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Technology Adoption and Optimal Policy

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Technology Adoption and Optimal Policy *

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Abstract

We study optimal policy in a dynamic general equilibrium model where heterogeneous monopolistic competitive firms pay a fixed cost to adopt an exogenously growing frontier technology. Using Mean Field Games tools, we show that the optimal policy consists of two time-invariant subsidies: one correcting static misallocation, and one correcting the dynamic under-incentive to adopt. This holds outside of balanced growth paths, for any initial distribution of technology gaps. We analyze a version of the model that aggregates to a Neoclassical Growth Model with an S-shaped production function whenever complementarities are strong, and fully characterize when the optimal policy uniquely implements the first best. When it does not, two novel results emerge: the efficient allocation prescribes escaping a *poverty trap*—providing an explicit optimality foundation for a Big Push—and escaping an *abundance trap*, where dismantling adopted technologies is optimal. In both cases, a temporary, costless supplementary policy restores unique implementation.

Keywords: Technology Adoption, Big Push, Development Dynamics.

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1 Introduction

We view our paper as a direct descendant of the seminal work by Murphy et al. (1989). We adopt their conceptualization of the ideas of a Big Push, as introduced by Rosenstein-Rodan (1943). We embrace their call to study the efficient allocation and the role of policy to achieve it. We consider a model that incorporates elements commonly used in the recent industrial policy literature: A fixed cost of adoption, market power, and complementary technologies. We study the problem in a fully dynamic, infinite horizon, general equilibrium setup. In particular, we analyze the set of equilibrium paths and compare the allocations with the one that solves the social planner's problem, i.e., the first best. Although the model has complex dynamics, there is a simple policy that implements the first best. We characterize when the implementation is unique, i.e., when there is a unique equilibrium whose allocation coincides with the first best. Finally, we consider more sophisticated policies that uniquely implement the first best.

We consider a dynamic environment where firms choose whether and when to pay a fixed cost to adopt technologies from a exogenously growing frontier. Each firm produces a differentiated variety under monopolistic competition. The output of each firm is combined into an aggregate good with a CES technology; this good is used as final consumption, used by firms as an intermediate input in production, and used to produce the input, i.e., the fixed cost, used to adopt frontier technologies. Firms produce using labor and the aggregate good as inputs with a constant returns to scale technology. Adoption of the frontier technology by one firm increases demand of all other firms and, in this sense, we refer to these technologies as complementary. Finally, there is a representative household with standard CRRA preferences that owns all firms and supplies labor inelastically.

In a period, given the distribution of technologies, firm demand labor and intermediate goods, and set prices with a constant markup, determining total output, firms' profits, real wages and the labor allocation. Given this distribution, the period's final output can be used for consumption or investment, i.e., to pay the fixed cost of the firms adopting the frontier technology. Due to the fixed cost, firms adopt the frontier technology at discrete time intervals, doing so whenever their technology gap to the frontier is sufficiently large. In their decision firms compare the value of the fixed cost with the highest firm value that the new level of productivity yields. A dynamic equilibrium is characterized by a forward-looking optimality condition for the optimal frequency of adoption, and a backward-looking distribution of the technologies adopted in the past. Interest rates are determined using the household intertemporal preferences. We characterize the dynamic equilibria, as well as the balanced-growth paths. We show that complementarities generate aggregate increasing returns, which can result in multiple balance growth paths and dynamic equilibria. Complementarities are fully encoded in a simple statistic $\zeta \equiv [(\eta - 1)(1 - \nu)]^{-1}$, where η regulates the elasticity of substitution in the production of the aggregate good, and ν accounts for the elasticity of a firm's output to this aggregate good. For instance, when $\zeta > 1$ we find sufficient conditions for three balanced growth paths, one without (costly) adoption and two with interior levels of adoption.

Our first main result is that, despite the complexity of the dynamic problem, the optimal policy is remarkably simple. We consider a planner that controls everything: The assignment of labor and intermediate goods to different technologies, the dynamics of aggregate consumption, as well as which firms adopt the frontier technology at each moment of time, in order to maximize the discounted utility out of the representative household. The optimal choice of inputs across firms results in an aggregate production function, which is a function of the distribution of active technologies. We show that this function is concave in this distribution if and only if $\zeta \leq 1$. We also show that the equilibrium and the solution to the planning problem do not coincide. Market power generates static and dynamic inefficiencies. The static inefficiency stems from under-utilization of intermediate inputs, which lowers aggregate output; this results in "static" misallocation, and does not arise in the special case where production does not use intermediate inputs, i.e., when $\nu = 0$. The dynamic inefficiency stems from a firm's sub-optimal technology adoption decision rule. This decision is not optimal because firms profits do not capture the entire consumer surplus resulting from adoption (Arrow, 1962). Nevertheless, the optimal policy is very simple, a time-invariant subsidy s_e^* to either intermediate inputs or revenue corrects the static inefficiency, and a time-invariant subsidy s_d^* to either profit, revenue or the cost of the adoption good to correct the low level of adoption—firms wait too long to replace old technologies with frontier ones. To summarize, with these two time-invariant subsidies there is an equilibrium with an allocation that solves the planner's problem. Still, even with the optimal subsidies, there can be other equilibria with allocations different from the first best. A necessary condition for this multiplicity to occur is $\zeta > 1$, i.e., that the aggregate production function be non-concave.

A feature of the model is that the distribution of technologies used by firms is infinitely dimensional, constraining our ability to fully characterize dynamic aspects of the model. Thus, we make further progress by analyzing a simpler related economy. In this simpler economy, each firm is endowed with a baseline level of technology, i.e., a distance from the frontier, with an exogenously given distribution across firms. Again, firms choose whether to adopt the frontier technology, or keep using their own baseline level. We assume that the adoption is reversible, in the sense that firms can sell it, recoup the fixed cost, and return to their endowed baseline technology gap. In this simpler economy, all the results obtained in the general formulation of the model still apply, but the economy is substantially simpler to analyze. In particular, here the relevant state variable is one-dimensional, given by the fraction of firms that adopted the frontier technology. We refer to this state variable as aggregate "capital", or $k(t)$. Surprisingly to us, we show that the equilibrium of this economy, where heterogeneous firms make lumpy decisions and compete monopolistically, satisfy the same equations that a version of the Neoclassical Growth Model with an investment tax, but with a S-shaped aggregate production function. In this simpler economy we fully characterize the dynamic path of all the equilibria, as well as the path of the efficient allocation.

Our second main result concerns the implementation of the efficient allocation. We provide a complete characterization of when the optimal policy uniquely implements the efficient allocation. When $\zeta \leq 1$ the production function is concave and the optimal policy uniquely implements the first best for any initial condition, exactly as in the standard Neoclassical Growth Model. When $\zeta > 1$ the production function is S-shaped, and unique implementation depends on the interplay between the discount rate $\bar{\rho}$ and the intertemporal elasticity of substitution $1/\theta$. When the discount rate is very low or very high, unique implementation obtains straightforwardly. For intermediate values of the discount rate—so that there are two interior steady states and one steady state with no adoption—the intertemporal elasticity of substitution plays a crucial role. When it is sufficiently low, i.e., θ is large enough, a remarkable result obtains: despite the presence of multiple steady states with different consumption levels, the optimal policy uniquely implements the efficient allocation for *any* initial condition $k(0)$. The efficient allocation is history-dependent in a precise sense: if $k(0)$ is large, the economy converges to the high-adoption interior steady state k^*_H; if $k(0)$ is small, the economy converges to the steady state with no adoption. Strikingly, this means that if an economy starts with no capital, the efficient allocation dictates remaining there—it is not always optimal to push for development, even when a steady state with higher consumption exists. This result stands in sharp contrast to the Big Push literature, which typically presumes that escaping a low-development equilibrium is always desirable. When the intertemporal elasticity of substitution is sufficiently high, however, unique implementation breaks down, giving rise to two novel results. First, the economy can be stuck in a *poverty trap* even after the optimal subsidies are in place; crucially, the efficient allocation prescribes escaping it by transitioning to the high-adoption steady state k_H^*, providing an explicit dynamic optimality foundation for the Big Push idea of Murphy et al. (1989). What is novel here is not the existence of the poverty trap per se—which has been documented in the literature since Murphy et al. (1989)—but rather that it can persist even after the optimal subsidies s_e^* and s_d^* are in place. This provides a precise dynamic characterization of when and why a Big Push is optimal, which to our knowledge has not been done before in a fully dynamic general equilibrium setting. Second, and more surprisingly, the economy can be stuck in an *abundance trap*: the efficient allocation prescribes converging to the steady state with no adoption, but coordination failures sustain an inefficient equilibrium with high adoption. Both results are, to our knowledge, novel in the literature, and together they paint a rich picture of the coordination failures that can arise even under optimal policy.

In the cases where the optimal policy fails to uniquely implement the efficient allocation, we show how to supplement it with a temporary, costless additional policy that eliminates the undesired equilibria without introducing new distortions.[1] When the economy starts near a poverty trap, this temporary policy unlocks persistent development dynamics, transitioning the economy from stagnation to the high-adoption interior steady state. A key feature of this supplementary policy is that it only needs to be in place for a finite amount of time, until the economy's capital stock crosses the unstable interior steady state k_L^*, after which it becomes unnecessary. Crucially, this is not merely a feasible policy—it is the *optimal* one. The planner, given the choice, prefers to escape the poverty trap and converge to the high-adoption steady state, and the supplementary policy is precisely what implements this preference in equilibrium. In this sense, our analysis provides an explicit dynamic optimality foundation for the

[1] This is more complicated than the optimal time-invariant policy $\{s_e^*, s_d^*\}$ as it needs to deal with off-equilibrium paths, yet it is not 'sophisticated' policy in the language of Atkeson et al. (2010).

Big Push idea of Murphy et al. (1989): a temporary, well-designed intervention can be optimal, and can permanently shift an economy from a low- to a high-development equilibrium. To our knowledge, this is the first paper to establish this result in a fully dynamic general equilibrium setting with microfounded complementarities and optimal policy.

To keep transparent the analysis of the complex dynamics of the model, we made several simplifying assumptions that we want to highlight. First, firms are ex-ante identical: Among them, those that adopt the technology are those that are further away from the frontier. If there were to be ex-ante heterogeneity across firms, such as differential productivity, then firms adopting will tend to be larger and more productive than the rest. Second, we have a very stylized input-output structure implemented with a symmetric roundabout technology of production. As a result, we are abstracting from heterogeneity in the centrality of different industries. We conjecture that the nature of the optimal policy would be identical to what we found, but the analysis of dynamics implementation would be significantly more involved. Third, while we analyze a closed-economy, we consider the limit when θ goes to zero, which can be regarded as an approximation to a small open economy where the interest rates is exogenous.

Related Literature. The topics we study have a large tradition in several related areas. To start with, as mentioned in our first paragraph, we follow Murphy et al. (1989). They themselves were inspired by the ideas on Rosenstein-Rodan (1943), and the related work by Hirschman (1958). There is a large literature that analyze different topics, all of them under the umbrella of a "Big Push".

The elements of our static model are a simplified version of Buera et al. (2021) that studies how distortions are amplified in a model that also features the adoption of complementary technologies, but in a static framework. The study of complementarities has a large literature too, such as (Ciccone, 2002; Jones, 2011) among many others.

We study dynamic equilibrium paths of models which can have multiple equilibria, as well as the efficient allocation. In this area, the seminal works of Matsuyama (1991) and Krugman (1991) are closely related to ours. Their work focus on multiplicity that stems from external increasing returns to scale. This is a large area too with many contributions, such as (Adserà and Ray, 1998; Monte et al., 2023). Different from this literature, we characterize the efficient allocation and we study whether there is a unique equilibrium under the optimal policy. When this is not the case, and therefore there is no unique implementation, we also expand the set of policies to study equilibrium selection. This is related to the work by Sturm Becko (2023).

To study equilibrium and optimal dynamics we begin by noting that, in the simplified version of the model, the relevant state variable is unidimensional, and the resulting aggregate production function has a convex segment followed by concave segment as long as $\zeta > 1$. Moreover, the model aggregates to the Neoclassical Growth Model. As a result, we can rely on the literature that studies the planning problem for the Neoclassical Growth Model with a convex-concave production function (Skiba, 1978; Brock and Dechert, 1983; Dechert and Nishimura, 1983; Stachurski et al., 2012).

The study of the difference in the incentives to technology adoption between market allocation and the first best has a long tradition, such as the seminal work by Arrow (1962). The difference between market and planner investment incentives has also been the subject of study in some recent work, including Baqaee and Farhi (2021), Schaal and Taschereau-Dumouchel (2025), and Bertolotti et al. (2025).

Finally, our model has a vintage capital structure. As such, it follows a long tradition started by Johansen (1959) and Solow (1960). This literature can be grouped regarding those featuring a replacement problem and those that do not. The replacement problem is the fact that it is optimal to abandon an old vintage to adopt a new one, which follows from the assumption that there is a fixed factor, e.g., land, or the number of entrepreneurs or managers with embedded knowledge (e.g., Chari and Hopenhayn, 1991; Parente, 1994; Cooley et al., 1997). In our model, the fixed factor is the measure of differentiated varieties. Relative to this literature, our contribution is to consider the case where technologies are complementary, which can lead to aggregate increasing returns, and study the design of the optimal policy.[2]

2 Setup

We consider an economy with a measure 1 of households and firms. The household demands a composite of varieties and supplies labor to firms. Firms produce differentiated varieties using a constant returns to scale technology. Firms can upgrade

[2]Modern versions of vintage capital models without the replacement problem are models with investment-specific technological change such as Greenwood et al. (1997). For a broad review of this literature, see also Boucekkine et al. (2011).

their production techniques to the technological frontier at any point in time. The technological frontier is continuously evolving.

Preferences and Taxes. Household utility is given by $\int_0^\infty e^{-\rho t}\left[C(t)^{1-\theta}-1\right]/(1-\theta)\,dt$, where $C(t)$ denotes aggregate consumption at time t. The household also supplies an inelastic amount of labor normalized to one in every period t.

The government uses lump sum taxes/transfers $T(t)$ to households and the following gross subsidies: (i) a revenue subsidy $s_r(t)$, (ii) a subsidy for the use of labor by firms $s_l(t)$, (iii) a subsidy for the use of intermediate inputs by firms $s_x(t)$, (iv) a subsidy to a firm's operating profit $s_\pi(t)$, and (v) a subsidy to the use of goods in innovations $s_a(t)$. Our notational convention is that a value of s larger than one is a subsidy, and thus setting any of the s smaller than one corresponds to a tax.

Technology adoption. The technological frontier of the economy grows at an exogenous rate γ. Any firm can use $\Phi(t)$ aggregate final goods at time t and adopt the frontier technology, i.e., their neutral TFP becomes $e^{\gamma t}$. We refer to this as a costly adoption event. Furthermore, with probability q per unit of time, the firm can adopt the frontier technology without paying the fixed cost $\Phi(t)$. We refer to this event as a free adoption opportunity. The firm's productivity stays constant until the firm either pays a fixed cost again or a free adoption opportunity arrives. We assume that the adoption cost scales with the level of the technological frontier measured in goods, i.e., $\Phi(t)=\phi e^{\frac{\gamma}{1-\nu}t}$, with $\phi>0$. The neutral productivity of an individual firm is $e^{(t-g)\gamma}$, where g is the technology gap measured in units of time. In equilibrium, firms will pay the cost of adopting when g reaches a time-varying threshold $G(t)$.

Distribution of gaps. Let $m(g,t)$ be the density at time t of the firms indexed by the technology gap g. Let $G(t)$ be the threshold at time t at which firms will adopt the frontier technology, and therefore their gap will be zero. Let $m_0(g)$ be the initial density at time $t=0$. The p.d.e. for the density m is given, for $t\geq 0$, by

$$m_t(g,t)+m_g(g,t)+q\,m(g,t)=0\ \text{ for } 0\leq g\leq G(t),\ \text{ and } 1=\int_0^{G(t)} m(g,t)dg\ . \quad (1)$$

Differentiating this equation with respect to time we get the "mass preservation" condition, for all $t\geq 0$, $0=\int_0^{G(t)} m_t(g,t)dg+m(G(t),t)G'(t)$, or

$$0=m(0,t)-m(G(t),t)-q\int_0^{G(t)} m(g,t)dg+m(G(t),t)G'(t), \quad (2)$$

where $G'(t) \equiv \partial G(t)/\partial t$ and where we used (1). Because $G(t)$ is the upper bound of the support of g and because the gap of firms that did not adjust moves dt with time, it follows that $G'(t) \leq 1$ with strict inequality for $G(t)$ finite.

Feasibility. Aggregate consumption $C(t)$ is net final aggregate output $Y(t)$ minus adoption costs,

$$C(t) = Y(t) - \Phi(t) \left(m(0,t) - q \int_0^{G(t)} m(g,t)dg \right), \tag{3}$$

where $\Phi(t)$ accounts for the number of final goods used by each firm to adopt when paying the fixed cost, and where $m(0,t)$ is the rate per unit of time at which firms adopt the frontier technology. We subtract $q \int_0^{G(t)} m(g,t)dg$ from the adoption rate, since adoption is free in these instances.

The feasibility condition in (3) implies that the distribution of gaps cannot have any mass points, i.e. that $G(t)$ is differentiable. This follows as, given that $C(t)$, $Y(t)$ and $\Phi(t)$ are flows, there are not sufficient resources in an instant to pay for the stock of adoption costs implied by a mass point.

There are two types of firms: a representative firm producing the final good, and a continuum of firms producing differentiated varieties. Each differentiated firm produces a variety of goods using labor and intermediate aggregate goods. The production function for a firm with gap g to the frontier at time t, hiring $n(g,t)$ workers and using $x(g,t)$ aggregate intermediate inputs is given by $y(g,t) = e^{(t-g)\gamma}\, \tilde{a}\, x(g,t)^{\nu}\, n(g,t)^{1-\nu} \quad (4)$, where ν is the elasticity of firm's output to the intermediate input, and $1-\nu$ is the elasticity of firm's output to labor, and where $\tilde{a} \equiv a^{1-\nu}(1-\nu)^{\nu-1}\nu^{-\nu}$ is defined to simplify the notation, and where a is a gross output productivity shifter. At time t, the TFP of a firm that adopted the frontier technology g time ago is $e^{(t-g)\gamma}$. Net final aggregate output $Y(t)$ is given by gross output $Q(t)$, which is a CES aggregator across varieties with elasticity of substitution η, minus aggregate intermediate inputs $X(t)$,

$$Q(t) \equiv Y(t) + X(t) = \left[\int_0^{G(t)} y(g,t)^{1-\frac{1}{\eta}}\, m(g,t)\, dg \right]^{\frac{1}{1-1/\eta}}, \tag{5}$$

where $X(t) = \int_0^{G(t)} x(g,t)\, m(g,t)\, dg \quad (6)$. The expression for $Q(t)$ follows by noticing that each firm producing a differentiated variety is uniquely identified by its gap g to the technological frontier. Thus, given that $m(g,t)=0$ for all $g>G(t)$, integrating across g for $0\leq g\leq G(t)$ using the density $m(g,t)$ is the same as integrating across firms or varieties. The same argument provides that aggregate (exogenous) labor supply, which is assumed to be equal to one, equals labor demand across all firms, $1=\int_0^{G(t)} n(g,t)\,m(g,t)\,dg \quad (7)$.

3 Equilibrium

We describe the equilibrium of this economy in different blocks.

Household Problem. The problem of the household is simple, receiving the profit from the aggregate portfolio of firms and wages, and borrowing and lending at rate $r(t)$ at t. The household's Arrow-Debreu budget constraint is given by $0=\int_0^{\infty} e^{-\int_0^t r(s)ds}\,[P(t)C(t)-\Pi(t)-w(t)+T(t)]\,dt$, where $r(t)$ is the interest rate, $w(t)$ is the time t wage rate, $P(t)$ is the time t price of the aggregate final good, $\Pi(t)$ are the time t profits of the the portfolio of firms, and $T(t)$ are the time t lump sum transfers on the household. This implies the following first order condition, $e^{-\rho t}C(t)^{-\theta}=\lambda_H e^{-\int_0^t r(s)ds}P(t)$, where λ_H is the Lagrange multiplier of the Arrow-Debreu budget constraint. Taking logs, normalizing $w(t)=1$ for all t, and differentiating with respect to time,

$$r(t)=\rho+\theta\frac{\dot{C}(t)}{C(t)}+\frac{\dot{P}(t)}{P(t)}\ . \tag{8}$$

Firm producing final goods. The firm buys $y(g,t)$ differentiated goods from each of the density of firms $m(g,t)$ with gap g at t, and produce the final good that can be used for consumption, as the adoption good, or as an intermediate good in the production of the differentiated varieties. The standard input demand equation is $y(g,t)=\left(\frac{p(g,t)}{P(t)}\right)^{-\eta}Q(t) \quad (9)$, where $p(g,t)$ denotes the price of a differentiated variety with gap g to the frontier at time t, and where the price of the aggregate final good is given by $P(t)\equiv\left[\int_0^{G(t)} p(g,t)^{1-\eta}m(g,t)dg\right]^{\frac{1}{1-\eta}} \quad (10)$.

The static problem of a differentiated good firm. We assume monopolistic competition. Firms set their price to maximize static profits. They take as given the period price for aggregate output after tax/subsidy, denoted by $P(t)/s_x$, and the period wages after tax/subsidy, denoted by $w(t)/s_x$. Demand is given by (9), and the cost minimization problem is given by $\mathcal{C}(\frac{w}{s_l},\frac{P}{s_x},y,t)\equiv\min_{x,n} P/s_x x+$

$nw/s_l + \lambda_c \left(y - e^{\gamma(t-g)}\tilde{a}x^\nu n^{1-\nu}\right)$, providing $\frac{P}{s_x}x = \lambda_c \nu e^{\gamma(t-g)}\,\tilde{a}\,x^\nu n^{1-\nu}$ and $\frac{w}{s_l}n = \lambda_c(1-\nu)e^{\gamma(t-g)}\,\tilde{a}\,x^\nu n^{1-\nu}$ (11). We combine these to obtain that the marginal cost is $\lambda_c = \frac{\partial}{\partial y}\mathcal{C}(\frac{w}{s_l}, \frac{P}{s_x}, y, t) = \left(\frac{P}{s_x}\right)^\nu \left(\frac{w}{s_l}\right)^{1-\nu} e^{(g-t)\gamma}/a^{1-\nu}$. The larger the firm's gap to the frontier, the higher the marginal cost of the firm. Then, a firm's static profits are $\pi(g,t) \equiv \max_p s_\pi(t)\left(\frac{p}{P(t)}\right)^{-\eta} Q(t)\left[s_r(t)p - \frac{\partial}{\partial y}\mathcal{C}\left(\frac{w}{s_l}, \frac{P}{s_x}, y, t\right)\right]$ (12), from where we immediately obtain

$$p(g,t) = \frac{\eta}{\eta - 1}\frac{1}{s_r(t)}\frac{1}{a^{1-\nu}}e^{\gamma(g-t)}\left(\frac{w(t)}{s_l(t)}\right)^{1-\nu}\left(\frac{P(t)}{s_x(t)}\right)^\nu . \tag{13}$$

The technology adoption problem of a differentiated good firm. The adoption problem for the firm can be characterized using standard dynamic programming methods. The state of the firm problem is the pair (g,t). We let $V(g,t)$ denote a firm's value function given the state. The value function satisfies the following Hamilton-Jacobi-Bellman (HJB) equation,

$$\begin{aligned} r(t)V(g,t) = &\max\left\{r(t)[V(0,t) - \Phi(t)P(t)/s_a(t)], \pi(g,t) + V_g(g,t)\right. \\ &\left. + V_t(g,t) + q\left[V(0,t) - V(g,t)\right]\right\}, \end{aligned} \tag{14}$$

where $r(t)$ is the interest rate in (8), and $\pi(g,t)$ is the profit of firm at time t with gap g in (12). The term in the left hand side is the flow value of the firm. The left hand side shows that the firm must choose between paying for adoption or not. The first term in the right hand side gives the value of costly adoption. The second term gives the continuation with no costly adoption. The value function changes with both time t and gap g. The gap increases one by one with time. For costly adoption the firm pays $\Phi(t)$ final goods, with unit cost $P(t)$, and where the firm is subject to an adoption subsidy $s_a(t)$. Time is part of the state of the firm's problem because $r(t), P(t), s_a(t), \Phi(t)$ and $\pi(g,t)$ are all functions of time. The term $q\left[V(0,t) - V(g,t)\right]$ is the expected value of a free adoption opportunity.

We can cast the dynamic problem of the firm as an optimal stopping time problem,

$$\begin{aligned} V(0,t) = &\max_{\tau \geq t} \int_t^\tau e^{-\int_t^s q + r(u)\,du}\left[\pi(s-t,s) + qV(0,s)\right] ds \\ &+ e^{-\int_t^\tau q + r(s)\,ds}\left[V(0,\tau) - \Phi(\tau)P(\tau)/s_a(\tau)\right], \end{aligned} \tag{15}$$

where τ is characterized using the first order conditions of problem (15),

$$0 = \pi(\tau - t, \tau) + qV(0, \tau) - (q + r(\tau))\left[V(0, \tau) - \Phi(\tau)P(\tau)/s_a(\tau)\right] \\ + V_t(0, \tau) - \frac{d}{dt}\left[\Phi(\tau)P(\tau)/s_a(\tau)\right] \ . \tag{16}$$

Given that for all t the flow profit $\pi(g, t)$ is decreasing in the gap g, it can be shown that $V(g, t)$ is also decreasing in g. This implies that the optimal firm adoption policy is of a threshold type $G(t)$ for each t, such that for all $0 \leq g \leq G(t)$ we have that

$$r(t)V(g, t) = \pi(g, t) + V_g(g, t) + V_t(g, t) + q\left[V(0, t) - V(g, t)\right] \ , \tag{17}$$

and for all $g > G(t)$ we have that $V(g, t) = V(0, t) - \Phi(t)P(t)/s_a(t)$ and thus $0 = V_g(g, t)$. Since the value function is continuous, we evaluate the previous expression at $g = G(t)$ to obtain the value matching condition, and if the value function is differentiable at $g = G(t)$ we obtain the smooth pasting condition when $G(t)$ is finite.[3] That is,

$$V(G(t), t) = V(0, t) - \Phi(t)P(t)/s_a(t) \ , \ 0 = V_g(G(t), t) \ , \ \text{for } t > 0 \ . \tag{18}$$

3.1 Temporary Equilibrium

We fix $m(\cdot, t)$ and we characterize the equilibrium conditions only involving static decisions. We refer to this as a temporary equilibrium.

Definition 1 *Fix $m(\cdot, t)$ and $\{s_r(t), s_\pi(t), s_x(t), s_l(t)\}$. Then $\{Y(t), Q(t), w(t), P(t)\}$ satisfies the temporary equilibrium conditions if there exist $\{n(\cdot, t), x(\cdot, t), p(g, t)\}$ such that: (i) $p(g, t)$ maximizes the profit of the firm, i.e., $p(g, t)$ satisfies (13), and the choice of inputs minimizes cost, i.e., (11) hold; (ii) Given $\{p(\cdot, t)\}$, the aggregate price satisfies (10); (iii) Labor market clearing (7); (iv) Intermediate good market clearing holds, i.e., (6) holds; and (v) $Y(t)$ is given by (5).*

The next proposition characterizes the temporary equilibrium conditions.

[3] See the Supplemental Appendix for more details.

Proposition 1 *Fix a time $t > 0$, and consider the static equilibrium corresponding to $m(\cdot,t)$. Then, real wages $w(t)/P(t)$, gross output $Q(t)$, GDP $Y(t)$, employment $n(g,t)$, and intermediate input demand $x(g,t)$ are given by:* $\frac{w(t)/s_l(t)}{P(t)/s_x(t)} = a\left[s_r(t)s_x(t)\left(\frac{\eta-1}{\eta}\right)\right]^{\frac{1}{1-\nu}} e^{\frac{\gamma}{1-\nu}t}\left[\int_0^{G(t)} e^{-\gamma g(\eta-1)} m(g,t)dg\right]^{\frac{1}{(\eta-1)(1-\nu)}}$ *(19),*
$Q(t) = \frac{1}{1-\nu}\frac{1}{s_r(t)s_x(t)}\frac{\eta}{\eta-1}\frac{w(t)/s_l(t)}{P(t)/s_x(t)}$ *(20),* $Y(t) = \frac{1}{1-\nu}\left[\frac{1}{s_r(t)s_x(t)}\left(\frac{\eta}{\eta-1}\right)-\nu\right]\frac{w(t)/s_l(t)}{P(t)/s_x(t)}$ *(21),*
$n(g,t) = \frac{e^{-g\gamma(\eta-1)}}{\int_0^{G(t)} e^{-g\gamma(\eta-1)}m(g,t)dg}$ *(22),* $\frac{w(t)/s_l(t)}{P(t)/s_x(t)} n(g,t)\frac{\nu}{1-\nu} = x(g,t)$ *(23).*

All proofs, with the exception of those that are direct and thus not included, are relegated to the Supplementary Appendix. An important consequence of inelastic labor supply combined with the CES structure is that, given $m(\cdot,t)$, the equilibrium labor allocation is invariant to any subsidies. Combining (19) with (21) provides that aggregate output satisfies

$$Y(t) = e^{\frac{\gamma}{1-\nu}t}\mathcal{A}(s_e(t))aF(m(\cdot,t)) \ , \tag{24}$$

where

$$\mathcal{A}(s_e(t)) \equiv \frac{1}{1-\nu}\left[\frac{1}{s_e(t)}\frac{\eta}{\eta-1}-\nu\right]\left[s_e(t)\frac{\eta-1}{\eta}\right]^{\frac{1}{1-\nu}} \ , \ s_e(t) \equiv s_r(t)s_x(t) \ , \tag{25}$$

$$F(m(\cdot,t)) \equiv \left[\int_0^{G(t)} e^{-\gamma g(\eta-1)}m(g,t)dg\right]^{\frac{1}{(\eta-1)(1-\nu)}} \tag{26}$$

can be interpreted as the aggregate production function in the economy, and where a takes the role of an aggregate productivity shifter. Aggregate output at time t expands with the technology frontier at rate γ, and depends on the degree of static misallocation, given by $\mathcal{A}(\cdot)$, and on aggregate production $F(m(\cdot,t))$. Dynamic inefficiencies, through their effect on the optimal choice of the threshold $G(t)$, will affect the distribution of $m(\cdot,t)$ and thus aggregate production in a temporary equilibrium at t. Given $m(\cdot,t)$, $s_e(t)$ encodes all subsidies relevant for an allocation in a temporary equilibrium. Due to this property, we refer to $s_e(t)$ as the total static subsidy.[4]

Proposition 2 further characterizes the cost of adoption $P(t)\Phi(t)$, aggregate demand $P(t)Q(t)$, and a firm's profit $\pi(g,t)$ in any temporary equilibrium.

[4]To better recall the notation it is helpful to pronounce s_e with Spanish accent.

Proposition 2 *Using Proposition 1, normalizing wages, i.e. $w(t)/s_l(t) = 1 \ \forall \ t$, we obtain the following in a temporary equilibrium: (i) $\frac{P(t)}{s_x(t)} Q(t) = \frac{1}{1-\nu} \frac{1}{s_e(t)} \frac{\eta}{\eta-1}$, (ii) $\frac{P(t)}{s_x(t)} \Phi(t) = \phi \frac{1}{a} \left(\frac{1}{s_e(t)} \frac{\eta}{\eta-1} \right)^{\frac{1}{1-\nu}} \left[\int_0^{G(t)} e^{-\gamma g(\eta-1)} m(g,t) dg \right]^{-\frac{1}{(\eta-1)(1-\nu)}}$, and (iii) $\pi(g,t) = s_\pi(t) \frac{1}{(1-\nu)(\eta-1)} \frac{e^{-\gamma g(\eta-1)}}{\int_0^{G(t)} e^{-\gamma(\eta-1)g'} m(g',t) dg'}$.*

The proposition states that several important objects do not depend on time and, if they do, they do so only indirectly through $m(\cdot, t)$. It is interesting to note that only the subsidy $s_\pi(t)$ affects the firm's profit $\pi(g,t)$ when measured in units of labor cost.

3.2 Dynamic Equilibrium

In this section we define a dynamic equilibrium, clarify the role played by subsidies, and write a Mean Field Game formulation of the equilibrium. We start by using the expression for profits in any temporary equilibrium in Proposition 2 to show that if the subsidies $s_\pi(t)$ and $s_a(t)$ are constant, only its product matter for the optimal adoption decision of firms.

Lemma 1 *Consider a path of constant subsidies $s_l, s_x, s_r, s_\pi, s_a$. Let $\{m(\cdot, t)\}$ be a path of densities, and denote $\{P(t)\}$ be the corresponding path of prices in a temporal equilibrium, using the normalization $w(t)/s_l = 1$. The optimal adoption decision of the firms, given by $\{G(t)\}$ for $t \geq 0$, depends only on $\Phi(t)P(t)/(s_a s_\pi)$ and the path $\{r(t)\}$. Moreover, in a temporary equilibrium this ratio satisfy*

$$\frac{P(t)}{s_a s_\pi} \Phi(t) = \frac{\phi}{a} \frac{1}{\mathcal{A}(s_e) \mathcal{D}(s_e, s_d)} \left[\int_0^{G(t)} e^{-\gamma g(\eta-1)} m(g,t) dg \right]^{-\frac{1}{(\eta-1)(1-\nu)}} \quad , \textit{ where}$$

$$\mathcal{D}(s_e, s_d) \equiv \left(\frac{1-\nu}{\frac{1}{s_e} \frac{\eta}{\eta-1} - \nu} \right) \frac{s_d}{s_e}, \textit{ with } \ s_d \equiv s_a \, s_\pi \, s_r \ . \tag{27}$$

We show later that $\mathcal{D}(s_e, s_d)$ captures the dynamic distortion that stems from market power in the economy. The total subsidy s_d summarizes all subsidies affecting the incentives for technology adoption. Next we define a dynamic equilibrium.

Definition 2 *Let $m(\cdot, 0) = m_0(\cdot)$. A dynamic equilibrium is given by a flow of densities $\{m(\cdot, t)\}$, a path of thresholds $\{G(t)\}$, and a value function $\{V(\cdot, t)\}$, such that: (i) The allocation $\{Y(t), Q(t), P(t), C(t)\}$ is a temporary equilibrium given $m(\cdot, t)$ and* *subsidies for every t; (ii) The path of densities $\{m(\cdot,t)\}$ and thresholds $\{G(t)\}$ solves the p.d.e. and boundary conditions in (1) and (2); (iii) The path of threshold $G(t)$ solves the value function $V(t,g)$ given the path of $\{m(\cdot,t)\}$ and $\{r(t)\}$ and path of subsidies; (iv) The path of interest rates and prices $\{r(t),P(t)\}$ solve the Euler equation (8) of the households for $C(t)=Y(t)-\Phi(t)m(0,t)$; (v) The lump-sum taxes/subsidies $\{T(t)\}$ are chosen to finance the subsidies at all t.*

Combining Proposition 1 and Lemma 1 we obtain an important corollary for a dynamic equilibrium under constant subsidies.

Corollary 1 *Consider an equilibrium where subsidies are constant through time. Then, the dynamic equilibrium allocation $\{m(\cdot,t),G(t),C(t),Y(t)\}$ depends only on the values of s_e and s_d, and it is independent of the value of s_l.*

We have defined a dynamic equilibrium for a large set of policies, i.e., for five different subsidies. Nevertheless, Corollary 1 shows that, in the case of constant subsides, only two combinations of them matter. While dynamic equilibria with the same values of s_e and s_d have the same allocation, the markups and the profits of the firms can be quite different. The next lemma further characterizes the dynamic equilibrium interest rate at any $t>0$. The lemma shows that $r(t)$ depends on time t only through the distribution $m(\cdot,t)$.

Lemma 2 *In a dynamic equilibrium with constant subsidies, $C(t)=c(t)e^{\frac{\gamma}{1-\nu}t}$ and the interest rate $r(t)$ satisfies*

$$r(t)=\rho+\theta\frac{\dot{C}(t)}{C(t)}+\frac{\dot{P}(t)}{P(t)}=\bar{\rho}+\theta\frac{d}{dt}\log c(t)-\frac{1}{1-\nu}\frac{d}{dt}\ln Z(t)\ , \tag{28}$$

$$\bar{\rho}\equiv\rho+(\theta-1)\frac{\gamma}{1-\nu}\ , \tag{29}$$

$$Z(t)\equiv\left[\int_0^{G(t)}e^{-\gamma g(\eta-1)}m(g,t)dg\right]^{\frac{1}{\eta-1}}\quad \text{for } t>0\ . \tag{30}$$

The aggregate $Z(t)$ is the intermediate input productivity in the economy, depending on calendar time t solely through the effect of time on $m(\cdot,t)$.[5]

Mean Field Game Formulation. We can characterize a dynamic equilibrium as two p.d.e.'s coupled by the path of $r(t)$, $Z(t)$, $G(t)$. We assume that all subsidies are

[5]A proof of the lemma is straightforward, and follows from replacing $C(t)$ and $P(t)$ into (8).

constant and use the normalization that $w(t)/s_l = 1$. Then, the dynamic equilibrium of the Mean Field Game (MFG) with constant subsidies is given by two functions $\{\hat{V}(g,t), m(g,t)\}$ for all $t > 0$ and $g \in [0, G(t)]$, and paths $\{G(t), c(t), r(t)\}$ for all $t > 0$ such that for all $t \geq 0$:

$$r(t)\hat{V}(g,t) = \hat{\pi}(g,t) + \hat{V}_g(g,t) + \hat{V}_t(g,t) + q\left(\hat{V}(0,t) - \hat{V}(g,t)\right), g \in [0, G(t)],$$
$$\hat{V}(0,t) = \hat{V}(G(t),t) + \frac{\phi}{a}\frac{1}{\mathcal{A}(s_e)\mathcal{D}(s_e,s_d)}Z(t)^{-\frac{1}{1-\nu}}, \text{ and } \hat{V}_g(G(t),t) = 0\ ,$$

where $m(g,t)$ satisfies the p.d.e. and boundary condition in (1), $r(t)$ is given by (28), $\hat{V}(g,t) \equiv \frac{V(g,t)}{s_\pi}$, $\hat{\pi}(g,t) \equiv \frac{\pi(g,t)}{s_\pi}$, $c(t) = \mathcal{A}(s_e)aZ(t)^{\frac{1}{1-\nu}} - \phi\left[m(0,t) - q\right]$, and where $Z(t)$ and $\pi(g,t)$ are defined in Lemma 2 and Proposition 2. We remind the reader that $c(t), Z(t)$ and $\pi(g,t)$ are functions of time solely through the effect of time on $m(\cdot,t)$. Consistent with Corollary 1, the MFG formulation provides that only s_e and s_d matter for the dynamic equilibrium allocation. The following remark provides an equivalence result between static distortions $\mathcal{A}(s_e)$ and the level of exogenous productivity a. The remark follows immediately from the MFG formulation.

Remark 1 *The dynamic equilibrium is invariant to changes in static distortions $\mathcal{A}(s_e)$ and productivity shifter a as long as $\mathcal{A}(s_e)a$ remains constant.*

That is, distortions that stem from static misallocation are isomorphic to a lower level of the productivity shifter a. The next remark provides a useful normalization.

Remark 2 *Let $h > 0$. Then, if $\{\hat{V}(g,t), m(g,t)\}$ and $g \in [0, G(t)]$, and paths $\{G(t), c(t), r(t)\}$ for all $t > 0$ constitutes an equilibrium of the MFG when $h = 1$, then $\{\hat{V}(g,t), m(g,t)\}$ and $g \in [0, G(t)]$, and paths $\{G(t), hc(t), r(t)\}$ for all $t > 0$ constitutes a dynamic equilibrium of the MFG when the cost of adoption is $h\phi$ and the productivity shifter is ha.*

If the productivity shifter a and flow cost of adoption ϕ are doubled, then consumption is doubled, but the remaining objects defining an equilibrium are unchanged.

4 Planner's Problem

We first describe the static efficient conditions for the planner problem. Then we describe the full dynamic planning problem.

Temporary Planner's Problem. Fix the density $m(g,t)$ and maximize aggregate output $Y^P(m,t)$ by choosing $\{y(g,t), x(g,t), n(g,t)\}$ subject to (4), (5), (6) and (7). We use a Lagrange multiplier W^P for (7) and replace $X(t)$ from (6) into (5). Then, the aggregate output for the planner $Y^P(m,t)$ is given by

$$Y^P(m,t) \equiv \max_{y,n,x} Q(t) - X(t) + W^P \left[1 - \int_0^{G(t)} n(g,t)\, m(g,t)\, dg\right] . \tag{31}$$

where $Q(t)$ and $X(t)$ are defined in (5) and (6). The next proposition provides the solution to the temporary planner's problem.

Proposition 3 *The solution of the problem defined in* (31) *is given by:*

$$W^P(m,t) = e^{\frac{\gamma}{1-\nu}t} a \left[\int_0^{G(t)} e^{-g\gamma(\eta-1)}\, m(g,t)\, dg\right]^{\frac{1}{(\eta-1)(1-\nu)}} = e^{\frac{\gamma}{1-\nu}t} a Z(t)^{\frac{1}{1-\nu}} .$$

$$Q^P(t) = \frac{Y^P(m,t)}{1-\nu} = \frac{W^P(m,t)}{1-\nu} \text{ and } n(g,t) = \frac{1-\nu}{\nu}\frac{x(g,t)}{W^P(m,t)} = \frac{e^{-g\gamma(\eta-1)}}{\int_0^{G(t)} e^{-g\gamma(\eta-1)} m(g,t) dg} .$$

The next corollary follows from comparing the characterizations of the temporary equilibrium in Proposition 1 and of the static planner's problem in Proposition 3.

Corollary 2 *Fix $m(\cdot,t)$. The temporary equilibrium conditions coincide with the conditions for the static planner's problem if and only if $s_e(t) > 0$ if $\nu = 0$ and $s_e(t) = \frac{\eta}{\eta-1}$ if $\nu \in (0,1]$.*

Few remarks are in order. First, as it can be seen from Proposition 1 and Proposition 3 the labor allocation in the temporary planner's problem is the same as in a temporary equilibrium regardless of the level of subsidies. Hence, if $\nu = 0$, the allocation is temporally efficient, regardless of the firms' markup. Second, in the case of $\nu > 0$, the inefficiency in the laissez-faire economy stems from a sub-optimal use of the aggregate good as an intermediate input. In particular, when $\nu > 0$, in the laissez-faire markups decrease the price of labor relative to intermediate inputs, i.e., the value of $w(t)/P(t)$ is "too low". As it can be seen in Proposition 1, a higher value of the product $s_r s_x$ monotonically increases real wages and gross output Q, but it also increases the use of the aggregate good as intermediate X. The temporary efficient use of aggregate goods as intermediate inputs, which is finite, can be induced by any combination of s_r

and s_x so that its product s_e equals $\eta/(\eta-1)$. Third, this implies that $Y(t)$ depends only on the product $s_e(t)$. Fourth, this last observation implies that the statically efficient value of $s_e(t)$ is constant through time, independently of $m(\cdot,t)$.

The next proposition explores several properties of the solution of the temporary Planner's problem. The first result shows the semi-elasticity of the efficient net output with respect to reshuffling firms with respect to their gaps to the technological frontier. The second results states that the efficient static output is concave on the density m only if $1/((\eta-1)(1-\nu)) < 1$. The third result shows that profits are proportional to the consumer surplus. Informally, this alignment is because with a CES consumer surplus on a good equals revenue times $1/(\eta-1)$, and so does the static profit. Without subsidies, the temporary equilibrium revenues of the monopolist are depressed relative to the consumer surplus, since markups depresses demand. For the proposition, we begin with a density $m(\cdot,t)$, with strictly positive density in two gaps, g_1 and g_2. We pick two positive numbers α and ϵ, and define $m^{\epsilon,\alpha}$ as a perturbation on the original density m. In particular, we add a uniform density of height ϵ and width α, centered around gap g_1, and subtract a uniform density of the same height and width centered around gap g_2. For small α and ϵ it defines the following density: $m^{\epsilon,\alpha}(g,t) = m(g,t) + \epsilon\left(1_{\{|g-g_1|<\alpha/2\}} - 1_{\{|g-g_2|<\alpha/2\}}\right)$ for all g. Relative to m, the density $m^{\alpha,\epsilon}$ reshuffles probability density from a neighborhood of g_2 to a neighborhood of g_1. This experiment is of interest for the study of technology adoption, since we model adoption as an action where a number of firms discretely eliminate their technology gap with respect to the frontier, i.e., the firms move from $g = G(t)$ to $g = 0$.

Proposition 4 *Fix $m(\cdot,t)$ and two points g_1, g_2 with strictly positive density. Then,*

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\frac{d}{d\epsilon}\log Y^P(m^{\epsilon,\alpha})|_{\epsilon=0} = \frac{1}{(\eta-1)(1-\nu)}\frac{e^{-\gamma(\eta-1)g_1} - e^{-\gamma(\eta-1)g_2}}{\int_0^{G(t)} e^{-\gamma(\eta-1)g'} m(g',t)dg'} \ , \tag{32}$$

$$Y^P(m^{\epsilon,\alpha}) \text{ is concave in } \epsilon \iff \frac{1}{(\eta-1)(1-\nu)} \leq 1 \ , \tag{33}$$

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\frac{dY^P(m^{\epsilon,\alpha})}{d\epsilon} = \frac{1}{\mathcal{A}(s_e)\mathcal{D}(s_e,s_d)} \times \frac{\pi(g_1,t)-\pi(g_2,t)}{P(t)/s_a} \ . \tag{34}$$

We discuss the proposition using the following combination of parameters:

$$\zeta \equiv [(\eta-1)(1-\nu)]^{-1} \ . \tag{35}$$

The expression in (32) measures the increase in the social planner's final output due to the reshuffling of probability density. This reshuffling in density changes Y^P by the difference in productivity gaps g_1 and g_2, normalized by the average productivity gaps, multiplied by ζ. Whether this change is less than linear (i.e. concave in ϵ), linear, or more than linear (i.e. convex in ϵ) depends on whether $\zeta \geq 1$ or not, as also stated in (33). The convexity of this mapping is controlled by ζ through two mechanisms. The first mechanism is the roundabout nature of production, captured by ν. The strength of this mechanism increases as ν increases towards one. The second mechanism is that the economy features complementarities in the production of intermediate goods. These complementarities are controlled by η, and they increase as η decreases towards one. While both mechanisms operate through ζ in similar ways, they do have relevant differences. In particular, while the multiplier effect of ν as $\nu \uparrow 1$ is unbounded, the effect as $\eta \downarrow 1$ is bounded. In particular, as $\eta \downarrow 1$, we have $dY^P/d\epsilon = (g_2 - g_1)/(1-\nu)$.

Finally, (34) connects the increase in the social planner's final output due to the reshuffling of probability density with static profits, measured in units of final output. We note on three important implications of this expression. First, the relative valuation (i.e., across different g's) in the planning problem and in the equilibrium with monopolistic competition are proportional, inversely related to the value of $\mathcal{A}(s_e)\mathcal{D}(s_e, s_d)$. Second, with no subsidies, the gains accrued by the planner due to the reshuffling are larger than those perceived by the firm, by a factor $[\eta/(\eta-1)]^{1/(1-\nu)}$, as the firm pays the cost but does not capture all the gains that result from adoption. And third, it is possible to equalize the relative valuation in the planning problem and in the equilibrium with the subsidies s_e and s_d, as there exist values for these subsidies such that $\mathcal{A}(s_e)\mathcal{D}(s_e, s_d) = 1$.[6] A particular case of interest is when $g_1 = 0$ and $g_2 = G(t)$, so that we consider the case of adoption. Moreover, when we consider a one-period model, (34) provides that the social value accrued from adoption is larger than the private value, and that the two are equalized whenever $\mathcal{A}(s_e)\mathcal{D}(s_e, s_d) = 1$.

In the rest of the section we will analyze the optimal allocation in our dynamic model. But, to see how the results that we have so far can preview the final results, and hence develop the intuition, we consider first a simple one period model.

Efficient Allocations. Now we define and analyze the dynamic planner's problem, and the necessary conditions for its implementation as a dynamic equilibrium.

[6] In particular, for any value of s_d one can set $s_e = \left\{ s_d^{-1} [\eta/(\eta-1)]^{1/(1-\nu)} \right\}^{(1-\nu)/\nu}$.

Definition 3 *Let $m(\cdot,0)=m_0(\cdot)$. The planner chooses the evolution of $\{u(t)\}$, where $u(t)\equiv G'(t)$. The objective is to maximize $\int_0^\infty e^{-\rho t}[C(t)^{1-\theta}-1]/(1-\theta)dt$ subject to (i) $C(t)=Y^P(m(\cdot,t),t)-\Phi(t)\left(m(0,t)-q\int_0^{G(t)}m(g,t)dg\right)$ for all t, and (ii) the p.d.e. and boundary condition for $m(g,t)$ in (1) for all g and t.*

The planning problem restricts attention to adoption policy functions that are characterized by a threshold rule $G(t)$. However, in the Supplemental Appendix we study a relaxed planning problem and show, among other, that the optimal adoption policy is indeed a threshold rule $G(t)$. The next proposition characterizes the necessary first order conditions for the planner's problem.

Proposition 5 *Fix $m_0(\cdot)$. Let $e^{\bar{\rho}t}\lambda(g,t)$ be the Lagrange multiplier of the p.d.e. for the evolution of the density, i.e. (1) and $e^{\bar{\rho}t}\omega(t)$ the Lagrange multiplier of the mass preservation in (1). Let $\{C(t)\}$ the optimal path of consumption, let $\{G(t)\}$ the optimal path of for the adoption threshold, and let $\{m(g,t)\}$ the density of gaps in the optimal path. Then, λ and ω satisfy the following p.d.e. and boundary conditions, where $c(t)=aZ(t)^{\frac{1}{1-\nu}}-\phi\left[m(0,t)-q\int_0^{G(t)}m(g,t)dg\right]$ and $\hat{\pi}(g,t)=\frac{1}{(\eta-1)(1-\nu)}\frac{e^{-g\gamma(\eta-1)}}{\int_0^{G(t)}e^{-g\gamma(\eta-1)}m(g,t)\,dg}$:*

$$\bar{\rho}\lambda(g,t)=c(t)^{-\theta}aZ(t)^{\frac{1}{1-\nu}}\hat{\pi}(g,t)-\omega(t) \tag{36}$$
$$\qquad+\lambda_t(g,t)+\lambda_g(g,t)+q\left(\lambda(0,t)-\lambda(g,t)\right)\ \text{for } t\geq 0 \text{ and } g\in[0,G(t)]\ ,$$
$$\lambda(0,t)=c(t)^{-\theta}\phi,\ \text{for all } t>0\ , \tag{37}$$
$$\lambda(G(t),t)=0,\ \text{for all } t>0\ , \tag{38}$$
$$\lambda_g(G(t),t)=0,\ \text{for all } t>0\ , \tag{39}$$
$$0=\lim_{T\to\infty}e^{-\bar{\rho}T}\lambda(g,T)m(g,T)\ ,\ \text{for all } 0\leq g<\lim_{T\to\infty}G(T)\ . \tag{40}$$

Solving this problem is hard as the planner controls $G(t)$, its evolution $G'(t)$, and the evolution of the infinitely-dimensional distribution $m(g,t)$. MFG techniques allow us to handle this complicated control problem. In a nutshell, solving the planner's problem reduces to studying the planner's Hamiltonian. MFG allows us to characterize the necessary conditions for a dynamic efficient allocation, as stated in Proposition 5. Notice that if $\zeta\leq 1$, then $Y^P(m(\cdot,t),t)$ is concave (Proposition 4). In this case, the necessary conditions for an allocation to solve the dynamic planner's problem are also sufficient. In other words, if $\zeta\leq 1$, if an allocation satisfies the conditions in

Proposition 5, then the allocation is dynamically efficient. However, when $\zeta > 1$ and $Y^P(m(\cdot,t),t)$ is not concave everywhere, then it does not suffice for an allocation to satisfy the planner Hamiltonian's first order conditions for the allocation to be dynamically efficient: when complementarities are strong there is the potential for multiple dynamic equilibria satisfying the necessary conditions of the planner problem.

The next proposition states that an efficient allocation can be decentralized as a dynamic equilibrium by any combination of subsidies satisfying $s_e = s_e^* \equiv \eta/(\eta-1)$ and $s_d = s_d^* \equiv \eta/(\eta-1)$. The constant static subsidy s_e^* corrects the static distortion if $\nu > 0$ as highlighted in Proposition 3 and Corollary 2. This distortion makes the temporary equilibrium inefficient, which is unrelated to technology adoption by definition. The constant dynamic subsidy s_d^* corrects the dynamic distortion highlighted in Proposition 4. This distortion comes from the lack of perfect alignment between the profits of the firms adopting and the consumer surplus, both measured in units of the aggregate good. A remarkable feature of the optimal policy is that both s_e^* and s_d^* are constant, or time-invariant. This is a consequence of the CES structure of production. Without the proportionality that results from CES, the optimal policy may differ for firms with different gaps g, and may depend on the evolution of $m(\cdot,t)$.

Proposition 6 *Let $\{m, G, C, \lambda, \omega\}$ be the allocation and multipliers that solve the necessary first order conditions for the dynamic planner problem (Proposition 5). Setting a constant dynamic subsidy $s_d = s_d^*$, and in the case of $\nu > 0$, also setting a constant static subsidy $s_e = s_e^*$, provides that $\{m, G, C, V, r\}$ is a dynamic equilibrium where the firm's value function for all $g \in [0, G(t)]$ and $t \geq 0$ is given by*

$$V(g,t) = \frac{\lambda(g,t)}{c(t)^{-\theta} a Z(t)^{\frac{1}{1-\nu}}} + \int_0^\infty e^{-\int_t^{t+s} r(u)du} \frac{\omega(t+s)}{c(t+s)^{-\theta} a Z(t+s)^{\frac{1}{1-\nu}}} ds \;, \tag{41}$$

where $Z(t)$ and $r(t)$ are defined in Lemma 2 and $c(t)$ is given in Proposition 5.

This proposition is analogous to the Second Welfare Theorem. It is only analogous because the dynamic equilibrium is not competitive and we are not starting with an efficient allocation, but rather with an allocation that solves the planner's necessary first order conditions. The next proposition proves the converse of the last proposition. In this sense, it is an analogous to the First Welfare Theorem. It shows that, given a dynamic equilibrium, we can find the Lagrange multipliers solving the first order condition for an interior solution of the dynamic planner's problem.

Proposition 7 *Let $\{m, G, C, V, r\}$ be the allocation, value function, and interest rate in a dynamic equilibrium with a constant dynamic subsidy $s_d = s_d^*$ and, in the case of $\nu > 0$, also with a static subsidy $s_e = s_e^*$. Then, for this allocation, we can construct multipliers $\{\lambda, \omega\}$ that solve the necessary first order conditions for the planner's problem in Proposition 5.*

$$\lambda(g,t) = c(t)^{-\theta} a Z(t)^{\frac{1}{1-\nu}} \left(V(g,t) - V(G(t),t)\right)/s_\pi \text{ all } g \in [0, G(t)],\ t \geq 0\ , \tag{42}$$

$$\omega(t) = c(t)^{-\theta} a Z(t)^{\frac{1}{1-\nu}} r(t) V(G(t),t)/s_\pi \text{ all } t \geq 0\ , \tag{43}$$

for any s_π that satisfies that $s_d = s_d^*$. Moreover, if $\zeta \leq 1$ so that $F(m(\cdot,t))$ is concave, then this allocation is the solution to the dynamic planner's problem.

In general, as discussed earlier, the aggregate production function may not be strictly concave, and thus there can be multiple solutions to the necessary conditions of the planner's problem. Proposition 7 provides formulas to construct these allocations, by constructing the corresponding multipliers λ and ω. When the aggregate production function is concave, which occurs when $\zeta \leq 1$, the necessary conditions of the planner's problem are also sufficient for an efficient allocation. Therefore, when $\zeta \leq 1$, an equilibrium under the subsidies $s_e = s_e^*$ and $s_d = s_d^*$ is unique and optimal.

5 Balanced Growth Path(s)

In this section we analyze the equilibrium balanced growth paths. In a balanced growth path, the density $m(g,t)$ and the value function $V(g,t)$ stay constant through time, so we define $\bar{m}(g) \equiv m(g,t)$ and $\bar{V}(g) \equiv V(g,t)$. Likewise, the threshold $G(t)$ is also time-invariant, so with a slight abuse of notation we denote it by $G \equiv G(t)$. This implies that $Z(t)$ is time invariant, so $Z \equiv Z(t)$. Also, $P(t) = \bar{P}(Z) e^{-\frac{\gamma}{1-\nu} t}$. Likewise, we let $\bar{c}$ denote the detrended part of consumption $c(t)$. The interest rate $r(t)$ is constant through time and equal to $\bar{\rho}$. We denote profits by $\bar{\pi}(g; Z)$ as function of the two scalars g and Z, $\bar{\pi}(g; Z) = \zeta e^{-g\gamma(\eta-1)}/Z^{\eta-1}$ (44). The definition of a balanced growth path is standard. Mathematically, it boils down to deleting time from all expressions, transforming p.d.e.'s into o.d.e.'s. A balanced growth path can be studied as the composition of three blocks of equations: one block for firms, one block for aggregation, and one block for feasibility and prices.

The first block is, given Z, the value function V and G that solves the following o.d.e. and boundary conditions: $(\bar{\rho}+q)\bar{V}(g) = s_\pi \bar{\pi}(g;Z)+\bar{V}_g(g)+q\bar{V}(0)$ for $g \in [0,G]$ (45), $\bar{V}(G) = \bar{V}(0)-\phi\bar{P}(Z)/s_a$ (46), and $\bar{V}_g(G) = 0$ (47). For the case of a balance growth without costly adoption, (47) does not apply and (46) becomes an inequality, i.e., $\lim_{G\to\infty} \bar{V}(G) \geq \bar{V}(0) - \phi\bar{P}(Z)/s_a$ (48).

The second block provides, given G, the distribution of gaps $m(g)$ solves the following o.d.e. and boundary condition: $0 = \bar{m}_g(g) + q\bar{m}(g)$ for $g \in [0,G]$ (49), and $\bar{m}(0) = \bar{m}(G) + q\int_0^G \bar{m}(g)dg$ (50).

The third block provides, given Z, expressions for the price index $\bar{P}(Z)$, consumption $\bar{c}(Z)$, and given G, an expression for $\bar{Z}(G)$: $\bar{P}(Z) = \frac{s_a s_\pi}{\mathcal{A}(s_e)\mathcal{D}(s_e,s_d)}\frac{1}{a}Z^{-\frac{1}{1-\nu}}$ (51), $\bar{c}(Z) = \mathcal{A}(s_e)\, a\, Z^{\frac{1}{1-\nu}} - \phi\,(\bar{m}(0)-q)$ (52), and $\bar{Z}(G) = \left[\int_0^G e^{-\gamma g(\eta-1)}\bar{m}(g)dg\right]^{\frac{1}{\eta-1}}$ (53). We can now define an equilibrium balanced growth path.

Definition 4 *A balanced growth path with costly adoption is given by the value function of $\bar{V}(g)$, the density $\bar{m}(g)$, a threshold G^*, and intermediate input productivity Z^* such that (45), (46), (47), (49), (50), (51), (52) and (53) hold. In turn, a balance growth path without costly adoption is given by the value function of $\bar{V}(g)$, the density $\bar{m}(g)$, a threshold $G^* = \infty$, and intermediate input productivity Z^* such that (45), (48), (49), (50), (51), (52) and (53) hold.*

The next proposition shows that solving for a balanced growth path reduces to finding a solution to an equation in one unknown, G^*. The proposition characterizes this equation and, hence, the set of balanced growth paths.

Proposition 8 *A balanced growth path gap with costly adoption G^* is a solution of the equation $\bar{\phi} = \bar{H}(G) \equiv R(G)\,\zeta\,\bar{Z}(G)^{(\eta-1)(\zeta-1)}$, where $\bar{\phi} \equiv \frac{\phi}{a}\frac{1}{\mathcal{A}(s_e)\mathcal{D}(s_e,s_d)}$ is the adjusted fixed cost, $R(G) \equiv \frac{1}{q+\bar{\rho}}\left\{1 - e^{-\gamma(\eta-1)G} - \frac{\gamma(\eta-1)}{q+\bar{\rho}+\gamma(\eta-1)}\left[1 - e^{-(q+\bar{\rho}+\gamma(\eta-1))G}\right]\right\}$ and $\bar{Z}(G) = \left(\frac{q}{q+\gamma(\eta-1)}\frac{1-e^{-(q+\gamma(\eta-1))G}}{1-e^{-qG}}\right)^{\frac{1}{\eta-1}}$. The function $\bar{H}$ satisfies: $\bar{H}(0) = 0$, $\bar{H}'(0) > 0$, and has an asymptote: $\bar{H}(\infty) \equiv \lim_{G\to\infty}\bar{H}(G) = \left(\frac{q}{q+\gamma(\eta-1)}\right)^{\zeta-1}\frac{\zeta}{q+\bar{\rho}+\gamma(\eta-1)}$. We have:*

1. *If $\zeta \leq 1$, the function $\bar{H}(G)$ is monotone increasing in G. Then there is at most one balanced growth path. If $\bar{\phi} < \bar{H}(\infty)$ then there is unique $G^* < \infty$, which is increasing in the adjusted fixed cost $\bar{\phi}$.*

2. *If $\zeta > 1$ and ν is large enough, then $\bar{H}(G)$ is not monotone in G and there is a non-empty interval of $\bar{\phi}$ such that there are multiple balanced growth paths with costly adoption.*

When $\bar{H}(\infty) < \bar{\phi}$, there is a balanced growth path without costly adoption ($G^* = \infty$).

The proposition shows that if strategic complementarities are weak, i.e., if $\zeta \leq 1$, either there is a unique balanced growth path with costly adoption, or if $\bar{\phi}$ is large enough the balanced growth path has no costly adoption. The threshold for interior balanced growth path G^* is increasing in the adjusted adoption cost, $\bar{\phi}$. Moreover, note that the adjusted adoption cost $\bar{\phi}$ decreases with subsidy s_e and s_d, and thus G^* decreases, so that the balanced growth path has more adoption.

When strategic complementarities are strong, i.e., when $\zeta > 1$, while there is always a balanced growth path with costly adoption for low $\bar{\phi}$, there can be multiple balanced growth paths with costly adoption if $\bar{\phi}$ is in an intermediate range and ν is large. We stress that this is a sufficient condition for the existence of multiple balanced growth paths. It is easy to construct examples where $\zeta > 1$ and ν is small, and where multiple balanced growth paths occur.[7] The next Corollary provides a related important result.

Corollary 3 *Let $\bar{H}(\infty) < \frac{\phi}{a}\left(\frac{\eta}{\eta-1}\right)^{\frac{1}{1-\nu}}$ such that when $s_e = s_d = 1$ no costly adoption is a balance growth path, and assume that the laissez-faire economy is initially in the balanced growth path without costly adoption. Furthermore, assume that $\frac{\phi}{a} < \bar{H}(\infty)$. Then, setting $s_e = s_e^*$ and $s_d = s_d^*$ removes the economy from the balanced growth path without costly adoption.*

The Corollary notes cases where the optimal policy removes the economy from an undesirable balanced growth path. Interpreting the balanced growth path with no costly adoption as a 'poverty trap', the corollary shows that setting the optimal subsidies eliminates the poverty trap. Moreover, once the optimal the optimal subsidies are set, there is an equilibrium whose allocation that coincides with the efficient one.

[7] For example, choosing η close to one.

6 Model with Time-invariant Gaps

We consider a simplified version of the model with no free adoption opportunities, i.e., $q = 0$, and where a firm's technology grows at the same rate than the technological frontier, i.e., at rate γ per unit of time. If a firm wants to adopt the modern technology, it needs to pay the fixed cost $\Phi(t)$ measured in goods. As a result, as we discussed earlier, the adoption decision follows a threshold rule. That is, at time t all firms with gaps $g \leq G(t)$ are yet to adopt the frontier technology. Given the threshold rule, the state of the economy is characterized by an exogenous density $\underline{\mathrm{m}}(g)$ of firms with productivity gaps $g \in (0, G(t)]$, with $\underline{\mathrm{M}}(g)$ denoting its CDF, and a mass of firms operating the frontier technology $k(t)$; we assume that the economy starts with a distribution of productivity gaps $\underline{\mathrm{m}}(g)$ with support $g \in (0, \underline{\mathrm{G}}]$ and an initial mass point at $g = 0$ which we denote by $k(0)$. Finally, we allow firms to revert their technology adoption decisions at any point in time. That is, a firm that adopted the frontier technology can give up the acquired technique and return to its time-invariant gap g. This assumption mimics the idea in the time-varying gaps model above that firms can be drifting away from the frontier, as $G(t)$ can increase with t. The two versions of the model are very similar, exhibiting the same aggregate production function, and the same inefficiencies. As a result, the necessary conditions for efficiency are the same, and both economies have three stationary equilibria: one steady state with no adoption, one interior balanced growth path with little adoption, and one interior balanced growth path with high adoption. However, while the state of the economy with time-varying gaps is given by the multidimensional density $m(g,t)$, the state of the time-invariant gap economy is given by a unidimensional object, $k(t)$. This allows for a sharper characterization and analysis.

We begin the analysis by noting that there is a one-to-one relationship between the marginal adopter $G(t)$ and the mass of firms that already adopted by time t. This relationship satisfies:

$$k(t) = 1 - \int_0^{G(t)} \underline{\mathrm{m}}(g) dg \ . \tag{54}$$

Taking time derivatives, we obtain the evolution of the distribution of gaps g, which is characterized by the accumulation of mass at $g = 0$ and the evolution of the adoption threshold $G(t)$: $\dot{k}(t) = -\underline{\mathrm{m}}(G(t)) G'(t)$. For $k(t) = 1$ or, equivalently, $G(t) = 0$, $\dot{k}(t) \leq 0$. This follows from $G(t) \geq 0$.

As in the general formulation of the model, firms that adopt the frontier technology incur in a cost $\Phi(t)$, measured in units of output. When a firm adopts, it joins the mass of firms that adopted. Combining this with the aggregate feasibility constraint we obtain the counterpart to (3) in the general formulation of the model, $\Phi(t)\dot{k}(t) + C(t) = Y(t) \equiv e^{\frac{\gamma}{1-\nu}t}\mathcal{A}(s_e)\, a\, F(k(t))$ for $k(t) < 1$, or

$$\phi\dot{k}(t) + c(t) = \mathcal{A}(s_e)\, a\, F(k(t)), \text{ for } k(t) < 1, \tag{55}$$

where $C(t) = e^{\frac{\gamma}{1-\nu}t}c(t)$ and, analogous to the aggregate production function in the general formulation of the model in (26),

$$F(k) \equiv \left[\int_0^{\hat{G}(k)} e^{-g\gamma(\eta-1)}\underline{\mathrm{m}}(g)dg + k\right]^\zeta, \tag{56}$$

with $k = 1 - \int_0^{\hat{G}(k)} \underline{\mathrm{m}}(g)dg$, or $\hat{G}(k) = \underline{\mathrm{M}}^{-1}(1-k)$, where G is defined implicitly as a function of k in (54), $\mathcal{A}(s_e)$ is defined in (25), and a is an aggregate productivity shifter. When $k(t) = 1$, feasibility requires that $c(t) \leq \mathcal{A}(s_e)aF(1)$.

In this economy, the mass of adopters k can be interpreted as the productive capital stock. Its law of motion, and the intertemporal preferences, are akin to those in the Neoclassical Growth Model (NGM). A crucial difference with the standard formulation of the NGM is that in our setting the production function is only guaranteed to be concave if $\zeta \leq 1$. The next proposition gives a characterization of the shape of the aggregate production function $F(\cdot)$.

Proposition 9 *Assume that $\underline{\mathrm{m}}(g) > 0$ in its domain. Then, (1) $F(k) > 0$, $F'(k) > 0$, $F(k)$ is bounded, and $F(\cdot)$ is concave near $k = 1$, with $F'(1) = 0$; (2) Assume $\zeta \leq 1$ then $F(\cdot)$ is globally concave; and (3) Assume $\zeta > 1$. Let k^i be such that $F''(k) > 0$ if $k < k^i$, $F''(k) < 0$ if $k > k^i$, and $F''(k) = 0$ if $k = k^i$. If $e^{-g\gamma(\eta-1)}/\underline{\mathrm{m}}(g)$ is increasing in g, then $F(\cdot)$ has at most one inflection point k^i. Furthermore, if $\frac{F''(0)}{F'(0)} > 0$ then there is an interior inflection point k^i. A sufficient condition for $\frac{F''(0)}{F'(0)} > 0$ is that either ν or $\underline{G}$ are large enough.*

The proposition shows that there is one inflection point provided enough complementarities (ζ large). We now give an intuitive explanation of the forces in the model for concavity and convexity of F as a function of k. The force for concavity is simple, given the threshold nature of an equilibrium, in particular that $\hat{G}(k)$ is decreasing in k. For low values of k, i.e. when adoption is low, an increase in one unit of k moves firms with large gaps g to the frontier. Instead, for high k, an increase in one unit of k moves firms with small gaps g to the frontier. So, this force pushes the marginal productivity to be decreasing. The force for convexity, in the case of $\zeta > 1$ has been analyzed and explained in Proposition 4. Recall that there we fixed two gaps, and consider an arbitrary movement of density from one gap to another. While in the general formulation of the model, as described in Proposition 4, F is non-concave whenever $\zeta > 1$, in this simpler economy Proposition 9 provides extra conditions on exogenous parameters and functions for F being non-concave, exhibiting a convex region for low values of $k < k^i$ and a concave region otherwise. As a result, from now on when we refer to F being S-shaped, we refer to the conditions stated in Proposition 9 that guarantee this to occur.

6.1 Equilibrium

We now specialize the expressions for the different problems in the simplified economy. A firm's profit is given by

$$\pi\left(g,t\right) = s_{\pi}(t)\zeta \frac{e^{-g\gamma(\eta-1)}}{\int_0^{G(t)} e^{-g\gamma(\eta-1)}\underline{\mathrm{m}}(g)dg + k\left(t\right)} \ , \tag{57}$$

and the price of the final good is

$$P\left(t\right) = s_x(t)e^{-\frac{\gamma}{1-\nu}t}\frac{1}{a}\left[s_e(t)\frac{\eta-1}{\eta}\right]^{-\frac{1}{1-\nu}}\left[\int_0^{G(t)} e^{-g\gamma(\eta-1)}\underline{\mathrm{m}}(g)dg + k\left(t\right)\right]^{-\zeta} \ . \tag{58}$$

The value at t of a firm with gap g is given by

$$V\left(g,t\right) = \max_{\tau\geq t}\int_t^{\tau} e^{-\int_t^s r(\tilde{s})d\tilde{s}}\pi\left(g,s\right)ds + e^{-\int_t^{\tau} r(\tilde{s})d\tilde{s}}\left[V^0\left(g,\tau\right) - \frac{P\left(\tau\right)\Phi(\tau)}{s_a(\tau)}\right] \ . \tag{59}$$

Although the value function is indexed by (g,t), in this version of the model the first index is fixed unless the firm adopts. We let $V^0(g,t)$ the value of a g firm that has adopted the frontier technology, and can produce with it. The technology index of the firm stays with the firm forever. In particular, recall that in this version of the model we allow a g firm that has adopted the technology to disinvest, recovering the adoption cost, and use its original technology again. Then,

$$V^0(g,t) = \max_{\tau \geq t} \int_t^\tau e^{-\int_t^s r(\tilde{s})d\tilde{s}} \pi(0,s)\, ds + e^{-\int_t^\tau r(\tilde{s})d\tilde{s}} \left[V(g,\tau) + \frac{P(\tau)\Phi(\tau)}{s_a(\tau)} \right] . \quad (60)$$

The remaining equilibrium objects and definitions are the complete analog to the model with time-varying gaps. The refer to the balanced growth paths of the detrended economy as steady states. An equilibrium, given the density $\underline{m}$ and subsidies $\{s_e, s_d\}$, consists of paths of $\{k(t), c(t), G(t), r(t), P(t), V(g,t)\}$ for $t \geq 0$ and $g \in [0, G(t)]$ such that: (i) given $\{r(t), P(t), \pi(g,t)\}$ optimal adoption is given by $\{G(t)\}$, (ii) given $\{k(t), P(t)\}$, then $\pi(g,t)$ is given by (57), (iii) given $\{G(t), k(t)\}$ prices are given by (58), (iv) $\{c(t), r(t), P(t)\}$ satisfy the Euler equation in (8), and (v) the allocation $\{c(t), k(t)\}$ is a static equilibrium.

The next proposition gives a very simple characterization of the equilibrium as the solution of a system of two o.d.e.'s for $\{k(t), c(t)\}$ with an initial condition for $k(t)$, $k(0)$, and boundary conditions.

Proposition 10 *Fix $\{s_e, s_d\}$ and $k(0)$. A necessary and sufficient condition for an interior equilibrium is that the path $\{c(t), k(t)\}$ solves the following system of o.d.e.'s:*

$$\dot{k}(t) = \mathcal{A}(s_e)\, a\, F(k(t))/\phi - c(t)/\phi \ , \ \textit{for } 0 \leq k(t) \leq 1 \ , \quad (61)$$

where $\dot{k}(t) \leq 0$ if $k(t) = 1$ and $\dot{k}(t) \geq 0$ if $k(t) = 0$,

$$\theta \frac{\dot{c}(t)}{c(t)} = \mathcal{D}(s_e, s_d)\mathcal{A}(s_e) a F'(k(t))/\phi - \bar{\rho} \ , \textit{for all } t \geq 0 \ , \quad (62)$$

where $\mathcal{D}(s_e, s_d)\mathcal{A}(s_e) > 0$, and $0 = \lim_{T\to\infty} e^{-\bar{\rho}T} c(T)^{-\theta} \mathcal{A}(s_e) a F(k(T))$ (63). Also, the path must satisfy the boundary conditions: when $k(t) = 0$ then $0 \leq c(t) \leq a\mathcal{A}(s_e)F(0)$, and when $k(t) = 1$ then $c(t) \geq a\mathcal{A}(s_e)F(1)$.

The proposition shows that the economy aggregates to a version of the Neoclassical Growth Model with a couple of twists. First, recall that $\mathcal{A}(s_e)aF'(k)$ measures the marginal product of capital or, equivalently, the return to capital investment. While the static inefficiency depresses this return through its effect on $\mathcal{A}(s_e)$, the dynamic inefficiency manifests as a tax on this capital investment, i.e., $1 - \mathcal{D}(s_e, s_d)$. In the laissez-faire economy, where $s_e = s_d = 1$, the tax rate equals $[\eta(1-\nu)+\nu]^{-1} \in (0,1)$.

The investment tax persists even when the optimal static subsidy is implemented, i.e., when $s_e = s_e^*$ and $s_d = 1$; in this case, the tax equals η^{-1}. Second, depending on the strength of complementarities, $F'(k)$ may not be always negative. When this occurs, multiple interior steady states can emerge, and multiple dynamic equilibria can result from it. We will characterize these cases later in detail.

The representation shown in Proposition 10 consists of two o.d.e.s and a set of boundary conditions. The fact that the o.d.e. in (61) holds is relatively straightforward: It follows from aggregation of firms' actions, and that the optimal adoption is a threshold rule, which is implied in the definition of $k(t)$. What is remarkable is that the optimal adoption decision by firms, which inherently involves a discrete choice problem, is characterized by the o.d.e. in (62), which coincides with the Euler equation for the standard Neoclassical Growth model with an investment tax. The reason why this Euler equation summarizes the adoption decision of firms is that it captures the difference in the value of the discounted profit streams between adoption and no adoption for the marginal firm. This difference in the level of profit turns out to be proportional to the level of $F'(k)$, and the change in relative prices and interest rates turns out to be captured by $\dot{c}(t)/c(t)$.

For a given density $\underline{\mathrm{m}}$ and subsidies $\{s_e, s_d\}$, an interior steady state equilibrium is $k^* \in [0,1]$ such that $aF'(k^*) = \frac{\bar{\rho}\phi}{\mathcal{A}(s_e)D(s_e,s_d)}$ (64). Thus, the level of steady state capital combines two different technologies, i.e., the production technology aF and the cost adoption ϕ, distortions $\mathcal{A}(s_e)\mathcal{D}(s_e, s_d)$, and preferences as measured by $\bar{\rho}$. Moreover, notice that the steady state level of capital does not vary with $\bar{\rho}$ or ϕ as long as $\bar{\rho}\phi$ is constant: If an increase in discounting is offset by a decrease in the cost of investment, the economy's steady state is invariant. In fact, this observation is more general.[8] The next remark provides a useful normalization.

Remark 3 *Let $\hat{c}(t) \equiv c(t)/\phi$ and let $\hat{F}(k) \equiv F(k)/\phi$. Then, $\{c(t), k(t)\}$ is a dynamic equilibrium in a economy with aggregate output $\mathcal{A}(s_e)F(k(t))$ if and only if $\{\hat{c}(t), k(t)\}$ is a dynamic equilibrium in a economy with aggregate output $\mathcal{A}(s_e)\hat{F}(k(t))$.*

As a result of this remark, unless otherwise noted, we normalize $\phi = 1$. An important corollary of Proposition 9 is the following partial characterization of number of interior steady states.

[8] Let $\bar{\rho}\phi = \hat{\rho}$. For any pair of $\bar{\rho}$ and ϕ such that $\hat{\rho}$ is constant, the trajectories of the dynamical system in Proposition 10 are the same; trajectories only differ in the speed of time.

Corollary 4 *Consider the following two cases:*

1. *If $F(\cdot)$ is strictly concave, then there is at most one interior steady state. If in addition, $aF'(0) > \frac{\bar{\rho}}{\mathcal{A}(s_e)\mathcal{D}(s_e,s_d)}$, the interior steady state exists.*

2. *If $F(\cdot)$ is S-shaped, then there are at most two interior steady states. Moreover, (i) if $aF'(0) > \frac{\bar{\rho}}{\mathcal{A}(s_e)\mathcal{D}(s_e,s_d)}$ then there is exactly one interior steady state, (ii) if $aF'(k^i) > \frac{\bar{\rho}}{\mathcal{A}(s_e)\mathcal{D}(s_e,s_d)} > aF'(0)$ there are exactly two interior steady states, and (iii) if $\frac{\bar{\rho}}{\mathcal{A}(s_e)\mathcal{D}(s_e,s_d)} > aF'(k^i)$ there are no interior steady states.*

When F is concave the economy behaves exactly the same as in the Neoclassical Growth Model: Because $F''(k) < 0$ for all k, there can be at most one interior steady state, and if $\bar{\rho}$ is low, this interior steady state exists. When F is S-shaped, the same condition provides that there is just one interior steady state.[9] There are two interior steady states whenever $\bar{\rho}$ is intermediate, such that, once normalized by $\mathcal{A}(s_e)\mathcal{D}(s_e, s_d)$, $\bar{\rho}$ lies between the marginal product of capital at zero and the maximum value that the marginal product of capital can attain, which occurs at the inflection point k^i. Finally, there are no interior steady states when the marginal product of capital is below the normalized $\bar{\rho}$ for all k.

We next analyze the local behavior of any dynamic equilibrium around an interior steady sate (k^*, c^*). This local behavior is useful when later on we study global behavior of dynamic equilibria. Linearizing (61) and (62) around an interior steady state provides the following 'local' dynamical system,

$$\begin{bmatrix} \dot{k}(t) \\ \dot{c}(t) \end{bmatrix} = M \begin{bmatrix} k(t) - k^* \\ c(t) - c^* \end{bmatrix} \;,\; M \equiv \begin{bmatrix} \frac{\bar{\rho}}{\mathcal{D}(s_e,s_d)} & -1 \\ \bar{\rho}\frac{c^*}{\theta}\frac{F''(k^*)}{F'(k^*)} & 0 \end{bmatrix} .$$

The two eigenvalues of M are $\sigma_{1,2} = \frac{1}{2}\left[\frac{\bar{\rho}}{\mathcal{D}(s_e,s_d)} \pm \sqrt{\left(\frac{\bar{\rho}}{\mathcal{D}(s_e,s_d)}\right)^2 - 4\bar{\rho}\frac{c^*}{\theta}\frac{F''(k^*)}{F'(k^*)}}\right]$. The next proposition follows immediately.

Proposition 11 *Assume that $F(\cdot)$ is S-shaped and that the condition in Corollary 4 apply so that there are two interior steady states. Denote them by $0 < k_L^* < k_H^* < 1$, with corresponding consumption $0 < c_L^* < c_H^*$. Then,*

[9] There is another instance where there is only one interior steady state when F is S-shaped: when $aF'(k^i) = \frac{\bar{\rho}}{\mathcal{A}(s_e)\mathcal{D}(s_e,s_d)}$. This is a knife-edge case which we do not deem useful for our analysis.

1. *The steady state* (k_H^*, c_H^*) *is a saddle: Both eigenvalues of* M *are real with* $\sigma_1 < 0 < \rho < \sigma_2$.

2. *The steady state* (k_L^*, c_L^*) *is a source: Both eigenvalues of* M *have positive real part. In particular, there is a threshold* θ^**, given by* $\theta^* \equiv 4\, c_L^* \frac{\mathcal{D}(s_e,s_d)^2}{\bar{\rho}} \frac{F''(k_L^*)}{F'(k_L^*)} > 0$*, such that: (a) If* $\theta \geq \theta^*$ *both eigenvalues of* M *are real and strictly positive,* $0 < \sigma_1 < \sigma_2$*, and* (k_L^*, c_L^*) *is a nodal source; and (b) if* $\theta < \theta^*$ *the eigenvalues of* M *are complex conjugates, with strictly positive real part equal to to* $\frac{\bar{\rho}}{2\mathcal{D}(s_e,s_d)}$*, and* (k_L^*, c_L^*) *is a spiral source.*

Proposition 11 has important consequences. First, the high adoption interior steady state (k_H^*, c_H^*) is saddle-path stable: For $k(0)$ close to k_H^* one can construct a dynamic equilibrium path that converges to (k_H^*, c_H^*). Second, the interior steady state with low adoption (k_L^*, c_L^*) is not stable. This lack of stability provides that this equilibrium should not be observed in the long-run, since any perturbation from the steady state will make the dynamic equilibrium to move away from it. The way the allocation moves away depends on whether θ is above or below θ^*, where the condition for θ^* can be written as a critical value for the curvature of $U(\cdot)$ relative to the curvature on $F(\cdot)$. As it is well known in the case of concave $F(\cdot)$, the speed of convergence of the Neoclassical Growth Model depends on this ratio, as well as on the discount factor $\bar{\rho}$. Less well known is that when F is S-shaped, this quantity plays a crucial role on determining the nature of the unstable dynamics of the interior steady state with low adoption, i.e., whether (k_L^*, c_L^*) is a nodal or a spiral source. Third, there can be multiple equilibrium paths starting from the same initial condition, as a consequence of coordinated expectations about different future paths for prices. For instance, when $\theta < \theta^*$, some of these dynamic equilibria may have oscillatory initial behavior, limit cycles or, for or a given initial condition, one can construct two dynamic equilibria, one featuring $k(t) \to 0$, and one where $k(t) \to k_H^*$. The next lemma shows that there can be no limit cycles that stem from the unstable interior steady state.

Lemma 3 *Assume that* $F(\cdot)$ *is S-shaped and that the condition in Corollary 4 apply so that there are two interior steady states. For any* $\{s_e, s_d\}$ *there are no equilibrium limit cycles around the unstable interior steady state with low adoption.*

The proof is related to the analysis in Skiba (1978), and uses the Bendixson-Dulac theorem to show that, locally, for any connected set containing the unstable interior steady state (k_L^*, c_L^*), any dynamic equilibrium trajectory 'escapes' from this set. In other words, there is no equilibrium trajectory that can oscillate in a perpetual cycle. The next proposition analyzes the steady state with no adoption.

Proposition 12 *Assume that* $aF'(0) < \frac{\bar{\rho}}{\mathcal{A}(s_e)\mathcal{D}(s_e,s_d)}$. *Then* $k^* = 0$ *and* $c^* = \mathcal{A}(s_e)aF(0)$ *is a steady state equilibrium with no adoption (i.e., non-interior). Furthermore, the steady state is locally stable, i.e., there exists an* $\epsilon > 0$ *such that for all* $0 < k(0) \leq \epsilon$, *there is an equilibrium path* $\{c(t), k(t)\}_{t\in[0,T]}$ *for which* $c(T) = \mathcal{A}(s_e)aF(0)$ *and* $k(T) = 0$ *for* $0 < T < \infty$, *i.e., the convergence is in finite time.*

When $k = 0$ and $c(t) = \mathcal{A}(s_e)aF(0)$ for all t, no firm finds it suitable to adopt the modern technology. The condition $aF'(0) < \frac{\bar{\rho}}{\mathcal{A}(s_e)\mathcal{D}(s_e,s_d)}$ guarantees this. Moreover, the proposition also shows that the steady state is stable. The next proposition collects these last results and discusses that, in the long run, we should only observe the steady state with no adoption, or the stable interior steady state with high adoption.

Proposition 13 *In the long run, (i) if* $k(0) \neq k_L^*$ *then* $(k(t), c(t)) \to (0, \mathcal{A}(s_e)aF(0))$ *or* $(k(t), c(t)) \to (k_H^*, c_H^*)$, *(ii) if* $k(0) = k_L^*$ *then* $(k(t), c(t)) = (k_L^*, c_L^*)$ *is also possible.*

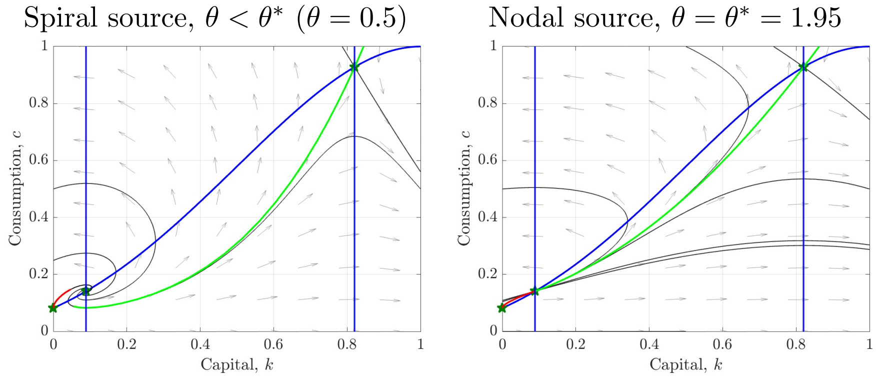
Consequently, all equilibrium paths must converge either to the no adoption steady state or the high-adoption interior steady state, unless if the initial level of adoption is exactly k_L^*. The proposition follows immediately from the results in Propositions 10, 11 and 12, together with Lemma 3 and Corollary 4. While the proposition states that in the long-run we should only observe either the steady state with no adoption or the interior steady state with high adoption, the nature of the unstable steady state has important consequences for equilibrium transitional dynamics.

6.2 Planner's problem

The sequence Planner's problem is given by

$$\max_{\{c(t)\}_{t\geq 0}} \int_0^\infty e^{-\bar{\rho}t} U(c(t))dt \text{ subject to } \phi \dot{k}(t) = a\, F(k(t)) - c(t)\ , \text{ with } k(0) \text{ given.}$$

Figure 1: Phase Diagram and Optimal Dynamic Path



Notes. We set $s_e = s_e^*$, $s_d = s_d^*$. The blue vertical lines denote the $\dot{c} = 0$ loci, while the the S-shaped blue line denotes the $\dot{k} = 0$ locus. For parameterization we assume that $\underline{m}$ is a truncated exponential $\underline{m}(g) = \chi e^{-\chi g}/(1 - e^{-\chi \underline{G}})$ with $\chi = 0.8$ and $\underline{G} = 7.5$. Other parameters values are $\eta = 3$, $N = 1$, $\gamma = 0$, $\nu = 0.75$, $\phi = 5$, $a = 1$, and $\rho = 0.15$.

Proposition 14 *Fix $k(0)$. An efficient allocation satisfies the following system:*

$$\dot{k}(t) = aF(k(t))/\phi - c(t)/\phi \;, \tag{65}$$

$$\theta \frac{\dot{c}(t)}{c(t)} = aF'(k(t))/\phi - \bar{\rho} \;, \tag{66}$$

with $0 = \lim_{T\to\infty} e^{-\bar{\rho}T} c(T)^{-\theta} k(T)$ *(67).*

When F is concave, if a dynamic allocation satisfies (65), (66) and (67) then the allocation is dynamically efficient. However, when F is S-shaped due to strong complementarities, satisfying this system of equations is only a necessary condition for efficiency. When $\zeta \leq 1$ and thus $F(k)$ is concave, the dynamical system corresponds to that one of the Neoclassical Growth Model. In this case, satisfying the system of equations is necessary and sufficient and, as long as the interior steady state exists, the unique solution of the dynamical system is given by the saddle path. A corollary of Propositions 10 and 14 is the following.

Corollary 5 *A necessary condition for an equilibrium to be efficient is that the subsidies satisfy $s_e = s_e^*$ $s_d = s_d^*$—so that $\mathcal{A}(s_e^*) = \mathcal{D}(s_e^*, s_d^*) = 1$; in this case, as in the general model, both static and dynamic distortions are corrected.*

Figure 1 shows the phase diagrams illustrating the equilibrium dynamics of an economy featuring strong complementarities, i.e., $\zeta > 1$, once we set $s_e = s_e^*$ and

$s_d = s_d^*$, so that $\mathcal{A}(s_e^*) = \mathcal{D}(s_e^*, s_d^*) = 1$. In each plot, the blue line depicts aggregate output $aF(k)$, the green filled stars account for steady state equilibria, the gray arrows provides the direction that the system $\{k, c\}$ moves at a given point, and the black lines account for different examples of dynamic equilibrium paths. There are three stationary equilibria: (i) one steady state with no adoption, $k = 0$, (ii) an interior steady state with low adoption, $k = k_L^*$, and (iii) an interior steady state with high adoption, $k = k_H^*$. The left panel considers the case where the elasticity of intertemporal substitution is high ($\theta = 0.5$), and below the threshold θ^*, while the right panel considers the case where the elasticity of intertemporal substitution is low ($\theta = 1.95$), and above the threshold θ^*. Both cases share the same stationary equilibria, which depends only on production parameters and the discount factor $\bar{\rho}$, as evident from (64). As discussed, the no-adoption steady state and the interior steady state with high adoption are stable. Furthermore, the interior high adoption steady state is a saddle: locally, there is one equilibrium path that converges to it. While in both figures the low adoption interior steady state is unstable—evident by the fact that the arrows push the equilibrium away from it, the case with low θ presented in the left panel exhibits a spiral source, while the case with high θ displays a nodal source. While we do not care per se about the low adoption interior steady state, the differential equilibrium dynamics sourcing from it are relevant when studying optimal paths and which policies may implement these paths.

When $\zeta > 1$ and the conditions are such that $F(\cdot)$ is S-shaped, there can be multiple solutions satisfying (65), (66) and (67) for a given initial condition $k(0)$. Indeed, in the left panel of Figure 1 you see that for $k(0)$ in the neighborhood of k_L^* there are multiple solutions. Away from an exceptional value for $k(0)$—known as the Skiba point, only one of those trajectories solves the planner's problem.

Important for our analysis is the early study of Skiba (1978), which provides the earliest characterization of the solution to the Neoclassical Growth Model with an S-shaped production function. A further characterization is available in Dechert and Nishimura (1983) and in Brock and Dechert (1983).[10] We restrict our attention to the case where $aF'(k^i) > \bar{\rho}$, where k^i is the inflection point of $F(k)$, so that there exist at least one interior steady state. The Supplemental Appendix summarizes the

[10]The setup in these papers is almost identical to the one for the efficient allocation in the model of this section. There are minor differences, for instance, these papers typically have $F(0) = 0$ and Dechert and Nishimura (1983) uses a discrete time version.

relevant results of these papers for our purposes.

6.2.1 Implementation of the Efficient Allocation

The next proposition collects results and shows that when $s_e = s_e^*$ and $s_d = s_d^*$ there exists a dynamic equilibrium which allocation corresponds to the efficient one.

Proposition 15 *Fix $k(0)$, and let $s_e = s_e^*$ and $s_d = s_d^*$ for all t. These policies implement the dynamically efficient allocation, i.e., there exists a path $\{c(t), k(t)\}$ that satisfy Propositions 10 and 14. We refer to these policies as the optimal policies.*

The optimal policy is such that, in equilibrium, agents internalize the whole surplus of their actions. While, given the optimal policy there is always a dynamic equilibrium that implements the dynamic efficient allocation, in some cases there can be multiple dynamic equilibria under the optimal policy. In these instances there can be a coordination failure and a dynamic equilibrium, with an allocation different from the efficient one, exists. We say that the optimal policy uniquely implements the efficient allocation when there is a unique dynamic equilibrium under the optimal policy. Likewise, if there are more than one dynamic equilibria under the optimal policy we say that the optimal policy does not uniquely implement the efficient allocation. Lack of unique implementation occurs when there are multiple steady states with ranked consumption, which is the extension to a dynamic general equilibrium model of Murphy et al. (1989).[11] Combining Propositions 14 and 15 imply the following corollary.

Corollary 6 *Fix $k(0)$. The optimal policy uniquely implements the efficient allocation if and only if there is a unique solution to the system (65), (66) and (67).*

In the remainder of this section we use this corollary to find sufficient conditions for unique implementation, or the failure of it. In particular, the way we use this corollary is by studying the phase diagram and properties of the efficient allocation. We provide several cases with sufficient conditions under which there is unique implementation, as well as cases where we give sufficient conditions where there is no

[11] Our definition of implementation is closer to the one used in Game Theory, which stresses uniqueness of equilibrium. Instead, Chari and Kehoe (1999) sidesteps this issue with their definition: "If there are multiple competitive equilibria associated with some policies, our definition of a Ramsey equilibrium requires that a selection be made from the set of competitive equilibria... In this chapter, we focus in the Ramsey equilibrium that yields the highest utility for the government."

unique implementation. The sufficient conditions we provide are intuitive and simple to interpret and are functions of ζ, $\bar{\rho}$ and θ.

Proposition 16 *Fix $k(0)$. Let $\zeta \leq 1$ so that F is concave. Then, under the optimal policy there is a unique dynamic equilibrium, and thus there is unique implementation. Moreover, if $\bar{\rho} \geq aF'(0)$ then the dynamic efficient allocation converges to the steady state with no adoption, and if $\bar{\rho} < aF'(0)$ the dynamic efficient allocation converges to the unique interior steady state.*

The proof of the proposition is straightforward. As stated earlier, when $\zeta \leq 1$ the economy behaves identically to the Neoclassical Growth Model. There is a unique dynamic equilibrium under the optimal policy, and whether the dynamic equilibrium converges to the steady state with no adoption or to the unique interior steady state depends on wether $\bar{\rho}$ is higher or lower than $aF'(0)$.

From now on in this section we assume that $\zeta > 1$ and $F(\cdot)$ is S-shaped. When $F(\cdot)$ is S-shaped the marginal product of capital is first increasing and then decreasing, giving rise to the possibility of two interior steady states. However, as in the case with $\zeta \leq 1$, if the discount rate is high the unique dynamic equilibrium converges to the steady state with no adoption.

Proposition 17 *Fix $k(0)$, and assume that $F(\cdot)$ is S-shaped and that $aF'(k^i) < \bar{\rho}$. Then, setting $s_e = s_e^*$ and $s_d = s_d^*$ uniquely implements the efficient allocation. Moreover, for any $k(0)$, the efficient allocation converges $k = 0$.*

Another case where there is unique implementation is when the discount rate is sufficiently low so that the steady state with no adoption does not exist and there is a unique interior steady state. Moreover, the interior steady state is a saddle. As a result, there is a unique dynamic equilibrium under the optimal policy.

Proposition 18 *Fix $k(0)$, and assume that $F(\cdot)$ is S-shaped, $aF'(0) > \bar{\rho}$ and that $aF'(k^i) > \bar{\rho}$. Then, setting $s_e = s_e^*$ and $s_d = s_d^*$ uniquely implements the efficient allocation. Moreover, the efficient allocation converges to the interior steady state.*

While $F(\cdot)$ is S-shaped, the assumption $aF'(0) > \bar{\rho}$ implies that there is a unique path that satisfies the equations in Proposition 14, i.e., the saddle path that converges to the interior steady state with high adoption. In this case, there is a unique interior steady state, and $k = 0$ is not a solution to the dynamical system. Therefore, for any $k(0)$, the optimal policy uniquely implements the dynamic efficient allocation.

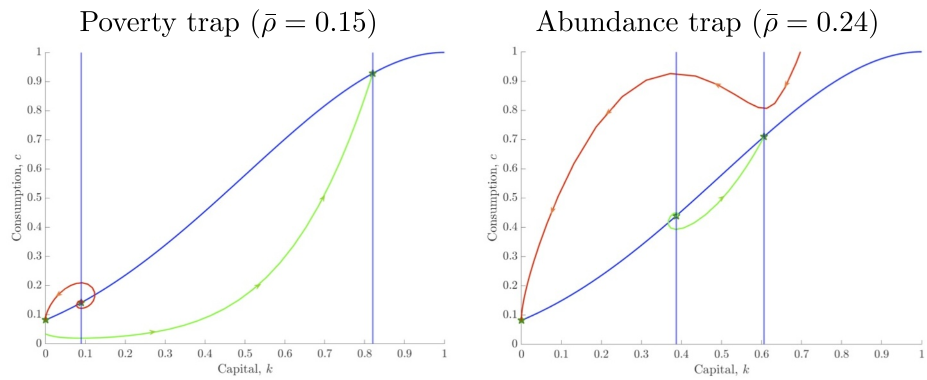
Next we assume that $\bar{\rho}$ is higher than in Proposition 18 and not very high—so that there are two interior steady states and one steady state with no adoption. In this case, at least for $k(0)$ close to k_H^*, the optimal policy uniquely implements the first best allocation.

Proposition 19 *Define $\tilde{k} \equiv \arg\max_{k>0} \left(aF(k) - aF(0)\right)/k$. Assume that (i) $F(\cdot)$ is S-shaped, (ii) $\bar{\rho} \in (aF'(0), aF'(k^i))$ and (iii) $\tilde{k} < k_H^*$. Then for any $k(0) > \tilde{k}$, setting $s_e = s_e^*$ and $s_d = s_d^*$ uniquely implements the dynamic efficient allocation, which converges monotonically to k_H^*.*

The logic of the proposition is the following. Consider a relaxed problem where the production function $F(\cdot)$ is replaced by $\tilde{F}(\cdot)$; $\tilde{F}(\cdot)$ is the smallest concave function satisfying $\tilde{F}(k) \geq F(k)$ for all k. The relaxed problem is a 'standard' problem. Now, given that k_H^* is a saddle, if there is a unique dynamic equilibrium converging to k_H^* in the relaxed problem, then this also must be true in the more constrained problem, at least locally. The assumption that $\tilde{k} < k_H^*$ guarantees that this is the case.

Now we study another case where there is unique implementation for intermediate values of the discount rate. However, while until now the elasticity of intertemporal substitution $1/\theta$ played no role in terms of implementation, here plays a crucial role in regulating dynamic paths. In particular, we show that, for high values of θ—or a low intertemporal elasticity of substitution, there is unique implementation of the efficient allocation for any $k(0)$, in spite of the fact that there are multiple steady state equilibria with different consumption under the optimal policy. The proof is involved, but the logic is the following. We study cases where θ is sufficiently large. We analyze separately initial conditions satisfying $k_L^* < k(0) \leq 1$, where we show that the economy converges to k_H^*, and initial conditions satisfying $0 \leq k(0) < k_L^*$, where the economy converges to the steady state with no adoption. In other words, we prove that there are two heteroclinic connections: one connecting the steady state with k_H^* with the the one with k_L^*, and one connecting this last one with the steady state with no adoption. In both cases, the proofs rely on showing that the saddle paths, i.e., the behavior of consumption with respect to capital, stay close to $\dot{k} = 0$ locus. This occurs when θ is high, i.e., the intertemporal elasticity of substitution is low, which provides that the household dislikes variations in consumption, and so capital,

Figure 2: The Traps



Notes. $\underline{\mathrm{m}}$ is a truncated exponential $\underline{\mathrm{m}}(g) = \chi e^{-\chi g}/(1 - e^{-\chi \underline{G}})$ with $\chi = 0.8$ and $\underline{G} = 7.5$. Other parameters: $\eta = 3$, $N = 1$, $s_e = s_e^*$ and $s_d = s_d^*$, $\gamma = 0$, $a = 1$, $\nu = 0.75$, $\phi = 5$, and $\theta = 0.2$.

and thus consumption, moves slowly through time. The relevance of this result for unique implementation is that once we constructed the path above, there are no other possible paths that solve the first order conditions of the planner's problem. The next proposition provides the formal statement.

Proposition 20 *Assume that $F(\cdot)$ is S-shaped, $\bar{\rho} \in (aF'(0), aF'(k^i))$. Let $s_e = s_e^*$ and $s_d = s_d^*$. Then, for any $k(0)$, there exists a $\tilde{\theta} > \theta^*$ such that for any $\theta \geq \tilde{\theta}$ there is a unique dynamic equilibrium which is efficient. Moreover, if $k(0) < k_L^*$ then $k(t) \downarrow 0$ and $c(t) \to aF(0)$, if $k(0) = k_L^*$ then $k(t) = k_L^*$ and $c(t) = c_L^*$ for all t, and if $k(0) > k_L^*$ then $k(t) \to k_H^*$ and $c(t) \to c_H^*$.*

We find interesting that the following three observations coexist under the assumptions of the proposition. First, there is a unique dynamic equilibrium which uniquely implements the dynamic efficient allocation. Second, the consumption of the three steady states is ranked, with the steady state with high adoption being the highest. Third, for any $k(0)$ there are feasible paths that converge to the steady state with high adoption. However, if the initial capital is less than k_L^*, the efficient allocation prescribes convergence to the steady state with the lowest consumption. The right panel of Figure 1 illustrates the results of the proposition, showcasing the two heteroclinic connections, one shown in red, and the other in green.

Until now we considered cases where there is unique implementation. Now we consider cases where there is no unique implementation, which corresponds to low values of θ, i.e., $\theta < \theta^*$ (Proposition 11). At the core of the multiplicity result lies the fact that θ is low. Technically, when $\theta < \theta^*$, the unstable interior balanced growth path with low adoption is a spiral source. In words, when θ is low, the households do not mind having paths with high variation in consumption. For example, in the extreme case when $\theta = 0$, all that matters for the value of a path is its present discounted value. Multiplicity of dynamic equilibria stems from the coordination of expectations of future profits by the marginal adopter that justify different current technology adoption policies. Together, these provide the building blocks for the coexistence of dynamic equilibria under the optimal policy for a given initial condition $k(0)$. We divide our analysis into two cases depending on the initial value of $k(0)$.

The first case occurs when $k(0)$ is close to k_L^*, as illustrated in the left panel of Figure 1. Several paths satisfy the necessary conditions of the planner's problem. Brock and Dechert (1983) provides that, for a given k, the optimal allocation corresponds to the largest or smallest consumption. As the figure shows, there are other trajectories consistent with a dynamic equilibrium. This always occur when $\theta < \theta^*$ since the interior steady state with low adoption is a spiral source. However, since all these trajectories converge to either $k = 0$ or k_H^* (Proposition 13), we do not focus on these cases as economies with k close to k_L^* should not be observed in the long run.

The second case occurs when the initial condition $k(0)$ is close to zero or close to k_H^*. We show that when the initial condition is the k of either the steady state with no adoption or the interior one with high adoption, the efficient allocation prescribes transitioning to a different steady state but there is a dynamic equilibrium that keeps the economy at the current steady state. Proposition 21 shows that when $k(0)$ is close to zero, the economy under the optimal policy is 'stuck' in a poverty trap. Perhaps more surprisingly, Proposition 22 shows that when $k(0)$ is close to k_H^* the economy under the optimal policy can be 'stuck' in an abundance trap. Each proposition provides sufficient conditions for these cases to occur.

The next proposition provides sufficient conditions under which, under the optimal policy: (a) if $k(0)$ is close to zero there are two dynamic equilibria, and (b) the efficient allocation corresponds to the dynamic equilibrium where $k(t)$ converges to k_H^*.

Proposition 21 *Let $s_e = s_e^*$ and $s_d = s_d^*$. Assume that $F(\cdot)$ is S-shaped and that $\bar{\rho} \in (aF'(0), aF'(k^i))$. Then, for $k(0)$ in the neighborhood of zero we have that*

1. *There is one dynamic equilibrium under the optimal policy that converges to the*

steady state with no adoption, i.e., $\{k(t), c(t)\} \to \{0, aF(0)\}$.

2. *If θ is positive and small, and $\frac{F(0)}{F(k_H^*)} < \left(1 + \frac{k_H^* F'(0)}{F(0)}\right)^{-\frac{\bar{\rho}}{aF'(0)}}$, then there is another dynamic equilibrium under the optimal policy where capital converges to the interior steady state with high adoption, i,e., $\{k(t), c(t)\} \to \{k_H^*, c_H^*\}$. This dynamic equilibrium implements the efficient allocation.*

Setting the subsidies at their optimal value, under the stated hypothesis, provides that the no adoption steady state is a locally stable equilibrium. Moreover, when θ is low and $k(0) = 0$, under extra technical conditions, there are exactly two dynamic equilibria: the first prescribes remaining in the steady state with no adoption, and the second one prescribes lower initial consumption and then convergence to the steady state k_H^*. The second dynamic equilibrium is the one that implements the efficient allocation. As a result, the steady state with no adoption can be understood as a 'poverty' trap. The left panel of Figure 2 illustrates this situation.

The next proposition considers the opposite end and provides sufficient conditions under which, under the optimal policy: (a) if $k(0)$ is close to the steady state k_H^* there are two dynamic equilibria, one converging to the interior steady state k_H^* and one converging to the steady state with no adoption, and (b) the efficient allocation corresponds to the dynamic equilibrium where $k(t)$ converges to zero.

Proposition 22 *Let $s_e = s_e^*$ and $s_d = s_d^*$. Assume that $F(\cdot)$ is S-shaped, $\bar{\rho} \in (aF'(0), aF'(k^i))$. Then, for $k(0)$ in the neighborhood of the high steady state k_H^* we have that*

1. *There is one dynamic equilibrium that converges to the interior steady state with high adoption, i.e., $\{k(t), c(t)\} \to \{k_H^*, c_H^*\}$.*

2. *If θ is positive but small enough and $\bar{\rho}$ is high enough so that $\left(aF(k_H^*) - aF(0)\right)/k_H^* < \bar{\rho}$, then there is another dynamic equilibrium where capital converges to the no adoption steady state, i.e., $\{k(t), c(t)\} \to \{0, aF(0)\}$. This dynamic equilibrium implements the efficient allocation.*

In this case, if $k(0)$ is close to k_H^*, there are two dynamic equilibria under the optimal policy. The first dynamic equilibrium has capital converging to k_H^* and its corresponding high consumption. The second dynamic equilibrium starts with very high consumption for a while, depletes capital, and then converges to zero in finite time. The second dynamic equilibrium is the one that implements the efficient allocation. As a result, the steady state with high adoption can be understood as an 'abundance' trap. The right panel of Figure 21 illustrates this situation.

6.2.2 Coordination Failures and Equilibrium Selection

Propositions 21 and 22 provide instances where there is no unique implementation of the efficient allocation under the optimal policy. In both cases, the source of the failure is a coordination problem: firms hold self-fulfilling beliefs about what other firms will do, and these beliefs sustain an equilibrium that differs from the efficient one. In this section we show how to supplement the optimal policy to resolve these coordination failures, and discuss the properties of the supplementary policy. We put special emphasis on eliminating the poverty trap, but we note that eliminating the abundance trap follows a similar logic.

Eliminating the poverty trap. When $k(0)$ is close to zero and θ is small, Proposition 21 establishes that there are two equilibria under the optimal policy: one converging to the no-adoption steady state, and one converging to k_H^*. The efficient allocation corresponds to the latter. The supplementary policy works by eliminating the former.

Remark 4 *Let all the conditions of Proposition 21 be satisfied so that, if $k(0) = 0$, there is no unique implementation. Let $c_p(k)$ denote consumption in the efficient allocation. Then, the following supplement to the policy provides unique implementation: $s_a(k) > \bar{\rho}/[aF'(0)] > 1$ for $c(k) > c_p(k)$ for all $k \leq k_L^*$, and $s_a(k) = 1$ otherwise.*

The intuition for why this policy works is the following. In the inefficient dynamic equilibrium converging to the no-adoption steady state, consumption falls as capital falls when $k < k_L^*$: firms do not adopt, capital declines, and households consume less and less as the economy contracts. The supplementary policy threatens to subsidize adoption whenever $k \leq k_L^*$ and consumption is above the efficient path $c^p(k)$. This threat makes the pessimistic equilibrium unsustainable: any path where consumption exceeds $c^p(k)$ for low k is eliminated, because households know that the adoption subsidy will be triggered, making it individually rational to adopt instead. As a result, the only remaining equilibrium is the one converging to k_H^*, which coincides with the efficient allocation.[12]

[12]The policy also avoids creating other equilibria.

Three properties of this supplementary policy deserve emphasis. First, it is costless: because it works through the threat of intervention rather than actual intervention, it is never triggered in equilibrium. The efficient allocation $c^p(k)$ is itself never above the threshold that would trigger the subsidy, so the policy has zero fiscal cost. For better or worst, this is reminiscent of the Taylor rule in models of monetary policy: just as a credible Taylor rule can pin down a unique equilibrium without ever being activated off-equilibrium, the supplementary policy here eliminates the undesired equilibrium through the credible threat of intervention. In particular, this policy is related to the one in Benhabib et al. (2002) which also analyzes a model with multiple steady states as well as multiple dynamic paths and a policy to provide unique implementation. Second, it introduces no new distortions: the subsidy $s_a(k)$ is set above one only off the equilibrium path, so it does not affect the equilibrium allocation, which continues to satisfy $s_e = s_e^*$ and $s_d = s_d^*$. The overall policy remains optimal. Third, and perhaps most importantly for policy, it is temporary: the supplementary policy $s_a(k) > 1$ only needs to be in place until $k(t)$ crosses k_L^*, which occurs in finite time t_p. We refer to this temporary intervention as a t_p-push: a supplementary, temporary policy that unlocks long-run development dynamics when the optimal policy alone fails to coordinate expectations.

Eliminating the abundance trap. When $k(0)$ is close to k_H^*, Proposition 22 establishes that there are two equilibria under the optimal policy: one converging to k_H^*, and one converging to the no-adoption steady state. The efficient allocation corresponds to the latter. The supplementary policy in this case works symmetrically to the poverty trap case, but in opposite direction: it threatens to tax adoption, eliminating paths where consumption rises with capital when k is close to k_H^*.

7 Implications for Development Dynamics

What do these results tell us about the role of optimal policy in unlocking persistent development dynamics? To be more precise, consider a laissez-faire economy, i.e., an economy with $s_e = s_d = 1$, and where strategic complementarities are strong enough so that the aggregate production function is S-shaped—so that there are three steady states. Let $k(0) = 0$ and the economy in the no-adoption steady state. Now assume that the optimal policy is adopted. Will this policy start a transition from the no-adoption to the high-adoption steady state? we discuss the conditions for this to occur using the characterization in the previous sections.

A simple case where persistent development dynamics are unlocked is when the optimal policy rules out the steady state with no adoption, which occurs when $aF'(0) > \bar{\rho}$. This relates to Proposition 18.

The following cases are such that even when setting $s_e = s_e^*$ and $s_d = s_d^*$ there are three steady states with ranked consumption. This occurs when $aF'(0) < \bar{\rho}$. Depending of the household's willingness to intertemporally substitute consumption, the optimal policy unlocks persistent development dynamics.

In spite of having the lowest consumption among all steady states, when the elasticity of intertemporal substitution is sufficiently low, remaining in the steady state with no adoption is optimal (see Proposition 20). In this case, the optimal policy only has level effect on output as the policy corrects for static misallocation for the case where $\nu > 0$, i.e., $A(1)$ increases to $A(s_e^*) = 1$. In other words, when θ is high, while feasible, it is not optimal to unlock long-run development dynamics.

Finally, when the elasticity of intertemporal substitution is sufficiently high and the discount factor is low, the optimal policy can unlock persistent development dynamics, as prescribed by the efficient allocation (Proposition 21). However, even in this case, setting $s_e = s_e^*$ and $s_d = s_d^*$ does not guarantee the occurrence of persistent development dynamics, as there are multiple dynamic equilibria even after distortions are corrected—the poverty trap is a consequence of self-fulfilling pessimistic expectations. For the efficient development dynamics to occur, the optimal policy needs to be supplemented with a policy that eliminates the trap by specifying subsidies that would be active in such a path. This policy selects the dynamic equilibrium implementing the efficient allocation (Remark 4), and it needs to be implemented for a finite amount of time. In this case, unlocking efficient long-run development dynamics results from the implementation of an optimal temporary Big Push.

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Supplemental Appendix

A Derivations

Derivation of Smooth Pasting Condition. We can use the expression characterizing τ in (16) to rewrite τ it in terms of $G(t)$. We then use a change of variables as $t \to t + \tau$ and $\tau \to G(t)$ to write: $0 = \pi(G(t), t) + qV(0, t) - (r(t) + q)[V(0, t) - \Phi(t)P(t)/s_a(t)] + V_t(0, t) - \frac{d}{dt}(\Phi(t)P(t)/s_a(t))$. Then, assume value matching, i.e., $V(G(t), t) = V(0, t) - \Phi(t)P(t)/s_a(t)$. Moreover, if V is differentiable at this point, we have $V_g(G(t), t)G'(t) + V_t(G(t), t) = V_t(0, t) - \frac{d}{dt}(\Phi(t)P(t)/s_a(t))$, so that $r(t)V(G(t), t) = \pi(G(t), t) + q[V(0, t) - V(G(t), t)] + V_g(G(t), t)G'(t) + V_t(G(t), t)$. Let's subtract the p.d.e. in (17) evaluated at $\{t, G(t)\}$ to obtain $0 = V_g(G(t), t)[G'(t) - 1]$, which gives smooth pasting given that $G'(t) < 1$ as $G(t)$ is finite.

Discrete time, discrete state version. We begin by discretizing time and gaps. We let $i = 1, 2, 3, \ldots$ index time, and $j = 1, 2, 3, \ldots$ index gap g. Both have a step size Δ. In particular, we let $t = (i-1)\Delta$ and $g = (j-1)\Delta$ for all i and j. We let $m_{i,j}\Delta$ be fraction of firms with gap $g = (j-1)\Delta$ at time $t = (i-1)\Delta$, so that $m_{i,j}$ has the units of a density. The law of motion of $m_{i,j}$ is given by $m_{j+1,i+1} = (1 - q\Delta)(m_{j,i} - a_{j,i})$ for $i, j = 1, 2, \ldots$ (68), where $a_{j,i}\Delta$ is the number of firms of type j at time i that adopt, and so $a_{j,i}$ has the units of a density. Moreover, $0 \leq a_{j,i} \leq m_{j,i}$ for $i, j = 1, 2, \ldots$ (69), which states that the density of adopters for a given gap at a given time cannot be larger that the density of firms with that given gap and time.[13] Finally, it must also be the case that, at any time $i = 1, 2, \ldots$, adding up all fraction of firms with different gaps $j = 1, 2, \ldots$ equals one, $\sum_j m_{j,i}\Delta = 1$ for all i (70). Using that this condition provides that $m_{1,i+1}\Delta = 1 - \sum_{j\geq 2} m_{j,i+1}\Delta$, we can use (68) to get $m_{1,i+1}\Delta = 1 - \sum_{j\geq 1}(1 - q\Delta)(m_{j,i} - a_{j,i})\Delta = \sum_{j\geq 1}(1 - q\Delta) a_{j,i}\Delta + q\Delta\sum_{j\geq 1} m_{j,i}\Delta$, where we used (70). Finally, dividing by Δ, $m_{1,i+1} = (1 - q\Delta)\sum_{j\geq 1} a_{j,i} + q\sum_{j\geq 1} m_{j,i}\Delta$ (71). Note that $m_{1,i+1}\Delta$ is the number of firms that adopt the technology in a period of length Δ between times indexed by i and $i+1$. The fraction of firms that adopt the technology in a costly manner during this period is thus given by

[13] We can rewrite the difference equation in (68), by subtracting $m_{i,j+1}$ from both sides and dividing by Δ, $\frac{m_{j+1,i+1} - m_{j+1,i}}{\Delta} = \frac{m_{j,i} - m_{j+1,i}}{\Delta} - qm_{j,i} - a_{j,i}(1 - q\Delta)/\Delta$, and if we let $\Delta \downarrow 0$ we obtain that $m_t(g, t) = -m_g(g, t) - qm(g, t)$ if $a(g, t) = 0$, and $m_t(g, t) = -\infty$ or not defined if $a(g, t) > 0$.

$\left(m_{1,i+1} - q\sum_{j\geq 1} m_{j,i}\right)\Delta = (1-q\Delta)\sum_{j\geq 1} a_{j,i}\Delta$.

Feasibility. For a period Δ,$c_i\Delta = [\sum_{j\geq 1} e^{-\gamma(\eta-1)(j-1)\Delta}\, m_{j,i}\,\Delta]^{\zeta}\Delta - \phi[m_{1,i+1} - q\sum_{j\geq 1} m_{j,i}\Delta]\Delta$, where $[\sum_{j\geq 1} e^{-\gamma(\eta-1)(j-1)\Delta}\, m_{j,i}\,\Delta]^{\zeta}\Delta$ is the output in a period of length Δ, and where $\sum_{j\geq 1} a_{j,i}\Delta$ is the fraction of firms going over a costly adoption of the technology in a period of length Δ. Then, it is immediate to obtain that $c_i = [\sum_{j=1,2...} e^{-\gamma(\eta-1)(j-1)\Delta}\, m_{j,i}\,\Delta]^{\zeta} - \phi[m_{1,i+1} - q\sum_{j=1}^{\infty} m_{j,i}\Delta]$ (72).

Planner's problem. The planner's problem is given by $\max_{\{a_{j,i}\}}\sum_{i\geq 1}\beta^i\frac{c_i^{1-\theta}}{1-\theta}\Delta$, where $\beta \equiv \frac{1}{1+\bar{\rho}\Delta}$, given an initial $\{m_{j,1}\}$ and subject to (68), (69), (70), (71), and (72). Note that in the objective function we adjust the discount factor β to be consistent with the discount rate $\bar{\rho}$, and use Δ so that the discounted sum of utilities converges to an integral of the discounted utilities for the continuum case. The next proposition shows that the solution of the planner problem is of the threshold type.

Proposition 23 *Let $\{m_{j,i}, a_{j,i}\}$ be the solution of the planner's problem. If $a_{j,i} > 0$, then for all integers $k > 0$ we have that $a_{j+k,i} = m_{j+k,i}$.*

Proof. The proof proceeds by a contradiction argument. Suppose that $a_{i,j} > 0$ and there is an integer $k > 0$ for which $a_{j+k,i} < m_{j+k,i}$. We will find an alternative policy with the same measure of adoption at all times but with higher consumption in period $i+1$. We will denote the new policy by $\{\tilde{a}_{j,i}\}$. To set this policy, let the strictly positive scalar ϵ be defined as $\epsilon = \frac{1}{2}\min\{a_{j,i}\,,\, m_{j+k,i} - a_{j+k,i}\} > 0$. The policy $\tilde{a}$ is identical to a at all times and gaps, except for $(j,i), (j+1,i+1), (j+k,i)$ and $(j+k+1,i+1)$. In particular we let $\tilde{a}_{j,i} = a_{j,i} - \epsilon$, $\tilde{a}_{j+1,i+1} = a_{j+1,i+1} + \epsilon(1-\Delta q)$, $\tilde{a}_{j+k,i} = a_{j+k,i} + \epsilon$, and $\tilde{a}_{j+k+1,i+1} = a_{j+k+1,i+1} - \epsilon(1-\Delta q)$. Letting $\tilde{m}$ the new process for the fraction of firms, we have that $\tilde{m}_{j+1,i+1} = m_{j+1,i+1} + (1-\Delta q)\epsilon$ and $\tilde{m}_{j+k+1,i+1} = m_{j+k+1,i+1} - (1-\Delta q)\epsilon$, but for all other j', i' we have $\tilde{m}_{j',i'} = m_{j',i'}$. Clearly $Y_{i+1}(\hat{m}_{i+1}) > Y_{i+1}(m_{i+1})$, and thus $\hat{c}_{i+1} > c_{i+1}$. ■

We can write the Lagrangian of the planner's problem $\mathcal{L}(a, m, \omega, \lambda)$ as follows,

$$\sum_{i=1}^{\infty}\beta^i\frac{[(\sum_{j=1}^{\infty} e^{-\gamma(\eta-1)(j-1)\Delta}\, m_{j,i}\,\Delta)^{\zeta} - \phi(m_{1,i+1} - q\sum_{j=1}^{\infty} m_{j,i}\Delta)]^{1-\theta}}{1-\theta}\Delta$$

$$+\sum_{i=1}^{\infty}\beta^i\omega_i[1 - \sum_{j=1}^{\infty} m_{j,i}\Delta]\Delta + \sum_{i=1}^{\infty}\beta^i(\sum_{j=1}^{\infty}\frac{\lambda_{j,i}\,[(1-\Delta q)\,(m_{j,i} - a_{j,i}) - m_{j+1,i+1}]}{\Delta}\Delta)\Delta\ .$$

Here we use $\beta^i\omega\Delta$ as the multiplier of (70) at time i and use $\beta^i\lambda_{j,i}\Delta^2$ as the multiplier of (68). Note that in general we have added the strictly positive multiplicative constant $\Delta > 0$ within the sums. Relative to the continuum case, the sum correspond to integrals, and Δ takes the place of both dt and dg. Further, we have written the law of motion for m dividing it by Δ because this is the object that converges to a non-trivial expression as Δ shrinks. Canceling Δ's, $\mathcal{L}(a, m, \omega, \lambda)$ is equal to

$$\sum_{i=1}^{\infty} \beta^i \frac{((\sum_{j=1}^{\infty} e^{-\gamma(\eta-1)(j-1)\Delta} m_{j,i} \Delta)^{\zeta} - \phi(m_{1,i+1} - q \sum_{j=1}^{\infty} m_{j,i}\Delta))^{1-\theta}}{1-\theta} \Delta$$
$$+ \sum_{i=1}^{\infty} \beta^i \omega_i [1 - \sum_{j=1}^{\infty} m_{j,i}\Delta]\Delta + \sum_{i=1}^{\infty} \beta^i (\sum_{j=1}^{\infty} \lambda_{j,i} [(1-\Delta q)(m_{j,i} - a_{j,i}) - m_{j+1,i+1}])\Delta \ .$$

The first order condition of $m_{j,i}$ for $j = 2, 3, \ldots$ and $i = 1, 2, \ldots$, in the case where $m_{j,i} > 0$ is $0 = \beta^i c_i^{-\theta} \frac{\partial Y_i}{m_{i,j}} \Delta + \beta^i c_i^{-\theta} \phi q \Delta^2 - \beta^i \omega_i \Delta^2 + \beta^i \lambda_{j,i}(1 - q\Delta)\Delta - \beta^{i-1} \lambda_{j-1,i-1} \Delta$, where $\frac{\partial Y_i}{m_{i,j}} = \frac{Y_i}{(\eta-1)(1-\nu)} \frac{e^{-\gamma(\eta-1)(j-1)\Delta}}{\sum_{j'=1}^{\infty} e^{-\gamma(\eta-1)(j'-1)\Delta} m_{j',i} \Delta} \Delta$. Dividing by $\beta^i \Delta$, using that $1/\beta = 1 + \Delta\bar{\rho}$ and rearranging, $\Delta(\lambda_{j-1,i-1}\bar{\rho} + \lambda_{j,i}\bar{\rho}) = c_i^{-\theta}(\frac{Y_i}{(\eta-1)(1-\nu)} \frac{e^{-\gamma(\eta-1)(j-1)\Delta}}{\sum_{j'=1}^{\infty} e^{-\gamma(\eta-1)(j'-1)\Delta} m_{j',i} \Delta} + \phi q)\Delta - \omega_i \Delta + \lambda_{j,i} - \lambda_{j-1,i-1}$. Finally, adding and subtracting $\lambda_{j,i-1}$, dividing by Δ, and taking the limit as $\Delta \downarrow 0$, $(\bar{\rho} + q)\lambda(g,t) = c(t)^{-\theta}(\frac{\partial Y(t)}{\partial m(g,t)} + \phi q) - \omega(t) + \lambda_g(g,t) + \lambda_t(g,t)$ (73). The first order condition of $\mathcal{L}(a, m, \omega, \lambda)$ with respect to $m_{1,i+1}$ for $i = 1, 2, \ldots$ is $0 = -\beta^i c_i^{-\theta} \phi \Delta + \beta^{i+1} \lambda_{1,i+1}(1 - \Delta q)\Delta - \beta^{i+1} \omega_{i+1} \Delta^2 + \beta^{i+1} c_{i+1}^{-\theta} \frac{\partial Y_{i+1}}{\partial m_{1,i+1}} \Delta + \beta^{i+1} c_{i+1}^{-\theta} q \phi \Delta^2$. Following similar steps as before, as taking $\Delta \downarrow 0$ we get $\lambda(0,t) = c(t)^{-\theta}\phi$ (74). Combining (73) and (74),

$$(\bar{\rho} + q)\lambda(g,t) = c(t)^{-\theta} \frac{\partial Y(t)}{\partial m(g,t)} + q\lambda(0,t) - \omega(t) + \lambda_g(g,t) + \lambda_t(g,t) \qquad (75)$$

The first order condition for $a_{j,i}$, if for that (j,i) then $0 < a_{j,i} \le m_{j,i}$ is interior, is $0 \le \frac{\partial \mathcal{L}(a,m,\omega,\lambda)}{\partial a_{j,i}} = -\beta^i \lambda_{j,i}(1 - q\Delta)\Delta$, or $\lambda_{j,i} \le 0$. If $a_{j,i} < m_{i,j}$ then this has to hold with equality, i.e.. if $0 < a_{j,i} < m_{j,i}$, then $\lambda_{j,i} = 0$.

For those $0 = a_{j,i} < m_{j,i}$ then $0 \ge \frac{\partial \mathcal{L}(a,m,\omega,\lambda)}{\partial a_{j,i}} = -\beta^i \lambda_{j,i}(1 - q\Delta)$, or $0 \ge \frac{\partial \mathcal{L}(a,m,\omega,\lambda)}{\partial a_{j,i}} = -\beta^i \lambda_{j,i}(1 - q\Delta)\Delta$, or $\lambda_{j,i} \ge 0$.

These three cases fimply that if $a(G,t) > 0$ then $\lambda(G(t),t) = 0$ (76).

Concavity of the planner's period return function. Let $F(m, m_1')$ be defined as the period return function of the planner problem, i.e.,

$$F(m,m_1') = \frac{\left(\left(\sum_{j=1}^{\infty} ye^{-\gamma(\eta-1)(j-1)\Delta}\, m_j\, \Delta\right)^{\zeta} - \phi\left(m_1' - q\sum_{j=1}^{\infty} m_{j,i}\Delta\right)\right)^{1-\theta}}{1-\theta}$$

Note that, the constraints of the planner defined a convex set, so the planner problem is a convex if F is concave. We can write the recursive equation for the planner using a distribution $m = \{m_1, m_2, \dots\}$ as the current state state and m' as the next period state, and u as the value function, as $u(m) = \max_{m'\in\Gamma(m)} \Delta\, F(m,m_1') + \beta\, u(m')$, where $\Gamma(m) = \{\, m' \,:\, 0 \le m'_{j+1} \le (1-\Delta q)m_j$ for $j = 1,2,3,\dots$ and $0 \le m_1'\,\Delta = 1 - \sum_{j=1}^{\infty} m_j'\,\Delta$.

Proposition 24 *F is concave if and only if $\zeta \le 1$.*

Proof. If $\zeta \le 1$, and $\theta \ge 0$, then F is the composition of concave functions, and hence concave. If $\zeta > 1$, then F is not concave, regardless of θ. In particular, we can show that the function is not quasi-concave. Recall that F is quasi-concave if the upper contour set is convex. To show that if $\zeta > 1$ the function is not quasi-concave, define $Q(M,x) = \frac{\left(M^{\zeta}-\phi x+\phi q\right)^{1-\theta}}{1-\theta}$. To find the upper contour set fix $\bar{c} > 0$, define $Q^{\bar{c}} \equiv \{M, x : \frac{\left(M^{\zeta}-\phi(x-q)\right)^{1-\theta}}{1-\theta} \ge \frac{\bar{c}^{1-\theta}}{1-\theta}\}$ or, equivalently, $Q^{\bar{c}} = \left\{M, x : M^{\zeta} - \phi(x-q) \ge \bar{c}\right\} = \left\{M, x : M \ge \psi(x;\bar{c}) \equiv (\bar{c} + \Phi(x-q))^{1/\zeta}\right\}$. Since $\psi(\cdot;\bar{c})$ is strictly concave for $\psi > 1$, then $Q^{\bar{c}}$ is not convex. ■

Relevant results from the convex-concave neoclassical growth model. To state some of the properties it is useful to define the maximized Hamiltonian. Let $\mathcal{H}(k,\lambda) \equiv \max_c u(c) + \lambda \frac{aF(k)-c}{\phi}$ denote the maximized Hamiltonian of the planning problem, where optimality implies that $\lambda = \phi u'(c)$. Here the relevant results of these papers, applied to our setting: (1) In an efficient allocation, $k(t)$ is either weakly increasing or weakly decreasing. Thus, $k(t)$ is a monotone function of time; (2) Take a trajectory $\{k(t), c(t)\}$ for all t that satisfies (65), (66) and (67). Then, the value of this trajectory is equal to the value of the maximized Hamiltonian evaluated at $t = 0$, i.e., $\mathcal{H}(k(0), \lambda(0))$, where $\lambda(0) = u'(c(0))$; (3) Since that the maximized Hamiltonian is convex in λ, if for $k(0)$ there are several trajectories $\{k(t), c(t)\}$ for all t that satisfy (65), (66) and (67), the optimal must be the trajectory with either the smallest or largest $c(0)$; and (4) If for $k(0) = k^*$ then (k^*, c^*) is a stationary solution to (65), (66) and there is non-stationary trajectory $\{k(t), c(t)\}$ with $k(0) = k^*$ that solves (65), (66) and (67), then the efficient one is the non-stationary one.

B Proofs

Proof of Proposition 1. We replace the optimal price (13) into $P(t)$ in (10). Then, immediate calculations provide (19). The first order conditions of the problem in (11) give $\frac{w(t)/s_l(t)}{P(t)/s_x(t)} n(g,t) \frac{\nu}{1-\nu} = x(g,t)$ (77) which is (23). We can replace the optimal x into the production function to obtain $y(g,t) = e^{\gamma(t-g)}\, n(g,t) a^{1-\nu} \frac{1}{1-\nu} \left(\frac{w(t)/s_l(t)}{P(t)/s_x(t)} \right)^{\nu}$. Taking a ratio for $g_1 = g$ and $g_2 = 0$, $\frac{y(g,t)}{y(0,t)} = e^{-\gamma g} \frac{n(g,t)}{n(0,t)}$. Using (13), we obtain an expression for $y(g,t)$. Then, evaluating at $g_1 = g$ and $g_2 = 0$, $\frac{y(g,t)}{y(0,t)} = e^{-\eta\gamma g}$. As a result, $n(g,t) = n(0,t) e^{-(\eta-1)\gamma g}$. Then, using market clearing, $1 = \int_0^{G(t)} n(g,t) m(g,t) dg = n(0,t) \int_0^{G(t)} e^{-(\eta-1)\gamma g} m(g,t) dg$, and using this result in the expression for $n(g,t)$ provides $n(g,t) = \frac{e^{-(\eta-1)\gamma g}}{\int_0^{G(t)} e^{-(\eta-1)\gamma g} m(g,t) dg}$, which is (22). Using the expression for $Q(t)$ in (5) together with (77), (22) and (19) provides (20). As an intermediate step to obtain an expression for $Y(t)$ we compute $X(t)$. Given (6), $X(t) = \frac{w(t)/s_l(t)}{P(t)/s_x(t)} \frac{\nu}{1-\nu} \int_0^{G(t)} n(g,t) m(g,t) dg = \frac{w(t)/s_l(t)}{P(t)/s_x(t)} \frac{\nu}{1-\nu}$. Then, given that $Y(t) = Q(t) - X(t)$, it is now immediate to obtain $Y(t)$ given the expression for $Q(t)$ in (20).

Proof of Proposition 3. Using the first order conditions of (31) we note that we can write $x(g,t) = B(m,y,n,t)\tilde{a}\nu$ and $n(g,t) W^P = B(m,y,n,t)\tilde{a}(1-\nu)$, so that $x(g,t) = \frac{\nu}{1-\nu} n(g,t) W^P$. Thus,

$$Y^P(m,t) = \max_n e^{\gamma t} \tilde{a} \left(\frac{\nu}{1-\nu} \right)^{\nu} \left(W^P \right)^{\nu} \left[\int_0^{G(t)} \left(e^{-g\gamma}\, n(g,t) \right)^{1-\frac{1}{\eta}} m(g,t)\, dg \right]^{\frac{1}{1-1/\eta}}$$
$$+ W^P N - W^P \left(1 + \frac{\nu}{1-\nu} \right) \int_0^{G(t)} n(g,t)\, m(g,t)\, dg \; ,$$

where we allow total labor supply to be N, and later evaluate the expressions at $N = 1$. Also, we use that $\tilde{a} \left(\frac{\nu}{1-\nu} \right)^{\nu} = a^{1-\nu} (1-\nu)^{\nu-1} \nu^{-\nu} \left(\frac{\nu}{1-\nu} \right)^{\nu} = a^{1-\nu} \frac{1}{1-\nu}$. Notice that this problem is homogeneous of degree one in N i.e., $Y^P(m,t,N) = N\, Y^P(m,t)$. Then, the Lagrange multiplier W^P is independent of N. Setting $N = 1$ and taking the first order condition with respect to n,

$$\left(W^P\right)^{1-\nu} = e^{\gamma t} a^{1-\nu} \left[\int_0^{G(t)} \left(e^{-g\gamma}\, n(g,t)\right)^{1-\frac{1}{\eta}} m(g,t) dg\right]^{\frac{1}{1-1/\eta}-1} \left(e^{-g\gamma}\, n(g,t)\right)^{-\frac{1}{\eta}} e^{-g\gamma} \; .$$

Take any two g_1 and g_2 satisfying $0 \le g_1 < g_2 \le G(t)$. The above expression implies that $n(g_1,t)^{-\frac{1}{\eta}} e^{-g_1\gamma\left(1-\frac{1}{\eta}\right)} = n(g_2,t)^{-\frac{1}{\eta}} e^{-g_2\gamma\left(1-\frac{1}{\eta}\right)}$, or $n(g_1,t) = n(g_2,t) e^{(g_2-g_1)\gamma(\eta-1)}$, or $n(g,t) = n(0,t)e^{-g\gamma(\eta-1)}$. Then, feasibility gives $N = 1 = \int_0^{G(t)} n(g,t)m(g,t)dg = n(0,t)\int_0^{G(t)} m(g,t)e^{-g\gamma(\eta-1)}dg$, or $n(g,t) = N\frac{e^{-g\gamma(\eta-1)}}{\int_0^{G(t)} e^{-g\gamma(\eta-1)} m(g,t)dg} = \frac{e^{-g\gamma(\eta-1)}}{\int_0^{G(t)} e^{-g\gamma(\eta-1)} m(g,t)dg}$. Replacing into the objective function and operating,

$$Y^P(m,t) = \frac{a^{1-\nu} e^{\gamma t} W^P(m,t,1)^{\nu} \left[\int_0^{G(t)} e^{-g\gamma(\eta-1)}\, m(g,t)\, dg\right]^{\frac{1}{\eta-1}} - \nu W^P(m,t,1)}{1-\nu}. \tag{78}$$

Next, we solve for $W^P(m,0,1)$. On the one hand, we use the envelope in the definition of Y^P in (31) to get $\frac{d}{dN} Y^P(m,t) = W^P(m,t,1)$. On the other hand, we also differentiate the expression in (78). Equating both expressions and rearranging we get $W^P(m,t) = e^{\frac{\gamma}{1-\nu}t} a \left[\int_0^{G(t)} e^{-g\gamma(\eta-1)}\, m(g,t)\, dg\right]^{\frac{1}{(\eta-1)(1-\nu)}}$.Then, replacing W^P into (78) and using $N = 1$, $Y^P(m,t) = W^P(m,t,1) = e^{\frac{\gamma}{1-\nu}t} a \left[\int_0^{G(t)} e^{-g\gamma(\eta-1)}\, m(g,t)\, dg\right]^{\frac{1}{(\eta-1)(1-\nu)}}$. Finally, using the expressions above, we can obtain $Q^P(t) = \frac{1}{1-\nu} W^P(m,t,1)$.

Proof of Proposition 4. From the definition of $Y^P(m,t)$ in Proposition 3, after differentiating with respect to ϵ and evaluating at $\epsilon = 0$,

$$\left.\frac{d}{d\epsilon} \log Y^P(m^{\epsilon,\alpha})\right|_{\epsilon=0} = \frac{1}{(\eta-1)(1-\nu)} \frac{\int_{g_1-\alpha/2}^{g_1+\alpha/2} e^{-\gamma(\eta-1)g} dg - \int_{g_2-\alpha/2}^{g_2+\alpha/2} e^{-\gamma(\eta-1)g} dg}{\int_0^G e^{-\gamma(\eta-1)g)} m(g) dg} \; .$$

Dividing both sides by α and taking the limit as α approaches zero provides (32). For (33) we begin by computing the second derivative of Y^P with respect to ϵ and evaluate it $\epsilon = 0$,

$$\begin{aligned}\left.\frac{d^2}{d\epsilon^2} Y^P(m^{\epsilon,\alpha})\right|_{\epsilon=0} =& e^{\frac{\gamma}{1-\nu}t} a \left[\frac{1}{(\eta-1)(1-\nu)} - 1\right] \frac{\left[\int_0^G e^{-\gamma(\eta-1)g} m(g) dg\right]^{\frac{1}{(\eta-1)(1-\nu)}-2}}{(\eta-1)(1-\nu)} \\ & \times \left(\int_{g_1-\alpha/2}^{g_1+\alpha/2} e^{-\gamma(\eta-1)g} dg - \int_{g_2-\alpha/2}^{g_2+\alpha/2} e^{-\gamma(\eta-1)g} dg\right)^2 \; .\end{aligned}$$

It is now immediate to see that Y^P is concave in ϵ only if $\frac{1}{(\eta-1)(1-\nu)} \leq 1$, which is the condition presented in (33). Following similar steps as above,

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \frac{d}{d\epsilon} Y^P(m^{\epsilon,\alpha}) \Big|_{\epsilon=0} = \frac{e^{\frac{\gamma}{1-\nu}t} a}{(\eta-1)(1-\nu)} \frac{e^{-\gamma(\eta-1)g_1} - e^{-\gamma(\eta-1)g_2}}{\left[\int_0^G e^{-\gamma(\eta-1)g} m(g) dg\right]^{1-\frac{1}{(\eta-1)(1-\nu)}}} .$$

Then, using (19), the expression for profit in Proposition 2, and the normalization $w(t)/s_l(t) = 1$, we can rewrite it as

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \frac{d}{d\epsilon} Y^P(m^{\epsilon,\alpha}) \Big|_{\epsilon=0} s_\pi(t) \left[s_r(t) s_x(t) \left(\frac{\eta-1}{\eta} \right) \right]^{\frac{1}{1-\nu}} = \frac{\pi(g_1,t) - \pi(g_2,t)}{P(t)/s_x(t)} .$$

Finally, direct calculations and the definitions of $\mathcal{A}(s_e)$ and $\mathcal{D}(s_e, s_d)$ provide (34).

Proof of Proposition 5. As a preliminary step we write the objective function as $\int_0^\infty e^{\left(-\rho+(1-\theta)\frac{\gamma}{1-\nu}\right)t} \frac{\left(Y^P(m(\cdot,t)) - \phi(m(0,t) - \int_0^{G(t)} m(g,t)dg)\right)^{1-\theta} - 1}{1-\theta} dt$, where $Y^P(m(\cdot,t)) = a\, Z(t)^{\frac{1}{1-\nu}}$ follows from $Y^P(m(\cdot,t),0) \equiv a \left[\int_0^{G(t)} e^{-g\gamma(\eta-1)}\, m(g,t)\, dg\right]^{\frac{1}{(\eta-1)(1-\nu)}}$. We attach a Lagrange multiplier $e^{-\bar{\rho}t}\lambda(g,t)$ to the constraint (1) for each (g,t), a Lagrange multiplier $e^{-\bar{\rho}t}\omega(t)$ to the mass preservation constraint in (1) for each t, and a Lagrange multiplier $e^{-\bar{\rho}t}\xi(t)$ to the constraint that $u(t) = G'(t)$. The solution of the planner problem is attained by computing $\max_{u,G,m} \min_{\lambda,\omega,\xi} \mathcal{L}(m, G, u\lambda, \omega, \xi)$, where the Lagrangian is defined as:

$$\begin{aligned}
\mathcal{L} = & \lim_{T\to\infty} \int_0^T e^{-\bar{\rho}t} \frac{\left(Y^P(m(\cdot,t),0) - \phi\left(m(0,t) - q\int_0^{G(t)} m(g,t)dg\right)\right)^{1-\theta} - 1}{1-\theta} dt \\
& - \lim_{T\to\infty} \int_0^T \int_0^{G(t)} e^{-\bar{\rho}t} \lambda(g,t) \left[m_t(g,t) + m_g(g,t) + q\, m(g,t)\right] dg dt \\
& + \lim_{T\to\infty} \int_0^T e^{-\bar{\rho}t}\omega(t) \left[1 - \int_0^{G(t)} m(g,t) dg\right] dt \\
& + \lim_{T\to\infty} \int_0^T e^{-\bar{\rho}t}\xi(t) \left[u(t) - G'(t)\right] dt .
\end{aligned}$$

We use integration by parts to rewrite the Lagrangian as

$$\mathcal{L} = \lim_{T\to\infty}\int_0^T e^{-\bar{\rho}t}\frac{\left(Y^P(m(\cdot,t),0)-\phi(m(0,t)-q\int_0^{G(t)}m(g,t)dg)\right)^{1-\theta}-1}{1-\theta}dt$$
$$-\lim_{T\to\infty}\int_0^T\int_0^{G(t)} e^{-\bar{\rho}t}m(g,t)\left[q\lambda(g,t)+\bar{\rho}\lambda(g,t)-\lambda_t(g,t)-\lambda_g(g,t)\right]dgdt$$
$$-\lim_{T\to\infty}\int_0^{G(t)} e^{-\bar{\rho}t}\lambda(g,t)m(g,t)|_{t=0}^{t=T}dg-\lim_{T\to\infty}\int_0^T e^{-\bar{\rho}t}\lambda(g,t)m(g,t)|_{g=0}^{g=G(t)}dt$$
$$+\lim_{T\to\infty}\int_0^T e^{-\bar{\rho}t}\omega(t)\left[1-\int_0^{G(t)}m(g,t)dg\right]dt$$
$$+\lim_{T\to\infty}\int_0^T e^{-\bar{\rho}t}\left[\left(\xi_t(t)-\bar{\rho}\xi(t)\right)G(t)+\xi(t)u(t)\right]dt-\lim_{T\to\infty}e^{-\bar{\rho}t}\xi(t)G(t)|_{t=0}^{t=T}\ .$$

The first order conditions with respect to $m(g,t)$ for $t>0$ and $g\in(0,G(t))$ are:

$$\frac{\partial\mathcal{L}}{\partial m(g,t)} = e^{-\bar{\rho}t}\left(C(t)\right)^{-\theta}\frac{\partial Y^P(m(\cdot,t),0)}{\partial m(g,t)}dt+q\phi C(t)^{-\theta}dt$$
$$-e^{-\bar{\rho}t}\left[q\lambda(g.t)-\omega(t)+\bar{\rho}\lambda(g,t)-\lambda_t(g,t)-\lambda_g(g,t)\right]dg\,dt\le 0\ ,$$

where $c(t)=Y^P(m(\cdot,t),0)-\phi(m(0,t)-q)$ and $\frac{\partial Y^P(m(\cdot,t))}{\partial m(g,t)}=aZ(t)^{\frac{1}{1-\nu}}\hat{\pi}(g,t)$, and where $\frac{\partial\mathcal{L}}{\partial m(g,t)}=0$ if $m(g,t)>0$. Thus, when $m(g,t)>0$, this implies that

$$\bar{\rho}\lambda(g,t)=c(t)^{-\theta}aZ(t)^{\frac{1}{1-\nu}}\hat{\pi}(g,t)-\omega(t)+\lambda_g(g,t)+\lambda_t(g,t)+q[\phi c(t)^{-\theta}-\lambda(g,t)]\ .$$

The first order conditions of $\mathcal{L}$ with respect to the boundary terms are

$$0=\frac{\partial\mathcal{L}}{\partial m(0,t)}=e^{-\bar{\rho}t}\left[\lambda(0,t)-\phi c(t)^{-\theta}\right]dt\ ,\ 0=\frac{\partial\mathcal{L}}{\partial m(G(t),t)}=-e^{-\bar{\rho}t}\lambda(G(t),t)dt\ .$$

From where we immediately obtain (37) and (38). Also, combining with $\frac{\partial\mathcal{L}}{\partial m(g,t)}=0$ when $m(g,t)>0$ provides (36). The first order conditions of $\mathcal{L}$ with respect to $u(t)$ are $\frac{\partial\mathcal{L}}{\partial u(t)}=e^{-\bar{\rho}t}\xi(t)dt=0$ so that $\xi(t)=0$ for all t. The first order condition with respect to $G(t)$ requires $\frac{\partial \mathcal{L}}{\partial G(t)} = 0$ or,

$$
\begin{aligned}
0 =& e^{-\bar{\rho}t}c(t)^{-\theta}\frac{\partial Y^P(m(\cdot,t),0)}{\partial G(t)}dt + e^{-\bar{\rho}t}c(t)^{-\theta}\phi\, q\, m(G(t),t)dt \\
& - e^{-\bar{\rho}t}m(G(t),t)\left[q\lambda(G(t),t) + \omega(t) + \bar{\rho}\lambda(G(t),t) - \lambda_t(G(t),t) - \lambda_g(G(t),t)\right] dt \\
& - e^{-\bar{\rho}t}\left[\lambda_g(G(t),t)m(G(t),t) + \lambda(G(t),t)m_g(G(t),t)\right] dt + e^{-\bar{\rho}t}\left[\xi_t(t) - \bar{\rho}\xi(t)\right] dt\ ,
\end{aligned}
$$

where $\frac{\partial Y^P(m(t))}{\partial G(t)} = m(G(t),t)aZ(t)^{\frac{1}{1-\nu}}\,\hat{\pi}(G(t),t)$. Then, using the first order conditions with respect to $m(g,t)$, $\lambda_g(G(t),t)m(G(t),t) + \lambda(G(t),t)m_g(G(t),t) = \xi_t(t) - \bar{\rho}\xi(t)$, which provides (39), given that $\xi(t) = 0$ for all t and (38).

Proof of Proposition 6. We leverage Proposition 5. Define $\Lambda \equiv \frac{\lambda(g,t)}{c(t)^{-\theta}aZ(t)^{\frac{1}{1-\nu}}} + \chi(t)$ and $\Omega(t) = \frac{\omega(t)}{c(t)^{-\theta}aZ(t)^{\frac{1}{1-\nu}}}$. Notice that (37) and (38) provide that $\Lambda(G(t),t) - \Lambda(0,t) = \phi a^{-1}Z(t)^{-\frac{1}{1-\nu}}$, and (39) provides that $\Lambda_g(G(t),t) = 0$. Moreover we can compute $\Lambda_t(g,t) = \frac{\lambda_t(g,t)}{c(t)^{-\theta}aZ(t)^{\frac{1}{1-\nu}}} - [\Lambda(g,t) - \chi(t)]\frac{d}{dt}\log\left(c(t)^{-\theta}aZ(t)^{\frac{1}{1-\nu}}\right) + \chi_t(t)$. Also, by Lemma 2 we have $\bar{\rho} - \frac{d}{dt}\log\left(c(t)^{-\theta}aZ(t)^{\frac{1}{1-\nu}}\right) = \bar{\rho} + \theta\frac{d}{dt}\log C(t) - \frac{1}{1-\nu}\frac{d}{dt}\log Z(t) = r(t)$. Then, we divide (36) by $c(t)^{-\theta}aZ(t)^{\frac{1}{1-\nu}}$ and use all of these expressions to obtain, $r(t)\Lambda(g,t) = \hat{\pi}(g,t) + \Lambda_g(g,t) + \Lambda_t(g,t) + r(t)\chi(t) - \chi_t(t) - \Omega(t) + q(\Lambda(0,t) - \Lambda(g,t))$. Letting $\chi_t(t) = r(t)\chi(t) - \Omega(t)$, or $\chi(t) = \int_0^\infty e^{-\int_t^{t+s}r(u)du}\Omega(t+s)ds$, we then get,[14]

$$
r(t)\Lambda(g,t) = \hat{\pi}(g,t) + \Lambda_g(g,t) + \Lambda_t(g,t) + q(\Lambda(0,t) - \Lambda(g,t))\ .
$$

Thus, letting $V(g,t) = s_\pi\Lambda(g,t)$ we have found that this function solves the p.d.e for the HJB equation for $g \in [0, G(t)]$ in (17) as well as value matching and smooth pasting in (18). In particular, letting $s_e = s_d = \eta/(\eta-1)$, by Lemma 1 value matching holds when $\Phi(t)P(t)/(s_a s_\pi) = \phi/a\left[\int_0^{G(t)} e^{-\gamma g(\eta-1)}m(g,t)dg\right]^{-\frac{1}{(\eta-1)(1-\nu)}}$, where $w(t)/s_l$ is normalized to one. Moreover, for a given $m(g,t)$, Corollary 2 provides that this choice for s_e attain static efficiency. Finally, consumers choices are optimal given $r(t)$ constructed above.

Proof of Proposition 7. We start with the p..d.e. and boundary conditions for the

[14] Differentiating under the integral sign is justified here by the fact that $\Omega(t)$ is bounded and continuous along any optimal path, and the integrand $e^{-\bar{\rho}s}\Omega(t+s)$ is dominated by an integrable function uniformly in t. An application of the Dominated Convergence Theorem then gives $\chi_t(t) = \bar{\rho}\chi(t) - \Omega(t)$.

firm's value function in an equilibrium. We construct λ and ω satisfying the p.d.e. and boundary conditions described in Proposition 5. Procedurally, we guess, and proceed to verify, that the expressions in (42) and (43) solve the relevant p.d.e. and boundary conditions.

Using value matching and smooth pasting for V in (18), the expression in (42) for λ immediately implies that (37), (38) and (39) hold. It remains to be shown that the p.d.e. for V in (17) implies that λ and ω defined in (42) and (43) solve the p.d.e. in (36). We turn to that next. As an intermediate step differentiate (42) with respect to time, and use the guess for λ and the expression for $r(t)$ in Lemma 2 to get

$$c(t)^{-\theta} a Z(t)^{\frac{1}{1-\nu}} \left(\hat{V}_t(g,t) - \hat{V}_t(G(t),t) \right) = \lambda_t(g,t) - \left(\lambda(g,t) - \lambda(G(t),t) \right) \left(\bar{\rho} - r(t) \right) \; .$$

Now, multiply both sides of the p.d.e. for V by $c(t)^{-\theta} a Z(t)^{\frac{1}{1-\nu}}$ and using this last expression we obtained,

$$\begin{aligned} \bar{\rho}\lambda(g,t) =& c(t)^{-\theta} a Z(t)^{\frac{1}{1-\nu}} \bar{\pi}(g,t) + \lambda_g(g,t) + \lambda_t(g,t) + q\left(\lambda(0,t) - \lambda(g,t)\right) \\ &+ (r(t) - \bar{\rho})\lambda(G(t),t) - c(t)^{-\theta} a Z(t)^{\frac{1}{1-\nu}} r(t) \hat{V}(G(t),t) \end{aligned}$$

Finally, using that in our guess $\lambda(G(t),t) = 0$, define $\omega(t)$ as in (43). Then, (42) follows.

Proof of Proposition 8. We begin by providing a useful lemma.

Lemma 4 *The optimal threshold for a firm G in a BGP facing a price $\bar{P}(Z)$ is obtained by solving the o.d.e and boundary conditions for $\bar{V}$ described in equations (44) to (52). The threshold G satisfies $R(G)\,\zeta\, Z^{-(\eta-1)} = \frac{\phi}{a} \frac{1}{\mathcal{A}(s_e)\mathcal{D}(s_e,s_d)} Z^{-\frac{1}{1-\nu}}$, where $R(G) \equiv \frac{1}{q+\bar{\rho}} \left\{ 1 - e^{-\gamma(\eta-1)G} - \frac{\gamma(\eta-1)}{q+\bar{\rho}+\gamma(\eta-1)} \left[1 - e^{-(q+\bar{\rho}+\gamma(\eta-1))G} \right] \right\}$ (79).*

Proof. We start by solving the ODE for the value function in the domain $[0,G]$. Its solution is the sum of the particular solution using $\bar{\pi}$, plus a constant A times the homogeneous solution. We then impose value matching and smooth pasting to obtain a system of two equations in two unknowns, namely (A,G). Solving for A in terms of G gives the desired solution. The particular solution satisfies $(\bar{\rho}+q)V^p(g) = s_\pi \bar{\pi}(g,\bar{Z}) + V^p_g(g) + q\bar{V}(0)$. One can then obtain that $V^p(g) = s_\pi \frac{\zeta}{(\bar{\rho}+q+\gamma(\eta-1))Z^{\eta-1}} e^{-g\gamma(\eta-1)} + \frac{q\bar{V}(0)}{\bar{\rho}+q}$. The homogeneous solution is: $V^h(g) = A\, e^{(\bar{\rho}+q)g}$ for some constant A. Using the smooth pasting condition, $A = \frac{\zeta s_\pi \gamma(\eta-1) e^{-G(\gamma(\eta-1)+\bar{\rho}+q)}}{(\bar{\rho}+q)(\bar{\rho}+q+\gamma(\eta-1))Z^{\eta-1}}$.

Using this expression and the value matching condition, together with the definitions of $\bar{P}(Z)$ in (51) and s_e and s_d, we obtain the expressions in the statement of the lemma. ■

The next lemma characterizes $R(G)$.

Lemma 5 *The function R in (79) satisfies: (i) $R(0) = 0$, (ii) $R(G) > 0$ for $G > 0$, (iii) R is strictly increasing in G with $0 < R'(G) < \gamma(\eta-1)/(q+\bar{\rho})e^{-\gamma(\eta-1)G}$, (iv) $R(G) = \frac{\gamma(\eta-1)}{2}G^2 + o(G)$, and (v) R has an asymptote equal to $1/(q+\bar{\rho}+\gamma(\eta-1))$. Then, the function $R(G)$ has an inverse $R^{-1} : [0, \frac{1}{q+\bar{\rho}+\gamma(\eta-1)}] \to [0, \infty]$.*

Proof. (i), (ii) and (v) follow immediately. For (iii) and (iv) write R as:

$$\begin{aligned} R(G) =& \frac{1}{q+\bar{\rho}+\gamma(\eta-1)} \left(1 - e^{-\gamma(\eta-1)G} + \frac{\gamma(\eta-1)}{q+\bar{\rho}} e^{-\gamma(\eta-1)G} \left(e^{-(q+\bar{\rho})G} - 1\right)\right) \ , \\ =& \frac{1}{d_3+d_4} \left(\frac{d_3}{d_4} e^{-d_3 G} \left(e^{-d_4 G} - 1\right) + \left(1 - e^{-d_3 G}\right)\right) \ . \end{aligned}$$

Then,

$$\begin{aligned} \frac{d}{dG}\frac{1}{(d_3+d_4)} \left(\frac{d_3}{d_4} e^{-d_3 G} \left(e^{-d_4 G} - 1\right) + \left(1 - e^{-d_3 G}\right)\right) &= \frac{d_3}{d_4}\left(e^{-d_3 G} - e^{-G(d_3+d_4)}\right) > 0 \\ \frac{1}{d_3+d_4} \left(\frac{d_3}{d_4} e^{-d_3 G} \left(e^{-d_4 G} - 1\right) + \left(1 - e^{-d_3 G}\right)\right) &= \frac{1}{2} d_3 G^2 + o(G^2) \ , \end{aligned}$$

so that $0 < R'(G) < \frac{\gamma(\eta-1)}{q+\bar{\rho}} e^{-\gamma(\eta-1)G}$. ■

The next lemma uses the function $R(G)$ to define and characterize $\bar{G}(Z)$.

Lemma 6 *The optimal threshold is $\bar{G}(Z) = R^{-1}\left(\frac{\phi}{a}\frac{s_e}{s_d}\left(\frac{1}{s_e}\frac{\eta}{\eta-1}\right)^{\frac{1}{1-\nu}} \frac{1}{\zeta} Z^{(\eta-1)(1-\zeta)}\right)$, as long as the adoption cost is not large. When the adoption cost is large, i.e., satisfying $\frac{\phi}{a}\frac{s_e}{s_d}\left(\frac{1}{s_e}\left(\frac{\eta}{\eta-1}\right)\right)^{\frac{1}{1-\nu}}\left(\frac{1}{\zeta}\right) Z^{(\eta-1)(1-\zeta)} > \frac{1}{q+\bar{\rho}+\gamma(\eta-1)}$, then $\bar{G}(Z) = +\infty$. When $\bar{G}(Z)$ is finite, (i) $\bar{G}$ is decreasing in Z if $\zeta > 1$, and increasing if $\zeta < 1$, and (ii) $\bar{G}(Z)$ is strictly increasing in ϕ and decreasing in s_d and s_e.*

The proof is a direct computation and thus omitted. The next lemma uses aggregation to find the relationship between the intermediate input productivity Z and the threshold G, defining a function $\bar{Z}(G)$, and provides a close-form expression for the stationary distribution of technology gaps.

Lemma 7 *Given a threshold G, we have that $\bar{m}(g) = \frac{qe^{-qg}}{1-e^{-qG}}$ for $g \in [0, G]$, and $\bar{Z}(G) = \left(\frac{q}{q+\gamma(\eta-1)} \frac{1-e^{-(q+\gamma(\eta-1))G}}{1-e^{-qG}}\right)^{\frac{1}{\eta-1}}$. The function $\bar{Z}$ is decreasing in G.*

The proof is a direct computation so it is omitted. Given Lemmas 6 and 7, a balance growth path with costly adoption is given by the solution of the fixed point $G^* = \bar{G}(\bar{Z}(G^*))$ and $Z^* = \bar{Z}(G^*)$. We rewrite the fixed point as the zero of an equation to analyze the existence and uniqueness of balanced growth paths. Direct use of the previous results provide the following proposition. In particular, using the functions $R(G)$ and $\bar{Z}(G)$ we define a new function $\bar{H}(G)$, so that a balance growth path is given by the solution of $\bar{\phi} = \bar{H}(G^*)$, where $\bar{\phi}$ is an adjusted fixed cost.

Proof of Proposition 9. First, we compute the marginal product of capital, $F'(k) = \zeta \left[\int_0^{\hat{G}(k)} e^{-g\gamma(\eta-1)}\underline{\mathrm{m}}(g)dg + k\right]^{\zeta-1} \left[1 - e^{-\hat{G}(k)\gamma(\eta-1)}\right] \geq 0 \quad (80)$. At $k = 0$ gives $F'(0) = F(0)\frac{\zeta\left[1-e^{-G_0\gamma(\eta-1)}\right]}{\int_0^{G_0} e^{-g\gamma(\eta-1)}\underline{\mathrm{m}}(g)dg} \quad (81)$, where G_0 is the upper bound of the support of m_0. We also compute the curvature of F,

$$\frac{F''(k)}{F'(k)} = \frac{\zeta-1}{\gamma} \frac{1 - e^{-\hat{G}(k)\gamma(\eta-1)}}{\int_0^{\hat{G}(k)} e^{-g\gamma(\eta-1)}\underline{\mathrm{m}}(g)dg + K} - (\eta-1)\frac{e^{-\hat{G}(k)\gamma(\eta-1)}}{1 - e^{-\hat{G}(k)\gamma(\eta-1)}} \left[\underline{\mathrm{m}}\left(\hat{G}(K)\right)\right]^{-1}.$$

Then, $\frac{F''(1)}{F'(1)} = -\infty$ and $\frac{F''(0)}{F'(0)} = \frac{\zeta-1}{\gamma} \frac{1-e^{-G_0\gamma(\eta-1)}}{\int_0^{G_0} e^{-g\gamma(\eta-1)}\underline{\mathrm{m}}(g)dg} - (\eta-1)\frac{e^{-G_0\gamma(\eta-1)}}{1-e^{-G_0\gamma(\eta-1)}} \left[\underline{\mathrm{m}}\left(G_0\right)\right]^{-1}$.

The proof of (1) follows directly from (80) and (81). The proof of the remaining parts follow from utilizing the expressions for the curvature as a function of k and in the extremes. For $F'(1) = 0$ we use that at $k = 1$ it is the case that $\hat{G}(1) = 0$. (2) follows directly from the expression for $\frac{F''(k)}{F'(k)}$. (3) uses the assumptions to show that the right hand side of $\frac{F''(k)}{F'(k)}$ is monotone increasing in k, and hence $F''(k)$ can cross zero only once. The last part of the item gives sufficient conditions for $F''(0)/F'(0) > 0$.

Proof of Proposition 10. The ODE (61) is feasibility. Next we turn to the necessity of (62) and of the terminal condition (63). We will return to sufficiency below.

Necessity. We obtain the ODE in (62) from the first order condition for adoption of the marginal firm at each time, replacing the expressions for $\pi(g, t)$ and $\pi(0, t)$, and using the Euler equation to replace $r(t)$. The boundary condition (63) is necessary for the finite value of $V(0, t)$. We expand into the details to obtain (62). We consider the case of adoption. The first order condition with respect τ of (59) is:

$$e^{-\int_t^{t+\tau} r(\tilde{s})d\tilde{s}}\pi\left(g,t+\tau\right)-r\left(t+\tau\right)e^{-\int_t^{t+\tau} r(\tilde{s})d\tilde{s}}\left[V^0\left(g,t+\tau\right)-P\left(t+\tau\right)\Phi(t+\tau)/s_a\right]$$
$$+\,e^{-\int_t^{t+\tau} r(\tilde{s})d\tilde{s}}\left[V_t^0\left(g,t+\tau\right)-\tfrac{d}{dt}[P\left(t+\tau\right)\Phi(t+\tau)]/s_a\right]=0\ ,$$

which, letting $P_{nt}(t)\equiv P(t)e^{\frac{\gamma}{1-\nu}t}$, simplifies to $r\left(t+\tau\right)\left[V^0\left(g,t+\tau\right)-\frac{P_{nt}(t)\phi}{s_a}\right]=\pi\left(g,t+\tau\right)+\left[V_t^0\left(g,t+\tau\right)-\frac{\dot{P}_{nt}(t+\tau)\phi}{s_a}\right]$ (82). A firm with gap g at time t finds it optimal to adopt at time $t+\tau$. Defining the value of g for the marginal adopter at time t as $G\left(t\right)$ gives: $\pi\left(G\left(t\right),t\right)-r\left(t\right)\left[V^0\left(g,t\right)-\frac{P_{nt}(t)\phi}{s_a}\right]+V_t^0\left(g,t\right)-\frac{\dot{P}_{nt}(t)\phi}{s_a}=0$ (83). Using that the time derivative of the value function is $V_t^0\left(g,t\right)=-\pi\left(0,t\right)+r\left(t\right)V^0\left(g,t\right)$ for all g and t, evaluated at $g=0$, and after canceling terms we get $\pi\left(0,t\right)-\pi\left(G\left(t\right),t\right)=r\left(t\right)P_{nt}\left(t\right)\phi/s_a-\dot{P}_{nt}\left(t\right)\phi/s_a$. Dividing by $P_{nt}(t)\phi/s_a$ and rearranging: $r\left(t\right)=\frac{s_a}{\phi}\frac{\pi(0,t)-\pi(G(t),t)}{P_{nt}(t)}+\frac{\dot{P}_{nt}(t)}{P_{nt}(t)}$. Notice that the Euler equation in (8) provides that $r(t)=\bar{\rho}+\theta\frac{\dot{C}(t)}{C(t)}+\frac{\dot{P}(t)}{P(t)}=\bar{\rho}+\theta\frac{\dot{c}(t)}{c(t)}+\frac{\dot{P}_{nt}(t)}{P_{nt}(t)}$. Then, combining these expressions and simplifying, $\theta\frac{\dot{c}(t)}{c(t)}=\frac{s_a}{\phi}\frac{\pi(0,t)-\pi(G(t),t)}{P_{nt}(t)}-\bar{\rho}$. Also, using the expression for $\pi(g,t)$ for $g=G(t)$ and $g=0$ and the one for $P_{nt}(t)$ we obtain that $\frac{\pi(0,t)-\pi(G(t),t)}{P_{nt}(t)}=\frac{s_\pi}{s_x}\zeta a\left(s_e\frac{\eta-1}{\eta}\right)^{\frac{1}{1-\nu}}\frac{1-e^{-G(t)\gamma(\eta-1)}}{\left[\int_0^{G(t)}e^{-g\gamma(\eta-1)}\underline{\mathrm{m}}(g)dg+k(t)\right]^{1-\zeta}}$. Replacing this expression into the one for $\dot{c}(t)/c(t)$, $\theta\frac{\dot{c}(t)}{c(t)}=\frac{s_a s_\pi}{s_x}\frac{a}{\phi}\zeta\left(s_e\frac{\eta-1}{\eta}\right)^{\frac{1}{1-\nu}}\frac{1-e^{-G(t)\gamma(\eta-1)}}{\left[\int_0^{G(t)}e^{-g\gamma(\eta-1)}\underline{\mathrm{m}}(g)dg+k(t)\right]^{1-\zeta}}-\bar{\rho}$. Finally, using the definition of s_e, s_d, and the expression for $F'(k)$ in (80), provides (62). The case of dis-investment corresponding to problem (60) is analogous and thus we do not include it here.

Now we study the boundary condition in (63). For $V(0,t)$ to be finite we require the tail of the integral to go to zero, i.e., $0=\lim_{T\to\infty}e^{-\int_t^T r(s)ds}\pi\left(0,T\right)$. Since $\pi(0,T)$ is bounded below and above, we require that $0=\lim_{T\to\infty}e^{-\int_t^T r(s)ds}$. Replacing $r(s)$ from the Euler equation we have $-\int_t^T r\left(s\right)ds=-\int_t^T\left(\bar{\rho}+\theta\frac{\dot{c}(s)}{c(s)}+\frac{\dot{P}_{nt}(s)}{P_{nt}(s)}\right)ds=-\bar{\rho}(T-t)-\theta\log\frac{c(T)}{c(t)}-\log\frac{P_{nt}(T)}{P_{nt}(t)}$. Then, $e^{-\int_t^T r(s)ds}=e^{-\bar{\rho}(T-t)-\theta\log\frac{c(T)}{c(t)}-\log\frac{P_{nt}(T)}{P_{nt}(t)}}=e^{-\bar{\rho}(T-t)}\left(\frac{c(T)}{c(t)}\right)^{-\theta}\frac{P_{nt}(t)}{P_{nt}(T)}$. Using $P_{nt}(T)$ and $F(k)$: $a\mathcal{A}(s_e)F\left(k\right)\frac{1-\nu}{\frac{1}{s_e}\frac{\eta}{\eta-1}-\nu}\frac{s_r}{s_e}=\frac{1}{P_{nt}(T)}$. Thus, $0=\lim_{T\to\infty}e^{-\int_t^T r(s)ds}\pi\left(0,T\right)$, is equivalent to $0=\lim_{T\to\infty}e^{-\bar{\rho}T}\left(c(T)\right)^{-\theta}\frac{1}{P_{nt}(T)}$, which is then equivalent to $0=\lim_{T\to\infty}e^{-\bar{\rho}T}\left(c(T)\right)^{-\theta}\mathcal{A}(s_e)aF(k(T))$.

Finally, the boundary conditions when $k=0$ and $k=1$ follow as, otherwise, an equilibrium fails to satisfy jointly feasibility and that prices are consistent with both the optimal consumption plan of the household and the adoption decisions of firms. *Sufficiency.* We show that these conditions are sufficient for an equilibrium. We take a path that solves the ODE system and relevant boundary conditions and recover the path of prices $P_{nt}(t)$ and interest rates $r(t)$ using (58) and the Euler equation in (8), respectively. By construction the path solves the first order conditions for the household and firms problem. The first order condition for the household problem, together with the boundary condition (63) are sufficient for the path to be optimal. The first order condition for the adoption (or dis-investment) are also satisfied by construction. The next lemma shows that, when $\{P(t), r(t)\}$ are constructed as indicated above, the corresponding optimal stopping times solve the firm's problem.

Lemma 8 *Let $\{k(t), c(t)\}$ solve the o.d.e's (61) and (62). Let $\{P(t), r(t)\}$ be defined using (58) and (8). Then the threshold $G(t)$ that solves the first order condition of the firms adoption or dis-investment problem achieves the optima.*

Proof. We consider the objective function for the adoption problem $f(g,t)$. The case of dis-investment is analogous and thus not included. We will let $t^*(g)$ a time for which $f_t(g, t^*(g)) = 0$, i.e., $t^*(g)$ satisfies the first order condition of the firms' problem. We will restrict ourselves to objective functions evaluated at path of prices and interest rates which comes from solution to the ODE's. We will show that such objective function is strictly locally concave at $t^*(g)$. Because this property holds at any solution of the first order condition, the firms' objective function is single-peaked, and hence the time that satisfies the first order condition is an optimum.[15]

A solution of the o.d.e.'s that is continuous in time can either start at a steady state (and stay there), or it cannot reach a steady state in finite time. Because of this we will treat two cases separately, first the case where $\dot{k}(t) \neq 0$ for almost all times, and then the case where the economy starts at steady state. For the first case we will show that $f_{tt}(G(t), t) = -(\eta - 1)\gamma\pi(G(t), t)\frac{|\dot{k}(t)|}{\underline{\mathrm{m}}(G(t))}$. We will treat the stationary solutions separately. Let $f(t, g)$ be the objective of a firm in the adoption problem: $f(t, g) = \int_0^t e^{-\int_0^s r(\tilde{s})d\tilde{s}}\pi(g, s)\, ds + e^{-\int_0^t r(\tilde{s})d\tilde{s}}\left[V^0(g,t) - P(t)\,\Phi(t)/s_a\right]$. Differentiating, $f_t(t, g) = e^{-\int_0^t r(\tilde{s})d\tilde{s}}\{\pi(g,t) - r(t)[V^0(g,t) - \frac{P_{nt}(t)\phi}{s_a}] + [V_t^0(g,t) - \frac{\dot{P}_{nt}(t)\phi}{s_a}]\}$. Using that $V_t^0(g,t) = -\pi(0,t) + r(t)\, V^0(g,t)$, $f_t(t,g) = e^{\int_0^t r(\tilde{s})d\tilde{s}}[\pi(g,t) -$

[15] To see that the objective function is single-peaked, note that $f_t(t, g)$ is positive for $g > G(t)$ and negative for $g < G(t)$. Since π is strictly decreasing in g, there is a unique t at which $f_t = 0$, confirming that the critical point is a global maximum.

$\pi(0,t) + \frac{r(t)P_{nt}(t)\phi}{s_a} - \frac{\dot{P}_{nt}(t)\phi}{s_a}]$. Using the Euler equation, $P_{nt}(t)\phi r(t)/s_a = \bar{\rho}P_{nt}(t)\phi/s_a + \theta\frac{\dot{c}(t)}{c(t)}P_{nt}(t)\phi/s_a + \dot{P}_{nt}(t)\phi/s_a$. Then, $f_t(t,g) = e^{-\int_0^t r(\tilde{s})d\tilde{s}}\{\pi(g,t) - \pi(0,t) + \frac{P_{nt}(t)\phi}{s_a}[\bar{\rho} + \theta\dot{c}(t)/c(t)]\}$. The o.d.e which is a necessary condition in equilibrium gives $[\theta\frac{\dot{c}(t)}{c(t)} + \bar{\rho}]\frac{P_{nt}(t)\phi}{s_a} = \pi(0,t) - \pi(G(t),t)$. Then, $f_t(t,g) = e^{-\int_0^t r(\tilde{s})d\tilde{s}}[\pi(g,t) - \pi(G(t),t)]$. So, $f_{tt}(t,g) = -r(t)e^{-\int_0^t r(\tilde{s})d\tilde{s}}[\pi(g,t) - \pi(G(t),t)] + e^{-\int_0^t r(\tilde{s})d\tilde{s}}[\pi_t(g,t) - \pi_g(G(t),t)\dot{G}(t) - \pi_t(G(t),t)]$. Evaluating it at $g = G(t)$ and using that $\dot{G}(t) = -\frac{\dot{k}(t)}{\underline{\mathrm{m}}(G(t))}$, $f_{tt}(t,G(t)) = e^{-\int_0^t r(\tilde{s})d\tilde{s}}\pi_g(G(t),t)\frac{\dot{k}(t)}{\underline{\mathrm{m}}(G(t))}$. Finally, using that $\pi_g(g,t) = -(\eta-1)\gamma\pi(g,t)$, provides $f_{tt}(t,G(t)) = -(\eta-1)\gamma e^{-\int_0^t r(\tilde{s})d\tilde{s}}\pi(G(t),t)\frac{\dot{k}(t)}{\underline{\mathrm{m}}(G(t))}$, where $\dot{k}(t) > 0$, as we are considering the adoption case.

Now we consider the case where $k(0) = k^*$ and $c(t) = c^*$ are a stationary interior solution of the o.d.e.'s. In this case the derivative of the objective function gives: $f_t(t,g) = e^{-\bar{\rho}t}[\pi^*(g) - \pi^*(0) + \bar{\rho}P^*_{nt}\phi/s_a]$, where $P^*_{nt}, \pi^*(g)$ and $\pi^*(0)$ are the steady state versions of the price and profit functions. It is immediate that $f(z,t)$ is either strictly increasing in time (for low g), strictly decreasing in time (for high g), and constant for a particular g. ■

Proof of Lemma 3. We now demonstrate the absence of limit cycles originating from the unstable interior steady state. Let $\mathcal{S} = \{(k,c) \mid k \in [0,1] \text{ and } c \in [0,+\infty)\}$, and for ϵ small define a compact subset around (k^*_L, c^*_L). For example, construct the compact subset $\mathcal{S}_\epsilon \in \mathcal{S}$ as $\mathcal{S}_\epsilon = \{(k,c) \,\middle|\, \frac{\epsilon}{2} \geq |k - k^*_L| \text{ and } \frac{\epsilon}{2} \geq |c - c^*_L|\}$.

Consider $k(0) = k^*_L$ and let $c(0)$ be arbitrarily close to c^*_L. Using the Bendixson-Dulac theorem with the Dulac function $\beta(c,k) = 1/c$, and using the differential equations in Proposition 10, we evaluate the divergence $d\mathcal{V}(c,k) \equiv \frac{\partial[\beta(c,k)\dot{c}]}{\partial c} + \frac{\partial[\beta(c,k)\dot{k}]}{\partial k} = \frac{\mathcal{D}(s_e,s_d)\mathcal{A}(s_e)}{\phi}\frac{aF'(k)}{c} > 0$, provided that $\mathcal{D}(s_e,s_d)\mathcal{A}(s_e) = \left(s_e\frac{\eta-1}{\eta}\right)^{\frac{1}{1-\nu}}\frac{s_d}{s_e} > 0$ for all s_e and s_d. Since $dV(c,k)$ is strictly positive within $\mathcal{S}_\epsilon$, no limit cycles can exist; trajectories necessarily 'escape' the set.

Proof of Proposition 12. We begin by showing that the no-adoption steady state, $c(t) = \mathcal{A}(s_e)aF(0)$ satisfies all the relevant conditions if $aF'(0) < \frac{\bar{\rho}}{\mathcal{A}(s_e)\mathcal{D}(s_e,s_d)}$ holds. For this we need to go back to the proof of Proposition 10 and modify it suitably. The first order condition for τ in (82) holds with inequality, so that (83) becomes $\pi(1,t) - r(t)[V(0,t) - P_{nt}(t)\phi/s_a a] + V_t(0,t) - \dot{P}_{nt}(t)\phi/s_a \leq 0$, since $G(t) = \infty$. The Euler equation for $\dot{c}$ and the equations for profits and $F'(k)$ still hold with equality so, after replacing them, we obtain that $\theta\frac{\dot{c}(t)}{c(t)} \leq \mathcal{D}(s_e,s_d)\mathcal{A}(s_e)aF'(0) - \bar{\rho}$. Since by feasibility at $k = 0$ we have that $\dot{k} \geq 0$, if $F'(0) \leq \bar{\rho}/[\mathcal{D}(s_e,s_d)\mathcal{A}(s_e)a]$ then $\dot{c}(t) = 0$

satisfies all the sufficient conditions.

On the stability, the proof follows from examining the phase diagram. In particular, given the assumptions, we can show that there exists a $\epsilon > 0$ such that for all $0 < k(0) \leq \epsilon$, there is a path $\{c(t), k(t)\}$ for $t \in [0, T]$ with $T < \infty$ which satisfy the ODE given by (61) and (62) for $0 \leq t < T$ for which $c(T) = \mathcal{A}(s_e)aF(0)$ and $k(T) = 0$. To see this, we reverse time defining $\tau = t_0 - t$ or $t = t_0 - \tau$ and define $\tilde{c}(\tau) \equiv c(t_0 - \tau)$ and $\tilde{k}(\tau) \equiv k(t_0 - \tau)$. The ODEs for $\{\tilde{c}, \tilde{k}\}$ are obtained by multiplying by minus one the expression for the time derivatives of c, K. We then run the system for $\{\tilde{c}, \tilde{k}\}$ with initial conditions $\tilde{c}(0) = \mathcal{A}(s_e)aF(0)$ and $\tilde{k}(0) = 0$. At $\tau = 0$, in the $\tilde{c}, \tilde{k}$ plane, the system starts vertically up. Since the $\frac{d}{dt}\tilde{k} = 0$ locus is upward slopping with finite slope, for any $\tau > 0$ the system is in the quadrant for which $\frac{d}{d\tau}\tilde{c}(\tau) > 0$ and $\frac{d}{d\tau}\tilde{k}(\tau) > 0$. By reversing time again, so that we go back to t instead of τ we found the desired path.

Proof of Proposition 14. We begin by writing the Hamiltonian as $H(k, c, \lambda) = e^{-\bar{\rho}t}\frac{c(t)^{1-\theta}-1}{1-\theta} + \lambda(t)\frac{aF(k(t))-c(t)}{\phi}$. Notice that $H_c = 0 :\ e^{-\bar{\rho}t}c(t)^{-\theta} = \lambda(t)/\phi$ and $H_k = -\dot{\lambda}(t) :\ \lambda(t)\frac{aF'(k(t))}{\phi} = -\dot{\lambda}(t)$. Log-differentiation of the $H_c = 0$ equation provides that $\dot{\lambda}(t)/\lambda(t) = -\bar{\rho} - \theta\frac{\dot{c}(t)}{c(t)}$. Using this expression in the equation for $H_k = 0$ provides $\theta\frac{\dot{c}(t)}{c(t)} = \frac{aF'(k(t))}{\phi} - \bar{\rho}$, which is the expression in (66). The boundary condition is given by $\lim_{T\to\infty}\lambda(T)k(T) = \lim_{T\to\infty}e^{-\bar{\rho}T}c(T)^{-\theta}k(T) = 0$, where we used the expression for $\lambda(t)$ above. Notice that this expression is (67). This completes the proof.

Proof of Proposition 19. First we prove that for $k(0) > \tilde{k}$, the first best allocation consists on the saddle-path that converges to k_H^*. This follows from considering a relaxed planning problem with a higher production function $\tilde{F}$ which is the smallest concave function that satisfies $\tilde{F} \geq F$ for all $k \geq 0$. Given that F is S-shaped, then, (a) $\tilde{F}(0) = F(0)$, (b) $\tilde{F}$ is linear in $(0, \tilde{k})$, and (c) $\tilde{F} = F$ in $k \geq \tilde{k}$. But since $\tilde{F}$ is concave, the saddle-path passing through k_H^* is the optimal trajectory, since it satisfies the sufficient conditions. Since this saddle path is a solution for a relaxed planing problem, and the trajectory is feasible for the original problem provided that $k(0) > \tilde{k}$, then it is the optimal for the original planning problem.

Now we argue that there can't be any other trajectory solving the Euler equations, Feasibility, and satisfying Transversality different from the saddle path. If such trajectory were to exist, consider $k(0) = k_H^*$. Then the Hamiltonian will have a minimum at $k = k_H^*$, since it is a steady state. Thus, by Theorem 1 in Brock and Dechert

(1983), which establishes that among all trajectories satisfying the necessary conditions, the optimal one maximizes the Hamiltonian at t=0, this alternative trajectory would itself be efficient, yielding a contradiction. Thus, there is no other trajectory solving the Euler equations, Feasibility and satisfying Transversality starting from $k(0) > \tilde{k}$, and hence there is no other equilibrium with $s_e = s_e^*$ and $s_d = s_d^*$. This shows that the optimal subsidies uniquely implement the first best.

Proof of Proposition 20. The proof of the proposition uses the next two lemmas.

Lemma 9 *Assume that $F(\cdot)$ is S-shaped, $aF'(0) \leq \bar{\rho}$ and $aF'(k^i) > \bar{\rho}$. Let $s(k;\theta)$ be the consumption of the saddle path that converges to k_H^* when the intertemporal elasticity of substitution is $1/\theta$. There exists a $\bar{\theta} \geq \theta^*$, such that for all $\theta \geq \bar{\theta}$, $s(k;\theta) \in (k_L^*, 1]$ and $c_L^* = s(k_L^*;\theta)$.*

The lemma provides that the saddle path converging to k_H^* is connected to the unstable steady state k_L^* (heteroclinic connection). Second, the lemma also provides that $k > k_L^*$. In other words, it never crosses the $\dot{k}(t) = 0$ or $\dot{c}(t) = 0$ locus.
Proof. The proof proceeds in several steps. We note that, in this proof, k_L^* and k_H^* denote the values under the optimal subsidies. We provide the proof for the case when $k \in (k_L^*, k_H^*]$. The case when $k > k_H^*$ is exaclty analogous and thus ommited.

1. Consider the saddle path s for any θ and reverse the flow of time, so that it starts at (c_H^*, k_H^*). We have that $s(k;\theta) < aF(k)$ for $k \in (k_L^*, k_H^*)$.

2. Consider $\theta_1 < \theta_2$. Then $s(k;\theta_1) < s(k;\theta_2) < aF(k)$ for all $k \in (k_L^*, k_H^*)$.

3. Let $D(\theta) = \max_{k \in [k_H^*, k_L^*]} F(k) - s(k;\theta)$, i.e., the Hausdorff distance between F and s in $[k_H^*, k_L^*]$. Define $\bar{F}' \equiv \max_{k \in [k_L^*, k_H^*]} F'(k)$. Then, $D(\theta) \leq (a\frac{\bar{F}'}{\bar{\rho}} - 1)\frac{c_H^*}{\theta}$.

4. There is a heteroclinic connection, i.e., for θ large enough, $s(k_L^*, \theta) = c_L^*$.

That step 1 holds follows by contradiction. If the saddle were to cross the $\dot{k} = 0$ in between the steady states, then there will be two orbits that share one point, i.e., the point where the saddle cross the $\dot{k} = 0$ locus. Note that $s(k;\theta) < aF(k)$ for all $k \in (k_L^*, k_H^*)$ by Step 1, since the saddle path lies strictly below the $\dot{k} = 0$ locus between the two interior steady states.

For step 2 we use two different results. First the slope of the saddle path at $k = k_H^*$ obtained by linearizing the ODE's around this steady state. This slope is one of the roots of a quadratic equation, it is positive decreasing in θ. Moreover, as $\theta \to \infty$, this slope converges to $\bar{\rho}$, which is the slope of the $\dot{k} = 0$ at that point. Thus, at least close to $k = k_H^*$ the monotonicity results holds, i.e., $s(\hat{k};\theta_2) > s(\hat{k};\theta_1)$ for $k < k_H^*$ and sufficiently close to it. Now we show that this is true for any $k \in [k_L^*, k_H^*)$. Suppose, by contradiction, that there exists $\hat{k} < k_H^*$ for which $s(\hat{k};\theta_2) \leq s(\hat{k};\theta_1)$. But, given the behavior close to k_H^*, it must the case that the slope at $\hat{k}$ of $s(\hat{k};\theta_2)$ is higher than the slope $s(\hat{k};\theta_1)$. But the slope of the saddle at point is given by $\frac{ds(k;\theta)}{dk} = \frac{\dot{c}}{\dot{k}} = \frac{c}{\theta}\frac{aF'(k)-\bar{\rho}}{aF(k)-c}$, where $c = s(k,\theta)$. Since then $s(\hat{k};\theta_2) = s(\hat{k};\theta_1)$, we have that $\frac{\frac{ds(\hat{k};\theta_2)}{dk}}{\frac{ds(\hat{k};\theta_1)}{dk}} = \frac{\theta_1}{\theta_2} < 1$, which is a contradiction. Hence, the monotonicity result is established.

Now we show that 3 holds. Let $k^*(\theta) = \arg\max_{k\in[k_L^*,k^*H]} aF(k) - s(k;\theta)$ and note $aF(k^*(\theta) - s(k^*(\theta);\theta) = D(\theta)$. Note that $k^*(\theta) \in [k_L^*, k_H^*)$ since the saddle converges to the steady state k_L^*. For any $\omega \in (0,1)$, we can find a $k^{**}(\theta) \in (k^*(\theta), k_H^*)$ for which $aF(k^{**}(\theta)) - s(k^{**}(\theta);\theta) = \omega D(\theta)$ and where $aF(k) - s(k,\theta) \geq \omega D(\theta) > 0$ for all $k \in [k^*(\theta), k^{**}(\theta)]$. Thus, using that $(aF(k^*(\theta)) - s(k^*(\theta),\theta)) - (aF(k^{**}(\theta)) - s(k^{**}(\theta),\theta)) = \omega D(\theta)$, the intermediate value theorem for integrals implies that there must be $\bar{k}(\theta) \in [k^*(\theta), \bar{k}(\theta)]$ such that $aF'(\bar{k}(\theta)) - \frac{ds(\bar{k}(\theta),\theta)}{dk} = aF'(\bar{k}(\theta)) - \frac{s(\bar{k}(\theta),\theta)}{\theta}\frac{aF'(\bar{k}(\theta))-\bar{\rho}}{aF(\bar{k}(\theta)-s(\bar{k}(\theta),\theta)} = -\frac{\omega D(\theta)}{(k^{**}(\theta)-k^*(\theta))} < 0$. From these inequalities we use $\frac{\theta}{s(\bar{k}(\theta),\theta)}aF'(\bar{k}(\theta)) < \frac{aF'(\bar{k}(\theta))-\bar{\rho}}{aF(\bar{k}(\theta)-s(\bar{k}(\theta),\theta)}$. But for any $k \in [k^*(\theta), k^{**}(\theta)]$ we have that $aF(k) - s(k,\theta) \geq \omega D(\theta)$, thus $\frac{\theta}{s(\bar{k}(\theta),\theta)}aF'(\bar{k}(\theta)) < \frac{aF'(\bar{k}(\theta))-\bar{\rho}}{\omega D(\theta)}$. Multiplying by $\omega D(\theta)$ and using that $s(k,\theta)) < c_H^*$ for all $k \leq k_H^*$: $\frac{\theta\omega D(\theta)}{c_H^*}aF'(\bar{k}(\theta)) < aF'(\bar{k}(\theta)) - \bar{\rho}$. Using that $aF'(k) \geq \bar{\rho}$ in $k \in [k_L^*, k_H^*]$ then we can write $\omega\theta\frac{\bar{\rho}D(\theta)}{c_H^*} < aF'(\bar{k}(\theta)) - \bar{\rho}$. Using the definition of $\bar{F}'$, $\omega\theta < (a\bar{F}' - \bar{\rho})\frac{c_H^*}{\bar{\rho}D(\theta)}$. But since ω can be taken to be arbitrarily close to one, then $\theta < \left(\frac{a\bar{F}'}{\bar{\rho}} - 1\right)\frac{c_H^*}{D(\theta)}$, or $D(\theta) < \left(\frac{a\bar{F}'}{\bar{\rho}} - 1\right)\frac{c_H^*}{\theta}$.

To show that step 4 holds, set $\theta \geq \theta^*$, so (c_L^*, k_L^*) is a non-spiraling source, i.e., both eigenvalues are positive and real. Invert the direction of time, so that this steady state becomes a sink with both eigenvalues positive and negative, and suppose that regardless of θ, the saddle path—with the time direction inverted—does not converge to the (c_L^*, k_L^*). But step 3 shows that for θ large, $s(k_L^*, \theta)$ is arbitrarily close to c_L^*. ■

Lemma 10 *Assume that $F(\cdot)$ is S-shaped, $aF'(0) \leq \bar{\rho}$ and $aF'(k^i) > \bar{\rho}$. Let $\hat{s}(k;\theta)$ be the consumption of to the equilibrium that converges to $k^* = 0$ when the intertemporal elasticity of substitution is $1/\theta$. There exists a $\hat{\theta} \geq \theta^*$, such that for all $\theta \geq \hat{\theta}$, $\hat{s}(k;\theta) \in [0, k_L^*)$ and $c_L^* = \hat{s}(k_L^*;\theta)$.*

This lemma provides that, for high θ, there is a heteroclinic connection between the stable, no adoption steady state and the, unstable, interior steady state with low adoption. Moreover, the connection is such that $k(t) < k_L^*$.

Proof. The proof of the lemma follows closely the proof of Lemma 9, but noting that the argument requires that the path lies above the $\dot{k}$ locus. The only relevant difference is that in the proof of Lemma 9 we could easily obtain the slope of the saddle around the high adoption steady state, and see that the slope is finite. Here, this is not immediate, as a direct evaluation of the linearized system around the steady state with no adoption would provide an infinite slope. A result of this is that we could not conclude that we can order the slope of the paths by the value of θ. We overcome this issue by looking, locally around the steady state with no adoption, at the behavior the ratio of the slopes of the equilibrium paths for different values of θ. We refer by $\hat{s}(k;\theta)$ as the path that solves the dynamical system and converges to the steady state with no adoption. Direct computation provides that $\hat{s}'(k;\theta) = \dot{c}/\dot{k}$, or $\hat{s}'(k;\theta) = \frac{\hat{s}(k;\theta)}{\theta}\frac{aF'(k)-\bar{\rho}}{aF(k)-\hat{s}(k;\theta)}$. Recall that $\hat{s}(0;\theta) = F(0)$ for all θ, and hence $\hat{s}'(k;\theta) \to +\infty$ as $k \downarrow 0$. Nevertheless, for two values of θ, $\frac{\hat{s}'(k;\theta_1)}{\hat{s}'(k;\theta_2)} = \frac{\theta_2}{\theta_1}\frac{\hat{s}(k;\theta_1)}{\hat{s}(k;\theta_2)}\frac{aF(k)-\hat{s}(k;\theta_2)}{aF(k)-\hat{s}(k;\theta_1)}$. Taking $k \downarrow 0$ and applying L'Hopital's rule, $\frac{\hat{s}'(0;\theta_1)}{\hat{s}'(0;\theta_2)} = \frac{\theta_2}{\theta_1}\frac{aF'(0)-\hat{s}'(0;\theta_2)}{aF'(0)-\hat{s}'(0;\theta_1)}$, or $\frac{\hat{s}'(0;\theta_1)}{\hat{s}'(0;\theta_2)}\frac{aF'(0)-\hat{s}'(0;\theta_1)}{aF'(0)-\hat{s}'(0;\theta_2)} = \frac{\theta_2}{\theta_1}$, or $\frac{\hat{s}'(0;\theta_1)}{\hat{s}'(0;\theta_2)} = \sqrt{\frac{\theta_2}{\theta_1}}$. In other words, $\lim_{k\downarrow 0}\frac{\hat{s}'(k;\theta_1)}{\hat{s}'(k;\theta_2)} = \sqrt{\frac{\theta_2}{\theta_1}}$. That is, even though $\hat{s}'(k;\theta)$ diverges to infinity as k goes to zero for any θ, for k close to zero the slope is higher the lower is θ. This implies that, while $\hat{s}(0;\theta_1) = \hat{s}(0;\theta_2) = aF(0)$, $\hat{s}(k;\theta_1) < \hat{s}(k;\theta_2)$ for $\theta_1 > \theta_2$ when k is close to zero. Away from $k = 0$, the proof is analogous to the proof of Lemma 9 and thus it is not included. ■

The two lemmas directly imply the following augmented version of Proposition 20. **Proposition** *Assume that $F(\cdot)$ is S-shaped, $\bar{\rho} \in (aF(0), aF(k^i)]$. Let $s_e = s_e^*$ and $s_d = s_d^*$. Then, for any $k(0)$, for any $\theta \geq \max\{\bar{\theta}, \hat{\theta}\} > \theta^*$, there is a unique dynamic equilibrium characterized by $c(k) = \hat{s}(k;\theta)$ for $k \in [0, k_L^*)$, $c(k) = c_L^*$ for $k = k_L^*$, and $c(k) = s(k;\theta)$ for $k \in (k_L^*, 1]$. The unique dynamic equilibrium is efficient. Moreover, if $k(0) < k_L^*$ then $k(0) \to 0$, if $k(0) = k_L^*$ then $k(t) = k_L^*$ for all t, and if $k(0) > k_L^*$ then $k(0) \to k_H^*$.*

Proof of Proposition 21. Part (1) of the proposition follows directly from Proposition 12. Any alternative candidate allocation to implement the efficient allocation must converge to k_H^*. This follows from Brock and Dechert (1983), as otherwise the path of capital would not be monotonic.

We now prove parts (2) and (3). Part (2) postulates that, under some parameter conditions, there is an alternative equilibrium under the subsidies $s_e = s_e^*$ and $s_d = s_d^*$ that converges to k_H^* and, when this equilibrium exists, part (3) postulates that this one implements the efficient allocation. We prove this through a sequence of results:

Result 1: The efficient allocation converging to k_H^* provides higher welfare than any other feasible plan available to the planner that also converges to k_H^*.

Proof. This is immediate: If there is a restricted plan available, the planner can achieve higher welfare if not subject to these restrictions. ■

Result 2: We consider the linear utility case, where $\theta = 0$. We construct a feasible plan and we show that, under some parameter restrictions, it provides higher welfare than converging to the steady state with no adoption. Thus, under $\theta = 0$ and the parameter restrictions, there is an equilibrium that converges to k_H^*. Because this equilibrium exists, this one implements the efficient allocation.

Proof. Consider the following plan: the household consumes zero while $k(t) < k_H^*$, and when $k(t) = k_H^*$ we have that $c(t) = c_H^*$. Moreover, instead of using $\dot{k}(t) = aF(k(t)) - c(t)$, we restrict the planner to use $\dot{k}(t) = aF(0) + aF'(0)k(t)$. Also notice that this alternative investment path is feasible, as $aF(0) + aF'(0)k(t) < aF(k(t))$. Let T denote the time at which the economy reaches k_H^* using the alternative investment plan. Given linear utility of consumption, i.e., $\theta = 0$, the alternative path dominates the alternative of converging to the steady state with no adoption when $aF(0)/\bar{\rho} < e^{-\bar{\rho}T} aF(k_H^*)/\bar{\rho}$ or,[16] $F(0) < e^{-\bar{\rho}T} F(k_H^*)$ (84). We can use the evolution of $k(t)$ in the alternative plan to find that $k_H^* = \frac{F(0)}{F'(0)}\left(e^{aF'(0)T} - 1\right)$, given that $k(0) \approx 0$. This provides that $e^{aF'(0)T} = \frac{k_H^* F'(0)}{F(0)} + 1$. Then, using that $e^{-\bar{\rho}T} = \left(e^{aF'(0)T}\right)^{-\bar{\rho}/[aF'(0)]}$ we obtain that $e^{-\bar{\rho}T} = [1 + k_H^* F'(0)/F(0)]^{-\frac{\bar{\rho}}{aF'(0)}}$. Then, replacing in (84), $F(0)/F(k_H^*) < [1 + k_H^* F'(0)/F(0)]^{-\frac{\bar{\rho}}{aF'(0)}}$. If this condition is satisfied the alternative plan is feasible and provides higher welfare than converging to the steady state with no adoption. As a result, an allocation that actually solves the planner's problem exists and implements the efficient allocation when $\theta = 0$. ■

Result 3: We show by a continuity argument that Result 2 applies for small θ.

Proof. Converging to the steady state with no adoption provides value $aF(0)/\bar{\rho}$ when $\theta = 0$ and $[aF(0)]^{1-\theta}/[\bar{\rho}(1-\theta)]$ when $\theta > 0$. The alternative plan converging to the interior steady state with high adoption provides value $e^{-\bar{\rho}T} aF(k_H^*)/\bar{\rho}$ when

[16] Because $k(0)$ is close to zero, we are not accounting for the welfare differences accrued from transitioning from $k(0)$ to $k = 0$.

$\theta = 0$ and value $e^{-\bar{\rho}T}[aF(k_H^*)]^{1-\theta}/[\bar{\rho}(1-\theta)]$ otherwise. Let $F(T, \bar{\rho}, F(0), k_H^*, \theta) \equiv e^{-\bar{\rho}T}[aF(k_H^*)]^{1-\theta}/[\bar{\rho}(1-\theta)] - aF(0)^{1-\theta}/[\bar{\rho}(1-\theta)]$ be the net gain from the alternative plan with $\theta > 0$ and $\hat{F}(T, \bar{\rho}, F(0), k_H^*) \equiv e^{-\bar{\rho}T} aF(k_H^*)/\bar{\rho} - aF(0)/\bar{\rho}$. In both cases, the alternative plan is preferred if these functions are positive. Notice that $F(T, \bar{\rho}, F(0), k_H^*, \theta)$ is a composition of continuous functions of θ. Thus, $F(T, \bar{\rho}, F(0), k_H^*, \theta)$ is well approximated by $\hat{F}(T, \bar{\rho}, F(0), k_H^*)$ when θ is small. ■

Proof of Proposition 22. First we show that if $\theta = 0$ and if $k(0) = k_H^*$, we can construct a feasible plan, with consumption $c(t) = \bar{c} > aF(c_H^*)$ for $0 \le t < T$, with decreasing capital, and with $k(t) = 0$ for $t \ge T$, and with $c(t) = aF(0)$ for $t \ge T$. We will show that by making $\bar{c}$ large enough, if $a\left(F(k_H^*) - F(0)\right)/k_H^* \equiv \omega < \bar{\rho}$ holds, the utility from this plan is higher than the one with $c(t) = aF(k_H^*)$ for all $t \ge 0$. For $t \in [0, T)$ we have $\dot{k}(t) = aF(k(t)) - \bar{c}$, so that $k(T) - k_H^* = \int_0^T aF(k(t))dt - T\bar{c}$. But T is defined so that $k(T) = 0$, then $k_H^* + \int_0^T aF(k(t))dt = T\bar{c}$, and since $F(k(t)) > 0$, then $T\,\bar{c} \ge k_H^*$. Also, since $F(k(t)) \le F(k_H^*)$, then as $\bar{c} \to \infty$, then $T \to 0$. Now we compute a lower bound to the utility of this plan. The utility is: $U_1(\bar{c}) \equiv \bar{c}\int_0^T e^{-\bar{\rho}t}dt + e^{-\bar{\rho}T}\frac{aF(0)}{\bar{\rho}} > \frac{k_H^*}{T}\int_0^T e^{-\bar{\rho}t}dt + e^{-\rho T}\frac{aF(0)}{\bar{\rho}} = \frac{k_H^*}{T}\frac{1-e^{-\bar{\rho}T}}{\bar{\rho}} + e^{-\bar{\rho}T}\frac{aF(0)}{\bar{\rho}}$. Using that $(1 - \exp(-\bar{\rho}T))/(\rho T) = 1 + o(\bar{\rho}T)/(\bar{\rho}T)$, then: $\lim_{\bar{c}\to\infty} U_1(\bar{c}) \ge k_H^* + \frac{aF(0)}{\bar{\rho}}$. The utility of the alternative plan with $c(t) = aF(0)$ for all $t \ge 0$ is $U_2 \equiv aF(k_H^*)/\bar{\rho}$, so for for any $\epsilon > 0$, there is a $\bar{c}$ large enough so that $U_1(\bar{c}) - U_2 \ge k_H^* + \frac{aF(0)}{\bar{\rho}} - \frac{aF(k_H^*)}{\bar{\rho}} - \epsilon = k_H^* + a\frac{F(0)-F(k_H^*)}{\bar{\rho}} - \epsilon = k_H^*\left(1 - \frac{\omega}{\bar{\rho}}\right) - \epsilon$. Thus, since by assumption $\omega < \bar{\rho}$, hence $(1 - \omega/\bar{\rho}) > 0$, and thus setting $\epsilon < (1 - \omega/\bar{\rho})$ we have shown that we can set $\bar{c}$ so that $U_1(\bar{c}) > U_2$.

Second, we shown that under our assumptions and for $\theta = 0$ that since the feasible plan with $\bar{c} > aF(k_H^*)$ for $t \in [0, T)$ and $c(t) = aF(0)$ for $t \ge T$ has higher utility then the one for $c(t) = aF(k_H^*)$ for $t \ge 0$. Then, as before, because $u(c)$ is continuous in θ, we have that for θ close to zero, and for fixed $T, \bar{k}_H^*, F(k_H^*), F(0)$ and $\bar{\rho}$, that the utility of the path going to $k = 0$ is higher than the utility of remaining at k_H^*.

Third, since we have shown that for $\theta > 0$ but low enough, and with initial condition $k(0) = k_H^*$ there is a feasible path with $k(t) \to 0$ with higher utility than staying with $c(t) = aF(k_H^*)$. Thus, there must be path that that starts with $c(0) > aF(k_H^*) = aF(k(0))$ and with $k(t)$ that converges to $k = 0$ in finite time. Since there is always a saddle path that crosses k_H^*. Hence there are two equilibria for initial conditions close to $k(0)$ closest to k_H^*.