,\boldsymbol{B}_k}. \tag{78}$$

Finally, sectoral gross output price changes follow $\Delta\boldsymbol{p}_t = \Delta \log \boldsymbol{\mu}_t + \mathbf{1}\Delta \log C_t$ so that, from the policy functions (62) and (63),

$$\begin{aligned} \Delta\boldsymbol{p}_t &= -\boldsymbol{\mathcal{L}}\Delta\boldsymbol{z}_t - \boldsymbol{\mathcal{L}}\boldsymbol{\alpha}_d\gamma_d\Delta\boldsymbol{k}_t + \mathbf{1}\boldsymbol{\theta}'\Delta\boldsymbol{c}_t \\ &= \left(\mathbf{1}\boldsymbol{\theta}' - \boldsymbol{I}\right)\boldsymbol{\mathcal{L}}\Delta\boldsymbol{z}_t + \left(\mathbf{1}\boldsymbol{\theta}' - \boldsymbol{I}\right)\boldsymbol{\mathcal{L}}\boldsymbol{\alpha}_d\gamma_d\Delta\boldsymbol{k}_t. \end{aligned}$$

Therefore, (log) changes in sectoral value added prices are given by

$$\Delta \boldsymbol{p}_t^Y = \gamma_d^{-1} \boldsymbol{\mathcal{L}}^{-1} \left(\mathbf{1}\boldsymbol{\theta}' - \boldsymbol{I}\right) \boldsymbol{\mathcal{L}} \Delta \boldsymbol{z}_t + \gamma_d^{-1} \boldsymbol{\mathcal{L}}^{-1} \left(\mathbf{1}\boldsymbol{\theta}' - \boldsymbol{I}\right) \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d \Delta \boldsymbol{k}_t. \tag{79}$$

Putting together equations (77), (78), and (79), we obtain an expression for real GDP growth in terms of the state variables, $\boldsymbol{k}_t$, and $\boldsymbol{z}_t^Y$,

$$\begin{aligned}
\Delta y_t = {} & \boldsymbol{\theta}' \boldsymbol{\mathcal{L}} \Delta \boldsymbol{z}_t + \boldsymbol{\theta}' \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d \Delta \boldsymbol{k}_t \\
& - \frac{\left(\boldsymbol{\theta} + \beta \boldsymbol{\Omega} \boldsymbol{\zeta}_d \boldsymbol{B}_k\right)' \boldsymbol{\mathcal{L}} \gamma_d}{1 + \beta \boldsymbol{\zeta}' \boldsymbol{B}_k} \gamma_d^{-1} \boldsymbol{\mathcal{L}}^{-1} \left(\mathbf{1}\boldsymbol{\theta}' - \boldsymbol{I}\right) \boldsymbol{\mathcal{L}} \Delta \boldsymbol{z}_t \\
& - \frac{\left(\boldsymbol{\theta} + \beta \boldsymbol{\Omega} \boldsymbol{\zeta}_d \boldsymbol{B}_k\right)' \boldsymbol{\mathcal{L}} \gamma_d}{1 + \beta \boldsymbol{\zeta}' \boldsymbol{B}_k} \gamma_d^{-1} \boldsymbol{\mathcal{L}}^{-1} \left(\mathbf{1}\boldsymbol{\theta}' - \boldsymbol{I}\right) \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d \Delta \boldsymbol{k}_t,
\end{aligned}$$

which simplifies to

$$\begin{aligned}
\Delta y_t = {} & \boldsymbol{\theta}' \boldsymbol{\mathcal{L}} \Delta \boldsymbol{z}_t + \boldsymbol{\theta}' \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d \Delta \boldsymbol{k}_t \\
& - \frac{\left(\boldsymbol{\theta} + \beta \boldsymbol{\Omega} \boldsymbol{\zeta}_d \boldsymbol{B}_k\right)'}{1 + \beta \boldsymbol{\zeta}' \boldsymbol{B}_k} \left(\mathbf{1}\boldsymbol{\theta}' - \boldsymbol{I}\right) \boldsymbol{\mathcal{L}} \Delta \boldsymbol{z}_t \\
& - \frac{\left(\boldsymbol{\theta} + \beta \boldsymbol{\Omega} \boldsymbol{\zeta}_d \boldsymbol{B}_k\right)'}{1 + \beta \boldsymbol{\zeta}' \boldsymbol{B}_k} \left(\mathbf{1}\boldsymbol{\theta}' - \boldsymbol{I}\right) \boldsymbol{\mathcal{L}} \boldsymbol{\alpha}_d \gamma_d \Delta \boldsymbol{k}_t,
\end{aligned}$$

or, finally,

$$\begin{aligned}
\Delta y_t &= \frac{\left(\boldsymbol{\theta} + \beta \boldsymbol{\Omega} \boldsymbol{\zeta}_d \boldsymbol{B}_k\right)' \boldsymbol{\mathcal{L}}}{1 + \beta \boldsymbol{\zeta}' \boldsymbol{B}_k} \Delta \boldsymbol{z}_t + \frac{\left(\boldsymbol{\theta} + \beta \boldsymbol{\Omega} \boldsymbol{\zeta}_d \boldsymbol{B}_k\right)' \boldsymbol{\mathcal{L}}}{1 + \beta \boldsymbol{\zeta}' \boldsymbol{B}_k} \gamma_d \boldsymbol{\alpha}_d \Delta \boldsymbol{k}_t \\
&= \boldsymbol{D}' \Delta \boldsymbol{z}_t + \boldsymbol{D}' \boldsymbol{\alpha}_d \Delta \boldsymbol{k}_t.
\end{aligned} \tag{80}$$

3.2 The Dynamics of Real GDP Growth and Hulten's Theorem

More generally, we can summarize the evolution of real GDP growth as,

$$\Delta y_t = \boldsymbol{D}' \Delta \boldsymbol{z}_t + \boldsymbol{D}' \gamma_d \boldsymbol{\alpha}_d \Delta \boldsymbol{k}_t, \tag{81a}$$

$$\Delta \boldsymbol{k}_{t+1} = \boldsymbol{\zeta}_d \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} \Delta \boldsymbol{z}_t + \left(\boldsymbol{I} - \boldsymbol{\zeta}_d + \boldsymbol{\zeta}_d \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} \gamma_d \boldsymbol{\alpha}_d\right) \Delta \boldsymbol{k}_t, \tag{81b}$$

where

$$\boxed{\boldsymbol{D} = \frac{\boldsymbol{\mathcal{L}}' \left(\boldsymbol{\theta} + \beta \boldsymbol{\Omega} \boldsymbol{\zeta}_d \boldsymbol{B}_k\right)}{1 + \beta \boldsymbol{\zeta}' \boldsymbol{B}_k}.}$$

We recover in these equations corollary 1[7],

$$\frac{\partial \Delta y_t}{\partial \Delta \boldsymbol{z}_t} = \boldsymbol{D}'. \tag{82}$$

[7] Alternatively, observe that $\partial \Delta y_t / \partial \Delta \boldsymbol{z}_t^Y = \boldsymbol{S}$.

Gabaix (2011), for example, approximates real GDP growth as an application of Hulten's theorem in every period, so just the first term in equation (81a), $\Delta y_t = \boldsymbol{D}'\Delta \boldsymbol{z}_t$.

Beyond the contemporaneous aggregate effects of sectoral disturbances, equations (81a) and (81b) describe the entire stochastic path of real GDP growth.

Observe also that, from equations (81a) and (81b),

$$\boxed{\frac{\partial \Delta y_t}{\partial \Delta \boldsymbol{k}_t} = \boldsymbol{D}'\boldsymbol{\gamma}_d\boldsymbol{\alpha}_d = \boldsymbol{S}\boldsymbol{\alpha}_d.} \tag{83}$$

In other words, beyond Hulten's insight regarding the aggregate implications of changes in sectoral productivity, Domar weights, adjusted by sectoral value added shares in gross output and sectoral capital shares, $D_i\gamma_i\alpha_i$ for sector i, also capture the aggregate contemporaneous effects of exogenous changes in the existing capital stock, e.g., an extreme weather event that damages or destroys structures.

More generally, we can characterize the dynamics of real GDP growth at date t in terms of the history of past sectoral TFP growth only,

$$\boxed{\Delta y_t = \boldsymbol{D}'\Delta \boldsymbol{z}_t + \sum_{h=0}^{\infty} \boldsymbol{D}'\boldsymbol{\gamma}_d\boldsymbol{\alpha}_d \left(\boldsymbol{I} - \boldsymbol{\zeta}_d + \boldsymbol{\zeta}_d\boldsymbol{\Omega}'\boldsymbol{\mathcal{L}}\boldsymbol{\gamma}_d\boldsymbol{\alpha}_d\right)^h \boldsymbol{\zeta}_d\boldsymbol{\Omega}'\boldsymbol{\mathcal{L}}\Delta \boldsymbol{z}_{t-1-h}.} \tag{84}$$

4 Unanticipated Sectoral TFP Shocks and GDP Growth

Equation (84) implies that the effects of sectoral productivity shocks at t on GDP growth at horizons $t+h, h = 0, 1, ...$ are given by the following impulse response functions,

$$\frac{\partial \Delta y_{t+h}}{\partial \Delta z_{it}} = \begin{cases} \boldsymbol{D}', & if\ h = 0 \\ \boldsymbol{D}'\boldsymbol{\gamma}_d\boldsymbol{\alpha}_d\boldsymbol{\mathcal{J}}^{h-1}\boldsymbol{\zeta}_d\boldsymbol{\Omega}'\boldsymbol{\mathcal{L}}, & if\ h > 0 \end{cases}, \tag{85}$$

where $\boldsymbol{\mathcal{J}} = (\boldsymbol{I} - \boldsymbol{\zeta}_d + \boldsymbol{\zeta}_d\boldsymbol{\Omega}'\boldsymbol{\mathcal{L}}\boldsymbol{\gamma}_d\boldsymbol{\alpha}_d)$. These impulse response functions then determine the forecast errors in real GDP growth associated with unanticipated sectoral productivity shocks, $\Delta \boldsymbol{z}_t = \boldsymbol{\epsilon}_t$, where $\boldsymbol{\epsilon}_t$ is i.i.d. $\sim\ N(0, \boldsymbol{\Sigma}_{\epsilon\epsilon})$, given by $\Delta y_{t+h} - E_{t-1}(\Delta y_{t+h}|\Delta z_{it} = \epsilon_{it}) = \frac{\partial \Delta y_{t+h}}{\partial \Delta z_{it}}\epsilon_{it}$.

We can use equation (85) to develop analytical intuition for the responses in GDP growth following sectoral productivity shocks. In particular, we focus on responses up to 2 years after the occurence of a shock in some sector, i, $\partial \Delta y_{t+h}/\partial \Delta z_{it}$, $h = 0, 1, 2$, where

$$\begin{aligned} \frac{\partial \Delta y_t}{\partial \Delta z_{it}} &= [\boldsymbol{D}]_i \\ \frac{\partial \Delta y_{t+1}}{\partial \Delta z_{it}} &= [\boldsymbol{D}'\boldsymbol{\gamma}_d\boldsymbol{\alpha}_d\boldsymbol{\zeta}_d\boldsymbol{\Omega}'\boldsymbol{\mathcal{L}}]_i \\ \frac{\partial \Delta y_{t+2}}{\partial \Delta z_{it}} &= [\boldsymbol{D}'\boldsymbol{\gamma}_d\boldsymbol{\alpha}_d \left(\boldsymbol{I} - \boldsymbol{\zeta}_d + \boldsymbol{\zeta}_d\boldsymbol{\Omega}'\boldsymbol{\mathcal{L}}\boldsymbol{\gamma}_d\boldsymbol{\alpha}_d\right) \boldsymbol{\zeta}_d\boldsymbol{\Omega}'\boldsymbol{\mathcal{L}}]_i. \end{aligned}$$

Given a shock to productivity growth in any sector, i, at date t, the response of GDP growth at date $t+1$ is a weighted sum of *all* Domar weights, $\boldsymbol{D}$, irrespective of the sector in which the shock takes place. Therefore, whereas $\partial \Delta y_t / \partial \Delta z_{it} = D_i$, in the following period,

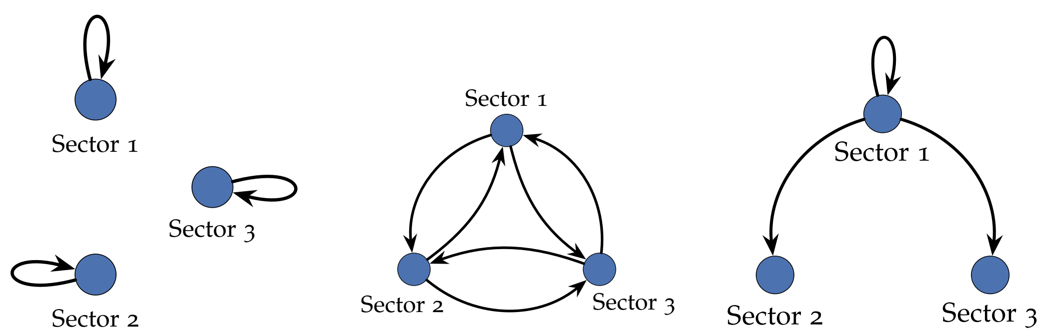
$$\frac{d\,\Delta y_{t+1}}{d\,\Delta z_{it}} = \sum_{j=1}^{N} \underbrace{\left(\zeta_j \gamma_j \alpha_j\right)}_{\substack{\text{production attributes} \\ \text{of downstream} \\ \text{sectors}}} \quad \underbrace{D_j}_{\substack{\text{relative size} \\ \text{of downstream} \\ \text{sectors}}} \quad \underbrace{\left[\Omega' \mathcal{L}\right]_{ji}}_{\substack{\text{dynamic} \\ \text{network effects}}} . \tag{86}$$

where

$$\boldsymbol{\mathcal{L}} = \left(\boldsymbol{I} - \left(\boldsymbol{I} - \gamma_d\right) \boldsymbol{\Phi}'\right)^{-1} = \begin{pmatrix} 1-(1-\gamma_1)\phi_{11} & -(1-\gamma_1)\phi_{21} & \dots & -(1-\gamma_N)\phi_{N1} \\ -(1-\gamma_2)\phi_{12} & \dots & & -(1-\gamma_2)\phi_{N2} \\ \dots-(1-\gamma_N)\phi_{1N} & \dots & & 1-(1-\gamma_N)\phi_{NN} \end{pmatrix}.$$

Recall also that $D_j\gamma_j$ is sector j' value added share in GDP, S_j, (or alternatively its relative size in the economy).

Figure 3: **Capital and Intermediate Input Flows, Ω and Φ**



(a) **Sector-Specific Inputs** (b) **Dense Production Networks** (c) **Star Supplier Networks**

4.1 Special Cases

To gain intuition, this section illustrates how the aggregate effects of sectoral shocks in the periods following the shock by way of special cases that cover extreme production networks. Given static production linkages in intermediate inputs, Φ, and dynamic production linkages in investment, Ω, it is sufficient to consider only 3 sectors. To help with transparency, we set $\alpha_i = \alpha$, $\gamma_i = \gamma$, $\zeta_i = \zeta \forall i$. Recall that in each sector, the Domar weight scales that sector's value added share in GDP by its ratio of gross output to value added, $D_i = (1/\gamma_i)S_i$. Then, under these assumptions, $\sum_i D_i = (1/\gamma)\sum_i S_i = 1/\gamma$.

1. **Sector-specific inputs:**

Figure 3a, illustrates the case where every sector produces its own inputs, both in terms of materials and investment goods. When inputs are sector-specific in both production networks, we have $\boldsymbol{\Phi} = \boldsymbol{\Omega} = \boldsymbol{I}$. Then, $\gamma_d \boldsymbol{\alpha}_d \boldsymbol{\zeta}_d \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} = \alpha\zeta \boldsymbol{I}$ so that

$$\frac{\partial \Delta y_{t+1}}{\partial \Delta z_{it}} = \alpha\zeta \sum_j D_j = \frac{\alpha\zeta}{\gamma} \ \forall i.$$

2. **Dense supplier networks:**

Figure 3b illustrates the case where every sector is equally important as an input supplier to every other sector, both in materials and investment. Then we have that $[\boldsymbol{\Phi}]_{ij} = [\boldsymbol{\Omega}]_{ij} = 1/3 \forall i, j$, and,

$$\left[\gamma_d \boldsymbol{\alpha}_d \boldsymbol{\zeta}_d \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}}\right]_{ij} = \frac{\alpha\zeta}{3} \ \forall i, j \ \text{ and } \ \frac{\partial \Delta y_{t+1}}{\partial \Delta z_{it}} = \frac{\alpha\zeta}{3} \sum_j D_j = \frac{\alpha\zeta}{3\gamma} \ \forall i.$$

Therefore, in an economy with completely dense production networks, the aggregate implications of sectoral shocks are similar to one in which sectors' operations are entirely independent of each other.

3. **Symmetric key supplier networks:**

At the opposite extreme of an economy where every sector is equally important to all other sectors, is an economy where a single sector is the key supplier of inputs to all other sectors, sometimes referred to as a star network. In particular, in this example, we assume that sector 1 is the sole supplier of both intermediate inputs and investment so that

$$\boldsymbol{\Phi} = \boldsymbol{\Omega} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case,

$$\gamma_d \boldsymbol{\alpha}_d \boldsymbol{\zeta}_d \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} = \begin{pmatrix} \alpha\zeta & 0 & 0 \\ \alpha\zeta & 0 & 0 \\ \alpha\zeta & 0 & 0 \end{pmatrix},$$

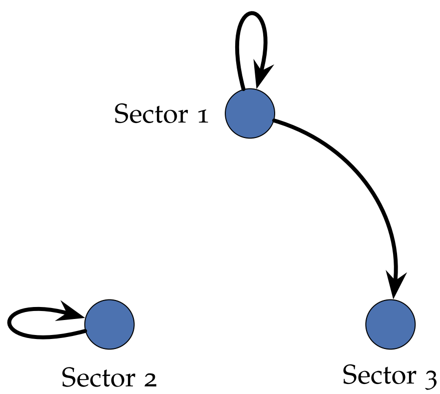
so that

$$\frac{\partial \Delta y_{t+1}}{\partial \Delta z_{it}} = \begin{cases} \frac{\alpha\zeta}{\gamma} & if\ i = 1, \\ 0 & if\ i = 2, 3 \end{cases}.$$

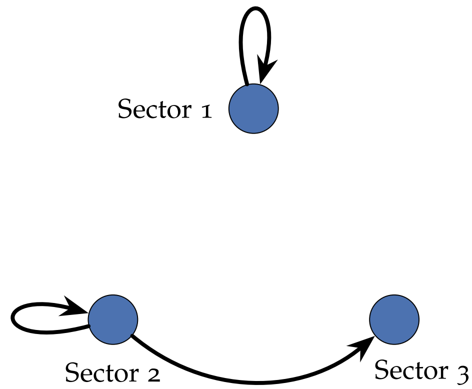
In summary, in all of these cases, no matter what the distribution of Domar weights is, and thus the distribution of current responses in aggregate GDP growth to sectoral shocks, the response of GDP growth in the following period is identical across sectors.

4. **Asymmetric key supplier networks:**

Figure 4: **Asymmetric Key Supplier Networks**



(a) **Capital Input Flows, Ω**



(b) **Intermediate Input Flows, Φ**

In this scenario, sector 1 is a key supplier of investment, not unlike the construction sector in practice, producing all capital goods for sector 3 and itself. Sector 2 is a key supplier of intermediate inputs, not unlike the Wholesale & Retail Trade in U.S. production, providing all intermediate inputs for sector 3 and itself. Finally, Sector 3 plays the role of a 'stand-in' for all other sectors, produces no inputs, and is responsible for all final demand in the economy. To further simplify the environment, we consider the limit case where $\beta \to 1$ and $\zeta_i \to 1 \ \forall i$.

These production networks imply sectoral input shares summarized by,

$$\boldsymbol{\Omega} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solving for the Domar weights:

$$\boldsymbol{D} = \frac{\boldsymbol{\mathcal{L}}'\left(\boldsymbol{\theta} + \beta \boldsymbol{\Omega} \boldsymbol{\zeta}_d \boldsymbol{B}_k\right)}{1 + \beta \boldsymbol{\zeta}' \boldsymbol{B}_k},$$

where

$$\boldsymbol{B}_k = \left[\boldsymbol{I} - \beta\left(\boldsymbol{I} - \boldsymbol{\zeta}_d + \gamma_d \boldsymbol{\alpha}_d \boldsymbol{\mathcal{L}}' \boldsymbol{\Omega} \boldsymbol{\zeta}_d\right)\right]^{-1} \gamma_d \boldsymbol{\alpha}_d \boldsymbol{\mathcal{L}}' \boldsymbol{\theta}$$

Under the maintained production network assumptions,

$$\boldsymbol{\mathcal{L}} = \begin{pmatrix} \frac{1}{\gamma} & 0 & 0 \\ 0 & \frac{1}{\gamma} & 0 \\ 0 & \frac{1-\gamma}{\gamma} & 1 \end{pmatrix},$$

while

$$\left[\boldsymbol{I} - \beta\left(\boldsymbol{I} - \boldsymbol{\zeta}_d + \gamma_d \boldsymbol{\alpha}_d \boldsymbol{\mathcal{L}}' \boldsymbol{\Omega} \boldsymbol{\zeta}_d\right)\right]^{-1} = \begin{pmatrix} \frac{1}{1-\alpha} & 0 & \frac{\alpha}{1-\alpha} \\ 0 & \frac{1}{1-\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \gamma_d \boldsymbol{\alpha}_d \boldsymbol{\mathcal{L}}' \boldsymbol{\theta} = \begin{pmatrix} 0 \\ \alpha(1-\gamma) \\ \alpha\gamma \end{pmatrix},$$

so that

$$\boldsymbol{B}_k = \begin{pmatrix} \frac{\alpha^2\gamma}{1-\alpha} \\ \frac{\alpha(1-\gamma)}{1-\alpha} \\ \alpha\gamma \end{pmatrix}.$$

Then, we have that

$$\begin{aligned} \boldsymbol{D} &= \boldsymbol{D}^c + \boldsymbol{D}^x \\ &= \frac{\boldsymbol{\mathcal{L}}'\boldsymbol{\theta}}{1+\beta\,\boldsymbol{\zeta}'\,\boldsymbol{B}_k} + \frac{\beta \boldsymbol{\mathcal{L}}' \boldsymbol{\Omega} \boldsymbol{\zeta}_d\, \boldsymbol{B}_k}{1+\beta\,\boldsymbol{\zeta}'\,\boldsymbol{B}_k} \end{aligned}$$

Observe that $1 + \beta \boldsymbol{\zeta}' \boldsymbol{B}_k = \frac{1}{1-\alpha}$ and, since $\boldsymbol{\theta} = (0\ 0\ 1)'$,

$$\boldsymbol{D}^c = \frac{\boldsymbol{\mathcal{L}}'\boldsymbol{\theta}}{1+\beta\,\boldsymbol{\zeta}'\,\boldsymbol{B}_k} = \begin{pmatrix} 0 \\ \frac{(1-\alpha)(1-\gamma)}{\gamma} \\ 1-\alpha \end{pmatrix}.$$

In addition,

$$\boldsymbol{\mathcal{L}}' \boldsymbol{\Omega} \boldsymbol{\zeta}_d \boldsymbol{B}_k = \begin{pmatrix} \frac{\alpha}{1-\alpha} \\ \frac{\alpha(1-\gamma)}{(1-\alpha)\gamma} \\ 0 \end{pmatrix},$$

and

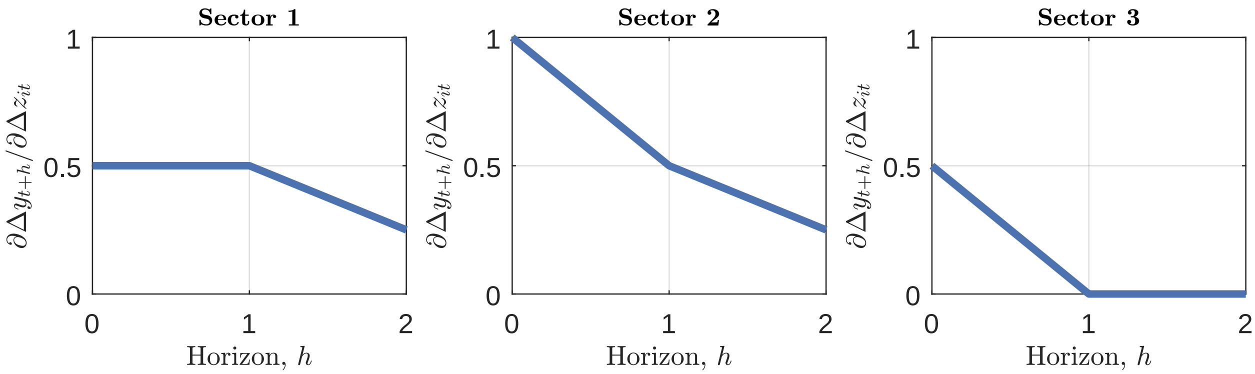
$$\boldsymbol{D}^x = \begin{pmatrix} \alpha \\ \frac{\alpha(1-\gamma)}{\gamma} \\ 0 \end{pmatrix}.$$

Distribution of Responses in GDP Growth to Sectoral Shocks:

It follows that

$$\begin{aligned} \frac{\partial \Delta y_t}{\partial \Delta \boldsymbol{z}_t} &= \boldsymbol{D}' \\ &= \boldsymbol{D}^{c\prime} + \boldsymbol{D}^{x\prime} \\ &= \begin{pmatrix} 0 & \frac{(1-\alpha)(1-\gamma)}{\gamma} & 1-\alpha \end{pmatrix} + \begin{pmatrix} \alpha & \frac{\alpha(1-\gamma)}{\gamma} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \frac{1-\gamma}{\gamma} & 1-\alpha \end{pmatrix}. \end{aligned}$$

Figure 5: Asymmetric Key Supplier Impulse Responses



Notes: Each subplot tracks the impulse response of aggregate GDP growth to a unit innovation in the TFP growth of each sector in the asymmetric key supplier network where $\alpha = \gamma = 1/2$, $\beta \to 1$ and $\zeta \to 1$. Sector 1 plays the role of a key supplier of investment goods, Sector 2 the role of a key supplier of intermediate inputs, and sector 3 the role of a downstream user that produces all final consumption. Compare with **??**.

Furthermore,

$$\frac{\partial \Delta y_{t+1}}{\partial \Delta \boldsymbol{z}_t} = \boldsymbol{D}' \gamma_d \boldsymbol{\alpha}_d \zeta_d \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} = \begin{pmatrix} \alpha & \frac{\alpha(1-\gamma)}{\gamma} & 0 \end{pmatrix}$$

while

$$\begin{aligned} \frac{\partial \Delta y_{t+2}}{\partial \Delta \boldsymbol{z}_t} &= \boldsymbol{D}' \gamma_d \boldsymbol{\alpha}_d \left(\boldsymbol{I} - \zeta_d + \zeta_d \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} \gamma_d \boldsymbol{\alpha}_d \right) \zeta_d \boldsymbol{\Omega}' \boldsymbol{\mathcal{L}} \\ &= \begin{pmatrix} \alpha^2 & \frac{\alpha^2(1-\gamma)}{\gamma} & 0 \end{pmatrix}. \end{aligned}$$

Consider the case $\alpha = \gamma = 1/2$, which turns out to be close to the empirical mean input shares for U.S. production at $\alpha = 0.43$ and $\gamma = 0.54$ across sectors. Then the impulse responses of real GDP growth to unit shocks in sectoral productivity growth are given by Figure 5. Moreover,

$$\begin{aligned} \frac{\partial \Delta y_{t+1} / \partial \Delta \boldsymbol{z}_t}{\partial \Delta y_t / \partial \Delta \boldsymbol{z}_t} &= \begin{pmatrix} 1 & 1/2 & 0 \end{pmatrix}, \\ \text{and } \frac{\partial \Delta y_{t+2} / \partial \Delta \boldsymbol{z}_t}{\partial \Delta y_t / \partial \Delta \boldsymbol{z}_t} &= \begin{pmatrix} 1/2 & 1/4 & n/a \end{pmatrix}. \end{aligned}$$

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