Interventions in markets with adverse selection: Implications for discount window stigma*

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Abstract

I study the implications for central bank discount window stigma of a workforce model of adverse selection in financial markets. In the model, firms (banks) need to borrow to finance a productive project. There is limited liability and firms have private information about their ability to repay their debts, which gives rise to the possibility of adverse selection. The central bank can ameliorate the impact of adverse selection by lending to firms. Discount window borrowing is observable and it may be taken as a signal of firms' credit worthiness. Under some conditions, firms borrowing from the discount window may pay higher interest rates to borrow in the market, a phenomenon often associated with the presence of stigma. I discuss these and other outcomes in detail and what they suggest about the relevance of stigma as an empirical phenomenon.

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1 Introduction

Discount window stigma is a relevant topic. It refers to the reluctance of banks to borrow from the central bank for fear of being regarded as in weak financial conditions as a result. Both policymakers and academic economists express concern about this issue on a regular basis. Former Federal Reserve Chair Ben Bernanke, for example, often cites stigma as an important consideration when designing policies (see also Fischer (2016)). He warns us that "the stigma problem is very real, with many historical illustrations" and suggests that, for example, Northern Rock was a victim of the kind of developments that give rise to stigma in financial markets (Bernanke (2015)).

As another example, Gorton (2015), in his review of U.S. Treasury Secretary Geithner's account of events during the 2007-08 financial crisis (Geithner (2014)), highlights the critical role played by stigma in the process of designing the policies that could address the perceived stresses in liquidity markets at the time. Both Geithner and Gorton, like Bernanke, believe that stigma was a real concern that could significantly compromise the effectiveness of interventions.

The effects of discount window stigma over outcomes in financial markets have received some attention in the academic literature. However, we are far from having a full understanding of the issues involved. In that context, any new opportunities to expand our stock of knowledge on the topic should be a welcomed development. A recent important article by Philippon and Skreta (2012) presents one such opportunity. The authors study optimal government interventions in financial markets suffering from adverse selection and, at the outset, emphasize that within the context of their model "taking part in a government program carries stigma" (see their abstract). However, in their paper, Philippon and Skreta (2012) do not explicitly address the issue of stigma. In this paper, I take on two tasks: first, I clarify the (mainly, "off equilibrium") implications for stigma in the model as exactly laid out by Philippon and Skreta and, second, I extend the model in relevant and important dimensions allowing me in this way to take a more explicit, "on equilibrium" approach to the issues associated with discount window stigma.

The basic framework used by Philippon and Skreta (2012) has a long tradition in financial economics and the study of security design (Myers and Majluf (1984)). For this reason, developing an understanding of the mechanisms that give rise to stigma in such model is particularly valuable. Put another way, using a workhorse model in financial economics to address an issue that is beyond the model's original intent can produce novel key insights, as it is shown in this paper.

Instead of the program-design (normative) approach taken by Philippon and Skreta (2012), I study perfect Bayesian equilibrium for a given discount window program in place. This different approach allows me to take a positive-economics perspective in understanding the nature of stigma within the context of the model. An equilibrium analysis is also more conducive to identifying equilibrium multiplicity and the role of off-equilibrium beliefs in determining equilibrium outcomes –both important aspects for a theory of discount window stigma.

In the model, a large set of firms (banks) own heterogeneous legacy assets and a new investment project. The project requires external funding and firms can borrow in the market to satisfy that need. However, the quality of the legacy assets is private information and an adverse selection problem arises: some firms are less likely to be able to repay their debt and, as a result, find borrowing more attractive. Without government intervention, adverse selection pushes market interest rates up and the level of investment that ensues is inefficient.

In order to increase borrowing and efficiency, the government can put in place a discount window and make loans at an administered (lower) interest rate. To be effective, the program has to attract a particular set of firms and leave only relatively better ones (those with higher repayment probability) to participate in the market. When this happens, borrowing from the discount window can be considered a signal of poor financial conditions, which is consistent with the idea of stigma.

If the loan taken at the discount window is sufficient to fund the new investment project (as in Philippon and Skreta (2012)), then no firm borrowing from the discount window also borrows from the market. As a result, stigma can only indirectly affect equilibrium outcomes. However, when discount window loans are smaller than what is needed to fund a new project (as in this paper), firms need to borrow from the market to complement their borrowing from the discount window. In such cases, depending on parameter values, some firms may borrow from the market at higher interest rates than the discount window rate, and firms borrowing from the discount window pay a higher rate in the market. These are both outcomes often regarded as evidence of stigma.

I also identify situations in the model where some firms choose to borrow from the discount window at interest rates that could be considered "penalty" rates. The interaction between repayment risk and interest rates is crucial for that result. By shifting repayment risk from the market to the discount window, firms can adjust their funding costs in ways that make the decision of borrowing from the discount window at a penalty rate consistent with equilibrium.

Defining discount window stigma is a non-trivial matter. Gorton (2015) calls stigma "a bank's reluctance to go to the discount window because of fears that depositors, creditors, and investors will view this as a sign of weakness, causing its borrowing costs to rise or maybe generating a bank run." This is broadly consistent with the interpretation of the term "stigma" used by Bernanke (2008) and, more recently, Armantier et al. (2015).[1] In terms of "observables," it is often taken as evidence of stigma the fact that some banks are willing to borrow from the market at rates (much) higher than the rates that they could obtain at the discount window (Furfine (2003)).

More generally, reference to stigma often comes in the form of a mixture of a set of observations

[1]Bernanke (2008) says: "the efficacy of the discount window has been limited by the reluctance of depository institutions to use the window as a source of funding. The "stigma" associated with the discount window, which if anything intensifies during periods of crisis, arises primarily from banks' concerns that market participants will draw adverse inferences about their financial condition if their borrowing from the Federal Reserve were to become known."

that would be associated with the phenomenon and an often partial explanation of the origin of those observations. Here, having an explicit model will allow us to map certain observables, such as interest rate differentials, to the mechanisms that generate those observables. Therefore, whether one decides to call that phenomenon "stigma" or something else becomes less relevant.

The paper is organized as follows. The rest of this section discusses some related literature. In Section 2, I introduce the economic environment and the equilibrium concept. In Section 3, I describe the equilibrium of the model when there is no discount window lending. After that, I analyze equilibria when the central bank makes discount window loans at a given (fixed) rate. I consider the case when the central bank restricts the size of the loans that is willing to provide to firms and the case when firms can choose how much to borrow. In each case, I discuss implications for the incidence of discount window stigma. I provide some concluding remarks in Section 4.

Related literature: The idea of discount window stigma has received some attention both in the empirical and in the theoretical literature. On the empirical side, the notable early work by Furfine (2003) uses data from before and after the Federal Reserve's move, in 2003, to change policy and transform the discount window into a standing facility (i.e., lending at a penalty rate with no questions asked) and finds that there was a lot less discount window borrowing happening after the change in policy than what one would have predicted by looking at the distribution of fed-funds trades before the change. Also, in the spirit of his earlier work (Furfine (2001)), he confirms that, under the new policy, the amount of borrowing in the market at rates higher than the discount window rate was still very significant. He concludes from these findings that there is unambiguous evidence of stigma at the Fed's discount window.[2]

The recent empirical work by Armantier et al. (2015) is especially valuable because, contrary to Furfine (2003), Armantier et al. (2015) do not rely on data on (supposed) interbank loans extracted, using Furfine's methodology, from the record of all daily Fedwire funds transfers. This is important since Furfine's strategy for identifying interbank loans has been recently shown to be not very reliable (Armantier and Copeland (2015)). Armantier et al. (2015), instead, use data from bids submitted by banks to the Term Auction Facility (TAF), a lending facility put in place by the Fed between December 2007 and March 2010. Using this data, they find strong evidence of discount window stigma during the financial crisis. Effectively, they find that many banks were willing to pay significantly higher interest rates to borrow from the TAF than the rate they would have had to pay to borrow from the discount window. As a result, banks were willing to accept (and indeed experience) significant extra cost in terms of interest payments in order to avoid the stigma associated with borrowing from the discount window.[3]

[2]Klee (2011) discusses selection effects that can complicate the measurement of discount window stigma using market interest rates, with an application to the 2007-08 financial crisis in the U.S.

[3]The paper by Gauthier et al. (2015) includes an empirical evaluation of the value of introducing the TAF as a sorting device and show that banks that borrowed at the TAF during the crisis received better future trading terms in the interbank market, relative to banks that borrow from the discount window.

Using data also from the crisis and the 2012 court-mandated disclosure of discount window activity, Kleymenova (2016) finds no evidence of discount window stigma affecting bank funding costs in capital markets (but she cannot rule out effects in the interbank market). She finds, though, that banks' behavior changed in ways consistent with stigma in response to newly imposed disclosure requirements of discount window activity.

Finally, Anbil (2018) and Vossmeyer (forthcoming) study empirically the incidence and impact of stigma in financial markets during the Great Depression. Anbil (2018) shows that borrowing from the RFC, when disclosed, was interpreted by depositors as a signal of bank financial weakness. The paper also identifies possible features of the organization of a lending facility that can help reduce stigma. Vossmeyer (forthcoming) approaches the issue from the perspective of banks willingness to access the lending facility and shows that, after disclosure started to happen, banks became reluctant to borrow form the RFC.

On the theoretical side, the work by Ennis and Weinberg (2013) focuses on the issue of stigma at the discount window and is aimed at identifying specific features of an economic environment where stigma, as is often described, can actually occur in equilibrium. As in Philippon and Skreta (2012), adverse selection plays a critical role in the paper by Ennis and Weinberg (2013). However, both the model and the mechanisms that give rise to stigma are quite different in the two papers.[4]

La'O (2014) studies a model of predatory trading (a la Brunnermeier and Pedersen (2005)) where banks are reluctant to borrow funds because such an action may become a signal of financial weakness: an illiquid bank seeking to take a loan fears that other traders, realizing that the bank is weak, would exploit that information to trade against it. Interestingly, stigma in La'O's model is associated not just with borrowing from the discount window, but also with borrowing from the interbank market. See Lowery (2014) for an interesting discussion of La'O's model.

Tirole (2012) studies interventions in markets with adverse selection in an environment closely related to the one studied here (and in Philippon and Skreta (2012)).[5] In both models, firms are in possession of a legacy asset and a new investment project, for which they need external funding. In Tirole (2012), the return on new investment is observable and verifiable, which allows firms to enter contracts that are contingent on it. This is the main difference with the model here, where only total income of the firm is observable; i.e., "the return of old and new projects are fungible" (Philippon and Skreta (2012, p. 3)). Che, Choe, and Rhee (2015) extend the model in Tirole (2012) to allow for several rounds of play and study stigma when the government intervention takes the form of an asset-purchases program.

Very recently, Gauthier et al. (2015), Li, Milne, and Qiu (2016), and Gorton and Ordoñez (2016)

[4] Corbett and Mitchell (2000) is an early precursor in this literature: they study a model that incorporates bank reputational concerns from accepting a recapitalization offer by the government during a financial crisis.

[5] Other examples of recent papers that discuss interventions in markets with adverse selection are Fuchs and Skrzypacz (2015), Moreno and Wooders (2016), Camargo, Kim, and Lester (2016), and Chiu and Koeppl (2016).

discuss models where discount window stigma plays a role. The models in Gauthier et al. (2015) and Li, Milne, and Qiu (2016) are very related to each other and, in both models, borrowing from the discount window may represent a signal of the inability of the bank in question to repay its debts. This is the case in the model of this paper, as well. In Gorton and Ordoñez (2016), discount window activity, if discovered, signals the quality of the asset-in-place held by a bank, which makes the bank more vulnerable to run-like phenomena in the future.

2 The model

I work with the same economic environment that Philippon and Skreta (2012) use in their paper. The main difference in the analysis is that Philippon and Skreta consider the optimal design of the intervention while I restrict attention to the perfect Bayesian equilibrium of the model, taking the structure of the discount window (i.e., the government program) as given. Notably, a discount window program in this setting is consistent with optimality.

2.1 Environment

There are three time periods $t = 0, 1, 2$, a set of risk-neutral investors who do not discount the future, a continuum of firms, and a central bank. In this context, firms should be thought of as banks. Each firm has cash c_0 at time 0 and an asset that pays a random return $a \in [0, \bar{A}]$ at time 2. The initial asset owned by firms is of heterogeneous quality. Let θ be the type of the asset and let the distribution of asset quality across firms be given by $H(\theta)$ for $\theta \in \Theta \subset [\underline{\theta}, \bar{\theta}]$, with density $h(\theta)$. An asset of type θ has a random return with distribution $F_A(a \mid \theta)$ and density $f_A(a \mid \theta)$. Firms privately know the type of their initial (legacy) asset. For simplicity, I refer to a firm that has initial assets of quality θ as a firm of type θ.

At time 1, each firm has an opportunity to make a new investment. The cost of the new investment is x, and it delivers a random return $v \in [0, \bar{V}]$ at time 2, independent of a and θ. Assume that $E[v] > x > c_0$, so that investing produces a positive expected net present value, and firms need external funding to be able to invest. At time 1, a market for funds opens where firms can borrow from investors. The market functions as follows: knowing their type θ, each firm proposes a debt contract (l, r), and any investor can accept to fulfill that contract by making a loan of size l to the firm at a (gross) interest rate r. Investors compete for contracts and have unlimited resources ("deep pockets").

At time 2, creditors of a firm only observe its total income. More specifically, creditors cannot observe whether the firm invested or not at time 1 and cannot discriminate between the income coming from new investment and other income of the firm.

One way to capture that the legacy asset of a type θ firm is "more productive" than the legacy asset of another firm of type θ' is by assuming that the distribution of cash flows for a firm with asset type θ first order stochastically dominates the distribution of cash flows for a firm with asset type θ'. In an environment closely related to this one, Nachman and Noe (1994) use an even stronger (in the sense of implying stochastic dominance) order of cash flows: conditional stochastic dominance, which allows them to establish the optimality of debt contracts.[6] Since I restrict attention to debt contracts, I do not need this stronger assumption.

Like Philippon and Skreta (2012), I adopt the approach of Nachman and Noe (1994) and assume stochastic dominance directly over the cash flow $y = a + v \in [0, \bar{A} + \bar{V}]$, where the distribution function of y is given by the convolution of the distributions of a and v. I also assume that $f_Y(y \mid \theta) > 0$ for all $(y, \theta) \in [0, \bar{A} + \bar{V}] \times \Theta$.[7]

To simplify notation, it is useful to define the function:

$$\rho(\theta, rl) = \int_Y \min(y, rl) f_Y(y \mid \theta) dy, \tag{1}$$

Since the *min* function inside the integrand is nondecreasing, and strictly increasing for some values of y that occur with positive probability, we have that the assumed stochastic dominance order implies that ρ is an increasing function of θ. This property will be important for characterizing equilibrium. Note also that $\rho(\theta, rl) < rl$ for all $rl > 0$ since $f_Y(y \mid \theta) > 0$ for all $(y, \theta) \in [0, \bar{A}+\bar{V}]\times\Theta$.

Now, let $l_0 = x - c_0$ and define $\underline{r}_0$ as the solution to $\rho(\underline{\theta}, \underline{r}_0 l_0) = l_0$ and $\bar{r}_0$ as the solution to:

$$l_0 = \int_\Theta \rho(\theta, \bar{r}_0 l_0) dH(\theta) \tag{2}$$

Clearly, we have that $\bar{r}_0 < \underline{r}_0$. Assume, also, that:

$$l_0 - x + E[v] - \rho(\bar{\theta}, \bar{r}_0 l_0) < 0. \tag{3}$$

As will become clear later, in the absence of a discount window, this last condition guarantees that there is not an equilibrium where all types invest. Since investment has positive net present value for all types, when not all types invest in equilibrium there is an economic inefficiency that the central bank may try to reduce by lending via a discount window. This possibility is the focus of attention in the paper by Philippon and Skreta (2012), and it is also the focus of attention in this paper.

Note that the cost x of undertaking the new investment plays the role of a liquidity "shock" for the banks in the model. The cash amount c_0 is the liquidity reserves held by each bank. Since $x > c_0$, banks need external funding to satisfy their liquidity demands, and can choose to access

[6] In Nachman and Noe (1994) firms do not have private information about the quality of legacy assets. In assuming private information, Philippon and Skreta (2012) and this paper follow Myers and Majluf (1984).

[7] Ideally, one would want to make assumptions over the distribution of a, the return on legacy assets, and then derive implications for the distribution of cash flows y. For simplicity, however, the literature has imposed assumptions directly over the distribution of y. This is also the approach followed here.

the central bank lending facility to cover (at least in part) the liquidity shortage. Alternatively, of course, banks can choose to obtain all its needed external funding from the private market.[8]

2.2 The discount window policy

The central bank can put in place a lending facility (discount window) that allows firms ("banks") to obtain loans from the central bank at time 0. A discount window loan is a pair (m, R), where m is the size of the loan and R is the (gross) interest rate to be paid back to the central bank at time 2. The central bank has the same information as the market about the quality of the legacy assets (and the loan repayment ability) of firms.[9]

The lending program we consider has a simple debt structure. In principle, the central bank could try to organize its lending so as to have different firms self-selecting into different loan contracts. Philippon and Skreta (2012) consider this possibility and show that there are no gains in this environment from offering menus of debt contracts if the objective of the central bank is to increase the level of investment at minimum cost. In fact, menus may induce unwelcome multiplicity. Here, for simplicity, we restrict attention to discount window policies that specify a unique interest rate for all loans granted by the central bank to indistinguishable borrowers. This is mainly consistent with common central-bank practices where discrimination, if it exists, tends to be very coarse. In the U.S., for example, there are three lending programs at the discount window and access to them depends on the borrower's standing in the supervisory process. The primary credit program is the main program for firms with supervisory ratings above a certain threshold. One can interpret the model as mainly dealing with firms that qualify for primary credit.

Again following Philippon and Skreta (2012), I assume that investors at time 1 can observe whether a given firm has borrowed from the discount window at time 0. In reality, discount window activity in the U.S. is not made public by the central bank. Instead, every two weeks, each Reserve Bank reports only the total amount borrowed in that period. However, it is often maintained that in many cases market participants are able to combine information from different sources to effectively identify discount window borrowers (see, for example, Duke (2010)).[10]

I assume that the objective of the central bank is (exclusively) to fund firms that are looking to

[8] It is often argued that the discount window can serve as a backup source of funding for solvent banks that loose access to market liquidity for reasons not closely linked to their financial conditions. The model can be extended to accommodate such situations by having banks access the market only with probability $\sigma < 1$. With probability $1 - \sigma$, the bank is limited to borrowing at the discount window. All the basic results remain the same in such case (see part 3 of the on-line appendix).

[9] See Rochet and Vives (2004) for a model of the discount window where the central bank has an information advantage over market lenders due to its supervisory powers.

[10] Armantier et al. (2015, p. 318) discuss in detail the various aspects that influence observability in the U.S. system. Ennis and Weinberg (2013) consider a model where discount window activity is observed only with some probability.

invest. Hence, any relevant discount window policy satisfies:

$$m \leq l_0 \equiv x - c_0, \tag{4}$$

and we restrict attention to these policies in the analysis below.

2.3 Payoffs

Firms need to decide whether to borrow from the discount window at time 0 and whether to borrow from the market and invest at time 1. Following Philippon and Skreta (2012), I assume that the discount window claim is junior to the claim originated from firms' borrowing in the market.[11]

The payoff of a firm that decides to borrow m from the discount window and l from the market is given by:

$$\int_A \int_V (c_0+m+l-x\cdot i+a+v\cdot i-\min\{c_0+m+l-x\cdot i+a+v\cdot i, Rm+rl\}) f_V(v) f_A(a \mid \theta) dv da, \tag{5}$$

where i takes values in the set $\{0,1\}$, with $i=1$ indicating that the firm decided to invest and $i=0$ indicating that the firm is not investing. Note here that the assumption is that firms cannot hide cash, and if they have cash at $t=2$, they have to use it to repay their debt. For this reason, if the firm does not spend the cash borrowed at $t=0$ and $t=1$, then those funds, m and l, become part of the observable cash flow at $t=2$ as indicated inside the bracket associated with the *min* sign in equation (5).

One possibility no being explicitly considered here (just as in Philippon and Skreta (2012)) is for firms to borrow from the discount window at time 0 and then lend some of those funds to other firms in the market at time 1. Whether firms would want to engage in this kind of intermediation is not obvious: lending firms would need to compete with deep-pocketed investors in the market at time 1. When the interest rate at the discount window is positive, the face value of the funding cost of those firms is higher than the investors' cost. However, in principle, firms intermediating funds may not be able to pay back their discount window loans in some situations, lowering the expected value of their funding cost.

2.4 Equilibrium concept

I study the Perfect Bayesian Equilibrium of the model. Define the functions $i(\theta)$, which maps the set Θ to $\{0,1\}$. When $i(\theta)=0$ the firm of type θ does not invest and when $i(\theta)=1$ the firm of type θ invests. Similarly, define the functions $m(\theta)$ and $l(\theta)$ mapping Θ to $\mathbb{R}_+$ that tells us how much a firm of type θ decides to borrow from the discount window and the market, respectively.

[11] In the U.S., discount window lending is collateralized and, in general, not the most junior claim in banks' portfolios. In footnote 15 of their paper, Philippon and Skreta (2012) argue that assuming that the government is a junior creditor is without loss of generality for their purposes. I will discuss below how this issue matters for stigma.

We denote with $B(\theta \mid l, m)$ the beliefs of the market (i.e., investors) about the value of θ when the firm borrows m from the central bank and l from the market.

Given a discount window policy (m, R), an equilibrium is a set of functions $\{i^*(\theta), l^*(\theta), m^*(\theta)\}$, market interest rates $r^*(l; m)$, and beliefs $B^*(\theta \mid l, m)$ such that the following conditions hold:

(1) Individual Rationality: The functions $i^*(\theta), l^*(\theta), m^*(\theta)$ maximize the objective of the firm given the interest rates $r^*(l; m)$ and R;

(2) Break-even: Given market beliefs, the interest rates $r^*(l; m)$ satisfies the condition:

$$\int_{\Theta} \rho(\theta, r^*(l; m)l) dB^*(\theta \mid l, m) = l; \tag{6}$$

(3) Belief consistency: Beliefs are consistent with Bayes' rule whenever the values of l and m are observed in equilibrium.

Condition (6) tells us that the expected repayment associated with a loan of size l in the market is equal to the value of the loan. This condition is the result of competition among risk-neutral investors who do not discount the future. The condition also reflects the fact that all investors share the same level of information and, hence, have the same (on equilibrium) beliefs.

The Perfect Bayesian Equilibrium concept places no constraints on off-equilibrium beliefs; that is, beliefs over θ when the values of l and m are not chosen in equilibrium. As it is well known, the freedom to set off-equilibrium beliefs in an unrestricted way can produce multiple equilibria. One approach often used in the signaling literature is to consider refinements, such as the Cho-Kreps intuitive criterion (Cho and Kreps (1987)). Nachman and Noe (1994) use the stronger D1 refinement and make it part of their definition of equilibrium. Philippon and Skreta (2012) do not discuss refinements in their paper.

3 Equilibrium

In this section, I study equilibrium with and without a discount window. In both situations, when there is some borrowing happening in the market, the equilibrium (net) interest rate in the market has to be positive. We state this result as a lemma.[12]

Lemma 1. *In any equilibrium with an active market for private loans, we have that $r^*(l, m) > 1$ for all $l > 0$ and all m.*

Even though there is no discounting, the fact that for small-enough values of y only partial repayment is possible ($\rho(\theta, rl) < rl$) implies that when full repayment happens, the interest rate has to over-compensate investors in order for them to break even (that is, $r^*(l, m)$ has to be greater than unity).

[12] Those proofs that do not appear in the main text are available in an on-line appendix.

Furthermore, given that cash-flows are observable at the time of debt repayment (and funds cannot be diverted), a firm planning to not invest would never borrow (from the market or from the government) at positive interest rates.

Lemma 2. *For any $\theta \in \Theta$, if $i^*(\theta) = 0$ then $l^*(\theta) = 0$ and, when $R > 1$, also $m^*(\theta) = 0$.*

The reason for this simple fact is that borrowing to not invest implies that the firm will have available for repayment the funds that borrowed. In consequence, when the net interest rate is positive, the firm will always be able to repay at least as much as it borrowed and, in most cases, strictly more. Hence, the strategy of borrowing and not investing is strictly dominated by the strategy of not borrowing and not investing, which is always available to firms.

3.1 Equilibrium without a discount window

When the central bank's discount window is not active, there is an equilibrium where all firms of types below a given threshold take a loan in the market and invest, and all firms of types above that threshold do not borrow and do not invest. Define the threshold value $\theta^* \in \Theta$ as the solution to the following equation:

$$l_0 - x + E[v] - \rho(\theta^*, r^* l_0) = 0, \tag{7}$$

where the interest rate r^* is the one that satisfies:

$$\int_{\underline{\theta}}^{\theta^*} \rho(\theta, r^* l_0) \frac{dH(\theta)}{H(\theta^*)} = l_0. \tag{8}$$

Figure 1 plots an example of the locus of values of θ^* (horizontal axis) and r^* (vertical axis) that satisfy conditions (7) and (8), separately. The intersection of the two curves identify the values of interest for θ^* and r^* in our equilibrium analysis.[13]

The conditions on parameters specified in Section 2.1 imply that the value of θ^* lies in the interior of the set Θ. Furthermore, from equations (7) and (8) we have that $r^* > 1$.

With the values of θ^* and r^* that solve equations (7) and (8), we are now ready to state the following proposition:

Proposition 1 (Equilibrium without discount window)**.** *When the discount window is not active, there is an equilibrium where: **(1)** $l^*(\theta) = l_0$ for all $\theta \leq \theta^* < \bar{\theta}$ and zero otherwise; **(2)** $i^*(\theta) = 1$ for all $\theta \leq \theta^* < \bar{\theta}$ and zero otherwise; **(3)** the market interest rate is equal to r^*; and **(4)** the market beliefs $B(\theta \mid l_0) = H(\theta)/H(\theta^*)$ for all $\theta \leq \theta^*$ and zero otherwise.*

[13] The example considers that $H(\theta)$ is a uniform distribution for values in the interval $[-0.8, 0.8]$ and y has a Beta distribution with parameters $2 + \theta$ and $2 - \theta$. The values for the other parameters are listed at the top of the figure: $l_0 = 0.25$, $x = 0.27$, and $E[v] = 0.285$.

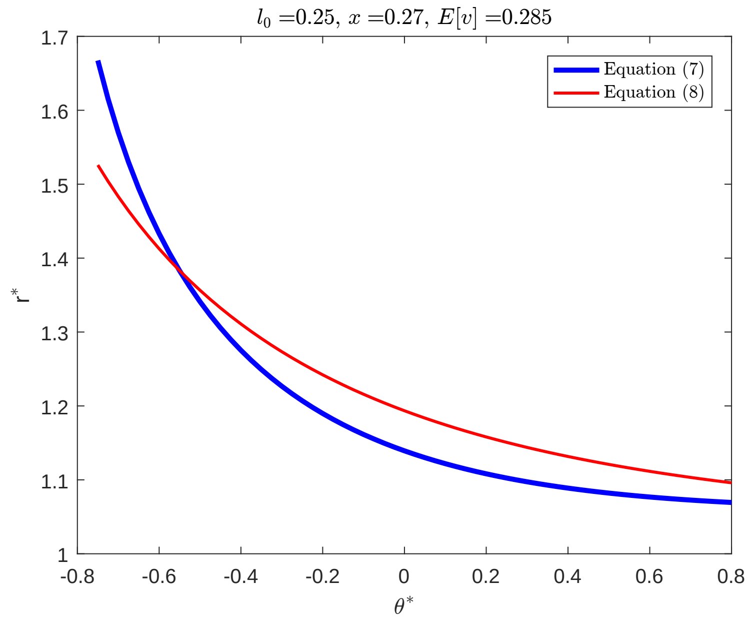

Figure 1: Equilibrium

The crucial step is to verify that $l^*(\theta)$ and $i^*(\theta)$ satisfy individual rationality given the interest rate r^*. Belief consistency is immediate given the strategies of firms and the break-even requirement follows directly from the definition of r^* in equation (8). Since $\theta^* < \bar{\theta}$ not all firms invest in equilibrium.

In principle, there could be more than one solution $\{\theta^*, r^*\}$ to equations (7) and (8). Using any of those solutions, we can construct an equilibrium like the one described in Proposition 1. This multiplicity is due to the adverse selection effects present in the model. The idea behind it is simple: when the interest rate is lower, more firms choose to borrow in the market and invest. This fact, in turn, improves the pool of firms borrowing in the market (more firms with higher values of θ decide to borrow), which justifies the lower interest rate.

Philippon and Skreta (2012), however, concentrate their attention on the equilibrium with the highest value of θ^*, denoted θ^D. The corresponding value of r, which together with θ^D, solves the system of equations (7) and (8) is denoted by r^D.

3.2 Equilibrium with a simple discount-window policy

Seeking to attain a higher level of investment than in the situation without intervention, suppose that the central bank sets the interest rate charged at the discount window $R < r^D$ and stands ready to make loans of size l_0 to any firm that wishes to borrow. This is consistent with how Berger et al. (2017) describe the use of central bank credit by banks during the recent financial crisis; that is, discount window loans supported investment in productive activities.

It is easy to see that if $R \leq 1$, then $m^*(\theta) = l_0$ for all θ and $i^*(\theta) = 1$ for all θ. In other words, when the central bank provides discount window loans at a negative net interest rate, all firms take the maximum loan at the discount window and all firms invest in equilibrium. While a discount window policy that sets its (net) interest rate to negative values ($R \leq 1$) attains the maximum level of investment, it also involves significant subsidies to borrowers. For this reason, the central bank may want to consider rates that increase investment without going all the way to the maximum amount. These policies involve interest rates in the range between unity and r^D.[14]

When $R \geq 1$ not all firms may borrow and invest. From Lemma 2 we know that when $R > 1$ no firm would borrow at the discount window with the intention of *not* investing. But, there are equilibria where some firms indeed borrow and invest and a subset of those firms may borrow from the government. The subset of firms borrowing from the government is different across equilibria and the implications for stigma naturally depend on this aspect of the equilibrium. We analyze first the equilibrium that was the main focus of attention in the implementation section in Philippon and Skreta (2012), and then we discuss other possible equilibria.

Suppose the central bank offers discount window loans of size l_0 at interest rate $R^T \in (1, r^D)$. Philippon and Skreta (2012) propose an equilibrium where firms with relatively low values of θ borrow from the government. Define θ^T as the solution to:

$$l_0 - x + E[v] - \rho(\theta^T, R^T l_0) = 0, \tag{9}$$

and θ^P as the solution to:

$$\int_{\theta^P}^{\theta^T} \rho(\theta, R^T l_0) \frac{dH(\theta)}{H(\theta^T) - H(\theta^P)} = l_0. \tag{10}$$

Note that such a $\theta^P \in [\underline{\theta}, \theta^T]$ exists because, on one side,

$$\lim_{\theta^P \to \theta^T} \int_{\theta^P}^{\theta^T} \rho(\theta, R^T l_0) \frac{dH(\theta)}{H(\theta^T) - H(\theta^P)} = \rho(\theta^T, R^T l_0) > l_0,$$

where the second inequality holds by equation (9) since we have that $\rho(\theta^T, R^T l_0) = l_0 - x + E[v] > l_0$ and, on the other side,

$$\int_{\underline{\theta}}^{\theta^T} \rho(\theta, R^T l_0) \frac{dH(\theta)}{H(\theta^T)} < l_0,$$

because condition (3) holds and, with $R^T \leq r^D$, we have that $\theta^T > \theta^D$. Since the left-hand side of equation (10) is continuous in θ^P, the intermediate value theorem implies that such a $\theta^P \in [\underline{\theta}, \theta^T]$ exists.

[14] Policies that do not achieve the maximum possible investment can be easily motivated by assuming that there is a cost involved in government revenue collection (such as, for example, distortionary taxes). See Tirole (2012) for an explicit treatment of such case.

Using the threshold values θ^P and θ^T we now construct an equilibrium where firms with low values of θ borrow from the discount window, firms with intermediate values of θ borrow from the market, and firms with high values of θ do not borrow. Only firms that borrow (from the discount window or the market) invest.

Proposition 2 (Philippon-Skreta equilibrium with a discount window)**.** *When the discount window offers loans of size l_0 at interest rate $R^T \in (1, r^D)$, there is an equilibrium where:* ***(1)*** *$m^*(\theta) = l_0$ for all $\theta < \theta^P$ and zero otherwise and $l^*(\theta) = l_0$ for all $\theta \in [\theta^P, \theta^T]$ and zero otherwise;* ***(2)*** *$i^*(\theta) = 1$ for all $\theta \leq \theta^T$ and zero otherwise;* ***(3)*** *the market interest rate equals R^T; and* ***(4)*** *the market beliefs $B(\theta \mid l_0, 0) = H(\theta)/[H(\theta^T) - H(\theta^P)]$ for all $\theta \in [\theta^P, \theta^T]$ and zero otherwise.*

For this equilibrium, it is important that the discount window offers loans *only* of size l_0. If firms could choose government loans of different sizes, then in principle there could be profitable deviations from the equilibrium strategies. Firms may be able to lower their total funding costs by borrowing less from the market. This is the case because the discount window rate is not adjusting to changes in the underlying probability of repayment. Then, a firm taking a smaller loan from the market may induce a higher probability of repayment for that loan. This, in turn, lowers the corresponding interest rate and may result in a reduction on the total interest cost from borrowing l_0. We study this case in more detail in section 3.3.2.

In the equilibrium of Proposition 2, firms are indifferent between borrowing from the discount window and from the market: the interest rate is the same in both cases. Versions of this indifference condition appear in later propositions and in Tirole (2012), which suggests a relatively robust property of equilibrium under the kind of interventions considered here.

The fact that firms borrowing from the market are less risky (in the sense that they are more likely to repay their debts) than firms borrowing from the discount window makes this equilibrium configuration partly consistent with the idea of stigma. However, since firms borrowing from the discount window pay the same interest rate as those borrowing from the market, stigma is not evident from prices. Furthermore, after a firm borrows from the discount window, it has no extra need to interact with the market, so stigma has no implications for its future actions. In a sense, there are no explicit stigma cost associated with borrowing from the discount window.

The way to generate more investment in the model is to get more risky firms to borrow from the discount window. This selection effect allows the composition of firms borrowing from the market to change in the direction of lower (repayment) risk — relative to a situation where all firms are borrowing from the market. In other words, the potential source of stigma in Proposition 2 is a reflection of the strategy used by the central bank to increase investment and enhance efficiency. In this sense, stigma could hardly be called an unintended consequence or an impediment to obtaining better policy results, which is often the argument made in policy circles. Interestingly, Gorton and

Ordoñez (2016) take a similar perspective on the "problem" of stigma and show that, in their model, stigma is desirable as it allows the government to implement the optimal policy during crisis.

There are other equilibria with similar characteristics to the one in Proposition 2 but where firms with higher values of θ than θ^P borrow from the central bank. Following Philippon and Skreta (2012), consider a function $p : \Theta \to [0, 1]$ and assume that for each value of θ a firm borrows from the discount window with probability $p(\theta)$. The case studied in Proposition 2 is that for which $p(\theta) = 1$ if $\theta < \theta^P$ and zero otherwise. However, there are many other possible functions $p(\theta)$ for which the break-even condition in the private market would be consistent with the interest rate R^T. Each of those different functions induce an equilibrium with a market interest rate R^T and some firms borrowing from the discount window. For the issue of stigma, all these equilibria have similar implications since the average "quality" of the pool of firms borrowing from the discount window is in each case the same.

Furthermore, under *the same* discount window policy described in Proposition 2, there is another equilibrium where all firms that borrow and invest get their funding from the discount window and the market for private loans is inactive (see the on-line appendix for details). In that equilibrium, of course, off-equilibrium beliefs play a critical role: How do private investors interpret an (off-equilibrium) request for a loan by a firm? The equilibrium concept does not place restrictions on off-equilibrium beliefs. So, if private investors believe that a firm asking for a private loan has relatively low value of θ, they would charge a relatively high interest rate (higher than R^T) and private investors will prefer to borrow from the discount window.[15] In this equilibrium, then, borrowing from the market is associated with being a firm with higher repayment risk (or, equivalently, in weak financial conditions). This seems counter to the idea of discount window stigma. It is important to note here that this equilibrium outcome is consistent with the model and the discount window policy as exactly specified by Philippon and Skreta (2012) and, in that way, the implications of the model for the issue of discount window stigma are not clear-cut.

3.3 Equilibrium under more general discount-window policies

The forces at work in the equilibrium of Proposition 2 could surface more clearly if some firms borrow from the central bank only a portion of the funds needed to invest. Those firms would later have to borrow from the market to fulfill their funding needs and the market may treat them differently as a result of their prior participation in the government program. Stigma, in this case, could play a more explicit role in equilibrium. With this in mind, I now consider policies that are consistent with such outcomes and I study their implications for market interest rates and for stigma-like effects in equilibrium.

[15] Note also that, as explained in the on-line appendix, this equilibrium cannot be refined away using the intuitive criterion.

3.3.1 Limited loan size

Suppose the central bank offers loans of size $\hat{m} < l_0$ at an interest rate $\hat{R}$, with $\hat{R} \in (1, r^D)$. Then, if a firm wants to borrow from the discount window and invest, it would have to complement that borrowing with a loan from the private market of size at least $l_0 - \hat{m}$.

For notational convenience, let $\hat{m} = \pi l_0$ with $\pi \in (0,1)$. Now, define r^S as the break-even interest rate for loans of size $(1-\pi)l_0$ when firms of type $\theta \leq \theta^P$ are expected to take such loans. Here, again, θ^P is the solution to equation (10) and so we have that r^S solves:

$$\int_{\underline{\theta}}^{\theta^P} \rho(\theta, r^S(1-\pi)l_0)\frac{dH(\theta)}{H(\theta^P)} = (1-\pi)l_0. \tag{11}$$

Note that the seniority of private debt is implicitly recognized in this formula – that is why the loan from the government does not enter this equation directly. It is clear, then, that in general r^S depends on the size of the private loan; and, hence, on π.

As before, the idea is to consider a situation where the government intends to increase investment by providing discount window loans of size πl_0, anticipating that the resulting configuration of interest rates and credit will generate a given, targeted amount of investment. In particular, assume that the government's target is that all firms with $\theta \leq \theta^T$ decide to invest. For the purpose of comparison, assume the value of θ^T is given by the solution to equation (9).

The following proposition shows that for an appropriate level of the discount rate, there is an equilibrium with the desired level of investment. In this equilibrium, firms with low values of θ ($\theta \leq \theta^P$) borrow from the discount window at time 0 and from the market at time 1; firms with intermediate values of θ (between θ^P and θ^T) borrow only from the market; and firms with relatively high values of θ ($\theta > \theta^T$) do not borrow any funds and hence do not invest.

Proposition 3 (Equilibrium with limited discount window lending)**.** *When the discount window offers loans of size $\hat{m} = \pi l_0$ with $\pi < 1$ at an interest rate $\hat{R} = [R^T - r^S(1-\pi)]/\pi$, there is an equilibrium where: **(1)** $m^*(\theta) = \hat{m}$ for all $\theta \leq \theta^P$ and zero otherwise; and $l^*(\theta) = l_0 - \hat{m}$ for all $\theta \leq \theta^P$, $l^*(\theta) = l_0$ for all $\theta \in (\theta^P, \theta^T]$, and $l^*(\theta) = 0$ for all $\theta > \theta^T$; **(2)** $i^*(\theta) = 1$ for all $\theta \leq \theta^T$ and zero otherwise; **(3)** there are two market interest rates, $r^*(l_0 - \hat{m}, \hat{m}) = r^S$ and $r^*(l_0, 0) = R^T$; **(4)** the market beliefs are: $B(\theta \mid (1-\pi)l_0, \pi l_0) = H(\theta)/H(\theta^P)$ for all $\theta \leq \theta^P$ and $B(\theta \mid l_0, 0) = H(\theta)/[H(\theta^T) - H(\theta^P)]$ for all $\theta \in (\theta^P, \theta^T]$.*

Proof. The key step in the proof is to verify that the proposed decision rules for firms satisfy individual rationality given the equilibrium interest rates. Belief consistency follows immediately from those decision rules, and the break-even conditions are implied by the fact that we consider a value of θ^P that satisfies equation (10) and a value r^S of the equilibrium interest rate on loans of size $(1-\pi)l_0$ that satisfies equation (11).

To show that $l^*(\theta)$, $m^*(\theta)$, and $i^*(\theta)$ are individually rational, first note that a firm deciding to invest needs to borrow a total amount of at least l_0. It can do that either by borrowing πl_0 from the discount window and the rest from the market or by borrowing only from the market.

Consider the case when firms investing borrow a total amount exactly equal to l_0. Since $\hat{R}\pi l_0 + r^S(1-\pi)l_0 = R^T l_0$ the funding costs are the same whether or not the firm borrows from the discount window. Then, using Lemma 2, we have that a firm of type θ would choose $i^*(\theta) = 1$ if and only if:

$$\int [y - \min(y, R^T l_0)] f_Y(y \mid \theta) dy \geq \int (c_0 + a) f_A(a \mid \theta) dy,$$

which is equivalent to:

$$E[v] - \rho(\theta, R^T l_0) \geq c_0 = x - l_0.$$

Hence, from the definition of θ^T in equation (9) and the fact that $\rho(\theta, R^T l)$ is increasing in θ, we have that $i^*(\theta) = 1$ for all $\theta \leq \theta^T$ and zero otherwise.

Since the funding costs associated with $(l, m) = (l_0, 0)$ and $(l, m) = (\pi l_0, (1-\pi)l_0)$ are the same, investing firms are indifferent between the two alternatives and we can have those firms with θ below θ^P choosing one alternative and those firms with values of θ between θ^P and θ^T choosing the other alternative. Consequently, the decision rules:

$$m^*(\theta) = \begin{cases} \pi l_0 & \text{for } \theta \leq \theta^P \\ 0 & \text{otherwise,} \end{cases}$$

and

$$l^*(\theta) = \begin{cases} (1-\pi)l_0 & \text{for } \theta \leq \theta^P \\ l_0 & \text{for } \theta \in (\theta^P, \theta^T] \\ 0 & \theta > \theta^T, \end{cases}$$

are compatible with individual rationality.

It remains to be shown that firms will not want to borrow an amount different than l_0 in equilibrium. First note that no firm would ask for a loan lower than l_0 because, then, investors would know that the firm is not investing. Since a non-investing firm has much lower expected future cash-flows, lenders would demand an even higher interest rate to break even and borrowing would not optimal for the firm.

There are several specifications of off-equilibrium beliefs that would justify firms not borrowing more than l_0 in equilibrium. A simple case is when beliefs are such that $B(\underline{\theta} \mid l, m) = 1$ for all l and m such that $l + m > l_0$. Consider first the case of a firm not borrowing from the discount window and borrowing from the markets an amount $l > l_0$. Given the proposed beliefs, the break-even condition implies that investors will charge an interest rate $\underline{r}_l$ that satisfies:

$$\int_Y \min(y + l - l_0, \underline{r}_l l) f_Y(y \mid \underline{\theta}) dy = l.$$

Using that $\rho(\underline{\theta}, \underline{r}_0 l_0) = l_0$, it is easy to show that $\underline{r}_0 l_0 = \underline{r}_l l_0 + (\underline{r}_l - 1)(l - l_0)$ and the payoff to a firm of taking a loan $l > l_0$ at rate $\underline{r}_l$ is the same as the payoff to a firm of taking a loan l_0 at rate $\underline{r}_0$. Now, from expressions (10) and the properties of the function ρ we can deduce that $\rho(\underline{\theta}, R^T l_0) < l_0$, which implies that $R^T < \underline{r}_0$. Hence, taking a loan l_0 at rate R^T gives a higher payoff to the firm than taking a loan $l > l_0$ at rate $\underline{r}_l$. A similar argument can be used to show that a firm borrowing from the discount window would not prefer to borrow more that $(1-\pi)l_0$ from the private market given the proposed off-equilibrium beliefs. We conclude, then, that the decision functions $m^*(\theta)$ and $l^*(\theta)$ in the statement of the proposition are individually rational under the proposed system of beliefs.[16] ■

Perhaps a natural question to ask is why would the central bank choose to limit the size of the loans provided to firms. A common consideration in policy circles when evaluating credit market interventions is the extent to which the proposed policy crowds out too much of private activity, creating what has been called a "footprint" concern (see, for example, Potter (2015)). In the simple model of this paper, unfortunately, justifications for the "footprint" concern cannot be explicitly evaluated.

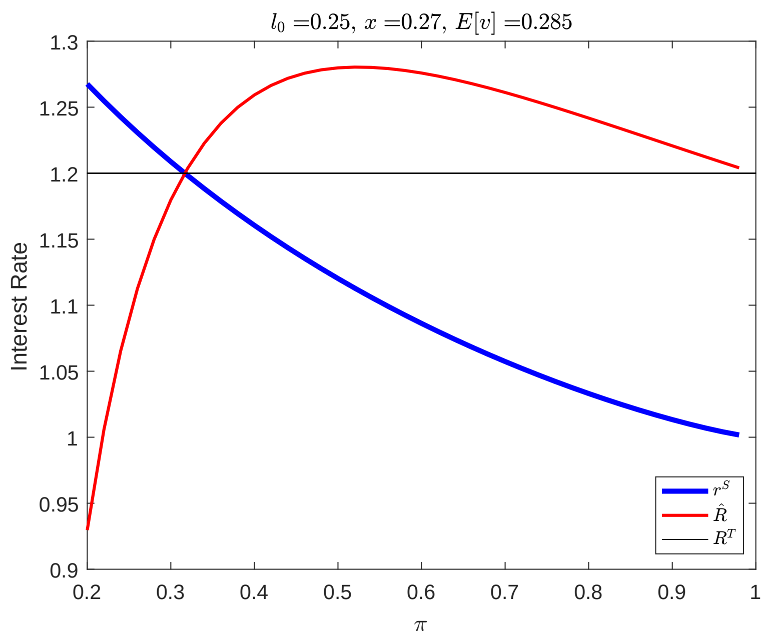

Figure 2: Interest rates

The equilibrium in Proposition 3 produces some interesting implications for thinking about discount window stigma. There are two situations to consider, depending on whether r^S is higher or lower than R^T. Using the example introduced in Section 3.1, Figure 2 plots r^S and $\hat{R}$ as a

[16] In the on-line appendix I discuss, within the context of Proposition 1, less extreme off-equilibrium beliefs that are also consistent with equilibrium. The same logic applies here.

function of π, where higher values of π correspond to larger discount window loans. As can be seen in the figure, for low values of π we have that r^S is greater than R^T and for high values of π the opposite is true. This gives rise to the two situations under consideration.

The dependence of $\hat{R}$ on π is more complicated because both direct and indirect effects (through r^S) play a role in this case. The function $\hat{R}(\pi)$ can be interpreted as the locus of central-bank policies $(\hat{R}, \pi)$ that are consistent with implementing a level of investment that has all firms with $\theta \leq \theta^T$ investing. In other words, if the central bank fixes a particular rate at the discount window, then the inverse of the function $\hat{R}(\pi)$ plotted in Figure 2 gives the size of the discount window loan that the central bank should offer to firms in order to implement the desired level of investment (that corresponds to θ^T). Interestingly, note that for high values of the discount window rate (values above R^T) there are two possible sizes of the discount window loan that implement the same level of investment in the economy.

Going back to the implications for stigma, we have that when $r^S > R^T$, firms borrowing at the discount window pay an interest rate in the market that is higher than the one paid by firms borrowing *only* in the market (R^T). Since r^S converges to a value higher than R^T when π converges to zero (compare expressions (10) and (11) with $\pi = 0$), we know that this case is possible when the central bank provides discount window loans that are relatively small.[17] Of course, the interest rate on the discount window loan, $\hat{R}$, would have to be lower than R^T for firms to remain indifferent between borrowing from the discount window or from (only) the market.

This situation, then, has firms that borrow from the discount window paying higher interest rates in the market than the rates paid by the firms not borrowing from the discount window. Also, firms that borrow only from the market borrow at a rate (R^T) that is higher than the rate they would obtain at the discount window ($\hat{R}$). These two outcomes are often associated with the perception that the discount window is subject to stigma.

An important aspect of U.S. discount window policy is that the discount rate is required to be a "penalty" rate: that is, a rate that is equal to a benchmark market interest rate (or policy rate) *plus* a positive premium.[18] For higher values of π, Proposition 3 is consistent with such a situation: the discount window rate is higher than the rate paid by firms that only borrow from the market (i.e., $\hat{R} > R^T$). At the same time, firms borrowing from the discount window actually pay in the market an interest rate that is *lower* than the rate paid by firms not borrowing from the discount window (i.e., $r^S < R^T$). So, some firms borrow from the discount window even though the interest rate associated to borrowing l_0 in the market would be lower.

[17] Philippon and Skreta (2012, p. 18) show that, for a given R^T, there is a minimum value of π consistent with $\hat{R} \geq 1$, which is required if the central bank wants to restrict borrowing only to firms that are planning to invest. This is also evident in Figure 2.

[18] At the time of writing, discount window lending in the U.S. (primary credit) was offered at an interest rate that is equal to (the top of the range for) the target policy rate plus 50 basis points.

The key to understanding this result is to note that, by borrowing from the discount window, a firm lowers the amount of funding that it needs to obtain from the market. Since private loans are senior claims, by reducing their size, the firm is able to reduce repayment risk and, consequently, reduce the interest rate paid on that portion of the total amount borrowed. Repayment risk is transferred from the market-loan to the discount-window-loan, but since the interest rate at the discount window is an administered rate — and does not adjust to changes in repayment risk — the shift in risk can reduce the total cost of borrowing for the firm.

This logic may help us understand why many banks borrowed from the U.S. Term Auction Facility (TAF) during the recent financial crisis (Armantier, Krieger, and McAndrews (2008)), even though such central-bank lending facility placed restrictions on the availability of borrowed funds, making it less appealing as an emergency source of funding. TAF auctions were conducted biweekly, at pre-announced fixed dates (not on demand) and the funds were credited to the auction-winners with a delay, two days after the results of the auction were announced.[19] These feature made TAF not well-suited to address unanticipated, immediate funding needs (and, hence, the observed borrowing is harder to justify that way). Yet, just as firms do in the model of this paper, it is possible that banks used the TAF facility as a complement to the market in a way that allowed them to better manage their repayment risk.

3.3.2 Flexible loan size

In many cases, central-bank lending programs do not impose constraints on the quantities borrow and, instead, let banks choose how much to borrow. For this reason, we now consider the case when the discount window offers loans of any size $m \leq l_0$ at a given interest rate R and firms decide how much to borrow from the market and how much from the discount window. To make this decision, firms have to know the interest rate that investors would charge for loans of different sizes. In technical terms, firms need to know a *price function* and, given that price function, they will choose the size for their private loan.

As is clear from the break-even condition (6), market interest rates depend on investors' beliefs. The perfect Bayesian equilibrium concept restricts only on-equilibrium beliefs, while the full system of beliefs (on and off equilibrium) determines the patterns of lending observed in equilibrium, through the price function. This delicate interaction between beliefs and equilibrium creates the possibility of multiple equilibrium configurations.

For concreteness, and to get a better sense of the forces at work in the model, we will study one particular equilibrium. In this equilibrium, investors believe that any firm borrowing from the market, regardless of how much it asks to borrow, is a random draw from the set of firms investing

[19] Gauthier et al. (2015) also emphasize these properties of the auction format in their study of the role of TAF during the financial crisis.

in equilibrium.

Suppose that all firms with legacy assets of type θ lower than a threshold value θ^{**} are expected to invest. Using the break-even condition, we obtain the interest rate function $r^{**}(m)$ that satisfies the equation:

$$\int_{\underline{\theta}}^{\theta^{**}} \rho[\theta, r^{**}(m)(l_0 - m)] \frac{dH(\theta)}{H(\theta^{**})} = l_0 - m, \tag{12}$$

for all values of $m \leq l_0$. Note that this equation is equivalent to equation (11) and, as illustrated in Figure 2, the function $r^{**}(m)$ is decreasing.

Suppose now that firms take as given the pricing function $r^{**}(m)$ when they decide how much to borrow from the discount window and the market. As we will confirm later, firms that decide to invest will borrow exactly the amount l_0; that is, $m + l = l_0$ where m is the amount borrowed at the discount window and l the amount borrow from the market. Then, we have that firms will choose m to solve:

$$\max_m \int_Y \{y - \min[y, Rm + r^{**}(m)(l_0 - m)]\} f_Y(y \mid \theta) dy,$$

which is equivalent to minimizing total funding costs (private plus discount window loans); that is:

$$\min_m Rm + r^{**}(m)(l_0 - m). \tag{13}$$

Denote by m^{**} the solution to this problem. Note that m^{**} is independent of θ, so all investing firms will choose to borrow the same amount from the discount window (and from the market).

Finally, the threshold value θ^{**} is given by the equation:

$$l_0 - x + E[v] - \rho[\theta^{**}, Rm^{**} + r^{**}(m^{**})(l_0 - m^{**})] = 0, \tag{14}$$

which is the counterpart of equation (7) in this case.

Consider a function $r^{**}(m)$ and values of θ^{**} and m^{**} that jointly solve equations (12), (13), and (14). Then, we have an equilibrium where all firms with $\theta \leq \theta^{**}$ borrow m^{**} from the central bank, $l_0 - m^{**}$ from the market, and invest. Firms with higher values of θ do not borrow and, hence, do not invest.

Proposition 4 (Equilibrium with flexible discount window lending)**.** *When the discount window offers loans of any size $m \leq l_0$ at an interest rate R, there is an equilibrium where:* ***(1)*** *$m^*(\theta) = m^{**}$ and $l^*(\theta) = l_0 - m^{**}$ for all $\theta \leq \theta^{**}$ and zero otherwise;* ***(2)*** *$i^*(\theta) = 1$ for all $\theta \leq \theta^{**}$ and zero otherwise;* ***(3)*** *the market interest-rate function is $r^*(l_0 - m, m) = r^{**}(m)$;* ***(4)*** *the market beliefs are: $B(\theta \mid l_0 - m, m) = H(\theta)/H(\theta^{**})$ for all $\theta \leq \theta^{**}$ and all $m \leq l_0$.*

Figure 3 plots an example of the objective function from problem (13): the total funding costs as a function of the size of the discount window loan m. When the optimal choice of m is interior

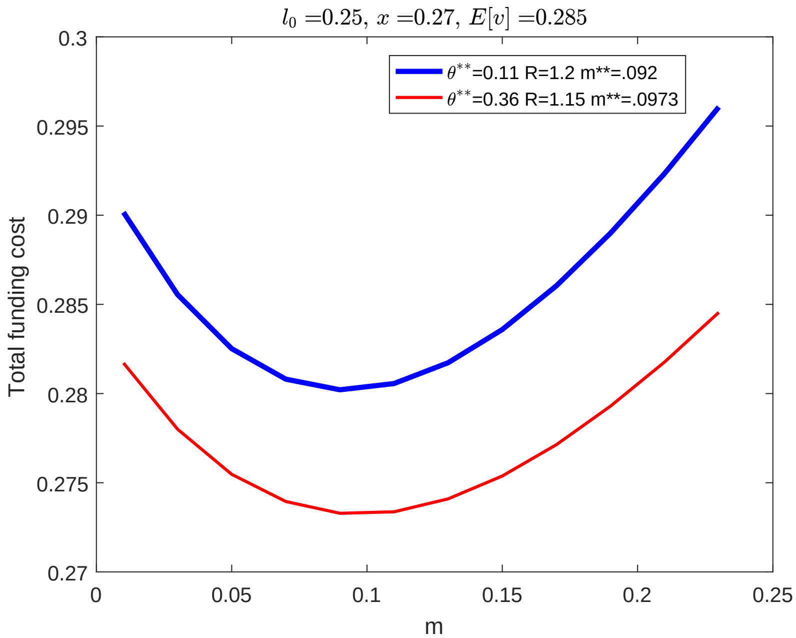

Figure 3: Optimal discount window loan

(as it is the case in the figure), we have that $R > r^{**}(m^{**})$ and the interest rate at the discount window could be seen as a "penalty" rate.

There are two forces at play in the determination of the optimal value of m. On one side, by borrowing more from the discount window and less from the market, firms can shift repayment risk away from market transactions and, in that way, lower the interest rate and the borrowing costs associated with private loans. On the other side, since discount window borrowing is more expensive, borrowing more from the discount window and less from the market tends to increase the total cost of borrowing.

Interestingly, then, when the central bank offers loans at a relatively high rate, firms may choose to borrow some from the market and some from the discount window as a way to manage their repayment risk in the dealings with private investors (the risk-sensitive counterparties in the model). An observer may wonder why firms are borrowing from the discount window at an interest rate higher than the one they are able to obtain in the market. The key to understand this outcome is to note that the interest rate on a market-loan is increasing in the amount of the loan. The ability to borrow from the discount window, then, gives firms flexibility to adjust the amount of their private borrowing so as to respond to those price effects.

When the solution m^{**} is interior, lower discount window interest rates are associated with higher discount window loans: an intensive margin effect. In the figure, when $R = 1.2$ we have that $m^{**} = 0.092$ and when $R = 1.15$ we have that $m^{**} = 0.0973$. There is also an extensive margin effect because the equilibrium value of θ^{**} also depends on the level of the discount window interest rate R. As shown in the figure, when the discount window rate decreases (from 1.2 to 1.15), it becomes less expensive to fund investment and more firms decide to invest; that is, the value of θ^{**} increases (from 0.11 to 0.36, in the figure). In this sense, both the intensive and the extensive margins move in the same direction: lower discount rates imply more discount window lending.

3.3.3 Seniority and "penalty" rates

As we saw in propositions 3 and 4, in principle, the equilibrium of the model can be consistent with situations where the interest rate charged at the discount window is higher than the rate charged in the market. However, it is clear from our discussion of those propositions that the results depend (critically) on the fact that discount window loans are junior claims relative to loans obtained in the market.[20]

In the U.S., discount window loans from the Federal Reserve are fully collateralized, which makes them relatively senior claims. However, in practical terms, seniority of claims is much less clear-cut when taking a consolidated-government perspective in the presence of deposit insurance. While discount window loans are fully collateralized, it is often the case that the losses experienced by the insurance fund depend on the ability of the failing bank to borrow from the discount window before failing (Goodfriend and Lacker (1999)). The extra liquidity available to the bank through the discount window is often used to pay back uninsured depositors, making them effectively senior claimants relative to the consolidated government. This discussion should make clear that seniority of claims is a delicate issue and the seniority assumption in the model should not be regarded as obviously inconsistent with the U.S. situation.

Assuming, alternatively, that the discount window loan is a senior claim relative to the private loan produces interesting additional implications. Suppose, as in Section 3.3.1, that the central bank is offering loans of size $\hat{m} < l_0$ at an interest rate $\hat{R}$ to target a level of investment consistent with all firms with $\theta \leq \theta^T$ investing. The main change of the equilibrium conditions occurs in the break-even condition for investors. In particular, when discount window loans are senior, the expected repayment function conditional on a given value of θ is:

$$\begin{aligned} \xi(\theta, Rm, rl) &= \int_Y \min[\max(0, y - Rm), rl] f_Y(y \mid \theta) dy \\ &= \int_{Rm}^{\bar{A}+\bar{V}} \min(y - Rm, rl) f_Y(y \mid \theta) dy, \end{aligned} \tag{15}$$

and this repayment function replaces $\rho(\theta, rl)$ in the investors' break-even condition (6). The other equilibrium conditions remain basically the same.

[20]Ennis and Weinberg (2013) generates equilibrium discount window lending at a penalty rate using a different mechanism.

Using condition (3) (see Section 2.1), it is possible to show that the (net) interest rate at the discount window would have to be negative for the central bank to pursue a policy that increases investment relative to laissez-faire (see the Appendix). In other words, the discount window would be channelling a direct subsidy to all possible participants. As a result, the managers of the program would need to be able to verify that the loans are being used for investment (and not just for simple arbitrage). This is an administrative task that was not needed when interest rates were positive.[21]

The fact that to increase investment discount window lending has to involve a form of subsidy is also reflected in the (related) results by Philippon and Skreta (2012): they demonstrate that the optimal intervention always involves a positive cost for the government. When discount window loans are junior, even if the interest rate charged at the facility is positive, it is not a break-even interest rate. When discount window loans are senior, the interest rate charged at the facility needs to be negative (and, again, not a break-even rate) in order to induce an increase in the equilibrium level of investment.

Interestingly, when the Federal Reserve tried to encourage participation at the discount window during the early stages of the 2007-2008 financial crisis, it modified the primary credit program in the direction suggested by these findings. In particular, the Fed reduced the "penalty" spread on the discount rate and increased the maturity term of loans (Klee (2011)). Given how fast conditions were changing during those critical times, the lengthening of the maturity of discount window loans could be interpreted as a move to make discount window loans more junior claims relative to other short-term liabilities. As suggested here, making discount window loans (effectively) junior claims could be a way to encourage borrowing when rates do not involve an evident subsidy.

4 Conclusions

There appears to be a relative consensus among policymakers that the Fed's discount window suffers from the ailment of stigma: the fear that financial market participants would regard discount window borrowing as a signal of financial weakness. From the way discount window stigma is being discussed in policy circles, one might conclude that we have a relatively good understanding of the theoretical underpinnings of the idea. However, I am aware of only very few papers in the literature that present models where the observable outcomes often associated with stigma can endogenously arise. One of those models is the one analyzed in this paper, which was recently also used by Philippon and Skreta (2012) to study the optimal design of programs aimed at intervening in credit markets.

I have investigated in detail the implications for discount window stigma of the Philippon-Skreta

[21] Before 2003, the discount window in the U.S. made loans at subsidized rates and discretion was used to select which loans were "justifiable." Since 2003, the Fed's discount window has been a "no-questions-asked" standing facility (see Madigan and Nelson (2002))

model. In the equilibrium of the model, the average "quality" of borrowers at the discount window can indeed be low – and in that sense borrowing from the discount window could be considered "a sign of financial weakness." Furthermore, after appropriate modifications, the model can produce the combination of observed interest rates (market and administered) that are often associated with the presence of stigma.

Beyond rationalizing the relevant outcomes, the analysis produced some interesting insights that can broaden our perspective when thinking about discount window activity and its implications. First, there is no clear sense in the model that stigma-like effects reduce "the efficacy of the discount window." In fact, "negative" selection at the discount window is the means by which the government enhances economic efficiency as it promotes more overall lending and investment. In other words, in the context of this model – just as is the case in the model by Gorton and Ordoñez (2016), for example – one is left thinking that stigma should perhaps be considered a *good* thing.

Second, the model highlights the subtle interactions between borrowing done at the discount window and (complementary) borrowing done in the market. Repayment risk, market interest rates, and the resulting funding costs depend crucially on the ability of firms to tap the discount window. This is an issue that has not received much attention in the prior literature and for which the analysis in this paper provides a valuable perspective.

A Appendix: Seniority of discount window loan

In the text of the paper, we have assumed, as did Philippon and Skreta (2012), that discount window loans are junior loans relative to private loans. Consider instead the case when the central bank wants to increase the level of investment and discount window loans are senior claims. We demonstrate here that to achieve that, the government would need to provide loans at negative (net) interest rates. Of course, negative interest rates create a different set of challenges for the central bank since all banks (regardless of their investment plans) would want to take loans at a subsidized rate.

Suppose the central bank wants to increase investment by moving the investment threshold from θ^D (the threshold under no intervention in Section 3.1) to some value $\theta^T > \theta^D$. Consider discount window loans of size $m < l_0$ so that the seniority of claims matters. The value of m could be fixed by the central bank (as in Section 3.3.1) or chosen by the borrowing bank (as in Section 3.3.2).

For concreteness, assume that $\Theta_{\mathcal{O}} = [\underline{\theta}, \theta^P]$ and $\Theta_{\mathcal{P}} = [\theta^P, \theta^T]$. This is not essential, but it simplifies the measurement of the relative sizes of the groups of investing firms that are participating, $\Theta_{\mathcal{P}}$, and not participating, $\Theta_{\mathcal{O}}$, in the government program. The arguments here apply for other sets $\Theta_{\mathcal{O}}$ and $\Theta_{\mathcal{P}}$, and even if $\Theta_{\mathcal{O}}$ is empty, as in Proposition (4).

The investment decision of firms participating in the program is given by a version of equation

(7) regardless of the relative seniority of claims. In consequence, to target a level of investment consistent with the threshold θ^T, the central bank has to choose a discount window policy (m, R) such that $Rm + r(l_0 - m) = R^T l_0$ where R^T solves equation (9). If $\Theta_{\mathcal{O}}$ is non-empty, then R^T is also the interest rate that non-participating (investing) firms obtain in the market when taking a loan of size l_0. In that case, investors' break-even implies that equation (10) holds.

Participating firms borrow $l_0 - m$ from the market and the relevant break-even condition is:

$$\int_{\underline{\theta}}^{\theta^P} \xi[\theta, Rm, r(l_0 - m)] \frac{dH(\theta)}{H(\theta^P)} = l_0 - m,$$

which is a version of equation (11) when discount window loans are senior claims. Manipulating this expression one can easily show that the following expression holds:

$$\int_{\underline{\theta}}^{\theta^P} \rho[\theta, Rm + r(l_0 - m)] \frac{dH(\theta)}{H(\theta^P)} > l_0 + (R - 1)m. \tag{16}$$

We are now ready to establish the key result: given the natural condition (3) (see Section 2.1), if the central banks wants to increase the level of investment in the economy it would need to offer negative net interest rates on discount window loans. To see this, note that for any value θ^T greater than θ^D conditions (3), (16), and investors' break-even imply:

$$\begin{aligned}
l_0 &> \int_{\underline{\theta}}^{\theta^T} \rho[\theta, R^T l_0] \frac{dH(\theta)}{H(\theta^T)} \\
&= \frac{H(\theta^P)}{H(\theta^T)} \int_{\underline{\theta}}^{\theta^P} \rho(\theta, R^T l_0) \frac{dH(\theta)}{H(\theta^P)} + \frac{H(\theta^T) - H(\theta^P)}{H(\theta^T)} \int_{\theta^P}^{\theta^T} \rho(\theta, R^T l_0) \frac{dH(\theta)}{H(\theta^T) - H(\theta^P)} \\
&> \frac{H(\theta^P)}{H(\theta^T)} [l_0 + (R-1)m] + \frac{H(\theta^T) - H(\theta^P)}{H(\theta^T)} l_0 \\
&= l_0 + \frac{H(\theta^P)}{H(\theta^T)} (R - 1)m,
\end{aligned}$$

which immediately implies that $R < 1$ must hold and the net interest rate on discount window loans has to be negative if the government is seeking to increase equilibrium investment. Note that these expressions hold even if $\theta^P = \theta^T$ and all firms that decide to invest borrow from the central bank (as in Proposition (4)).

References

ANBIL, S. (2018). "Managing Stigma during a Financial Crisis." *Journal of Financial Economics* 130 (1), pp. 166–181.

ARMANTIER, O. and A. COPELAND (2015). "Challenges in Identifying Interbank Loans." *Economic Policy Review* 21 (1), pp. 1–17.

ARMANTIER, O., S. KRIEGER, and J. MCANDREWS (2008). "The Federal Reserve's Term Auction Facility." *Current Issues in Economics and Finance* 14 (5), pp. 1–10.

ARMANTIER, O. et al. (2015). "Discount Window Stigma During the 2007 - 2008 Financial Crisis." *Journal of Financial Economics* 118 (2), pp. 317–335.

BERGER, A. N. et al. (2017). "Bank Loan Supply Responses to Federal Reserve Emergency Liquidity Facilities." *Journal of Financial Intermediation* 32 (October), pp. 1–15.

BERNANKE, B. S. (2008). "Liquidity Provision by the Federal Reserve." *Speech* (May).

— (2015). "Warren-Vitter and the lender of last resort." *Brookings Institution* (May).

BRUNNERMEIER, M. K. and L. H. PEDERSEN (2005). "Predatory Trading." *Journal of Finance* 60 (4), pp. 1825–1863.

CAMARGO, B., K. KIM, and B. LESTER (2016). "Information Spillovers, Gains from Trade, and Interventions in Frozen Markets." *Review of Financial Studies* 29 (5), pp. 1291–1329.

CHE, Y.-K., C. CHOE, and K. RHEE (2015). "Bailout Stigma." *manuscript* (Columbia University).

CHIU, J. and T. V. KOEPPL (2016). "Trading Dynamics with Adverse Selection and Search: Market Freeze, Intervention and Recovery." *Review of Economic Studies* 83 (3), pp. 969–1000.

CHO, I.-K. and D. M. KREPS (1987). "Signaling Games and Stable Equilibria." *Quarterly Journal of Economics* 102 (2), pp. 179–221.

CORBETT, J. and J. MITCHELL (2000). "Banking Crises and Bank Rescues: The Effect of Reputation." *Journal of Money, Credit, and Banking* 32 (3), pp. 474–512.

DUKE, E. (2010). "Unusual and Exigent: My First Year at the Fed." *Speech* (February).

ENNIS, H. M. and J. A. WEINBERG (2013). "Over-the-Counter Loans, Adverse Selection, and Stigma in the Interbank Market." *Review of Economic Dynamics* 16 (4), pp. 601–616.

FISCHER, S. (2016). "The Lender of Last Resort Function in the United States." *Speech* (February).

FUCHS, W. and A. SKRZYPACZ (2015). "Government interventions in a dynamic market with adverse selection." *Journal of Economic Theory* 158, pp. 371–406.

FURFINE, C. (2001). "The Reluctance to Borrow from the Fed." *Economic Letters* 72 (2), pp. 209–213.

— (2003). "Standing Facilities and Interbank Borrowing: Evidence from the Federal Reserve's New Discount Window." *International Finance* 6 (3), pp. 329–347.

GAUTHIER, C. et al. (2015). "Emergency Liquidity Facilities, Signaling, and Funding Costs." *Staff Working Paper* 2015-44, Bank of Canada.

GEITHNER, T. F. (2014). *Stress Test: Reflextions on Financial Crises.* Crown Publishers, New York.

GOODFRIEND, M. and J. M. LACKER (1999). "Limited Commitment and Central Bank Lending." Federal Reserve Bank of Richmond *Economic Quarterly* 85 (4), pp. 1–27.

GORTON, G. (2015). "Stress for Success: A Review of Timothy Geithner's Financial Crisis Memoir." *Journal of Economic Literature* 53 (4), pp. 975–995.

GORTON, G. and G. ORDOÑEZ (2016). "Fighting Crises." NBER Working Paper No. 22787 (October).

KLEE, E. (2011). "The First Line of Defense: The Discount Window during the Early Stages of the Financial Crisis." *Finance and Economics Discussion Series* 2011-23, Board of Governors of the Federal Reserve System (U.S.)

KLEYMENOVA, A. (2016). "Consequences of Mandated Bank Liquidity Disclosures." Chicago Booth Research Paper No. 16-04.

LA'O, J. (2014). "Predatory trading, Stigma and the Feds Term Auction Facility." *Journal of Monetary Economics* 65 (July), pp. 57–75.

LI, M., F. MILNE, and J. QIU (2016). "The signaling effect and optimal LOLR policy." *Department of Economics and Finance Discussion Paper* 2016-01, University of Guelph.

LOWERY, R. (2014). "Comment on: 'Predatory Trading, Stigma, and the Feds Term Auction Facility' by Jennifer La'O." *Journal of Monetary Economics* 65 (July), pp. 76–79.

MADIGAN, B. and W. NELSON (2002). "Proposed Revision to the Federal Reserve's Discount Window Lending Programs." *Federal Reserve Bulletin* 88 (7), pp. 312–319.

MORENO, D. and J. WOODERS (2016). "Dynamic markets for lemons: Performance, liquidity, and policy intervention." *Theoretical Economics* 11, pp. 601–639.

MYERS, S. C. and N. S. MAJLUF (1984). "Corporate financing and investment decisions when firms have information that investors do not have." *Journal of Financial Economics* 13 (2), pp. 187–221.

NACHMAN, D. C. and T. H. NOE (1994). "Optimal Design of Securities under Asymmetric Information." *Review of Financial Studies* 7 (1), pp. 1–44.

PHILIPPON, T. and V. SKRETA (2012). "Optimal Interventions in Markets with Adverse Selection." *The American Economic Review* 102 (1), pp. 1–28.

POTTER, S. (2015). "Money Markets and Monetary Policy Normalization." *Speech* (April).

ROCHET, J.-C. and X. VIVES (2004). "Coordination failures and the lender of last resort: Was Bagehot right after all?" *Journal of the European Economic Association* 2 (6), pp. 1116–1147.

TIROLE, J. (2012). "Overcoming Adverse Selection: How Public Intervention Can Restore Market Functioning." *The American Economic Review* 102 (1), pp. 29–59.

VOSSMEYER, A. (forthcoming). "Analysis of Stigma and Bank Credit Provision." *Journal of Money, Credit, and Banking.*

On-line appendix

1. Proofs

Lemma 1. In any equilibrium with an active market for private loans, we have that $r^*(l,m) > 1$ for all $l > 0$ and all m.

Proof. The result follows from applying the break-even condition and the fact that $\rho(\theta, rl) < rl$ whenever $rl > 0$ since we then have that:

$$l = \int_{\Theta} \rho[\theta, r^*(l,m)l]dB^*(\theta \mid l,m) < \int_{\Theta} r^*(l,m)l dB^*(\theta \mid l,m) = r^*(l,m)l,$$

which implies that $r^*(l,m) > 1$. ■

Lemma 2. For any $\theta \in \Theta$, if $i^*(\theta) = 0$ then $l^*(\theta) = 0$ and, when $R > 1$, also $m^*(\theta) = 0$.

Proof. From Lemma 1 we know that $r^*(l;m) \equiv r^*_{lm} > 1$ for all $l > 0$ and all m. Now suppose, by way of contradiction, that for some θ we have that $i^*(\theta) = 0$ while $l^*(\theta) > 0$ and/or $m^*(\theta) > 0$. In that case, the payoff of the firm is:

$$\int_{\underline{a}}^{\bar{A}} (c_0 + l + m + a - r^*_{lm}l - Rm) f_A(a \mid \theta) da,$$

where $\underline{a} = max\{0, r^*_{lm}l + Rm - c_0 + l + m\}$. But, then, we have that:

$$\int_{\underline{a}}^{\bar{A}} [c_0 + a - (r^*_{lm} - 1)l - (R-1)m] f_A(a \mid \theta) da < \int_{A} (c_0 + a) f_A(a \mid \theta) da,$$

where the last term is the payoff of the firm that does not borrow and does not invest. Hence, $l^*(\theta) = 0$ and $m^*(\theta) = 0$ when $i^*(\theta) = 0$ is a better option for firms. Since this strategy is always available to firms, the statement of the lemma must hold. ■

Proposition 1. Equilibrium without discount window. When the discount window is not active, there is an equilibrium where: **(1)** $l^*(\theta) = l_0$ for all $\theta \leq \theta^* < \bar{\theta}$ and zero otherwise; **(2)** $i^*(\theta) = 1$ for all $\theta \leq \theta^* < \bar{\theta}$ and zero otherwise; **(3)** the market interest rate is equal to r^*; and **(4)** the market beliefs $B(\theta \mid l_0) = H(\theta)/H(\theta^*)$ for all $\theta \leq \theta^*$ and zero otherwise.

Proof. First, we show that θ^* lies in the interior of the set Θ. Suppose this is not the case and instead $\theta^* = \underline{\theta}$, then by equation (8) we have that $r^* = \underline{r}_0$ and hence $\rho(\underline{\theta}, r^* l_0) = l_0$. But, then, since $E[v] > x$, this contradicts equation (7). Now suppose that $\theta^* = \bar{\theta}$, then equation (8) implies $r^* = \bar{r}_0$ and condition (3) (in Section 2.1) immediately implies a contradiction of equation (7). Clearly, for the pair $(\underline{\theta}, \underline{r}_0)$ we have that $l_0 - x + E[v] - \rho(\underline{\theta}, \underline{r}_0 l_0) > 0$ and for the pair $(\bar{\theta}, \bar{r}_0)$ we have that $l_0 - x + E[v] - \rho(\bar{\theta}, \bar{r}_0 l_0) < 0$. Since both expressions (7) and (8) are continuous in (θ^*, r^*), we have that there is a solution to the system of equations (7) and (8) with $\theta^* \in (\underline{\theta}, \bar{\theta})$ and $r^* \in (\bar{r}_0, \underline{r}_0)$.

To see that $l^*(\theta)$ and $i^*(\theta)$ are individually rational, first note that by Lemma 2 no firm would borrow to not invest. In consequence, the alternative to borrowing and investing is to not borrow and not invest. Now, note that to be able to invest, a firm needs to borrow from the market at least l_0. If the firm borrows exactly l_0 when it decides to invest, then it would decide to invest whenever the following inequality holds:

$$\int_A \int_V [c_0 + l_0 - x + a + v - \min(a + v, r^* l_0)] f_V(v) f_A(a \mid \theta) dv da \geq \int_A (c_0 + a) f_A(a \mid \theta) da,$$

which can be simplified to:

$$l_0 - x + E[v] - \rho(\theta, r^* l_0) \geq 0. \tag{17}$$

Recall that ρ is a strictly increasing function of θ. Then, by the definition of θ^* in equation (7) we have that equation (17) holds for all $\theta \leq \theta^*$ and does not hold for any $\theta > \theta^*$. This confirms that conditional on a firm borrowing l_0, the decision function $i^*(\theta)$ in the statement of the proposition satisfies individual rationality.

In principle, there are several specifications of off-equilibrium beliefs that can sustain $l^*(\theta)$ as an equilibrium. A simple case is when beliefs are such that for all $l > l_0$ we have that $B(\theta \mid l) = 1$ if $\theta = \underline{\theta}$ and zero otherwise. That is, if a firm were to ask for a loan greater than l_0, investors would believe that the firm is of type $\underline{\theta}$.[22] Given these beliefs, the break-even condition implies that investors will charge an interest rate $\underline{r}_l$ that satisfies:

$$\int_Y \min(y + l - l_0, \underline{r}_l l) f_Y(y \mid \underline{\theta}) dy = l.$$

Using that $\rho(\underline{\theta}, \underline{r}_0 l_0) = l_0$, it is easy to show that $\underline{r}_0 l_0 = \underline{r}_l l_0 + (\underline{r}_l - 1)(l - l_0)$ and the payoff to a firm of taking a loan $l > l_0$ at rate $\underline{r}_l$ is the same as the payoff to a firm of taking a loan l_0 at rate $\underline{r}_0$. Now, from expressions (7) and (8) we have that $\rho(\underline{\theta}, r^* l_0) < l_0$, which implies that $r^* < \underline{r}_0$. Hence, taking a loan l_0 at rate r^* gives a higher payoff to the firm than taking a loan $l > l_0$ at rate $\underline{r}_l$. We conclude that the decision function $l^*(\theta)$ in the statement of the proposition is individually rational under the proposed beliefs system. ■

Off-equilibrium beliefs are rather extreme in the proof of this proposition. In particular, investors believe that any firm asking for a loan greater than l_0 has legacy assets of the lowest type. This was used just for simplicity. The argument still goes through for many other systems of off-equilibrium beliefs. In fact, even if investors believe that any firm borrowing $l > l_0$ is a random draw from the relevant set of firms, the equilibrium configuration in Proposition 1 is still an equilibrium.

Formally, suppose that for any $l > l_0$ investors' beliefs are $B(\theta \mid l) = H(\theta)/H(\theta^*)$ if $\theta < \theta^*$ and zero otherwise. That is, investors believe that a firm borrowing $l > l_0$ is a random draw from the

[22] Note that no firm would ask for a loan lower than l_0 because then investors would know that the firm is not investing and would demand a high interest rate, making borrowing not optimal for the firm.

set of firms willing to borrow l and invest. Then, the break-even condition for investors is:

$$\int_{\underline{\theta}}^{\theta^*} \int_Y \min(y + l - l_0, r_l l) f_Y(y \mid \theta) dy \frac{dH(\theta)}{H(\theta^*)} = l.$$

Subtracting $l - l_0$ from both sides of the equation, we get:

$$\int_{\underline{\theta}}^{\theta^*} \int_Y \min(y, r_l l - (l - l_0)) f_Y(y \mid \theta) dy \frac{dH(\theta)}{H(\theta^*)} = l_0,$$

which can be compared with the break-even condition for loans of size l_0:

$$\int_{\underline{\theta}}^{\theta^*} \int_Y \min(y, r^* l_0) f_Y(y \mid \theta) dy \frac{dH(\theta)}{H(\theta^*)} = l_0,$$

to conclude that $r_l l - (l - l_0) = r^* l_0$.

The payoff of a firm borrowing $l > l_0$ at rate r_l is given by:

$$\int_Y (y + l - l_0 - \min\{y + l - l_0, r_l l\}) f_Y(y \mid \theta) dy = \int_Y (y - \min\{y, r_l l - (l - l_0)\}) f_Y(y \mid \theta) dy,$$

and the payoff of a firm borrowing l_0 at rate r^* is given by:

$$\int_Y [y - \min(y, r^* l_0)] f_Y(y \mid \theta) dy$$

but, since $r_l l - (l - l_0) = r^* l_0$, firms are indifferent between the two alternatives and choosing to borrow l_0 is an equilibrium.[23]

Any belief system that is more "pessimistic" about the types of those firms borrowing more than l_0, such as the one used in the proof of the proposition, will make firms no longer indifferent between borrowing l_0 or more. When this is the case, the equilibrium action in Proposition 1 becomes the strictly-preferred choice for firms.

Proposition 2. Philippon-Skreta equilibrium with a discount window. When the discount window offers loans of size l_0 at interest rate $R^T \in (1, r^D)$, there is an equilibrium where: **(1)** $m^*(\theta) = l_0$ for all $\theta < \theta^P$ and zero otherwise and $l^*(\theta) = l_0$ for all $\theta \in [\theta^P, \theta^T]$ and zero otherwise; **(2)** $i^*(\theta) = 1$ for all $\theta \leq \theta^T$ and zero otherwise; **(3)** the market interest rate equals R^T; and **(4)** the market beliefs $B(\theta \mid l_0, 0) = H(\theta)/[H(\theta^T) - H(\theta^P)]$ for all $\theta \in [\theta^P, \theta^T]$ and zero otherwise.

Proof. If the firm decides to invest, then it must pick l and m such that $l + m \geq l_0$. Consider the case when the firm investing chooses $l + m = l_0$. Given the equilibrium interest rate in the market, the payoff of a type-θ firm is:

$$\int [y - \min(y, R^T m + R^T l)] f_Y(y \mid \theta) dy,$$

[23]Note that considering an alternative equilibrium, where some randomly selected firms borrow l_0 and some borrow $l > l_0$, does not change equilibrium predictions in any significant way: it is still the case that firms with $\theta \leq \theta^*$ borrow and invest, and investors break even.

and, hence, the payoffs from choosing $m = l_0$ or $l = l_0$ (with $l + m = l_0$) are the same.
Assume, as we did in the proof of Proposition 1, that off-equilibrium beliefs for $l > l_0$ and $m = 0$ are given by $B(\underline{\theta} \mid l, 0) = 1$. Then, just as in the proof of Proposition 1, break-even conditions imply that $\underline{r}_l l - (l - l_0) = \underline{r}_0 l_0$ and, since $R^T < \underline{r}_0$, firms have no incentives to deviate and borrow more than l_0 when borrowing from the private market. When a discount window is available, we need to also consider the situation when $m = l_0$ and $l \neq l_0$. Again, assume that $B(\underline{\theta} \mid l, l_0) = 1$ in this case. Since the break-even condition implies that $r_{lm} > 1$, no firm will choose to deviate to $m = l_0$ and $l \neq l_0$.
Using Lemma 2, we have that a firm of type θ would choose $i^*(\theta) = 1$ if and only if:

$$\int [y - \min(y, R^T l_0)] f_Y(y \mid \theta) dy \geq \int (c_0 + a) f_A(a \mid \theta) dy,$$

which is equivalent to:

$$E[v] - \rho(\theta, R^T l_0) \geq c_0 = x - l_0.$$

Hence, from the definition of θ^T and the fact that $\rho(\theta, R^T l)$ is increasing in θ, we have that $i^*(\theta) = 1$ for all $\theta \leq \theta^T$ and zero otherwise.
Given that all firms with $\theta \leq \theta^T$ choose to invest and that firms that invest are indifferent between any feasible choice of l and m such that $l + m = l_0$, we have that $m^*(\theta) = l_0$ for $\theta \leq \theta^P$ and $l^*(\theta) = l_0$ for $\theta \in [\theta^P, \theta^T]$ satisfy individual rationality. Since only firms with $\theta \in [\theta^P, \theta^T]$ borrow from the market, belief consistency implies that $B(\theta \mid l_0, 0) = H(\theta)/[H(\theta^T) - H(\theta^P)]$ for all $\theta \in [\theta^P, \theta^T]$, as required. Finally, by the definition of θ^P in equation (10), the break-even condition is immediately satisfied. ■

Proposition 4. Equilibrium with flexible discount window lending. When the discount window offers loans of any size $m \leq l_0$ at an interest rate R, there is an equilibrium where: **(1)** $m^*(\theta) = m^{**}$ and $l^*(\theta) = l_0 - m^{**}$ for all $\theta \leq \theta^{**}$ and zero otherwise; **(2)** $i^*(\theta) = 1$ for all $\theta \leq \theta^{**}$ and zero otherwise; **(3)** the market interest-rate function is $r^*(l_0 - m, m) = r^{**}(m)$; **(4)** the market beliefs are: $B(\theta \mid l_0 - m, m) = H(\theta)/H(\theta^{**})$ for all $\theta \leq \theta^{**}$ and all $m \leq l_0$.

Proof. Consider the following system of beliefs: for all $m \geq 0$ and $l = l_0 - m$, let $B(\theta \mid l, m) = H(\theta)/H(\theta^{**})$ for all $\theta \leq \theta^{**}$ and zero otherwise; for all $m \geq 0$ and $l > l_0 - m$, let $B(\underline{\theta} \mid l, m) = 1$. Given these beliefs, for all $m \geq 0$ and $l = l_0 - m$, the break-even condition for investors (12) implies that $r^*(l_0 - m, m) = r^{**}(m)$. If $l \geq l_0 - m$, then the break-even condition for investors under the proposed beliefs is:

$$\int_Y min(y + l + m - l_0, \underline{r}_{lm} l) f_Y(y \mid \underline{\theta}) dy = l,$$

which determines the value of $\underline{r}_{lm}$. Following similar steps as in the proof of the previous propositions, we can show that $r^{**}(m)(l_0 - m) < r_{lm} l - (l + m - l_0)$, which implies that the funding cost associated with a loan of size $l > l_0 - m$ is higher than the funding cost of a loan of size $l = l_0 - m$. Hence, firms take loans in the private market of size $l_0 - m$. From this we conclude that, given the pricing function $r^{**}(m)$ and the fact that m^{**} solves equation (13), firms investing will choose $m^*(\theta) = m^{**}$ and $l^*(\theta) = l_0 - m^{**}$. Using Lemma 2 and equation (14), we have that all firms with $\theta \leq \theta^{**}$, and only those firms, will choose to invest (that is, $i^*(\theta) = 1$ for all $\theta \leq \theta^{**}$ and zero otherwise). Finally, given firms decisions, we have that the beliefs $B(\theta \mid l_0 - m^{**}, m^{**}) = H(\theta)/H(\theta^{**})$ for all $\theta \leq \theta^{**}$ satisfy Bayes rule. ■

2. Inactive-market equilibrium

Suppose again that the interest rate at the discount window is $R^T \in (1, r^D)$ and the central bank offers loans of size l_0. This discount window policy is *the same* as the one in place in the equilibrium described in Proposition 2. Interestingly, there is another equilibrium consistent with that policy, where all firms that borrow and invest get their funding from the discount window and the market for private loans is inactive.

Proposition. Equilibrium with an inactive private market. When the discount window offers loans of size l_0 at interest rate $R^T \in (1, r^D)$, there is an equilibrium where: **(1)** $m^*(\theta) = l_0$ for all $\theta \leq \theta^T$ and zero otherwise and $l^*(\theta) = 0$ for all $\theta \in \Theta$; **(2)** $i^*(\theta) = 1$ for all $\theta \leq \theta^T$ and zero otherwise; **(3)** the private market for loans is inactive.

Proof. Suppose firms borrowing in equilibrium only borrow from the discount window. We verify this is the case later in the proof. Since $R^T > 1$, firms only borrow from the discount window if they plan to invest. A firm planning to invest, then, borrows l_0 from the discount window at rate R^T. A firm would invest if and only if:

$$E[v] - \rho(\theta, R^T l_0) \geq c_0 = x - l_0.$$

Hence, given the definition of θ^T in equation (9) and the fact that $\rho(\theta, R^T l)$ is increasing in θ, we have that $i^*(\theta) = 1$ for all $\theta \leq \theta^T$ and zero otherwise.

It remains to verify that no firm would want to borrow from the private market. Assume that off-equilibrium beliefs are such that $B(\underline{\theta} \mid l, m) = 1$ for all $l > 0$ and all m. There are two cases to consider. First, if a firm borrows l_0 from the discount window and some extra funds l from the market, then the firm will be able to pay back the private loan with probability one and the break-even interest rate is equal to one. The firm, then, is indifferent between playing the equilibrium strategy or deviating to this alternative. The second case is when the firm does not borrow from the discount window and instead takes a loan of size $l \geq l_0$ from the market. Following similar steps as in the proof of Proposition 1 we can show that the firm would be indifferent between taking a loan of size $l > l_0$ at rate $\underline{r}_l$ and a loan of size l_0 at rate $\underline{r}_0$. Since $\underline{r}_0 > R^T$, we have that the firm would prefer to take a loan of size l_0 at rate R^T from the discount window. ∎

As before, there are other off-equilibrium beliefs consistent with this same equilibrium and, hence, it is not essential to have investors believing that any firm asking for a loan in the market has legacy assets of the lowest type. Even if investors believe that a firm borrowing from the market is a random draw from the relevant set of firms, the equilibrium configuration in the proposition is still an equilibrium. In this case, the "relevant set of firms" is those that would find the strategy of borrowing from the market and investing more attractive than not borrowing and not investing.

To understand this claim note that the break-even condition implies that the net interest cost for the firm of borrowing and investing is the same regardless of whether the firm borrows from the market $l > l_0$ or exactly l_0. From equations (7) and (8) in the paper, the relevant firms are those for which $\theta \leq \theta^D$ and the borrowing cost is $r^* l_0 = r^D l_0$. Given that $r^D > R^T$, firms will prefer to borrow from the discount window rather than from the market, which confirms the equilibrium of the proposition, where the private market is inactive.

Note, finally, that this equilibrium cannot be refined away using the intuitive criterion: if a firm deviates and borrows from the private market claiming to be a high-θ type and investors believe it, hence lowering the interest rate, then all other firms with lower values of θ would have similar incentives to deviate. This logic undermines the power of the intuitive criterion more generally in this model.

3. Limited market access

Consider an extension of the model where each firm can access the market for private funds at time 1 only with some probability $\sigma < 1$. With probability $1 - \sigma$ the firm can only obtain funding by borrowing from the discount window —if it chooses to do so. At time 0 each firm finds out if it will have access to the market at time 1.

Given a discount window policy (m, R), if $m < l_0$ then firms with no access to the market will not borrow at the discount window and the discount window could not serve as the backup source of funding that we are interested in considering here. So, assume that $m = l_0$.

If the discount rate is lower than the (expected) market interest rate then no firm borrows in the market and it makes no difference for firms to have or not have access to that market. On the other hand, if the discount rate is higher than the market interest rate, then, conditional on having access to the market, all firms with $\theta \leq \theta^*$ borrow from the market and θ^* solves equation (7) in the paper.

To determine the market interest rate, note that:

$$P(\theta \mid borrowing) = \begin{cases} \frac{\sigma P(\theta)}{P(borrowing)} & \text{for } \theta \leq \theta^* \\ 0 & \text{otherwise,} \end{cases}$$

where $P(borrowing) = \sigma P(\theta \leq \theta^*)$. This implies that the conditional probability over types participating in the market is independent of the probability of having access to the market (σ). Then, we have that r^* solves equation (8) in the paper.

Firms with no access to the market borrow from the discount window as long as $\theta \leq \theta^R$ where θ^R solves:

$$l_0 - x + E[v] - \rho(\theta^R, Rl_0) = 0,$$

which is equivalent to equation (9) in the paper.

If the discount rate is equal to the market interest rate, then firms that have access to the market are indifferent between borrowing in the market or at the discount window and we can construct an equilibrium like the one in Proposition (2). In particular, suppose that $R = R^T$ and consider θ^T and θ^P, the solutions to equations (9) and (10). Here, again, the conditional probabilities are independent of the probability of having access to the market. Then, all firms with $\theta \leq \theta^P$ borrow from the discount window regardless of their access to private liquidity and those firms with $\theta \in [\theta^P, \theta^T]$ borrow from the private market if they have access to it, and from the discount window if they do not.