The Past and Future of U.S. Structural Change: Compositional Accounting and Forecasting

Technical Appendix and Supplementary Material

Andrew T. Foerster
Federal Reserve Bank of San Francisco

Andreas Hornstein
Federal Reserve Bank of Richmond

Pierre-Daniel G. Sarte
Federal Reserve Bank of Richmond

Mark W. Watson
Princeton University and NBER

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Contents

We generically partition the set of all sectors, $\mathcal{N} = \{1, ..., n\}$, into two subsets. We denote by $\widetilde{\mathcal{N}}$ the set of sectors whose goods are assembled using CES aggregators and whose shares will vary with prices and quantities over time. We denote by $\bar{\mathcal{N}}$ the set of sectors whose goods are assembled using unit-elastic aggregators and whose shares will be constant. These two subsets cover all sectors, $\mathcal{N} = \bar{\mathcal{N}} \cup \widetilde{\mathcal{N}}$, and are disjoint, $\bar{\mathcal{N}} \cap \widetilde{\mathcal{N}} = \emptyset$.

The subset $\widetilde{\mathcal{N}}$ (or $\bar{\mathcal{N}}$) will not always include the same sectors depending on which aspect of the environment, production, or preferences is being addressed. For example, we denote by $\widetilde{\mathcal{N}}_j^x$ the set of sectors whose goods are assembled into investment, x, by sector j using a CES technology. Thus, $\widetilde{\mathcal{N}}_j^x$ would include Durable Goods and IPP Services (in this case $\forall j$).

1 Preferences

We distinguish between two sets of consumption goods, $j \in \widetilde{\mathcal{N}}^c$ and $j \in \bar{\mathcal{N}}^c$. Total household expenditures on consumption, e, reflect expenditures on both types of goods with expenditure on goods $j \in \widetilde{\mathcal{N}}^c$ and $j \in \bar{\mathcal{N}}^c$ denoted by $\widetilde{e}$ and $\bar{e}$ respectively, where $e = \widetilde{e} + \bar{e}$.

1.1 The Homothetic Bundle

Goods $j \in \bar{\mathcal{N}}^c$ are assembled into a homothetic consumption bundle, $\bar{c}$, according to the aggregator (abstracting from the time subscripts),

$$\bar{c} = \prod_{j \in \bar{\mathcal{N}}^c} \left(\frac{c_j}{\zeta_j} \right)^{\zeta_j}, \ \sum_{j \in \bar{\mathcal{N}}^c} \zeta_j = 1.$$

The Lagrangian associated with the corresponding expenditure minimization problem is

$$L^{\bar{c}} = \sum_{j \in \bar{\mathcal{N}}^c} p_j^y c_j + p^{\bar{c}} \left[\bar{c} - \prod_{j \in \bar{\mathcal{N}}^c} \left(\frac{c_j}{\zeta_j} \right)^{\zeta_j} \right].$$

Then, we have that,

$$p_j^y = p^{\bar{c}} \frac{\bar{c}}{c_j / \zeta_j},$$

$$p^{\bar{c}} = \prod_{j \in \bar{\mathcal{N}}^c} (p_j^y)^{\zeta_j}.$$

In addition,

$$p_j^y c_j = \zeta_j p^{\bar{c}} \bar{c} \Rightarrow \sum_{j \in \bar{\mathcal{N}}^c} p_j^y c_j = p^{\bar{c}} \bar{c},$$

which implies

$$\bar{e} = p^{\bar{c}}\bar{c} = \prod_{j\in\bar{\mathcal{N}}^c} (p^y_j)^{\zeta_j}\bar{c}.$$

Thus, expenditures on the homothetic bundle, $\bar{e}$, are the product of the utility aggregate, $\bar{c}$, and the price index for the utility aggregate.

1.2 The Non-Homothetic Bundle

Goods $j \in \widetilde{\mathcal{N}}^c$ are bundled using the aggregator, $\widetilde{c} = \mathcal{C}(c_j : j \in \widetilde{\mathcal{N}}^c)$, defined implicitly by

$$\sum_{j\in\widetilde{\mathcal{N}}^c} \Theta_j^{1/\sigma} \left[c_j/(\widetilde{c})^{\epsilon_j}\right]^{(\sigma-1)/\sigma} = 1, \tag{1}$$

where $\Theta_b = \epsilon_b = 1$ for some base commodity $b \in \widetilde{\mathcal{N}}^c$. All other preference parameters satisfy $\Theta_j, \epsilon_j \geq 0$ for $j \neq b$ and $\sigma \geq 0$. The associated expenditure minimization problem is

$$\min \widetilde{e} = \sum_{j\in\widetilde{\mathcal{N}}^c} p^y_j c_j \text{ subject to } \mathcal{C}(c_j : j \in \widetilde{\mathcal{N}}^c) = \widetilde{c}.$$

The corresponding Lagrangian is,

$$\min \sum_{j\in\widetilde{\mathcal{N}}^c} p^y_j c_j + \Lambda \left[1 - \sum_{j\in\widetilde{\mathcal{N}}^c} \Theta_j^{1/\sigma} \left[c_j/\widetilde{c}^{\epsilon_j}\right]^{(\sigma-1)/\sigma}\right].$$

The FOCs are

$$p^y_j = \frac{\sigma-1}{\sigma}\Lambda\Theta_j^{\frac{1}{\sigma}}\widetilde{c}^{\frac{\epsilon_j(1-\sigma)}{\sigma}}c_j^{\frac{-1}{\sigma}},$$

which we can solve for consumption,

$$c_j = \left(\frac{\sigma-1}{\sigma}\right)^{\sigma} \Lambda^{\sigma}\Theta_j (p^y_j)^{-\sigma}\widetilde{c}^{\epsilon_j(1-\sigma)}.$$

Total cost-minimizing expenditures are

$$\widetilde{e} = \sum_{j\in\widetilde{\mathcal{N}}^c} p^y_j c_j = \widetilde{\mathcal{E}}(p^y, \widetilde{c}) = \left(\frac{\sigma-1}{\sigma}\right)^{\sigma} \Lambda^{\sigma} \sum_{j\in\widetilde{\mathcal{N}}^c} \Theta_j (p^y_j)^{1-\sigma}\widetilde{c}^{\epsilon_j(1-\sigma)}.$$

Moreover, we can rewrite the expression for c_j as

$$c_j^{\frac{\sigma-1}{\sigma}} = \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \Lambda^{\sigma-1}\Theta_j^{\frac{\sigma-1}{\sigma}}(p_j^y)^{1-\sigma}\widetilde{c}^{\varepsilon_j(1-\sigma)\frac{\sigma-1}{\sigma}}$$
$$\Rightarrow \Theta_j^{1/\sigma}\frac{c_j^{\frac{\sigma-1}{\sigma}}}{\widetilde{c}^{\frac{\varepsilon_j(\sigma-1)}{\sigma}}} = \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \Lambda^{\sigma-1}\Theta_j(p_j^y)^{1-\sigma}\widetilde{c}^{\varepsilon_j(1-\sigma)}.$$

Summing this expression over all commodities and recalling the implicit definition of preferences in expression (1), we obtain

$$1 = \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \Lambda^{\sigma-1}\sum_{j\in\widetilde{\mathcal{N}}^c}\Theta_j(p_j^y)^{1-\sigma}\widetilde{c}^{\varepsilon_j(1-\sigma)}$$
$$\Rightarrow \Lambda^{(1-\sigma)}\left(\frac{\sigma-1}{\sigma}\right)^{1-\sigma} = \sum_{j\in\widetilde{\mathcal{N}}^c}\Theta_j(p_j^y)^{1-\sigma}\widetilde{c}^{\varepsilon_j(1-\sigma)}.$$

It follows that $\widetilde{\mathcal{E}}(p^y,\widetilde{c}) = \left(\frac{\sigma-1}{\sigma}\right)\Lambda$ so that the Hicksian demand functions are given by

$$c_j = C_j(p^y,\widetilde{c}) = \left(\frac{p_j^y}{\widetilde{\mathcal{E}}(p^y,\widetilde{c})}\right)^{-\sigma}\Theta_j\widetilde{c}^{\varepsilon_j(1-\sigma)},\ j\in\widetilde{\mathcal{N}}^c \tag{2}$$

and the expenditure function is

$$\widetilde{\mathcal{E}}(p^y,\widetilde{c}) = \left[\sum_{j\in\widetilde{\mathcal{N}}^c}\Theta_j(p_j^y)^{1-\sigma}\widetilde{c}^{\varepsilon_j(1-\sigma)}\right]^{\frac{1}{1-\sigma}}. \tag{3}$$

1.3 The Household Problem

The representative household solves the utility maximization problem,

$$\max\ \mathcal{U} = (\bar{c}_t)^{\rho_t}(\widetilde{c}_t)^{1-\rho_t} - \sum_{j\in\mathcal{N}}\frac{\varphi_{j,t}\ell_{j,t}^{1+\gamma_\ell}}{1+\gamma_\ell},$$
$$\text{subject to } p_t^{\bar{c}}\bar{c}_t + \mathcal{E}(p_t^y,\widetilde{c}_t) + \sum_{j\in\mathcal{N}}p_{j,t}^x\left[k_{j,t+1}-(1-\delta_j)k_{j,t}\right] = \sum_{j\in\mathcal{N}}w_{j,t}\ell_{j,t} + \sum_{j\in\mathcal{N}}u_{j,t}k_{j,t} \tag{4}$$

where $\ell_{j,t}$ and $\varphi_{j,t}$ denote, respectively, labor input and labor supply shifters in sector j while γ_ℓ is the Frisch elasticity of labor.

Let λ_t denote the Lagrange multiplier associated with the constraint (4). Turning our attention first to the consumption problem, the corresponding FOCs are

$$\rho_t u_t / \bar{c}_t = \lambda_t p_t^{\bar{c}},$$
$$(1-\rho_t) u_t / \widetilde{c}_t = \lambda_t \frac{\partial \mathcal{E}(p_t^y, \widetilde{c}_t)}{\partial \widetilde{c}_t},$$

where $u_t = (\bar{c}_t)^{\rho_t} (\widetilde{c}_t)^{1-\rho_t}$.

Taking the ratios of the first two FOCs gives

$$\frac{\rho_t}{(1-\rho_t)} \frac{\widetilde{c}_t}{\bar{c}_t} = \frac{p_t^{\bar{c}}}{\partial \mathcal{E}_t / \partial \widetilde{c}_t},$$

which we can solve for $p_t^{\bar{c}} \bar{c}_t$ in the budget constraint to obtain

$$e_t = \frac{\rho_t}{(1-\rho_t)} \frac{\partial \mathcal{E}(p_t^y, \widetilde{c}_t)}{\partial \widetilde{c}_t} \widetilde{c}_t + \mathcal{E}(p_t^y, \widetilde{c}_t).$$

It follows that the expenditure share of the non-homothetic bundle is given by

$$\begin{aligned} \frac{\mathcal{E}_t}{e_t} &= \left[\frac{\rho_t}{(1-\rho_t)} \frac{\partial \mathcal{E}(p_t^y, \widetilde{c}_t)}{\partial \widetilde{c}_t} \frac{\widetilde{c}_t}{\mathcal{E}_t} + 1 \right]^{-1} \\ &= \left[\frac{\rho_t \eta_t^{\mathcal{E}}}{(1-\rho_t)} + 1 \right]^{-1}, \text{ with } \eta_t^{\mathcal{E}} \equiv \frac{\partial \mathcal{E}(p_t^y, \widetilde{c}_t)}{\partial \widetilde{c}_t} \frac{\widetilde{c}_t}{\mathcal{E}_t}. \end{aligned}$$

Thus

$$\boxed{\frac{\mathcal{E}_t}{e_t} = \frac{(1-\rho_t)}{\rho_t \eta_t^{\mathcal{E}} + (1-\rho_t)},} \tag{5}$$

while the expenditure share of the homothetic bundle is

$$\frac{p_t^{\bar{c}} \bar{c}_t}{e_t} = \frac{\rho_t \eta_t^{\mathcal{E}}}{\rho_t \eta_t^{\mathcal{E}} + (1-\rho_t)}. \tag{6}$$

In the data, $\frac{p_t^{\bar{c}} \bar{c}_t}{e_t}$ has stayed approximately constant around 0.1 over the last 70 years. Thus, let

$$\frac{p_t^{\bar{c}} \bar{c}_t}{e_t} = s^c. \tag{7}$$

It follows that ρ_t must solve

$$\rho_t = \frac{s^c}{(1-s^c)\eta_t^{\mathcal{E}} + s^c}, \tag{8}$$

which requires knowing $\eta_t^{\mathcal{E}}$.

The expenditure function associated with the non-homothetic bundle, equation (3), repeated here for convenience,

$$\mathcal{E}(p_t^y, \widetilde{c}_t) = \left[\sum_{j \in \widetilde{\mathcal{N}}^c} \Theta_{j,t} (p_{j,t}^y)^{1-\sigma} (\widetilde{c}_t)^{\epsilon_j(1-\sigma)} \right]^{\frac{1}{1-\sigma}}$$

is homogeneous of degree one in prices, p_t^y, but not in the 'real' consumption index, $\widetilde{c}_t$. In other words, there is no separate price index for $\widetilde{c}_t$. The elasticity of non-homothetic expenditures, $\mathcal{E}_t$, with respect to the 'real' consumption index, $\widetilde{c}_t$, is

$$\begin{aligned}\frac{\partial \mathcal{E}(p_t^y, \widetilde{c}_t)}{\partial \widetilde{c}_t} \frac{\widetilde{c}_t}{\mathcal{E}(p_t^y, \widetilde{c}_t)} &= \frac{\mathcal{E}(p_t^y, \widetilde{c}_t)^{\sigma} \left[\sum_{j \in \widetilde{\mathcal{N}}^c} \Theta_{j,t} (p_{j,t}^y)^{1-\sigma} \epsilon_j (\widetilde{c}_t)^{\epsilon_j(1-\sigma)-1} \right] \widetilde{c}_t}{\mathcal{E}(p_t^y, \widetilde{c}_t)} \\ &= \mathcal{E}(p_t^y, \widetilde{c}_t)^{\sigma-1} \left[\sum_{j \in \widetilde{\mathcal{N}}^c} \Theta_{j,t} (p_{j,t}^y)^{1-\sigma} \epsilon_j (\widetilde{c}_t)^{\epsilon_j(1-\sigma)} \right],\end{aligned}$$

that is,

$$\eta_t^{\mathcal{E}} = \frac{\sum_{j \in \widetilde{\mathcal{N}}^c} \epsilon_j \Theta_{j,t} (p_{j,t}^y)^{1-\sigma} (\widetilde{c}_t)^{\epsilon_j(1-\sigma)}}{\sum_{j \in \widetilde{\mathcal{N}}^c} \Theta_{j,t} (p_{j,t}^y)^{1-\sigma} (\widetilde{c}_t)^{\epsilon_j(1-\sigma)}}. \tag{9}$$

This expenditure elasticity differs from 1 since for goods other than the base good, $\epsilon_j \neq 1$.

Turning our attention next to the labor supply problem, the corresponding FOC is

$$\varphi_{j,t} \ell_{j,t}^{\gamma_\ell} = \lambda_t w_{j,t}, \tag{10}$$

where λ_t solves

$$\rho_t \left(\frac{\widetilde{c}_t}{\bar{c}_t} \right)^{1-\rho_t} = \lambda_t p_t^{\bar{c}}. \tag{11}$$

2 Technology

2.1 Investment

The technology used to produce investment goods for a given sector j is given by

$$x_j = \mathcal{X}(x_{ij}).$$

We write $\mathcal{X}$ as a function of two investment bundles, $\widetilde{x}_j$ and $\bar{x}_j$, where

$$x_j = \left(\frac{\widetilde{x}_j}{\rho_j^x} \right)^{\rho_j^x} \left(\frac{\bar{x}_j}{1-\rho_j^x} \right)^{1-\rho_j^x}, \ \ \rho_j^x \in (0,1),$$

and

$$
\widetilde{x}_j = \left[\sum_{i \in \widetilde{N}_j^x} z_{ij}^x x_{ij}^{\frac{\epsilon_j^x - 1}{\epsilon_j^x}} \right]^{\frac{\epsilon_j^x}{\epsilon_j^x - 1}}, \sum_{i \in \widetilde{N}_j^x} z_{ij}^x = 1,\ \epsilon_j^x \in (0, \infty),
$$

$$
\bar{x}_j = \prod_{i \in \bar{N}_j^x} \left(\frac{x_{ij}}{\zeta_{ij}^x} \right)^{\zeta_{ij}^x}, \sum_{i \in \bar{N}_j^x}^{n} \zeta_{ij}^x = 1.
$$

Concretely, the sub-bundle $\widetilde{x}_j$ consists of Durable Goods and IPP in every sector j.

- The cost minimization problem associated with x_j is

$$
\min p_j^{\widetilde{x}} \widetilde{x}_j + p_j^{\bar{x}} \bar{x}_j \text{ subject to } x_j = \left[\widetilde{x}_j / \rho_j^x \right]^{\rho_j^x} \left[\bar{x}_j / \left(1 - \rho_j^x \right) \right]^{1 - \rho_j^x}.
$$

  The corresponding Lagrangian is

$$
L^{x_j} = p_j^{\widetilde{x}} \widetilde{x}_j + p_j^{\bar{x}} \bar{x}_j + p_j^x \left[x_j - \left(\frac{\widetilde{x}_j}{\rho_j^x} \right)^{\rho_j^x} \left(\frac{\bar{x}_j}{1 - \rho_j^x} \right)^{1 - \rho_j^x} \right].
$$

  The FOCs are

$$
p_j^{\widetilde{x}} = p_j^x \frac{x_j}{\widetilde{x}_j / \rho_j^x},
$$

$$
p_j^{\bar{x}} = p_j^x \frac{x_j}{\bar{x}_j / \left(1 - \rho_j^x \right)}.
$$

  Then

$$
p_j^x = (p_j^{\widetilde{x}})^{\rho_j^x} (p_j^{\bar{x}})^{1 - \rho_j^x}.
$$

  In addition,

$$
p_j^{\widetilde{x}} \widetilde{x}_j = \rho_j^x p_j^x x_j,
$$

$$
p_j^{\bar{x}} \bar{x}_j = (1 - \rho_j^x) p_j^x x_j.
$$

- The cost minimization problem associated with $\widetilde{x}_j$ is

$$
\min \sum_{i \in \widetilde{\mathcal{N}}_j^x} p_i^y x_{ij} \text{ subject to } \widetilde{x}_j = \left[\sum_{i \in \widetilde{\mathcal{N}}_j^x} z_{ij}^x x_{ij}^{\frac{\epsilon_j^x - 1}{\epsilon_j^x}} \right]^{\frac{\epsilon_j^x}{\epsilon_j^x - 1}}.
$$

The corresponding Lagrangian is

$$L^{\widetilde{x}_j} = \sum_{i \in \widetilde{\mathcal{N}}_j^x} p_i^y x_{ij} + p_j^{\widetilde{x}} \left\{ \widetilde{x}_j - \left[\sum_{i \in \widetilde{\mathcal{N}}_j^x} z_{ij}^x x_{ij}^{\frac{\epsilon_j^x - 1}{\epsilon_j^x}} \right]^{\frac{\epsilon_j^x}{\epsilon_j^x - 1}} \right\}.$$

The FOCs are

$$p_i^y = p_j^{\widetilde{x}} \left[\sum_{i \in \widetilde{\mathcal{N}}_j^x} z_{ij}^x x_{ij}^{\frac{\epsilon_j^x - 1}{\epsilon_j^x}} \right]^{\frac{1}{\epsilon_j^x - 1}} z_{ij}^x x_{ij}^{\frac{-1}{\epsilon_j^x}}, \;\; i \in \widetilde{\mathcal{N}}_j^x.$$

It follows that

$$\left(z_{ij}^x\right)^{\epsilon_j^x} \left(p_i^y\right)^{1-\epsilon_j^x} = \left(p_j^{\widetilde{x}}\right)^{1-\epsilon_j^x} \left[\sum_{i \in \widetilde{\mathcal{N}}_j^x} z_{ij}^x x_{ij}^{\frac{\epsilon_j^x - 1}{\epsilon_j^x}} \right]^{-1} x_{ij}^{\frac{\epsilon_j^x - 1}{\epsilon_j^x}} z_{ij}^x, \;\; i \in \widetilde{\mathcal{N}}_j^x.$$

Summing across $i \in \widetilde{\mathcal{N}}_j^x$ gives

$$\sum_{i \in \widetilde{\mathcal{N}}_j^x} \left(z_{ij}^x\right)^{\epsilon_j^x} \left(p_i^y\right)^{1-\epsilon_j^x} = \left(p_j^{\widetilde{x}}\right)^{1-\epsilon_j^x} \left[\sum_{i \in \widetilde{\mathcal{N}}_j^x} z_{ij}^x x_{ij}^{\frac{\epsilon_j^x - 1}{\epsilon_j^x}} \right]^{-1} \sum_{i \in \widetilde{\mathcal{N}}_j^x} z_{ij}^x x_{ij}^{\frac{\epsilon_j^x - 1}{\epsilon_j^x}},$$

so that

$$p_j^{\widetilde{x}} = \left[\sum_{i \in \widetilde{\mathcal{N}}_j^x} \left(z_{ij}^x\right)^{\epsilon_j^x} \left(p_i^y\right)^{1-\epsilon_j^x} \right]^{\frac{1}{1-\epsilon_j^x}}.$$

In addition, observe that

$$p_i^y x_{ij} = p_j^{\widetilde{x}} \left[\sum_{i \in \widetilde{\mathcal{N}}_j^x} z_{ij}^x x_{ij}^{\frac{\epsilon_j^x - 1}{\epsilon_j^x}} \right]^{\frac{1}{\epsilon_j^x - 1}} z_{ij}^x x_{ij}^{\frac{\epsilon_j^x - 1}{\epsilon_j^x}} \Rightarrow \sum_{i \in \widetilde{\mathcal{N}}_j^x} p_i^y x_{ij} = p_j^{\widetilde{x}} \widetilde{x}_j.$$

Moreover,

$$\frac{p_i^y x_{ij}}{p_j^{\widetilde{x}} \widetilde{x}_j} = \left(\widetilde{x}_j\right)^{\frac{1-\epsilon_j^x}{\epsilon_j^x}} \left(\widetilde{x}_{ij}\right)^{\frac{\epsilon_j^x - 1}{\epsilon_j}} z_{ij}^x \Rightarrow \sum_{i \in \widetilde{\mathcal{N}}_j^x} \frac{p_i^y x_{ij}}{p_j^{\widetilde{x}} \widetilde{x}_j} = 1.$$

In the special case where $\epsilon_j^x = 1$, $\frac{p_i^y x_{ij}}{p_j^{\widetilde{x}} \widetilde{x}_j} = z_{ij}^x$ (i.e., production is Cobb-Douglas).

- The cost minimization problem associated with $\bar{x}_j$ is

$$\min \sum_{i \in \bar{\mathcal{N}}_j^x} p_i^y x_{ij} \text{ subject to } \bar{x}_j = \prod_{i \in \bar{\mathcal{N}}_j^x} \left(\frac{x_{ij}}{\zeta_{ij}^x} \right)^{\zeta_{ij}^x}.$$

The corresponding Lagrangian is

$$L^{\bar{x}_j} = \sum_{i \in \bar{\mathcal{N}}_j^x} p_i^y x_{ij} + p_j^{\bar{x}} \left[\bar{x}_j - \prod_{i \in \bar{\mathcal{N}}_j^x} \left(\frac{x_{ij}}{\zeta_{ij}^x} \right)^{\zeta_{ij}^x} \right].$$

Following the derivations above, we have that

$$p_i^y = p_j^{\bar{x}} \frac{\bar{x}_j}{x_{ij}/\zeta_{ij}^x},$$

$$p_j^{\bar{x}} = \prod_{i \in \bar{\mathcal{N}}_j^x} (p_i^y)^{\zeta_{ij}^x}.$$

In addition,

$$p_i^y x_{ij} = \zeta_{ij}^x p_j^{x^n} \bar{x}_j \Rightarrow \sum_{i \in \bar{\mathcal{N}}_j^x} p_i^y x_{ij} = p_j^{\bar{x}} \bar{x}_j.$$

Putting these results together gives

$$p_j^x = \mathcal{P}_j^X(p^y) = \underbrace{\left[\sum_{i \in \widetilde{\mathcal{N}}_j^x} (z_{ij}^x)^{\epsilon_j^x} \left(p_i^y\right)^{1-\epsilon_j^x} \right]^{\frac{\rho_j^x}{1-\epsilon_j^x}}}_{(p_j^{\tilde{x}})^{\rho_j^x}} \underbrace{\prod_{i \in \bar{\mathcal{N}}_j^x} (p_i^y)^{\zeta_{ij}^x (1-\rho_j^x)}}_{(p_j^{\bar{x}})^{1-\rho_j^x}}.$$

2.2 Materials

The approach here is analogous to that of investment. Thus, we have

$$p_j^m = (p_j^{\widetilde{m}})^{\rho_j^m} (p_j^{\bar{m}})^{1-\rho_j^m},$$

$$p^{\widetilde{m}} = \left[\sum_{i \in \widetilde{\mathcal{N}}_j^m} (z_{ij}^m)^{\epsilon_j^m} \left(p_i^y\right)^{1-\epsilon_j^m} \right]^{\frac{1}{1-\epsilon_j^m}},$$

$$p_j^{\bar{m}} = \prod_{i \in \bar{\mathcal{N}}_j^m} (p_i^y)^{\zeta_{ij}^m}.$$

Together these expressions imply

$$p_j^m = \mathcal{P}_j^M(p^y) = \underbrace{\left[\sum_{i \in \widetilde{\mathcal{N}}_j^m} (z_{ij}^m)^{\epsilon_j^m} \left(p_i^y\right)^{1-\epsilon_j^m} \right]^{\frac{\rho_j^m}{1-\epsilon_j^m}}}_{(p_j^{\widetilde{m}})^{\rho_j^m}} \underbrace{\prod_{i \in \bar{\mathcal{N}}_j^m} (p_i^y)^{\zeta_{ij}^m (1-\rho_j^m)}}_{(p_j^{\bar{m}})^{1-\rho_j^m}}.$$

In addition, for $i \in \widetilde{\mathcal{N}}_j^m$,

$$p_i^y m_{ij} = p_j^{\widetilde{m}} \left[\sum_{i \in \widetilde{\mathcal{N}}_j^m} z_{ij}^m m_{ij}^{\frac{\epsilon_j^m - 1}{\epsilon_j^m}} \right]^{\frac{1}{\epsilon_j^m - 1}} z_{ij}^m m_{ij}^{\frac{\epsilon_j^m - 1}{\epsilon_j^m}} \Rightarrow \sum_{i \in \widetilde{\mathcal{N}}_j^m} p_i^y m_{ij} = p_j^{\widetilde{m}} \widetilde{m}_j,$$

and

$$\frac{p_i^y m_{ij}}{p_j^{\widetilde{m}} \widetilde{m}_j} = \left(\widetilde{m}_j\right)^{\frac{1-\epsilon_j^m}{\epsilon_j^m}} \left(\widetilde{m}_{ij}\right)^{\frac{\epsilon_j^m - 1}{\epsilon_j}} z_{ij}^m \Rightarrow \sum_{i \in \widetilde{\mathcal{N}}_j^m} \frac{p_i^y m_{ij}}{p_j^{\widetilde{m}} \widetilde{m}_j} = 1,$$

while for $i \in \bar{\mathcal{N}}_j^m$,

$$p_i^y = p_j^{\bar{m}} \frac{\bar{m}_j}{m_{ij}/\zeta_{ij}^m},$$

$$p_j^{\bar{m}} = \prod_{i \in \bar{\mathcal{N}}_j^m} (p_i^y)^{\zeta_{ij}^m},$$

and

$$p_i^y m_{ij} = \zeta_{ij}^m p_j^{\bar{m}} \bar{m}_j \Rightarrow \sum_{i \in \bar{\mathcal{N}}_j^m} p_i^y m_{ij} = p_j^{\bar{m}} \bar{m}_j.$$

2.3 Gross Output

Gross output in each sector is produced using value added (from capital and labor), v_j, and the materials bundle, m_j, whose price index is derived above. In all sectors $j \in \mathcal{N}$, gross output, y_j, is produced according to the Cobb-Douglas technology

$$y_j = \left(\frac{v_j}{\gamma_j}\right)^{\gamma_j} \left(\frac{m_j}{1-\gamma_j}\right)^{1-\gamma_j},$$

which implies a price index

$$p_j^y = \mathcal{P}_j^Y(p_j^v, p_j^m) = \left(p_j^v\right)^{\gamma_j} \left(p_j^m\right)^{1-\gamma_j}.$$

In addition,

$$\frac{p_j^v v_j}{p_j^y y_j} = \gamma_j,$$
$$\frac{p_j^m m_j}{p_j^y y_j} = (1-\gamma_j).$$

2.4 Value Added

Similarly, value added is produced using a unit-elastic technology in all sectors, $j \in \mathcal{N}$.

Let

$$v_j = z_j \underbrace{\left(\frac{k_j}{\alpha_j}\right)^{\alpha_j} \left(\frac{\ell_j}{1-\alpha_j}\right)^{1-\alpha_j}}_{\mathcal{V}_j(k_j,\ell_j)}.$$

Then the FOCs for capital and labor are

$$u_j = \alpha_j p_j^v \frac{z_j \mathcal{V}_j(k_j,\ell_j)}{k_j} = p_j^v z_j \left(\frac{k_j}{\alpha_j}\right)^{\alpha_j - 1} \left(\frac{\ell_j}{1-\alpha_j}\right)^{1-\alpha_j} = p_j^v z_j \left(\frac{k_j}{\ell_j}\right)^{\alpha_j - 1} \left(\frac{\alpha_j}{1-\alpha_j}\right)^{1-\alpha_j},$$
$$w_j = (1-\alpha_j) p_j^v \frac{z_j \mathcal{V}(k_j,\ell_j)}{\ell_j} = p_j^v z_j \left(\frac{k_j}{\alpha_j}\right)^{\alpha_j} \left(\frac{\ell_j}{1-\alpha_j}\right)^{-\alpha_j} = p_j^v z_j \left(\frac{k_j}{\ell_j}\right)^{\alpha_j} \left(\frac{\alpha_j}{1-\alpha_j}\right)^{\alpha_j}.$$

3 Modeling Past Structural Change as an Equilibrium

We interpret the low-frequency evolution of allocations and prices in the data through successive steady-states of the model and abstract from transition dynamics.

In this section, we take the exogenous drivers z_j, z_{ij}^m, z_{ij}^x, $\psi^{nx} + \psi^g$, and Θ_j as exogenous, obtained as described in the main text. Conditional on these exogenous drivers, we describe how to obtain a steady state equilibrium as a fixed point in the final goods price vector, p^y.[1]

Guess a price vector, $p^y = (p_1^y, ..., p_n^y)'$.

[1]For this approach, we take observations on labor, ℓ_j, as given (labor is exogenous) and use the equilibrium equations to infer labor supply shocks, φ_j, consistent with these observations given z_j (shocks to labor demand).

$$\textbf{STEP 1:}\ \ p_j^x = \mathcal{P}_j^X(p^y) = \underbrace{\left[\sum_{i\in\widetilde{\mathcal{N}}_j^x} (z_{ij}^x)^{\epsilon_j^x} \left(p_i^y\right)^{1-\epsilon_j^x}\right]^{\frac{\rho_j^x}{1-\epsilon_j^x}}}_{(p_j^{\tilde{x}})^{\rho_j^x}} \underbrace{\prod_{i\in\bar{\mathcal{N}}_j^x} (p_i^y)^{\zeta_{ij}^x(1-\rho_j^x)}}_{(p_j^{\bar{x}})^{1-\rho_j^x}},\ \forall j \in \mathcal{N}.$$

$$\textbf{STEP 2:}\ \ p_j^m = \mathcal{P}_j^M(p^y) = \underbrace{\left[\sum_{i\in\widetilde{\mathcal{N}}_j^m} (z_{ij}^m)^{\epsilon_j^m} \left(p_i^y\right)^{1-\epsilon_j^m}\right]^{\frac{\rho_j^m}{1-\epsilon_j^m}}}_{(p_j^{\widetilde{m}})^{\rho_j^m}} \underbrace{\prod_{i\in\bar{\mathcal{N}}_j^m} (p_i^y)^{\zeta_{ij}^m(1-\rho_j^m)}}_{(p_j^{\bar{m}})^{1-\rho_j^m}},\ \forall j \in \mathcal{N}.$$

$$\textbf{STEP 3:}\ \ p_j^v = \left[\frac{p_j^y}{(p_j^m)^{1-\gamma_j}}\right]^{\frac{1}{\gamma_j}},\ \forall j \in \mathcal{N}.$$

$$\textbf{STEP 4:}\ u_j = p_j^x\left(\frac{1}{\beta} - 1 + \delta_j\right),\ \forall j \in \mathcal{N}.$$

$$\textbf{STEP 5:}\ \ \frac{k_j}{\ell_j} = \left[\frac{u_j}{p_j^v z_j}\left(\frac{\alpha_j}{1-\alpha_j}\right)^{\alpha_j - 1}\right]^{\frac{1}{\alpha_j - 1}},\ \forall j \in \mathcal{N}.$$

$$\textbf{STEP 6:}\ \ w_j = p_j^v z_j \left(\frac{k_j}{\ell_j}\right)^{\alpha_j}\left(\frac{\alpha_j}{1-\alpha_j}\right)^{\alpha_j},\ \forall j \in \mathcal{N}.$$

STEP 7 serves as a check while **STEP 8** is simply

$$\textbf{STEP 8:}\ \ k_j = \left(\frac{k_j}{\ell_j}\right)\ell_j,\ \forall j \in \mathcal{N}.$$

$$\textbf{STEP 9:}\ \ x_j = \delta_j k_j,\ \forall j \in \mathcal{N}.$$

$$\textbf{STEP 10:}\ \ v_j = z_j \mathcal{V}_j(k_j, \ell_j) = \frac{z_j}{\alpha_j^{\alpha_j}(1-\alpha_j)^{(1-\alpha_j)}}\left(\frac{k_j}{\ell_j}\right)^{\alpha_j}\ell_j,\ \forall j \in \mathcal{N}.$$

STEP 11 follows from Shephard's lemma,

$$v_j = \frac{\partial \mathcal{P}_j^Y(p_j^v, p_j^m)}{\partial p_j^v} y_j \Rightarrow y_j = \left(\frac{\partial \mathcal{P}_j^Y(p_j^v, p_j^m)}{\partial p_j^v}\right)^{-1} v_j.$$

Then

$$\textbf{STEP 11:}\;\; v_j = \gamma_j \left(p_j^v\right)^{\gamma_j - 1} \left(p_j^m\right)^{1-\gamma_j} y_j \Rightarrow y_j = \frac{1}{\gamma_j} \left(p_j^v\right)^{1-\gamma_j} \left(p_j^m\right)^{\gamma_j - 1} v_j,\; \forall j \in \mathcal{N}.$$

Intermediate input use is determined in the next step, also based on Shephard's lemma,

$$\textbf{STEP 12:}\; m_j = \frac{\partial \mathcal{P}_j^Y(p_j^v, p_j^m)}{\partial p_j^m} y_j \Rightarrow m_j = (1-\gamma_j) \left(p_j^v\right)^{\gamma_j} \left(p_j^m\right)^{-\gamma_j} y_j.$$

and

$$\textbf{STEP 12:}\;\; m_{ij} = \rho_j^m (p_j^{\widetilde{m}})^{\rho_j^m - 1} (p_j^{\bar{m}})^{1-\rho_j^m} \left[\sum_{i \in \widetilde{\mathcal{N}_j^m}} (z_{ij}^m)^{\epsilon_j^m} \left(p_i^y\right)^{1-\epsilon_j^m}\right]^{\frac{\epsilon_j^m}{1-\epsilon_j^m}} (z_{ij}^m)^{\epsilon_j^m} (p_i^y)^{-\epsilon_j^m} m_j,\; i \in \widetilde{\mathcal{N}}^{m_j},$$

and

$$\textbf{STEP 12:}\;\; m_{ij} = (1-\rho_j^m)(p_j^{\widetilde{m}})^{\rho_j^m} (p_j^{\bar{m}})^{-\rho_j^m} \zeta_{ij}^m \frac{\prod_{i \in \mathcal{N}_j^m} (p_i^y)^{\zeta_{ij}^m}}{p_i^y} m_j,\; i \in \bar{\mathcal{N}}^{m_j}.$$

STEP 13 again follows from Shephard's lemma,

$$x_{ij} = \frac{\partial \mathcal{P}_j^X(p^y)}{\partial p_i^y} x_j.$$

$$\textbf{STEP 13:}\;\; x_{ij} = \rho_j^x (p_j^{\bar{x}})^{1-\rho_j^x} (p_j^{\widetilde{x}})^{\rho_j^x - 1} \left[\sum_{i \in \widetilde{\mathcal{N}_j^x}} (z_{ij}^x)^{\epsilon_j^x} \left(p_i^y\right)^{1-\epsilon_j^x}\right]^{\frac{\epsilon_j^x}{1-\epsilon_j^x}} (z_{ij}^x)^{\epsilon_j^x} (p_i^y)^{-\epsilon_j^x} x_j,\; i \in \widetilde{\mathcal{N}}^{x_j}.$$

and

$$\textbf{STEP 13:}\;\; x_{ij} = (1-\rho_j^x)(p_j^{\widetilde{x}})^{\rho_j^x} (p_j^{\bar{x}})^{-\rho_j^x} \zeta_{ij}^x \frac{\prod_{i \in \mathcal{N}_j^x}^n (p_i^y)^{\zeta_{ij}^x}}{p_i^y} x_j,\; i \in \bar{\mathcal{N}}^{x_j}.$$

STEP 14 finds, for some exogenous shift parameter, ψ_j, the residual supply of final goods available for consumption,

$$\textbf{STEP 14:}\; c_j^s = y_j - \sum_{i \in \mathcal{N}} m_{ji} - \sum_{i \in \mathcal{N}} x_{ji} - \underbrace{\psi_j \frac{p_j^v v_j}{p_j^y}}_{\Psi_j}.$$

STEP 15 then defines total nominal expenditures, e, (from the supply side)

$$\textbf{STEP 15:}\;\; e = \sum_{j \in \mathcal{N}} p_j^y c_j^s,$$

which we partition in terms of $\bar{e}$, nominal expenditures on the homothetic bundle, all goods except for Non-Durable Goods and Services, and $\widetilde{e}$, nominal expenditures on the non-homothetic bundle, Non-Durable Goods and Services.

STEP 16 then gives us total expenditures on goods in sectors $j \in \bar{\mathcal{N}}^c$,

$$\textbf{STEP 16:} \;\; \bar{e} = s^c e$$

and sectors $j \in \widetilde{N}^c$

$$\textbf{STEP 16:} \;\; \widetilde{e} = (1 - s^c)\, e$$

STEP 17 then gives us the consumption index for goods in sectors $j \in \bar{\mathcal{N}}^c$,

$$\textbf{STEP 17:} \;\; \bar{e} = \prod_{j \in \bar{\mathcal{N}}^c} (p_j^y)^{\zeta_j} \bar{c} \;\Rightarrow\; \bar{c} = \left(\prod_{j \in \bar{\mathcal{N}}^c} (p_j^y)^{\zeta_j} \right)^{-1} \bar{e},$$

and sectors $j \in \widetilde{N}^c$,

$$\textbf{STEP 17:} \;\; \widetilde{e} = \left[\sum_{j \in \widetilde{N}^c} \Theta_j (p_j^y)^{(1-\sigma)} (\widetilde{c})^{\epsilon_j (1-\sigma)} \right]^{\frac{1}{1-\sigma}}.$$

STEP 18 gives us the consumption demand for final goods in each sector, again using Shephard's lemma,

$$c_j^d = \frac{\partial \mathcal{E}}{\partial p_j^y}.$$

Thus, for sectors $j \in \bar{\mathcal{N}}^c$, we have,

$$\textbf{STEP 18:} \;\; c_j^d = \frac{\partial \mathcal{E}(p^y, \bar{c})}{\partial p_j^y} = \frac{\zeta_j \prod_{j \in \bar{\mathcal{N}}^c} (p_j^y)^{\zeta_j}}{p_j^y} \bar{c}, \; j \in \bar{\mathcal{N}}^c.$$

For sectors $j \in \widetilde{\mathcal{N}}^c$, we have,

$$\textbf{STEP 18:} \;\; c_j^d = \frac{\partial \mathcal{E}(p^y, \widetilde{c})}{\partial p_j^y} = \left[\sum_{j \in \widetilde{N}^c} \Theta_j (p_j^y)^{1-\sigma} (\widetilde{c})^{\epsilon_j (1-\sigma)} \right]^{\frac{\sigma}{1-\sigma}} \Theta_j (\widetilde{c})^{\epsilon_j (1-\sigma)} (p_j^y)^{-\sigma}, \; j \in \widetilde{N}^c.$$

The vector p^y is an equilibrium price when the goods market clears in each sector,

$$|c_i^d - c_i^s| < \epsilon, \;\; \forall i \in \mathcal{N}. \tag{12}$$

To find the equilibrium price vector, we implement a tâtonnement process that adjusts p^y in order to clear the market for consumption goods. That is, we set a new price $(p^y)'$ where

$$(p_i^y)' = p_i^y + b \times \underbrace{\left(c_i^d - c_i^s\right)}_{\textbf{Excess Demand for Consumption Goods}}, \; b > 0,$$

and iterate until $\|(p^y)^{m+1} - (p^y)^m\| < \epsilon$. This tâtonnement process implies that when excess demand is positive, $\left(c_i^d - c_i^s\right) > 0$, prices are adjusted upward, and when excess demand is negative, $\left(c_i^d - c_i^s\right) < 0$, prices are adjusted downward.

To find the parameters and labor supply shifters consistent with observed labor input as well as the fact that s^c is constant in the data, we solve the following steps,

$$\textbf{STEP 19: } \rho = \frac{s^c}{(1-s^c)\eta^{\tilde{\mathcal{E}}} + s^c}, \text{ with } \eta^{\tilde{\mathcal{E}}} = \frac{\sum_j \epsilon_j \Theta_j (p_j^y)^{1-\sigma} (\widetilde{c})^{\epsilon_j(1-\sigma)}}{\sum_j \Theta_j (p_j^y)^{1-\sigma} (\widetilde{c})^{\epsilon_j(1-\sigma)}}.$$

$$\textbf{STEP 20: } \lambda = \frac{\rho}{p^{\bar{c}}} \left(\frac{\widetilde{c}}{\bar{c}}\right)^{1-\rho}.$$

$$\textbf{STEP 21: } \varphi_j = \frac{\lambda w_j}{\ell_j^{\gamma_\ell}}.$$

An important property of the steady state solution is that it is homogeneous of degree one in p^y. Expression (12) describes n residual equations in n unknowns, the price vector p^y. Although non-homothetic preferences are not homogeneous of degree one in the utility index (after suitable transformation), they continue to be homogeneous of degree one in prices. In other words, there are only $n-1$ independent residual equations and we need to normalize the prices in one sector to 1, i.e, choose a numéraire good.

To see this, suppose that p^y is a solution to steps 1 through 18 above, and that we scale all prices and factor rentals by some factor $\lambda > 0$, $\widetilde{p}^y = \lambda p^y$, $\widetilde{p}^v = \lambda p^v$, $\widetilde{p}^m = \lambda p^m$, etc. Because the price functions are CRS, the expressions in steps 1 through 4 continue to hold. In step 5, the capital-labor ratio is independent of λ as the scale factors cancel. In steps 9 through 13, the quantities remain unaffected by λ since the demand functions are homogeneous of degree zero in prices. Therefore, consumption in step 14, c^s, is independent of λ. This means that total expenditures in steps 15 and 16 are scaled by exactly λ. In step 17, the scale factors cancel since the expenditure functions are homogeneous of degree one in prices in both the homothetic and non-homothetic cases. It follows that the utility indices, $\widetilde{c}$ and $\bar{c}$, in step 17 remain unchanged. Since the expenditure function is homogeneous of degree one in prices, the demand functions are homogeneous of degree zero in prices, and since the utility indices are unchanged, consumption demand, c^d, in (12) is unchanged. Therefore, all equations in steps 1 through 18 continue to hold. Put simply, if p^y is an equilibrium price such that excess demand is zero in equation (12), so is $\widetilde{p}^y = \lambda p^y$.

4 Counterfactual Allocations and Forecasts as Equilibrium Outcomes

In this section, we take the exogenous drivers z_j, z_{ij}^m, z_{ij}^x, Θ_j, ρ, $\psi_j = \psi_j^{nx} + \psi_j^g$, and φ_j as given. The exogenous drivers are obtained as described in the main text or as forecasts. The derivation of φ is described in the previous section. Conditional on these exogenous drivers, we describe how to obtain a steady state equilibrium as a fixed point in the extended final goods price vector, that now includes the shadow price of consumption, $p_\lambda^y = (p_1^y, ..., p_n^y, \lambda)'$.

Guess an extended price vector, p_λ^y.

Repeat **STEP 1** through **STEP 6**. **STEP 6'** then solves for labor supply.

$$\textbf{STEP 6':}\ \ell_j = \left(\frac{\lambda w_j}{\varphi_j}\right)^{\frac{1}{\gamma_\ell}},$$

Repeat **STEP 7** through **STEP 15**, which gives us e conditional on the guess $p_\lambda^y = (p_1^y, ..., p_n^y, \lambda)'$.

From the household problem, we have that

$$\left[\frac{\rho\eta^{\mathcal{E}}}{(1-\rho)} + 1\right]\frac{\widetilde{e}}{e} = 1,$$

or, alternatively,

$$\left[\frac{\rho\frac{\sum_j \epsilon_j \Theta_j (p_j^y)^{1-\sigma}(\widetilde{c})^{\epsilon_j(1-\sigma)}}{\sum_j \Theta_j (p_j^y)^{1-\sigma}(\widetilde{c})^{\epsilon_j(1-\sigma)}} + 1 - \rho}{(1-\rho)}\right]\frac{\left[\sum_j \Theta_j (p_j^y)^{(1-\sigma)}(\widetilde{c})^{\epsilon_j(1-\sigma)}\right]^{\frac{1}{1-\sigma}}}{e} = 1,\ j \in \widetilde{\mathcal{N}}^c.$$

Therefore, for sectors $j \in \widetilde{\mathcal{N}}^c$, we solve for $\widetilde{c}$ from

STEP 16:

$$\left[\frac{\rho\sum_j \epsilon_j \Theta_j (p_j^y)^{1-\sigma}(\widetilde{c})^{\epsilon_j(1-\sigma)} + (1-\rho)\sum_j \Theta_j (p_j^y)^{(1-\sigma)}(\widetilde{c})^{\epsilon_j(1-\sigma)}}{(1-\rho)}\right]$$
$$\times\frac{\left[\sum_j \Theta_j (p_j^y)^{(1-\sigma)}(\widetilde{c})^{\epsilon_j(1-\sigma)}\right]^{\frac{\sigma}{1-\sigma}}}{e} = 1, j \in \widetilde{\mathcal{N}}^c.$$

$$\textbf{STEP 17:}\ \widetilde{e} = \left[\sum_j \Theta_j (p_j^y)^{(1-\sigma)}(\widetilde{c})^{\epsilon_j(1-\sigma)}\right]^{\frac{1}{1-\sigma}},\ j \in \widetilde{\mathcal{N}}^c.$$

$$\textbf{STEP 18:}\ \eta^{\mathcal{E}} = \frac{\sum_j \epsilon_j \Theta_j (p_j^y)^{1-\sigma}(\widetilde{c})^{\epsilon_j(1-\sigma)}}{\sum_j \Theta_j (p_j^y)^{1-\sigma}(\widetilde{c})^{\epsilon_j(1-\sigma)}}, j \in \widetilde{\mathcal{N}}^c.$$

$$\textbf{STEP 19:}\ \ \bar{e} = \left[\frac{\rho\eta^{\mathcal{E}}}{\rho\eta^{\mathcal{E}} + (1-\rho)}\right] e.$$

$$\textbf{STEP 20:}\ \ \bar{e} = \prod_{j \in \bar{\mathcal{N}}^c} (p_j^y)^{\zeta_j} \bar{c} \ \Rightarrow \ \bar{c} = \left(\prod_{j \in \bar{\mathcal{N}}^c} (p_j^y)^{\zeta_j}\right)^{-1} \bar{e},\ j \in \bar{\mathcal{N}}^c.$$

As before, **STEP 21** gives us the demand for final consumption in each sector from Shephard's lemma,

$$c_j^d = \frac{\partial \mathcal{E}}{\partial p_j^y}.$$

Thus, for sectors $j \in \bar{\mathcal{N}}^c$, we have,

$$\textbf{STEP 21:}\ \ c_j^d = \frac{\partial \mathcal{E}(p^y, \bar{c})}{\partial p_j^y} = \frac{\zeta_j \prod_{j \in \bar{\mathcal{N}}^c} (p_j^y)^{\zeta_j}}{p_j^y} \bar{c},\ j \in \bar{\mathcal{N}}^c.$$

For sectors $j \in \widetilde{\mathcal{N}}^c$, we have,

$$\textbf{STEP 21:}\ \ c_j^d = \frac{\partial \mathcal{E}(p^y, \widetilde{c})}{\partial p_j^y} = \left[\sum_{j \in \widetilde{\mathcal{N}}^c} \Theta_j (p_j^y)^{1-\sigma} (\widetilde{c})^{\epsilon_j(1-\sigma)}\right]^{\frac{\sigma}{1-\sigma}} \Theta_j (\widetilde{c})^{\epsilon_j(1-\sigma)} (p_j^y)^{-\sigma},\ j \in \widetilde{\mathcal{N}}^c.$$

The vector, $(p_1^y, ..., p_n^y)'$, is an equilibrium price vector when the goods market clears in each sector,

$$|c_i^d - c_i^s| < \epsilon, \ \ \forall i \in \mathcal{N}.$$

To find the extended equilibrium price vector, including λ, we implement a tâtonnement process that adjusts p_λ^y in order to both clear the market for consumption goods and equate the shadow price of consumption to marginal utility. That is, we set a new price $(p^y)'$ where

$$(p_i^y)' = p_i^y + b \times \underbrace{\left(c_i^d - c_i^s\right)}_{\textbf{Excess Demand for Consumption Goods}},\ b > 0.$$

Similarly, we update the shadow price of consumption according to,

$$\lambda' = \lambda + b \times \left(\frac{\rho}{\prod_{j \in \bar{\mathcal{N}}^c} (p_j^y)^{\zeta_j}} \left(\frac{\widetilde{c}}{\bar{c}}\right)^{1-\rho} - \lambda\right),\ b > 0.$$

We then repeat these steps until $\|(p_\lambda^y)^{m+1} - (p_\lambda^y)^m\| < \epsilon$.

5 Quantifying the Model

5.1 Investment Goods

We observe (abstracting from the time subscripts) $\omega_{ij} = \frac{p_i^y x_{ij}}{p_j^x x_j}$ and need to assign values to ζ_{ij}^x, z_{ij}^x, ρ_j^x and ϵ_j^x.

Consider first the set of inputs $\bar{\mathcal{N}}_j^x$ with constant cost shares. We have that

$$\begin{aligned} p_j^{\bar{x}} \bar{x}_j &= \sum_{i \in \bar{\mathcal{N}}_j^x} p_i^y x_{ij}, \\ \Rightarrow (1-\rho_j^x) p_j^x x_j &= \sum_{i \in \bar{\mathcal{N}}_j^x} \omega_{ij} p_j^x x_j \\ \Rightarrow 1 - \rho_j^x &= \sum_{i \in \bar{\mathcal{N}}_j^x} \omega_{ij}. \end{aligned}$$

Moreover,

$$\begin{aligned} 1-\rho_j^x &= \frac{p_j^{x^n} x_j^n}{p_j^x x_j} = \frac{p_j^{x^n} x_j^n}{p_i^y x_{ij}} \frac{p_i^y x_{ij}}{p_j^x x_j} = \frac{1}{\zeta_{ij}^x} \omega_{ij} \\ \Rightarrow \zeta_{ij}^x &= \frac{\omega_{ij}}{1-\rho_j^x}. \end{aligned}$$

We can then match these shares in the data irrespective of equilibrium prices.

Next, consider the set of sectors $i \in \widetilde{\mathcal{N}}_j^x$ with time-varying factor shares. To assign values to ϵ_j^x and z_{ij}^x, recall from the FOCs associated with the cost minimization problem,

$$p_i^y = p^{\widetilde{x}_j} \left[\sum_{i \in \widetilde{\mathcal{N}}_j^x} z_{ij}^x x_{ij}^{\frac{\epsilon_j^x - 1}{\epsilon_j^x}} \right]^{\frac{1}{\epsilon_j^x - 1}} z_{ij}^x x_{ij}^{\frac{-1}{\epsilon_j^x}}.$$

so that

$$\frac{p_i^y}{p_m^y} = \frac{z_{ij}^x}{z_{mj}^x} \left(\frac{x_{ij}}{x_{mj}} \right)^{\frac{-1}{\epsilon_j^x}} \Rightarrow \frac{x_{ij}}{x_{mj}} = \left(\frac{p_i^y}{p_m^y} \right)^{-\epsilon_j^x} \left(\frac{z_{ij}^x}{z_{mj}^x} \right)^{\epsilon_j^x}, \quad i, m \in \widetilde{\mathcal{N}}_j^x$$

which implies

$$\log \frac{p_i^y x_{ij}}{p_m^y x_{mj}} = \log \frac{p_i^y x_{ij} / p_j^x x_j}{p_m^y x_{mj} / p_j^x x_j} = \log \frac{\omega_{ij}}{\omega_{mj}} = \epsilon_j^x \log \frac{z_{ij}^x}{z_{mj}^x} + (1-\epsilon_j^x) \log \frac{p_i^y}{p_m^y},$$

subject to $\sum_{i \in \widetilde{\mathcal{N}}_j^x} z_{ij}^x = 1$ and $\epsilon_j^x \geq 0$.

To obtain $z^x_{ij,t}$ given ϵ^x_j, recall that in equilibrium,

$$p^y_i = p^{\widetilde{x}_j} \left[\sum_{i \in \widetilde{\mathcal{N}}^x_j} z^x_{ij} x_{ij}^{\frac{\epsilon^x_j - 1}{\epsilon^x_j}} \right]^{\frac{1}{\epsilon^x_j - 1}} z^x_{ij} x_{ij}^{\frac{-1}{\epsilon^x_j}}.$$

so that

$$\frac{p^y_i}{p^y_m} = \frac{z^x_{ij}}{z^x_{mj}} \left(\frac{x_{ij}}{x_{mj}} \right)^{\frac{-1}{\epsilon^x_j}} \Rightarrow \frac{x_{ij}}{x_{mj}} = \left(\frac{p^y_i}{p^y_m} \right)^{-\epsilon^x_j} \left(\frac{z^x_{ij}}{z^x_{mj}} \right)^{\epsilon^x_j}, \quad i, m \in \widetilde{\mathcal{N}}^x_j$$

which implies

$$\log \frac{p^y_i x_{ij}}{p^y_m x_{mj}} = \log \frac{p^y_i x_{ij} / p^x_j x_j}{p^y_m x_{mj} / p^x_j x_j} = \log \frac{\omega_{ij}}{\omega_{mj}} = \epsilon^x_j \log \frac{z^x_{ij}}{z^x_{mj}} + (1 - \epsilon^x_j) \log \frac{p^y_i}{p^y_m},$$

Then, we recover z^x_{ij} as follows. Conditional on ϵ^x_j, and using the normalization, $z^x_{ij} + z^x_{mj} = 1$, we have that

$$\log \frac{z^x_{ij}}{z^x_{mj}} = \frac{1}{\epsilon^x_j} \left[\log \frac{\omega_{ij}}{\omega_{mj}} - (1 - \epsilon^x_j) \log \frac{p^y_i}{p^y_m} \right]$$

so that

$$\frac{z^x_{ij}}{1 - z^x_{ij}} = \exp \left\{ \frac{1}{\epsilon^x_j} \left[\log \frac{\omega_{ij}}{\omega_{mj}} - (1 - \epsilon^x_j) \log \frac{p^y_i}{p^y_m} \right] \right\} = Z \left(\epsilon^x_j, \frac{\omega_{ij}}{\omega_{mj}}, \frac{p^y_i}{p^y_m} \right).$$

Then

$$z^x_{ij} = \frac{Z \left(\epsilon^x_j, \frac{\omega_{ij}}{\omega_{mj}}, \frac{p^y_i}{p^y_m} \right)}{1 + Z \left(\epsilon^x_j, \frac{\omega_{ij}}{\omega_{mj}}, \frac{p^y_i}{p^y_m} \right)}. \tag{13}$$

5.2 Materials

We follow an analogous approach in quantifying the parameters associated with the production of materials, ζ^m_{ij}, z^m_{ij}, ρ^m_j.

5.3 Estimating Elasticities

This section discusses the estimation of the elasticities of substitution, ϵ^m_j and ϵ^x_j for $j \in \mathcal{N}$. We describe the methods for estimating ϵ^m_j; the methods for ϵ^x_j are analogous.

Let $m_{dur,j,t}$ denote the real value of durables supplied to sector j in year t and similarly for $m_{ipp,j,t}$. We estimate ϵ_j^m using the relationship

$$\Delta \log \left(\frac{m_{dur,j,t}}{m_{ipp,j,t}} \right) = -\epsilon_j^m \Delta \log \left(\frac{p_{dur,t}}{p_{ipp,t}} \right) + \epsilon_j^m \Delta \log \left(\frac{z_{dur,j,t}^m}{z_{ipp,j,t}^m} \right). \tag{14}$$

Given the linear structure of (14), one is tempted to estimate ϵ_j^m from a simple regression of $\Delta \log \left(m_{dur,j,t} / m_{ipp,j,t} \right)$ onto $\Delta \log \left(p_{dur,t} / p_{ipp,t} \right)$, but the potential correlation of $\Delta \log \left(p_{dur,t} / p_{ipp,t} \right)$ with $\Delta \log \left(z_{dur,j,t}^m / z_{ipp,j,t}^m \right)$ makes this estimator problematic. Instead, we use an IV estimator using $\Delta \log \left(z_{dur,t} / z_{ipp,t} \right)$, the relative growth rates in TFP, as an instrument. We have implemented both Bayes and frequentist estimators which are discussed in the following subsections.

Before describing the estimators, it is useful to streamline the notation.[2] Let

$$y = \Delta \log \left(\frac{m_{dur,j}}{m_{ipp,j}} \right), \; p = -\Delta \log \left(\frac{p_{dur}}{p_{ipp}} \right), a = \epsilon_j^m \Delta \log \left(\frac{z_{dur,j}^m}{z_{ipp,j}^m} \right), \text{ and } x = \Delta \log \left(\frac{z_{dur,j}^m}{z_{ipp,j}^m} \right)$$

so that (14) becomes

$$y = \beta p + a \tag{15}$$

where $\beta = \epsilon_j^m$ is the parameter to be estimated. Suppose that p and x are related via the reduced form relationship

$$p = \pi x + u \tag{16}$$

where π is an unknown constant and u is an error term. In (15) and (16) we assume that x is uncorrelated with a and u so that (15) and (16) comprise a standard linear simultaneous equation model.

As stressed in the paper, we view the model – and in particular the relationship (14) – as a steady-state relationship that characterizes the long-run trends in the data. Thus, we estimate ϵ_j^m using only the variation in these long-run trends. As described in Section 8.1 of this appendix, variation in these trends are associated with low-frequency cosine weighted averages of the data. (These "cosine transforms" of the data correspond to the OLS coefficient of the $\psi_{j,t}$ regressors introduced in Section 8.1). As discussed in Müller and Watson (2017) (also see Müller and Watson (2020)), linear relationships between variables such as (15) and (16) also hold for the cosine weighted averages, and when the variables are $I(0)$, these weighted averages are (approximately) i.i.d. zero-mean normally distributed random variables with variance given by the long-run variance of the process. Thus, with (y, p, x) denoting the cosine weighted averages, we have

$$\begin{pmatrix} y_i \\ p_i \\ x_i \end{pmatrix} \overset{a}{\sim} i.i.d.N(0, \Sigma) \text{ for } i = 1, ..., q \tag{17}$$

[2]Please note that in this context y, p, x, and ψ do not refer to final goods output, prices, investment, and exogenous final demand.

where q denotes the number of cosine weighted averages and Σ is the long-run variance of the (y, p, x) process. Throughout the paper we set $q = 7$, so (as discussed in Section 2.3 of this appendix) the analysis uses variation in the series for periods longer than $2 \times T/q = 144/7 \approx 20$ years.

Thus, the estimation problem corresponds to a canonical problem in econometrics: estimation of β in the simultaneous equation model (15) and (16) using $q = 7$ i.i.d. normal random variables. The simultaneous equation model (15)-(16) expresses the conditional distribution of $(y, p)|x$ as a function of the 5 parameters: β, π, and the unique elements of Ω, the covariance matrix of (a, u).

Because the effective size is $q = 7$, this is a small-sample estimation exercise. We now review standard frequentist and Bayes estimators for this problem.

5.3.1 Bayes Estimators and Credible Intervals

Given the small sample size, Bayes estimators are particularly attractive in this context. We have computed the posterior for β using the following priors for (β, π, Ω).

- $\beta \sim N(1, 4)$ truncated so that $\beta \geq 0$.
- $\pi \sim N(0, 2500)$ so the prior is essentially diffuse
- $\Omega \sim$ Wishart with covariance matrix I_2 and 0.001 degrees of freedom – again, a prior that is essentially diffuse.

The posterior is computed using a standard MCMC (Gibbs) algorithm.

5.3.2 Frequentist Confidence Intervals for ϵ^m

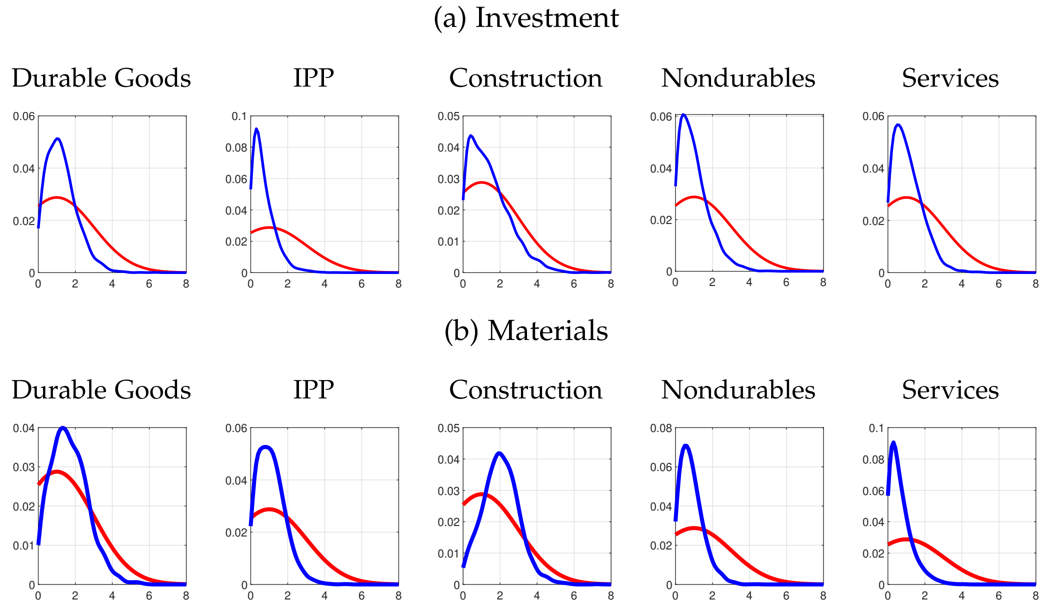
The standard frequentist estimator for this problem is 2SLS, which corresponds to the maximum likelihood estimator in this simple model. However, the number of observations is quite small (here $q =$ 7), so that the usual large-sample properties of 2SLS provide a poor guide for estimation and inference.

An alternative exploits the normal distribution of the data to carry out exact small-sample inference. This can be achieved using Anderson-Rubin (AR) methods. These methods proceed by regressing $y - bp$ onto x and constructing the t-statistic (say $\tau(b)$) for testing the null hypothesis that the coefficient on x is equal to zero. When $b = \beta_0$, the true value of β, the coefficient on x is equal to zero, and under normality (see (17)), $\tau(\beta_0) \sim$ Student-t with $q = 6$ degrees of freedom. The $100 \times (1 - \alpha)$ percent Anderson-Rubin confidence interval for β is

$$AR(\beta) = \{\beta || \tau(\beta)| \leq \tau_{1-\alpha/2}\}$$

where $\tau_{1-\alpha/2}$ is the $(1 - \alpha/2)$ percentile Student-t distribution with $q - 1 = 6$ degrees of freedom.

Figure 1: Priors and Posteriors for the Sectoral Production Elasticities



5.3.3 Results

Figure 1 plots the prior and posterior for β. Table 1, an extended version of Table 4 in the paper, summarizes the posterior and also includes 67 percent AR confidence intervals.

We highlight three results. First, there is considerable uncertainty in the value of the coefficients – this is unsurprising given the limited information in the small $(q = 7)$ number of observations characterizing the long-run (periodicities great than 20 years) variation in the series. That said, as shown in Figure 1, the posteriors are markedly different than the priors; thus the posterior is informed by the sample data. But, as shown in Table 1, the frequentist confidence intervals are quite wide indicating that there is a wide range of values of these elasticities that are generally consistent with the low-frequency variability in the sample data. The implications of the uncertainty about the values of these elasticities for the main quantitative conclusions in the paper are investigated in Section 9.2 in this appendix.

5.4 Value Added

We observe $\gamma_j = \frac{p_j^v v_j}{p_j^y y_j}$. Thus, for all sectors $j \in \mathcal{N}$, we simply use the sample average, γ_j.

Table 1: Estimates of Sectoral Production Elasticities

Sectors	Construction	Durables	IPP	Nondurables	Services
Investment: ϵ^x					
Posterior Mean	1.43	1.26	0.69	1.01	1.08
Posterior Median	1.24	1.14	0.52	0.84	0.93
67% Credible Interval	0.37 to 2.47	0.46 to 2.04	0.17 to 1.20	0.26 to 1.73	0.32 to 1.81
67% AR Confidence Interval	-1.93 to 8.91	-0.13 to 2.89	-2.82 to -0.13	-1.15 to 3.12	-0.08 to 4.36
Materials: ϵ^m					
Posterior Mean	2.03	1.69	1.11	0.86	0.69
Posterior Median	2.01	1.59	1.03	0.76	0.51
67% Credible Interval	1.08 to 2.92	0.69 to 2.61	0.37 to 1.79	0.27 to 1.43	0.14 to 1.21
67% AR Confidence Interval	1.44 to 5.22	-2.07 to 2.64	0.49 to 3.82	-0.26 to 1.89	-1.32 to 1.32

5.5 Sectoral Investment Shares in Value Added and Depreciation Rates

Since $\psi_j^x = \frac{p_j^x x_j}{p_j^v v_j}$, we use the analogous sample average in the data. Moreover,

$$\begin{aligned} \frac{p_j^x x_j}{p_j^v v_j} &= \left(\frac{\delta_j}{\frac{1}{\beta} - 1 + \delta_j} \right) \frac{u_j k_j}{p_j^v v_j} \\ &= \frac{\delta_j}{\frac{1}{\beta} - 1 + \delta_j} \alpha_j. \end{aligned}$$

Thus, we set

$$\delta_j = \frac{\frac{\psi_j^x}{\alpha_j} \left(\frac{1}{\beta} - 1 \right)}{1 - \frac{\psi_j^x}{\alpha_j}}.$$

5.6 Preferences

We observe $\theta_j = \frac{p_j^y c_j}{\sum_j p_j^y c_j}$ and take as given the homothetic consumption share in total expenditures, $s^c = \frac{\bar{e}}{e}$.

Recall that $e = \sum_{j \in \mathcal{N}} p_j^y c_j$ while for the homothetic aggregate, $\bar{e} = p^{\bar{c}} \bar{c} = \sum_{j \in \bar{N}^c} p_j^y c_j$. Therefore, for sectors $j \in \bar{N}^c$, we have

$$\theta_j = \frac{p_j^y c_j}{\sum_j p_j^y c_j} = \frac{p_j^y c_j}{\bar{e}} \frac{\bar{e}}{e} = \zeta_j s^c.$$

Then

$$\zeta_j = \frac{\theta_j}{s^c}.$$

For the non-homothetic aggregate preference parameters, we have that

$$\frac{p_j^y c_j}{e} = \frac{p_j^y c_j}{\widetilde{e}} \frac{\widetilde{e}}{e} = \omega_j^c (1 - s^c),$$

where

$$\ln \omega_j^c = (1-\sigma) \ln(p_j^y / p_b^y) + (1-\sigma) \left(\epsilon_j - 1\right) \ln(\widetilde{e}/p_b^y) + \epsilon_j \ln \omega_b^c + \ln \Theta_j,$$

and $\omega_j^c = p_j^y c_j / \widetilde{e}$. Equivalently, we have

$$\ln \left(\frac{\omega_j^c}{\omega_b^c} \right) = (1-\sigma) \ln \left(\frac{p_j^y}{p_b^y} \right) + (1-\sigma) \left(\epsilon_j - 1\right) \ln \left(\frac{\widetilde{e}}{p_b^y} \right) + (\epsilon_j - 1) \ln \omega_b^c + \ln \Theta_j.$$

We set ϵ_j and σ as in Comin, Mestieri, and Lashkari (2021). We then treat Θ_j as a preference shifter obtained as a residual,

$$\ln \Theta_j = \ln \omega_j^c - (1-\sigma) \ln(p_j^y / p_b^y) - (1-\sigma) \left(\epsilon_j - 1\right) \ln(\widetilde{e}/p_b^y) - \epsilon_j \ln \omega_b^c.$$

Then,

$$\omega_j^c = \Theta_j \left(\frac{p_j^y}{p_b^y} \right)^{(1-\sigma)} \left(\frac{\widetilde{e}}{p_b^y} \right)^{(1-\sigma)(\epsilon_j - 1)} (\omega_b^c)^{\epsilon_j},$$

where $\omega_b^c = 1 - \sum_{j \in \widetilde{\mathcal{N}}^c} \omega_j^c$ at observed prices, p^y and ratio, $\frac{\widetilde{e}}{p_b^y}$. Thus, for the non-homothetic consumption sectors, it follows that

$$\frac{p_j^y c_j}{e} = \frac{p_j^y c_j}{\widetilde{e}} \frac{\widetilde{e}}{e} = \omega_j^c (1 - s^c).$$

Observe that to obtain the non-homothetic consumption shares, $\theta_{j,t}$, $j \in \widetilde{\mathcal{N}^c}$, at observed prices, p^y, as an equilibrium of the model, this equilibrium needs to deliver both the observed p^y and the observed ratio, $\frac{\widetilde{e}}{p_b^y}$. Thus, let $h = \frac{\widetilde{e}}{p_b^y}$ (as observed in the data) which implies $\widetilde{e} = h \times p_b^y$. Then, in a 'data matching' exercise, as described in Section 3, we skip **STEP 14** and, since $\widetilde{e} = (1 - s^c)\, e$, we choose e in **STEP 15** such that

$$e = \frac{h \times p_b^y}{1 - s^c}, \tag{18}$$

where the RHS is all data. By construction, this will ensure that i) $\frac{\widetilde{e}}{p_b^y} = h$ as in the data, and ii) that $\frac{\widetilde{e}}{p_j^y}$ matches their counterpart in the data in all sectors since $\frac{\widetilde{e}}{p_j^y} = \frac{\widetilde{e}}{p_b^y} \frac{p_b^y}{p_j^y}$. The goal is then eventually to return to **STEP 14** and use $\psi_{d,t}^g + \psi_{d,t}^{nx}$ to clear sectoral goods markets in order to support observed prices, p_t^y.

5.7 Total Factor Productivity and the Scale of the Model Economy in the Initial Period

We define a steady state equilibrium as a gross output price p_t^y that clears goods markets for given exogenous drivers, $\left(z_t, z_t^m, z_t^x, \psi_t^{nx} + \psi_t^g, \Theta_t\right)$. Recall that the goods market clearing condition can be rewritten as our compositional identity

$$\left(\left[I - \Phi_t(I - \Gamma_{d,t})\right]\Gamma_{d,t}^{-1} - \Omega_t\psi_{d,t}^x - \psi_{d,t}^g - \psi_{d,t}^{nx}\right)\frac{(p_t^v. \times v_t)}{e_t} = \theta_t. \tag{19}$$

Letting $\eta_t = \frac{p_t^v \times v_t}{e_t} = \left(\left[I - \Phi_t(I - \Gamma_{d,t})\right]\Gamma_{d,t}^{-1} - \Omega_t\psi_{d,t}^x - \psi_{d,t}^g - \psi_{d,t}^{nx}\right)^{-1}\theta_t$, then

$$s_t^v = \frac{\eta_t}{\mathbf{1}'\eta_t}. \tag{20}$$

At observed prices, p_t^y, subsections 5.1 through 5.5 show how to match (adjusted versions of) Φ_t, Γ_t, Ω_t, $\psi_{d,t}^x$, and θ_t in part through choice of z_t^m, z_t^x, and Θ_t.[3] The model then needs to deliver the only remaining unknown in equation (19), the observed ratio of nominal value added to total consumption expenditures, $\eta_{j,t} = \frac{p_{j,t}^v v_{j,t}}{e_t}$, as part of the equilibrium to yield bserved prices p_t^y equilibrium prices.

We obtain sectoral TFP, z_t, from KLEMS data. Note that productivity accounting delivers productivity growth rates in a sector, but not the scale of productivity. In a one-sector economy, the choice of scale is a simple normalization. In our multi-sector setting with non-homothetic preferences, the scale of the economy, however, matters for shares. For the initial period, $t = 1$, we thus choose z_1 to match η_1 in the data, which amounts to matching the scale of the model economy in the initial period. For all subsequent periods, $t > 1$, sectoral TFP is not chosen to obtain market clearing, but the level of sectoral TFP is determined by the cumulative sectoral TFP growth calculated from KLEMS data, starting from the just determined initial TFP, z_1. This also means that for $t > 1$, the model will not exactly yield observed prices, p_t^y, as equilibrium prices.[4]

Abstracting from the time subscripts, given observed prices p^y (and the partial model inversion described in subsections 5.1 through 5.5), we first follow **STEP 1** through **STEP 4** in section 3.

Then, from **STEP 4**, we have

$$u_j = \left(\frac{1}{\beta} - 1 + \delta_j\right)p_j^x = \Delta_j p_j^x.$$

Let $\widetilde{k_j} = k_j/\ell_j$ denote the capital-labor ratio in sector j. We have expressions for real value-added and the

[3] These subsections assume that some rows of Φ_t, Ω_t, and θ_t are constant so that the implied value added shares in equation (20) will not match their counterparts in the data exactly.

[4] Alternatively, one can choose z_t to match η_t in the data period by period. In this case, the model is used to infer TFP.

FOC for optimal capital,

$$v_j = z_j \left(\frac{k_j}{\alpha_j}\right)^{\alpha_j} \left(\frac{\ell_j}{1-\alpha_j}\right)^{1-\alpha_j} = z_j \kappa_j \tilde{k}_j^{\alpha_j} \ell_j \;\text{ with } \kappa_j = \frac{1}{\alpha_j^{\alpha_j}(1-\alpha_j)^{1-\alpha_j}},$$
$$u_j = p_j^v \alpha_j z_j \kappa_j k_j^{\alpha_j - 1} \ell_j^{1-\alpha_j} = z_j p_j^v \alpha_j \kappa_j \tilde{k}_j^{\alpha_j - 1}.$$

We can then solve for the optimal capital-labor ratio,

$$\tilde{k}_j = \left(\frac{z_j p_j^v \alpha_j \kappa_j}{u_j}\right)^{1/(1-\alpha_j)},$$

and substitute this result into the expression for the price of value-added times real value-added to obtain nominal value-added, $p_j^v v_j$,

$$\begin{aligned} p_j^v v_j &= p_j^v z_j \kappa_j \left(\frac{z_j p_j^v \alpha_j \kappa_j}{u_j}\right)^{\alpha_j/(1-\alpha_j)} \ell_j \\ &= \left(p_j^v z_j \kappa_j\right)^{1/(1-\alpha_j)} \left(\frac{\alpha_j}{u_j}\right)^{\alpha_j/(1-\alpha_j)} \ell_j \\ &= \left(z_j\right)^{1/(1-\alpha_j)} \left[\left(p_j^v \kappa_j\right)^{1/(1-\alpha_j)} \left(\frac{\alpha_j}{u_j}\right)^{\alpha_j/(1-\alpha_j)} \ell_j\right] \\ &= \left(z_j\right)^{1/(1-\alpha_j)} \widetilde{V}_j, \end{aligned}$$

where $\widetilde{V}$ denotes normalized or unit nominal value added.

Let $\eta_j = \frac{p_j^v v_j}{e}$ as observed in the data. Then we need z_j to satisfy

$$\eta_j = \frac{p_j^v v_j}{e} = \frac{\left(z_j\right)^{1/(1-\alpha_j)} \widetilde{V}_j}{\frac{h \times p_b^y}{1-s^c}},$$

or

$$\widetilde{z}_j = \frac{\eta_j \times \frac{h \times p_b^y}{1-s^c}}{\widetilde{V}_j}, \tag{21}$$

where $\widetilde{z}_j = \left(z_j\right)^{1/(1-\alpha_j)}$.

6 Allocations, Prices, and Model-Implied Shares

6.1 Investment Input Shares, Ω

Investment input shares in sector j are given by

$$\frac{p_i^y x_{ij}}{p_j^x x_j},\ i \in \mathcal{N},$$

which we denoted in the data by $\omega_{ij} = \frac{p_i^y x_{ij}}{p_j^x x_j}$.

At the model solution, we should have for sectors $i \in \bar{\mathcal{N}}$,

$$\frac{p_i^y x_{ij}}{p_j^x x_j} = \frac{p_i^y x_{ij}}{p_j^{\bar{x}} \bar{x}_j} \frac{p_j^{\bar{x}} \bar{x}_j}{p_j^x x_j} = \zeta_{ij}^x (1 - \rho_j^x),$$

and given our calibration, $1 - \rho_j^x = \sum_{i \in \bar{\mathcal{N}}} \omega_{ij}$, $\zeta_{ij}^x = \omega_{ij} / 1 - \rho_j^x$,

$$\frac{p_i^y x_{ij}}{p_j^x x_j} = \omega_{ij}.$$

In other words, for the sectors with unit elasticity of substitution, the model-implied investment input shares should match their counterpart (average) in the data exactly with $\sum_{i \in \bar{\mathcal{N}}} \omega_{ij} = 1 - \rho_j^x$.

For sectors $i \in \widetilde{\mathcal{N}}$, the model solution implies

$$\frac{p_i^y x_{ij}}{p_j^x x_j} = \frac{p_i^y x_{ij}}{p_j^{\widetilde{x}} \widetilde{x}_j} \frac{p_j^{\widetilde{x}} \widetilde{x}_j}{p_j^x x_j} = \left(\widetilde{x}_j\right)^{\frac{1-\epsilon_j^x}{\epsilon_j^x}} \left(\widetilde{x}_{ij}\right)^{\frac{\epsilon_j^x - 1}{\epsilon_j^x}} z_{ij}^x \rho_j^x.$$

Observe that for those sectors, $\sum_{i \in \widetilde{\mathcal{N}}} \frac{p_i^y x_{ij}}{p_j^x x_j} = \rho_j^x$ so that across all sectors, investment input shares sum to 1, $\sum_{\mathcal{N}} \frac{p_i^y x_{ij}}{p_j^x x_j} = \sum_{i \in \bar{\mathcal{N}}} \frac{p_i^y x_{ij}}{p_j^x x_j} + \sum_{i \in \widetilde{\mathcal{N}}} \frac{p_i^y x_{ij}}{p_j^x x_j} = \rho_j^x + (1 - \rho_j^x) = 1$.

6.2 Materials Input Shares, Φ

Materials input shares in sector j are given by

$$\frac{p_i^y m_{ij}}{p_j^m m_j},\ i \in \mathcal{N},$$

which we denote in the data by $\phi_{ij} = \frac{p_i^y m_{ij}}{p_j^m m_j}$.

At the model solution, we should have for sectors $i \in \bar{\mathcal{N}}$,

$$\frac{p_i^y m_{ij}}{p_j^m m_j} = \frac{p_i^y m_{ij}}{p_j^{\bar{m}} \bar{m}_j} \frac{p_j^{\bar{m}_j} \bar{m}_j}{p_j^m m_j} = \zeta_{ij}^m (1 - \rho_j^m),$$

and given our calibration, $1 - \rho_j^m = \sum_{i \in \bar{\mathcal{N}}} \phi_{ij}$, $\zeta_{ij}^m = \phi_{ij}/1 - \rho_j^m$,

$$\frac{p_i^y m_{ij}}{p_j^m m_j} = \phi_{ij}.$$

In other words, for the sectors with unit elasticity of substitution, the model-implied materials input shares should match their counterpart (average) in the data exactly with $\sum_{i \in \bar{\mathcal{N}}} \phi_{ij} = 1 - \rho_j^m$.

For sectors $i \in \widetilde{\mathcal{N}}$, the model solution implies

$$\frac{p_i^y m_{ij}}{p_j^m m_j} = \frac{p_i^y m_{ij}}{p_j^{\widetilde{m}} \widetilde{m}_j} \frac{p_j^{\widetilde{m}} \widetilde{m}_j}{p_j^m mj} = \left(\widetilde{m}_j\right)^{\frac{1-\epsilon_j^m}{\epsilon_j^m}} \left(\widetilde{m}_{ij}\right)^{\frac{\epsilon_j^m - 1}{\epsilon_j^m}} z_{ij}^m \rho_j^m.$$

Observe that for those sectors, $\sum_{i \in \widetilde{\mathcal{N}}} \frac{p_i^y m_{ij}}{p_j^m m_j} = \rho_j^m$ so that across all sectors, materials input shares sum to 1, $\sum_{i \in \mathcal{N}} \frac{p_i^y m_{ij}}{p_j^m m_j} = \sum_{i \in \widetilde{\mathcal{N}}} \frac{p_i^y m_{ij}}{p_j^m m_j} + \sum_{i \in \bar{\mathcal{N}}} \frac{p_i^y m_{ij}}{p_j^m m_j} = \rho_j^m + (1 - \rho_j^m) = 1$.

6.3 Value Added Shares in Gross Output, Γ

By construction, at the model solution, we have for for sectors $i \in \mathcal{N}$,

$$\frac{p_j^v v_j}{p_j^y y_j} = \gamma_j,$$

the sample average of $\frac{p_j^v v_j}{p_j^y y_j}$ observed in the data.

6.4 Consumption Shares, θ

For the sectors associated with the homothetic consumption index, $j \in \bar{\mathcal{N}}^c$ we have that

$$\frac{p_j^y c_j}{e} = \zeta_j s^c = \theta_j.$$

For the sectors associated with the non-homothetic index, $j \in \widetilde{\mathcal{N}}^c$, we have

$$\frac{p_j^y c_j}{e} = \frac{p_j^y c_j}{\widetilde{e}} \frac{\widetilde{e}}{e} = \omega_j^c (1 - s^c),$$

where

$$\omega_j^c = \Theta_j \left(\frac{p_j^y}{p_b^y} \right)^{(1-\sigma)} \left(\frac{\widetilde{e}}{p_b^y} \right)^{(1-\sigma)(\epsilon_j - 1)} (\omega_b^c)^{\epsilon_j},$$

with $\omega_b^c = 1 - \sum_{j \in \widetilde{\mathcal{N}}^c} \omega_j^c$.

By construction, we have that $\sum_{j \in \mathcal{N}^c} \frac{p_j^y c_j}{e} + \sum_{j \in \widetilde{\mathcal{N}}^c} \frac{p_j^y c_j}{e} = s^c + (1 - s^c) = 1$.

7 Data

We construct a five-sector decomposition of the US economy based on detailed industry KLEMS data, input-output data, and investment flow data.

7.1 KLEMS Data

The KLEMS dataset contains quantity and price indices for inputs and outputs of 44 sectors from 1948 to 2022. The growth rate of an industry quantity index is defined as a Divisia index given by the value-share weighted average of its disaggregated component growth rates. Labor input is differentiated by gender, age, education, and labor status. Labor input growth is then defined as a weighted average of growth in annual hours worked across all labor types using labor compensation shares of each type as weights. Similarly, intermediate input growth reflects a weighted average of the growth rate of all intermediate inputs averaged using payments to those inputs as weights. Finally, capital input growth reflects a weighted average of growth rates across 53 capital types using payments to each type of capital as weights. Capital payments are based on implicit rental rates consistent with a user-cost-of-capital approach. Total payments to capital are the residuals after deducting payments to labor and intermediate inputs from the value of production. Put another way, there are no economic profits. An industry's TFP growth rate is defined in terms of its Solow residual, specifically output growth less the revenue-share weighted average of input growth rates. This approach is consistent with the maintained assumption of this paper: that markets are competitive and production is constant returns-to-scale.

Our calculations are based on the official 2023 version of the ILPA KLEMS dataset which covers the period 1987-2021, and the experimental ILPA KLEMS dataset for the period 1947-2016.[5] The experimental ILPA data from 1947-1963 cover 42 SIC private industries, the federal government, and state and local governments, while the experimental ILPA data from 1963-2016, and the official ILPA data from 1987-2018, cover 61 private NAICS industries, the federal government, and state and local governments. Thus, at the most detailed level of disaggregation for the full sample 1947-2016 we have 44 sectors: 42 private

[5] The official ILPA dataset for 1987-2021 is downloaded from `https://www.bea.gov/data/special-topics/integrated-industry-level-production-account-klems` and the experimental ILPA dataset for 1947-2016 is downloaded from `https://www.bls.gov/mfp/special_requests/tables_detail.xlsx`. See Fleck et al. (2014) and Corby et al. (2020) for a detailed description of the official ILPA data, and Eldridge et al. (2020) for the experimental ILPA data.

and 2 government. We splice the experimental and official ILPA data in 1987, using experimental data before 1987 and official data from 1987 onwards.

We consolidate the detailed sectoral data into 5 aggregate sectors that produce structures, durable goods, Intellectual Property Products (IPP), nondurable goods, and services. The construction sector produces structures. The sector producing durable goods contains the durable goods manufacturing sectors from wood products to apparel. The sector producing IPP contains two service industries: information and professional, scientific, and technical services. The sector producing non-durable goods covers the non-durable goods manufacturing industries, agriculture and forestry, mining, and utilities. The service sector covers trade, housing, government, and all service industries, excluding IPP services. Aggregates of quantity indices are constructed as Divisia indices from the underlying sectoral series.

For the five consolidated sectors we have real quantity indices and implied price indices for gross output, (y_j, p_j^y), value-added, (v_j, p_j^v), intermediate aggregates, (m_j, p_j^m), capital services, (k_j, u_j), and labor services, (ℓ_j, w_j). Value-added shares in gross output are $\gamma_j = p_j^v v_j / p_j^y y_j$, capital income shares are $\alpha_j = u_j k_j / p_j^v v_j$, and industry TFP, z_j, is the Solow residual from industry value-added and capital and labor services. Since sector quantity indices are Divisia indices, their growth rates are defined, but their initial levels are arbitrary normalizations. The same applies to the implied price indices.

7.2 Input-Output Tables

We use the BEA input-output tables to parameterize the industry use of intermediate goods and the industry sources of final demand. The use tables describe how commodities (rows) are used as intermediate inputs in industries and final demand (columns). The make table describes which industries (rows) produce what commodities (columns). The requirement table is the transformed make table where each commodity column consists of the industries' shares in its production.

In our model, we do not distinguish between commodities and industries in the production of goods, rather industries produce distinct goods for intermediate and final use. To match our model to the input-output data, we transform the commodity-by-industry use table into an industry-by-industry use table through the application of the make table. We pre-multiply the use tables with the requirement table to obtain a mapping from industry production to intermediate input and final use.

We start with the BEA use and make tables from 1947 to 2021.[6] The most detailed information on intermediate input use and final demand for the full sample covers 47 industries and 13 final demand categories. We consolidate the 47 industries into the five sectors defined for the KLEMS data, and final demand into private and government consumption, investment, and net exports plus inventory investment.

From the consolidated industry-by-industry use tables we calculate intermediate input shares, ϕ_{ji}, industry source shares of private consumption, θ_j, and the ratios of government consumption and net-

[6] The use and make tables are downloaded as spreadsheets from the BEA's Interactive Data Tables, https://www.bea.gov/itable/input-output.

exports to value-added, ψ_j^g and ψ_j^{nx}.

A comment on our use of input-output data, which includes imports and exports, to parameterize a closed-economy model. Imports are implicit in the use table to the extent that industries use commodities that are not domestically produced but imported. Effectively, we assume that imported and domestically produced commodities are perfect substitutes and that the input-output production structure is independent of the source of the commodities.

7.3 Investment Flows

Our investment flows, that is, industry sources of industry investment are based on vom Lehn and Winberry (2021) (vLW). The vLW investment flows have the input-output use structure, that is, they are commodity-by-industry, but they include only the purchase of new capital goods and ignore transactions involving used capital goods. We modify vLW industry investment flows to incorporate the purchase of used capital goods. We then transform the commodity-by-industry investment flows to industry-by-industry investment flows by pre-multiplying them with the industry requirement table from the input-output tables.

The investment category of final demand in the input-output use tables includes used capital goods transactions in the two commodity categories 'scrap' and 'noncomparables'. We modify vLW industry investment flows by allocating total investment in used capital goods to industries subject to the constraints that (1) an industry's total investment adds up to that industry's total investment in the Fixed Asset Tables, and (2) the commodity sources across all industries' investment add up to the commodity source in final demand investment. We choose industry investment flows to minimize the difference between the vLW commodity shares in industry investment and the commodity shares in new investment from the modified investment flows.

The vLW investment flows cover 41 private sectors and 2 government sectors from 1947-2018. We again consolidate the modified industry-by-industry investment flows into the five sectors defined in the KLEMS section. By construction, the total investment is consistent with investment in final demand of the input-output tables. From the investment flows we calculate the industry source shares of investment, ω_{ji}.

8 Trend Estimation and Forecasting

This section augments the discussion in Sections 2.3 and 3.2 and Appendices A and B in the paper that describes our approach to estimating and forecasting the long-run trends in sectoral shares and other time series used in the analysis. The discussion proceeds in four steps. Section 8.1 defines the long-run trends used in our analysis. Section 8.2 describes how we use in-sample values of long-run trends to construct long-horizon forecasts (and prediction intervals) for future values of these trends. Section 8.3

describes how the methods can be adapted to construct in-sample and forecasts of long-run trends for vectors of shares – that is, vectors of variables that are bounded between zero and one, and sum to unity. Additional formulas are provided in Section 8.4

The analysis in Sections 8.1, 8.2 and 8.4 is largely based on methods presented in Müller and Watson (2020), and much of the notation is borrowed from that paper.

8.1 Estimating the Long-Run Trend Value of a Time Series

We are interested in constructing a long-run trend for an $n \times 1$ vector-valued time series x_t. By 'long-run trend,' we mean a version of x_t that includes its 'level,' any 'linear trend' component, and the series' low-frequency oscillations around this level/trend. To simplify the presentation, we ignore the linear trend throughout much of this section; Section 8.5 describes the necessary modifications required for its inclusion.

With this background, we begin by representing x_t in terms of its level, the $n \times 1$ vector μ, and residual, e_t

$$x_t = \mu + e_t. \tag{22}$$

Suppose we have observations on x_t for $t = 1, ..., T$ and we are interested in extracting the 'low-frequency' or long-run components of x_t over this sample period. We do this by regressing x_t onto a constant and q periodic regressors, $\psi_{j,t}$ for $j = 1, ..., q$, where $\psi_{j,t} = \sqrt{2}\cos(j\pi(t-0.5)/T)$. The regressor $\psi_{j,t}$ has period $2T/j$, so that by including $\psi_{j,t}$ for $j = 1, ..., q$ the regression captures periodicities longer than $2T/q$. In our application, the sample covers 1947-2018, so that $T = 72$ years, we use $q = 7$, and so the regression capture periodicities longer than $2 \times 72/7 \approx 20$ years.

The fitted values from this regression (estimated by OLS) are the long-run trends used in our analysis. We denote these trend values as $\hat{x}_{t,(1:T)}$ where the (cumbersome, but useful) notation emphasizes that the trend value of x_t is computed from the sample data available from time period 1 through T.

8.2 Forecasting the Long-Run Trend Value of a Time Series

Suppose that interest is focused on $\hat{x}_{t,(1:T)}$ for values of t and T that extend beyond the end of sample period. For example, in our application, the sample ends in 2018 and in various places in the paper, we present forecasts for trend values in $t = 2038$, computed from hypothetical regressions that extend from 1947 through $T = 2090$. In this section we describe how we construct a forecast (and predictive distribution) for this trend value.

Some notation will prove useful. Let T_{IS} denote the "in-sample" value of T; in the example $T_{IS} = 72$ and includes the sample period from 1947 through 2018. Let T_{FS} denote the "full-sample" value of T; in the example $T_{FS} = 144$ and includes the sample period from 1947 through 2090. Let t^* denote the forecast period of interest; in the example $t^* = 2038$. The goal is to construct a forecast of $\hat{x}_{t^*,(1:T_{FS})}$ using the in-sample trend values $\hat{x}_{t:(1:T_{IS})}$ for $t = 1, ..., T_{IS}$.

The forecasting problem is simplified by two key features of the problem.

1. First, $\hat{x}_{t,(1:T)}$ are fitted values from an OLS regression where the regressors (1 and $\psi_{j,t}$) are deterministic functions of time. Thus, the randomness and uncertainty in $\hat{x}_{t,(1:T)}$ arises from the OLS coefficients, not the regressors. Using notation from Müller and Watson (2020), let $\mathbf{X}_T^0$ denote the OLS coefficients used to construct $\hat{x}_{t,(1:T)}$. The in-sample value of the OLS coefficient can be calculated from the in-sample data, that is, $\mathbf{X}_{T_{IS}}^0$ is known. Because T_{FS} extends beyond the in-sample period, the full-sample value $\mathbf{X}_{T_{FS}}^0$ is unknown. The problem is to forecast $\mathbf{X}_{T_{FS}}^0$ given $\mathbf{X}_{T_{IS}}^0$.

2. The second key feature of the problem is that, in setting like those considered in our analysis, $(\mathbf{X}_{T_{IS}}^0, \mathbf{X}_{T_{FS}}^0)$ are approximately normally distributed when T_{IS} and $(T_{FS} - T_{IS})$ are large. The distribution of $\mathbf{X}_{T_{FS}}^0$ given $\mathbf{X}_{T_{IS}}^0$ is then readily calculated from the standard multivariate normal formula.

8.3 Long-Run Trends for Shares

The analysis in Sections 8.1 and 8.2 did not impose constraints on the support of x_t. However, a time series of "shares" contains series that are non-negative and that sum to unity for each date t; these are sometimes called "compositional" variables. A standard approach to modelling and forecasting compositional variables is through the use of functions, such as the logit, that map series with unbounded support into series that satisfy the support constraints for compositional data. (See Aitchison (1986) for a textbook discussion.) We provide an overview here.

Let y_t denote an $(n+1) \times 1$ vector of shares, that is $y_{i,t} \geq 0$ with $\sum_{i=1}^{n+1} y_{i,t} = 1$ for all t. Let x_t denote an $n \times 1$ vector of variables with unconstrained support. Then y_t are derived from x_t via the function $y_t = F(x_t)$ where $F : \mathbb{R}^n \mapsto \Delta^n$.

With this background, we construct long-run trends for y_t in three steps: (1) Compute $x_t = F^{-1}(y_t)$; (2) compute $\hat{x}_{t,(1:T)}$ as described in Section 8.1; construct the long-run trend in y_t as $\hat{y}_t = F(\hat{x}_{t,(1:T)})$. Forecasts and predictive distributions for $\hat{y}_t$ are computed from the predictive distribution for $\hat{x}_{t:(1:T)}$ which in turn is computed using the procedure outlined in Section 8.2.

A standard choice for F is the logit function where

$$y_{i,t} = \frac{\exp(x_{i,t})}{1 + \sum_{j=1}^{n} \exp(x_{j,t})} \text{ for } i = 1, ..., n$$

and

$$y_{n+1,t} = 1 - \sum_{i=1}^{n} y_{i,t}.$$

Inverting this function yields

$$x_{i,t} = \ln\left(\frac{y_{i,t}}{y_{n+1,t}}\right) \text{ for } i = 1, ..., n.$$

The reduced-form trends and forecasts shown in Section 2 of the paper use the logit function. The structural model in Sections 3 of the paper provides a model-based version of F, where the variables in x_t are the exogenous variables in the structural model.

8.4 Additional Formula

This subsection provides further details underlying the estimation and forecasting of long-run trends. Let $\mathbf{x}_{1:T} = (x_1'|x_2'|...|x_T')$ denote the $T \times n$ matrix of sample observations and where $|$ denotes vertical concatenation. With $\mathbf{e}_{1:T}$ defined analogously, then (from (22)) the T observations can be written as

$$\mathbf{x}_{1:T} = (\mathbf{1}_T \otimes \mu') + \mathbf{e}_{1:T}.$$

where $\mathbf{1}_T$ is a $T \times 1$ vector of 1s.

Let $\Psi_j(s) = \sqrt{2}\cos(js\pi)$ denote a cosine function on $s \in [0, 1]$ with period $2/j$. Let $\mathbf{\Psi}(s) = [\Psi_1(s), \Psi_2(s), \ldots, \Psi_q(s)]'$ denote a vector of these functions with periods 2 through $2/q$. Let $\mathbf{\Psi}_T$ denote the $T \times q$ matrix with t-th row $\mathbf{\Psi}\left((t-1/2)/T\right)'$. Let $\mathbf{\Psi}_T^0 = [\mathbf{1}_T, \mathbf{\Psi}_T]$. The long-run trend is defined as the fitted value from the regression of $\mathbf{x}_{1:T}$ onto $\mathbf{\Psi}_T^0$. For the n variables in x_t, these are given by the columns of the $T \times n$ matrix

$$\hat{\mathbf{x}}_{1:T} = \mathbf{\Psi}_T^0(\mathbf{\Psi}_T^{0\prime}\mathbf{\Psi}_T^0)^{-1}\mathbf{\Psi}_T^{0\prime}\mathbf{x}_{1:T}. \tag{23}$$

The construction of the weights $\mathbf{\Psi}_T^0$ lead to a convenient formula for $\hat{\mathbf{x}}_{1:T}$:

$$\begin{aligned}\hat{\mathbf{x}}_{1:T} &= \mathbf{\Psi}_T^0(\mathbf{\Psi}_T^{0\prime}\mathbf{\Psi}_T^0)^{-1}\mathbf{\Psi}_T^{0\prime}\mathbf{x}_{1:T} \\ &= \mathbf{\Psi}_T^0\mathbf{X}_T^0\end{aligned}$$

where $\mathbf{X}_T^0$ are the regression coefficients $\mathbf{W}_T'\mathbf{x}_{1:T}$ with $\mathbf{W}_T' = (\mathbf{\Psi}_T^{0\prime}\mathbf{\Psi}_T^0)^{-1}\mathbf{\Psi}_T^{0\prime}$.

For forecasting, partition the data into in-sample and out-of-sample observations: say $\mathbf{x}_{1:T_{IS}}$ and $\mathbf{x}_{T_{FS}+1:T_{FS}}$ with $\mathbf{x}_{1:T_{FS}} = (\mathbf{x}_{1:T_{IS}}|\mathbf{x}_{T_{IS}+1:T_{FS}})$. Recall that q, the number of cosine terms in the regression reflects the long-run periodicity, and we set these periodicities to be (approximately) equal for the in-sample and full-sample trends. Thus, let q_{IS} denote the in-sample value of q and q_{FS} denote the full-sample value. Let $\mathbf{\Psi}_{T_{IS}}^0$,$\mathbf{W}_{T_{IS}}$, $\mathbf{X}_{T_{IS}}^0$, etc. denote the values of these variables constructed using the in-sample observations using q_{IS}. Note that

$$\begin{aligned}\mathbf{X}_{T_{IS}}^0 &= \mathbf{W}_{T_{IS}}'\mathbf{x}_{1:T_{IS}} \\ &= \tilde{\mathbf{W}}_{T_{FS}}'\mathbf{x}_{1:T_{FS}}\end{aligned}$$

where

$$\widetilde{W}_{T_{FS}} = \begin{bmatrix} \mathbf{W}_{T_{IS}} \\ \mathbf{0} \end{bmatrix}$$

where $\mathbf{0}$ is $(T_{FS} - T_{IS}) \times (1 + q_{IS})$ matrix of zeros.

As described in Section 8.2, we construct forecasts of $\mathbf{X}^0_{FS}$ given $\mathbf{X}^0_{T_{IS}}$ based on a joint-normal distribution of $(\mathbf{X}^0_{T_{IS}}, \mathbf{X}^0_{FS})$. This normal distribution follows from large-sample results discussed in Müller and Watson (2020). Essentially, these results imply that, when e_t follows an $I(0)$ process with long-run covariance matrix Σ, the large sample approximation to the distribution of $(\mathbf{X}^0_{T_{IS}}, \mathbf{X}^0_{FS})$ is the same as the exact distribution that obtains when $e_t \sim i.i.d.N(0,\Sigma)$, and when $\Delta e_t \sim I(0)$ with long-run covariance matrix Σ, the large-sample approximation coincides with the exact distribution for $\Delta e_t \sim i.i.d.N(0,\Sigma)$. To simplify the presentation we make these *i.i.d* Gaussian assumptions here and refer the reader to Müller and Watson (2020) for the associated large-sample approximations. Thus, we assume

$$e_t \sim i.i.d.N(0,\Sigma) \; (e_t \sim I(0)) \tag{24}$$

or

$$\Delta e_t \sim i.i.d.N(0,\Sigma) \; (e_t \sim I(1)) \tag{25}$$

We need an assumption about the values of the level parameter μ. In particular, we assume

$$\mu \sim N(0, \kappa\Sigma) \tag{26}$$

When κ is large, this yields a diffuse prior for the elements of μ.[7]

Under these assumptions, straightforward calculations show that

$$\begin{bmatrix} \text{vec}(\mathbf{X}^0_{T_{IS}}) \\ \text{vec}(\mathbf{X}^0_{T_{FS}}) \end{bmatrix} \sim N(0, \Omega) \tag{27}$$

where $\Omega = \Sigma \otimes \Upsilon$ with

$$\Upsilon = \begin{bmatrix} \tilde{\mathbf{W}}'_{T_{FS}} \\ \mathbf{W}'_{T_{FS}} \end{bmatrix} \left[\kappa \mathbf{1}_{T_{FS}} \mathbf{1}'_{T_{FS}} + \Lambda\right] \begin{bmatrix} \tilde{\mathbf{W}}'_{T_{FS}} \\ \mathbf{W}'_{T_{FS}} \end{bmatrix}' \tag{28}$$

where Λ is $T_{FS} \times T_{FS}$ with $\Lambda = I_{T_{FS}}$ under (24) and $\Lambda(i,j) = \min(i,j)$ under (25).

Partitioning Ω conformably with $\begin{bmatrix} \text{vec}(\mathbf{X}^0_{T_{IS}}) \\ \text{vec}(\mathbf{X}^0_{T_{FS}}) \end{bmatrix}$ as $\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}$, (27) yields the predictive distribution for $\mathbf{X}^0_{T_{FS}}$:

$$\text{vec}(\mathbf{X}^0_{T_{FS}}) | \text{vec}(\mathbf{X}^0_{T_{IS}}) \sim N(\Omega_{21}\Omega_{11}^{-1}\text{vec}(\mathbf{X}^0_{T_{IS}}), \Omega_{22} - \Omega_{21}\Omega_{11}^{-1}\Omega_{12}).$$

8.5 Time Trends and Implementation Details

Many of the long-run trends and associated forecasts in the paper use a modified version of the formulas presented above that incorporates a linear trend in x_t. That is, it allowed Δx_t to have a non-zero

[7] Müller and Watson (2020) discuss the implied equivariance properties of forecasts and prediction intervals using such diffuse priors for μ.

mean. In this case, equation (22) becomes $x_t = \mu' z_t + e_t$, where $z_t = (1\ t)'$ and μ' is $n \times 2$, with columns that include intercepts and trend slopes. This leads to three changes in the formulae given above.

- First, the periodic functions, $\Psi_j(s)$, are changed. As noted in Müller and Watson (2008), the cosine functions used above are the eigenvectors of the covariance matrix of a demeaned random walk, where the demeaning eliminates the constant term in the regression. With a time trend included, we instead use the eigenvectors of a detrended random walk. Analytic formulae for these eigenvectors are given in Müller and Watson (2008), but they are readily computed numerically. Periodicities longer than $2T/q$ are captured using the eigenvectors corresponding to the $q-1$ largest eigenvalues, so the regression continues to include $q+1$ regressors.

- Second, the matrix of regressors Ψ_T^0 now includes the linear trend, so that $\mathbf{\Psi}_T^0 = [\mathbf{Z}_{1:T}, \mathbf{\Psi}_T]$ where $\mathbf{Z}_{1:T} = (z_1'|z_2'|...|z_T')$.

- Third, the analysis requires an assumption about the trend coefficient: it replaces (26) with $\text{vec}(\mu) \sim N(0, \Sigma_{\text{vec}(\mu)})$ where $\Sigma_{\text{vec}(\mu)} = \kappa(\Sigma \otimes I_2)$. This changes (28) to $\text{Y} = \begin{bmatrix} \tilde{\mathbf{W}}'_{T_{FS}} \\ \mathbf{W}'_{T_{FS}} \end{bmatrix} \left[\kappa \mathbf{Z}_{1:T_{FS}} \mathbf{Z}'_{1:T_{FS}} + \Lambda\right] \begin{bmatrix} \tilde{\mathbf{W}}'_{T_{FS}} \\ \mathbf{W}'_{T_{FS}} \end{bmatrix}$.

With these changes, the analysis proceeds as described above.

We end with a few additional details describing the calculations reported in the paper:

- Unless noted otherwise in the paper, the results used the linear trend specification with $I(1)$ errors.

- We use $q_{IS} = 7$ and $T_{FS} = 2 \times T_{IS} = 2 \times 72 = 144$ so that $q_{FS} = 2 \times q_{IS} = 14$.

- The forecasting procedure requires an estimate of Σ, the long-run covariance matrix of e ($I(0)$ model) or Δe ($I(1)$ model). We used a Newey-West estimator (i.e., a Bartlett kernel) with two lags applied to the demeaned value of x_t ($I(0)$ model) or Δx_t ($I(1)$ model).

9 Additional Results

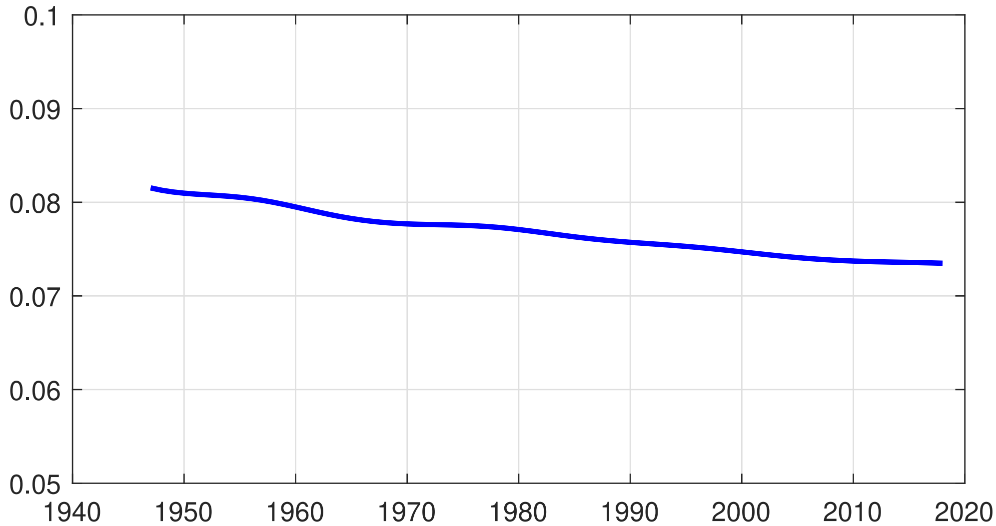
9.1 Preference Bundle Shift

Figure 2 plots the time-variation in the relative weight of the homothetic bundle of consumption in preferences, ρ_t^c. As noted in the paper, this compensating variation in bundle weights is small and assures that the share of the homothetic bundle in total consumption is constant, as indicated in the empirical results.

9.2 Robustness of Production Elasticities

Table 4 of the paper indicates that the elasticities that we estimate have wide 68% posterior intervals. Figures 3 and 4 repeat Figures 10 and 12 from the paper, assuming that all elasticities are at the low end

Figure 2: **Shift in Preference Bundle**
Homothetic versus Non-Homothetic Consumption, ρ_t^c



of the range or at the high end of the range. The top panel of each Figure shows the benchmark results, the middle panel shows the results if all elasticities are on the low end of the range, and the bottom panel shows the results if all elasticities are on the top end of the range. Note that these are extreme exercises since they assume that all elasticities are jointly higher or lower than the benchmark. The figures are on the whole similar across panels, both in terms of how the model matches the data (Figure 3) and the relative importance of each channel (Figure 4). That is, despite the wide range of uncertainty in our elasticity estimates, our model results are largely robust to alternative parameterizations.

Figure 3: **Structural Change: Model vs. Data with Difference Elasticities**
Consumption (red), Consumption and Production (cyan), Value Added (blue)

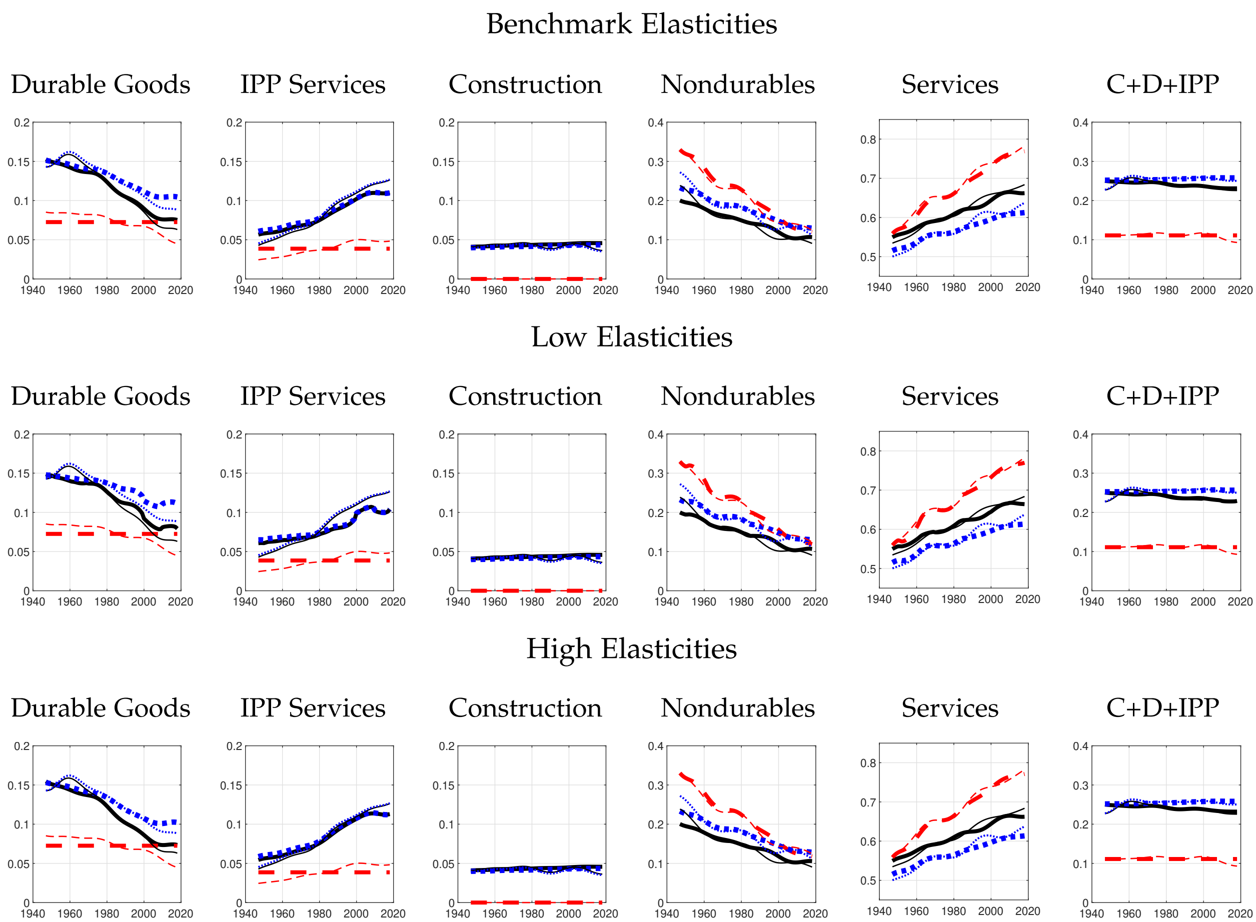
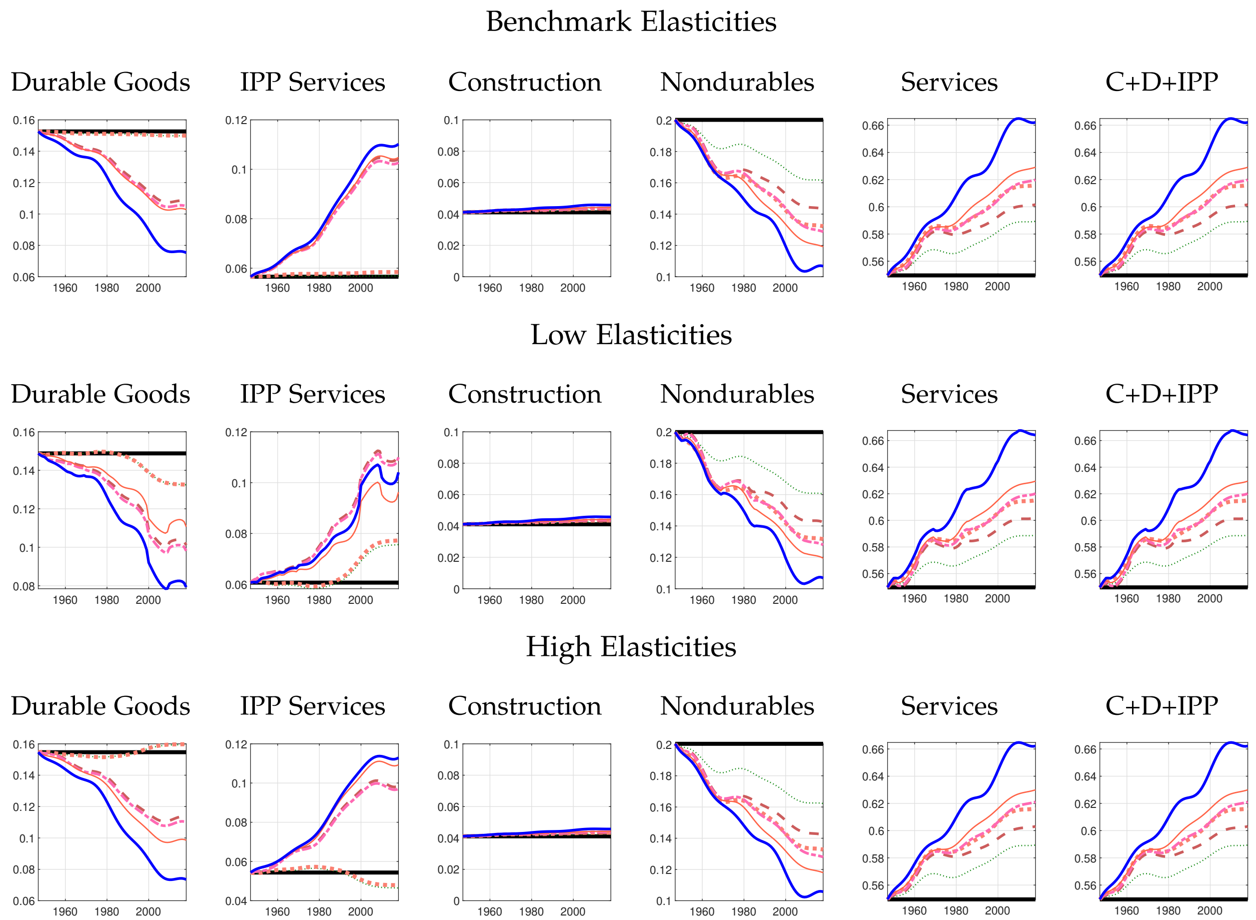


Figure 4: **Cumulative Decomposition of Value Added Shares with Different Elasticities**
Homothetic Preferences with TFP (thin dots), Non-homothetic Preferences with TFP (thick dots), with IBTC (dash), with Preference Shifts (dash-dot), with Labor Supply Shifts (thin), with Government and Net Exports (thick)



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