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Currency Quality and Changes in the Behavior of Depository Institutions

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The Federal Reserve System distributes currency to and accepts deposits from Depository Institutions (DIs). In addition, the Federal Reserve maintains the quality level of currency in circulation by inspecting all deposited notes. Notes that meet minimum quality requirements (fit notes) are bundled to be reentered into circulation while old and damaged notes are destroyed (shredded) and replaced by newly printed notes.

Between July 2006 and July 2007, the Federal Reserve implemented a Currency Recirculation Policy for $10 and $20 notes. Under the new policy, Reserve Banks will generally charge DIs a fee on the value of deposits that are subsequently withdrawn by DIs within the same week. In addition, under certain conditions the policy allows DIs to treat currency in their own vaults as reserves with the Fed. It is reasonable to expect that the policy change will result in DIs depositing a smaller fraction of notes with the Fed. While the policy is aimed at decreasing the costs to society of currency provision, it may also lead to deterioration of the quality of notes in circulation since notes that are deposited less often are inspected less often.

This article analyzes the interaction between deposit behavior of DIs and the shred decision of the Fed in determining the quality distribution of currency. For a given decrease in the rate of DIs' note deposits with the Fed, absent any change in the Fed's shred decision, what effect would there be on the quality

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distribution of currency in circulation? What kind of changes in the shred criteria would restore the original quality distribution?

To answer these questions, we use the model developed by Lacker and Wolman (1997).[1] In the model, the evolution of the currency quality distribution over time is governed by (i) a quality transition matrix that describes the probabilistic deterioration of notes from one period to the next, (ii) DIs' deposit probabilities for notes at each quality level, (iii) the Fed's shred decision for notes at each quality level, (iv) the quality distribution of new notes, and (v) the growth rate of currency.

We estimate three versions of the model for both \$5 and \$10 notes. We have not estimated the model for \$20 notes because they were redesigned recently, and the new notes were introduced in October 2003. The transition from old to new notes makes our estimation procedure impractical; we discuss this further in the Conclusion.[2] Although the policy affects \$10 and \$20 notes only, we also estimate the model for \$5 notes because the policy change initially proposed in 2003 included \$5 notes. (It is possible that at some point the recirculation policy might be expanded to cover that denomination.) Also, it is likely that the reduced deposits of \$10 and \$20 notes may induce DIs to change the frequency of transporting notes to the Fed and, hence, affect the deposit rate of other denominations. The model predicts roughly comparable results for both denominations.

In each version of our model, we choose parameters so that the model approximates the age and quality distributions of U.S. currency deposited at the Fed. For each estimated model, we describe the deterioration of currency quality following decreases in DI deposit rates of 20 and 40 percent, and we provide examples of Fed policy changes that would counteract that deterioration. As described in more detail below, we view a 40 percent decrease in deposit rates as an upper bound on the change induced by the recirculation policy.

According to the model(s), a 20 percent decrease in the DI deposit rate would eventually result in an increase in the number of poor quality (unfit) notes of between 0.8 and 2.5 percentage points. While this range corresponds to different specifications of the model, not to a statistical confidence interval, it should be interpreted as indicating the range of uncertainty about our results. For \$10 notes, very small changes in shred policy succeed in preventing a significant increase in the fraction of unfit notes.[3] Slightly larger changes in

[1] The Appendix to Lacker (1993) contains a simpler model of currency quality that shares some basic features with the model here.

[2] New \$10 notes were introduced in March 2006 and new \$5 notes are expected to be introduced in 2008; our data were collected in 2004 and early 2006.

[3] We view "fit notes" as referring to any notes that meet a fixed quality standard determined by the Federal Reserve. Prior to a decrease in deposit rates, a fit note is synonymous with a note that meets the Fed's quality threshold for not shredding. If the Fed adjusts its shred policy in

shred policy are required to keep the fraction of unfit $5 notes from increasing in response to a 20 percent lower deposit rate. Naturally, a 40 percent decrease in deposit rates would cause a larger increase in the number of unfit notes, although the greatest increase we find is still less than 6 percentage points. And even in that case there are straightforward changes in shred policy that would be effective in restoring the level of currency quality.

1. INSTITUTIONAL BACKGROUND

Federal Reserve Banks issue new and fit used notes to DIs and destroy previously circulated notes of poor quality. In order to maintain the quality level of currency in circulation, the Fed uses machines to inspect currency notes deposited by DIs at Federal Reserve currency processing offices. These machines inspect each note to confirm its denomination and authenticity, and measure its quality level on many dimensions. The dimensions that are measured include soil level, tears, graffiti or marks, and length and width of the currency notes. Fit notes are those that pass the threshold quality level on all dimensions. Once sorted, the fit notes are bundled and then recirculated when DIs request currency from the Reserve Banks. To replace destroyed notes and accommodate growth in currency demand, the Federal Reserve orders new notes from the Bureau of Engraving and Printing (B.E.P.) of the U.S. Department of Treasury. The Fed purchases the notes from B.E.P. at the cost of production.[4] In 2006, the Federal Reserve ordered 8.5 billion new notes from the B.E.P., at a cost of $471.2 million (Board of Governors of the Federal Reserve System 2006a)—approximately 5.5 cents per note.

In 2006, the Federal Reserve took in deposits of 38 billion notes, paid out 39 billion notes, and destroyed 7 billion notes (Federal Reserve Bank of San Francisco 2006). Of the 19.9 million pounds of notes destroyed every year, approximately 48 percent are $1 notes, which have a life expectancy of about 21 months. The $5, $10, and $20 denominations last roughly 16, 18, and 24 months, respectively (Bureau of Engraving and Printing 2007). Each day of 2005, the Federal Reserve's largest cash operation, in East Rutherford, New Jersey, destroyed approximately 5.2 million notes, worth $95 million (Federal Reserve Bank of New York 2006).

response to a decrease in deposit rates, then it will shred some notes that were fit according to this fixed standard.

[4] Thus, seigniorage for notes accrues initially to the Federal Reserve. In contrast, the Fed purchases coins from the U.S. Mint (a part of the Department of Treasury) at face value, so that seigniorage for coins accrues directly to the Treasury.

Costs and Benefits of Currency Processing and Currency Quality

The Federal Reserve's operating costs for currency processing in 2006 were $319 million (Federal Reserve Bank of San Francisco 2006). DIs benefit from the Fed's currency processing services in at least two ways. First, the Federal Reserve ships out only fit currency, whereas DIs accumulate a mixture of fit and unfit currency; to the extent that DIs' customers—and their ATMs—demand fit currency, DIs benefit from the Fed's sorting of currency. Second, while DIs need to hold currency to meet their customers' withdrawals, they also incur costs by holding inventories of currency in their vaults. Currency inventories take up valuable space and require expenditures on security systems; in addition, currency in the vault is "idle," whereas currency deposited with the Fed is eligible to be lent out in the federal funds market at a positive nominal interest rate. Thus, the Fed's currency processing services amount to an inventory management service for DIs. The benefits DIs accrue from currency processing may not coincide exactly with the benefits to society. On one hand, positive nominal interest rates make the inventory-management benefit to DIs of currency processing exceed the social benefit (Friedman 1969). On the other hand, the social benefits of improved currency quality may exceed the quality benefits that accrue to DIs: for example, maintaining high currency quality may deter counterfeiting by making counterfeit notes easier to detect (Klein, Gadbois, and Christie 2004). On net, it seems unlikely that the social benefit of currency processing greatly (if at all) exceeds the private benefit. This implies that it would be optimal for DIs to face some positive price for currency processing. Lacker (1993) discusses in detail the policy question of whether the Federal Reserve should subsidize DIs' use of currency.

Historically, the Federal Reserve did not charge DIs for processing currency deposits and withdrawals.[5] Policy did prohibit a DI's office from cross-shipping currency; cross-shipping is defined as depositing fit currency with the Fed and withdrawing currency from the Fed within the same five-day period. However, as explained in the Federal Reserve Board's request for comments that introduced the proposed recirculation policy (Board of Governors of the Federal Reserve System 2003a), the restriction on cross-shipping was not practical to enforce. Thus, overall the Federal Reserve cash services policy clearly subsidized DIs' use of currency.

Policy Revision

By 2003, the Federal Reserve had come to view existing policy as leading DIs to overuse the Fed's currency processing services (Board of Governors of the

[5] Note, however, that DIs do pay for transporting currency between their own offices and Federal Reserve offices.

Federal Reserve System 2003b). Factors contributing to this situation included an increase in the number of ATM machines and a decrease in the magnitude of required reserves. The former likely increased the value of the Fed's sorting services, and the latter meant that for a given flow of currency deposits and withdrawals by the DIs' customers, there would be greater demand by DIs to transform vault cash into reserves with the Fed—which requires utilizing the Fed's processing services. In October 2003, the Federal Reserve proposed and requested comments on changes to its cash services policy, aimed at reducing DIs' overuse of the Fed's processing services (Board of Governors of the Federal Reserve System 2003a). In March of 2006, a modified version of the proposal was adopted as the Currency Recirculation Policy (Board of Governors of the Federal Reserve System 2006b).

The Recirculation Policy has two components, both of which cover only \$10 and \$20 denominations. The first component is a custodial inventory program. This program enables qualified DIs to hold currency at the DI's secured facility while transferring it to the Reserve Bank's ledger—thus making the funds available for lending to other institutions but avoiding both the transportation cost and the Fed's processing cost. DIs must apply to be in the custodial inventory program. One criterion for qualifying is that a DI must demonstrate that it can recirculate a minimum of 200 bundles (of 1,000 notes each) of \$10 and \$20 notes per week in the Reserve Bank zone. The policy's second component is a fee of approximately \$5 per bundle of cross-shipped currency. While this new policy is aimed at reducing the social costs incurred because of cross-shipping currency, absent changes in shred policy it is likely to lower the quality of currency in circulation through reduced deposits and thus reduced shredding of unfit currency.[6] The primary concerns of our study are the effect on currency quality of the anticipated decrease in deposit rates, and the measures the Fed can take to offset that decrease in quality. To address these issues we construct a model of currency quality. We assume that shredding policy is aimed at restoring or maintaining the original quality distribution. If the cost of maintaining quality at current levels exceeds the social benefits of doing so, it would be optimal to let the quality of currency deteriorate somewhat.

2. THE MODEL

The model applies to one denomination of currency.[7] Time is discrete, and a time period should be thought of as a month. For the purposes of this

[6] Federal Reserve Banks have estimated that over 10 years, the recirculation policy could reduce their currency processing costs by a present value of \$250 million. Taking into account increased DI costs, the corresponding societal benefit is estimated at \$140 million (Board of Governors of the Federal Reserve System 2006b).

[7] By changing the parameters appropriately, it can be applied *separately* to more than one denomination; indeed we will do just that.

study, there are three major dimensions to currency quality: soil level front (we will use the shorthand "soil level" or SLF), ink wear worst front ("ink wear" or IWWF), and graffiti worst front ("graffiti" or GWF). There are also at least 18 minor dimensions to currency: soil level back, graffiti total front, etc. For a given denomination, we have separate models for each major dimension.[8] Those models describe, for example, how the distribution over soil level evolves over time. For each of those models, however, we use data on the other dimensions to more accurately describe the probability that a note of a particular major-dimension quality level will be shredded.[9]

The basic structure of the model is as follows. At the beginning of each period, banks deposit currency with the Fed; their deposit decision may be a function of quality in the major dimension (that is, banks may sort for fitness). The Fed processes deposited notes, shredding those deemed unfit and recirculating the rest at the end of the period. The shred decision is based on quality level in whatever major dimension the model is specified. However, notes that are fit according to their quality level in the major dimension are nonetheless shredded with positive probability; this is to account for the fact that they may be unfit along one of the other (major or minor) dimensions in which the model is not specified. The stock of currency is assumed to grow at a constant rate. Banks make withdrawals from the Fed at the end of the period but these are not specified explicitly; instead, withdrawals can be thought of as a residual that more than offsets deposits in order to make the quantity of currency grow at the specified rate. In order to accommodate growth in currency and replace shredded notes, the Fed must introduce newly printed notes. Meanwhile, the notes that were not deposited with the Fed deteriorate in quality stochastically. The quality of notes in circulation at the end of a period, and thus at the beginning of the next period, is determined by the quality of notes that have remained in circulation and the quality of notes withdrawn from the Fed.

Formal Specification of the Model

Time is indexed by a subscript $t = 0, 1, 2, ...$ Soil level can take on values $0, 1, 2, ..., n_s - 1$; ink wear can take on values $0, 1, 2, ..., n_i - 1$; and graffiti can take on values $0, 1, 2, ..., n_g - 1$; in general, larger numbers denote poorer quality.[10] We will use q to denote a particular (arbitrary) quality level.

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8 The models for the three major dimensions are truly separate, in that they will yield different predictions.

9 As mentioned earlier, the model was first developed in Lacker and Wolman (1997). That article studied a different policy question, namely expanding the dimensions of quality measurements to include limpness.

10 The exception is soil level zero, which is assumed to describe currency that has been laundered (i.e., has been through a washing machine) and is deemed unfit.

For the DIs' deposit decision, the vector ρ contains in its q^{th} element the probability that a DI will deposit a note conditional on that note being of quality level q. The vector ρ has length Q, where $Q = n_s$ or n_i or n_g, depending on the particular model in question. For the Fed's fitness criteria, the $Qx1$ vector α contains in its q^{th} element the probability that a deposited note of quality q is put back into circulation. If the model were specified in terms of every quality characteristic—so that Q were a huge number describing every possible combination of "soil level front," "soil level back," etc.—then the elements of α would each be zero or one and they would be known parameters, taken from the machine settings. Because the model is specified in terms of only one characteristic, the elements of α that would be one according to q are adjusted downward to account for the fact that some quality-q notes are unfit according to other dimensions of quality. The values of α must then be estimated, and we describe in Section 4 how they are estimated.

The net growth rate of the quantity of currency is γ; that is, if the quantity of currency is M in period t, then it is $(1 + \gamma) M$ in period $(t + 1)$. The $Qx1$ vector g describes the distribution of new notes; its q^{th} element is the probability that a newly printed note is of quality q.[11] The deterioration of non-deposited notes is described by the QxQ matrix π; the row-r column-c element of π is the probability that a non-deposited note will become quality r next period, conditional on it being quality-c this period.[12] Note that each column of π sums to one, because any column q contains the probabilities of all possible transitions from quality level q.

The model's endogenous variables are the numbers of notes of different quality levels, i.e., the quality distribution of currency. At the beginning of period t, the $Qx1$ vector m_t contains in its q^{th} element the number of notes in circulation of quality q. The total number of notes in circulation is $M_t = \sum_{q=1}^{Q} m_{q,t}$, where $m_{q,t}$ denotes the q^{th} element of the vector m_t.

[11] We allow for new notes to have some variation in quality. However, by choosing g appropriately we can impose the highest quality level for all new notes.

[12] We assume that the number of notes is sufficiently large that the probability that a quality c note makes a transition to quality r is the same as the fraction of type c notes that make the transition to type r. That is, the law of large numbers applies.

Combining these ingredients, the number of notes at each quality level evolves as follows:

$$\begin{aligned} \underset{Qx1}{m_{t+1}} = & \underset{QxQ}{\pi} \cdot \left((1-\underset{Qx1}{\rho}) \odot \underset{Qx1}{m_t} \right) \\ & + \underset{Qx1}{\alpha} \odot \underset{Qx1}{\rho} \odot \underset{Qx1}{m_t} \\ & + \underset{1x1}{\left(\sum_{q=1}^{Q} (1-\alpha_q)\rho_q m_{q,t} \right)} \underset{Qx1}{g} \\ & + \underset{1x1}{(\gamma M_t)} \underset{Qx1}{g} . \end{aligned} \tag{1}$$

The symbol $\odot$ denotes element-by-element multiplication of vectors or matrices.[13]

Equation (1) is the model, although we will rewrite it in terms of fractions of notes instead of numbers of notes. On the left-hand side, m_{t+1} contains the number of notes at each quality level at the beginning of period $t + 1$. The right-hand side describes how m_{t+1} is determined from the interaction of m_t (the number of notes at each quality level at the beginning of period t) with the model's parameters. The first term on the right-hand side is

$$\underset{QxQ}{\pi} \cdot \left((1-\underset{Qx1}{\rho}) \odot \underset{Qx1}{m_t} \right) . \tag{2}$$

This term accounts for the fractions $(1 - \rho)$ of notes at each quality level that are not deposited. These notes deteriorate according to the matrix π, and thus the first term is a $Qx1$ vector containing in its q^{th} element the number of circulating notes that were not deposited in period t and that begin period $t+1$ with quality q. If banks were to sort for fitness, then the notes that remain in circulation and deteriorate during the period would be relatively high quality notes, otherwise they would be a random sample of notes. The matrix π has Q^2 elements; assigning numbers to those elements will be the key difficulty we face in choosing parameters for the model.

The second term is

$$\underset{Qx1}{\alpha} \odot \underset{Qx1}{\rho} \odot \underset{Qx1}{m_t} . \tag{3}$$

This term accounts for the fractions $\alpha \odot \rho$ of notes at each quality level that are deposited and not shredded—that is, $\alpha \odot \rho \odot m_t$ comprises the deposited notes at each quality level that are fit and will be put back into circulation at the end of period t. If banks were to sort for fitness in a manner consistent

[13] For example, if $a = [1, 2]$ and $b = [3, 4]$, then $a \odot b = [3, 8]$.

with the Fed's fitness definitions, and if banks possessed enough unfit notes to meet their deposit needs, then this term would disappear—all deposited notes would be shredded.

The third term, $\left(\sum_{q=1}^{Q}(1-\alpha_q)\rho_q m_{q,t}\right)\underset{Qx1}{g}$, represents replacement of shredded notes. The object in parentheses is the number of unfit notes that are processed (and shredded) each period. Multiplying by the distribution of new notes g gives the vector of new notes at each quality level that are added to circulation at the end of period t to replace shredded notes.

The fourth term, $\underset{1x1}{(\gamma M_t)}\ \underset{Qx1}{g}$, represents growth in the quantity of currency. The number of new notes added to circulation to accommodate growth (as opposed to shredding) is γM_t, and the distribution of new notes is g, so this term is a vector containing the numbers of new notes at each quality level added to circulation at the end of period t to accommodate growth.

We noted above that withdrawals are not treated explicitly in the model. The quantity of withdrawals can, however, be calculated. The number of notes withdrawn in period t must be equal to the sum of deposits and currency growth. That is, withdrawals equal

$$\left(\sum_{q=1}^{Q}\rho_q m_{q,t}\right)+\gamma M_t. \tag{4}$$

Note that the model does not incorporate currency inventories at the Fed. New notes materialize as needed, and fit notes deposited at the Fed are recirculated at the end of the period.

The evolution of currency quality over time is determined entirely by equation (1). Given a vector m_t describing the distribution of currency quality at the beginning of any period t, equation (1) determines the vector m_{t+1} describing the distribution of currency quality at the beginning of period $t+1$. The law of motion is determined by the parameters π, ρ, g, γ, and α.[14]

The Model in Terms of Fractions of Notes

The model has been expressed in terms of the numbers of notes at each quality level. To express the model in terms of fractions of notes at each quality level, we first define f_t to be the vector of fractions, that is the $Qx1$ vector of numbers of notes at each quality level divided by the total number of notes:

$$f_t \equiv \left(\frac{1}{M_t}\right)\cdot m_t. \tag{5}$$

[14] We have written the model as if all parameters are constant over time. We maintain that assumption for the quantitative results described in this report. The model remains valid if the parameters change over time, although estimation becomes more challenging.

Likewise, the fraction of notes at a particular quality level is

$$f_{q,t} \equiv \left(\frac{1}{M_t}\right) \cdot m_{q,t}. \tag{6}$$

Note that the elements of f_t sum to one, because $M_t = \sum_{q=1}^{Q} m_{q,t}$. Using these definitions, we can rewrite the model (1) by dividing both sides by M_t and recalling that $M_{t+1} = (1+\gamma)\, M_t$:

$$\begin{aligned}(1+\gamma)\, f_{t+1} &= \pi \cdot ((1-\rho) \odot f_t) \\ &+ \alpha \odot \rho \odot f_t \\ &+ \left(\textstyle\sum_{q=1}^{Q} (1-\alpha_q)\rho_q f_{q,t}\right) g \\ &+ \gamma g.\end{aligned} \tag{7}$$

With this formulation it will be straightforward to study the model's steady state with currency growth.

The Steady-State Distribution of Notes

Under certain conditions, the distribution of currency quality converges to a steady state with the distribution f_t, which is constant over time (see, for example, Stokely, Lucas, with Prescott, chap.11). Assuming that a unique steady-state distribution exists, we will denote it by f^*. In the steady state, the law of motion (7) becomes

$$\begin{aligned}(1+\gamma)\, f^* &= \pi \cdot ((1-\rho) \odot f^*) + \alpha \odot \rho \odot f^* \\ &+ \left(\textstyle\sum_{q=1}^{Q} (1-\alpha_q)\rho_q f_q^*\right) g + \gamma g.\end{aligned} \tag{8}$$

Our method of choosing the model's parameters will require us to compute the steady-state distribution—we will assume that our data are generated in a steady-state situation. One way to compute the steady state is to simply iterate on (7) from some arbitrary initial distribution f_0 and hope that the iterations converge. If they converge, we have found the steady state. Alternatively, we can use matrix algebra to solve directly for the steady state from (8). Ultimately, we want to rewrite (8) in the form

$$\Gamma \cdot f^* = \gamma g, \tag{9}$$

where Γ is a QxQ matrix. If we can rewrite (8) in this way, then the steady-state distribution is $f^* = \Gamma^{-1} \cdot (\gamma g)$. The first step is to note that for any $Qx1$ vector v, we have $v \odot f^* = \text{diag}(v) \cdot f^*$, where $\text{diag}(v)$ denotes the QxQ matrix with the vector v on the diagonal and zeros, elsewhere. Using this fact, we can rewrite (8) as

$$\begin{aligned}(1+\gamma)\, f^* &= \pi \cdot \mathrm{diag}(1-\rho)\cdot f^* + \mathrm{diag}\,(\alpha \odot \rho)\cdot f^* \\ &\quad + \left(\textstyle\sum_{q=1}^{Q}(1-\alpha_q)\rho_q f_q^*\right) g + \gamma g.\end{aligned} \tag{10}$$

Next, note that the scalar $\left(\sum_{q=1}^{Q}(1-\alpha_q)\rho_q f_q^*\right)$ can be rewritten as $((1-\alpha)\odot\rho)'\, f^*$, where " $'$ " denotes transpose. Using this fact, we have

$$\left(\sum_{q=1}^{Q}(1-\alpha_q)\rho_q f_q^*\right) g = \underset{Qx1}{g}\, \underbrace{((1-\alpha)\odot\rho)'\, \underset{Qx1}{f^*}}_{1x1}. \tag{11}$$

Now we can express (8) in the same form as (9), $\Gamma \cdot f^* = \gamma g$, where

$$\Gamma \equiv \left[(1+\gamma)\, I - \pi \cdot \mathrm{diag}(1-\rho) - \mathrm{diag}\,(\alpha\odot\rho) - g\,((1-\alpha)\odot\rho)'\right]^{-1}. \tag{12}$$

Thus, the steady state can be computed directly as

$$f^* = \Gamma^{-1}\cdot(\gamma g)\,.$$

The steady-state distribution f^* contains in its q^{th} element the fraction of notes with quality q, corresponding to a particular measurement of soil level, graffiti or ink wear. Thus, f^* can be thought of as the *marginal distribution* over soil level, graffiti or ink wear. When comparing the model to data, we will use the marginal distributions for each major quality dimension and the distribution of notes by age. We use the age distribution because the quality distribution alone puts few restrictions on the matrix π: we can match a given quality distribution with many π matrices, each implying a different age distribution.

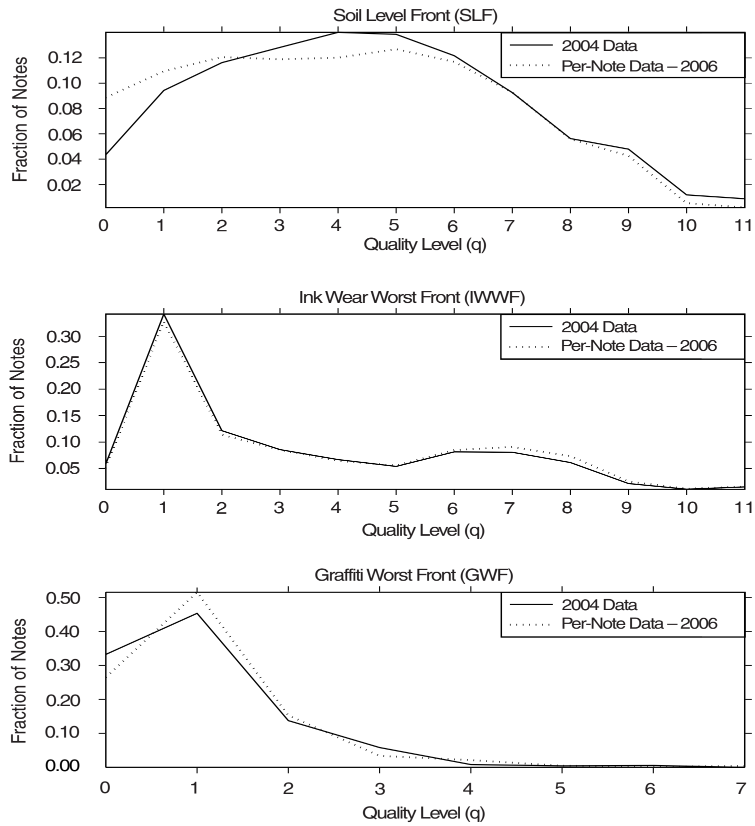
The Appendix contains a detailed description of how to calculate the steady-state age distribution of notes. For now, we simply state the notation: $h_{q,k}$ denotes the fraction of notes that are quality q and age k, and h_k denotes the Q by 1 vector of age k notes, the q^{th} element of which is $h_{q,k}$.

3. THE DATA

The model's predictions will depend on the numerical values we assign to the matrix π describing deterioration of notes, the vector ρ of deposit probabilities, the vector α of shred probabilities, the quality distribution of new notes g, and the currency growth rate γ. This section describes the basic data whose features we attempt to match in choosing the model's parameters.

The ideal data set for our purposes would be one with a time series of observations on a large number of currency notes, with observations each month on the quality of every note. Data of this sort would allow for nearly

Figure 1 Marginal Quality Distributions, $5 Notes

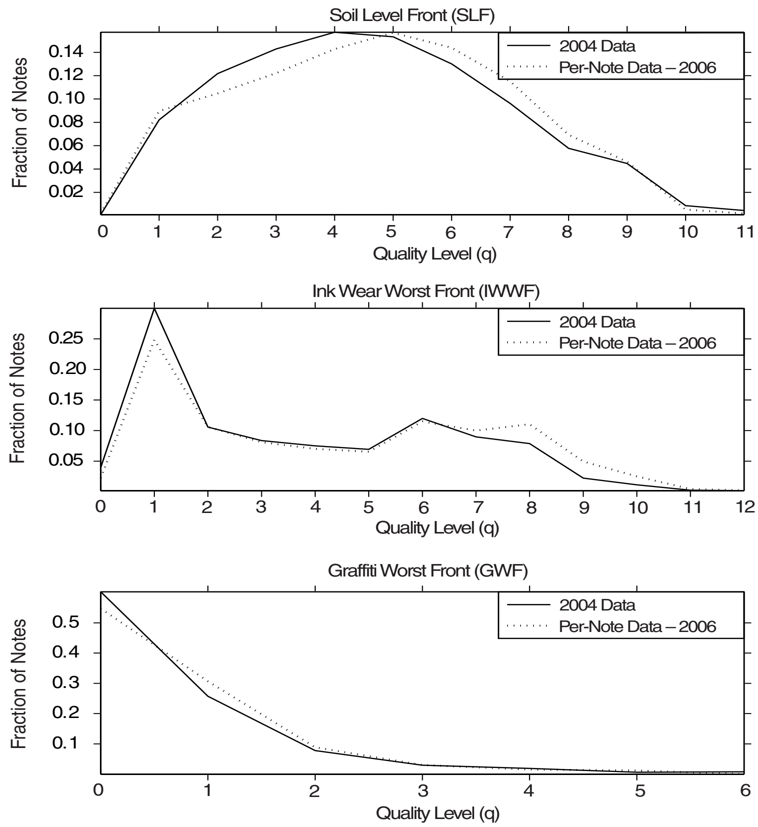


direct measurement of the matrix π. Of course such data does not exist, and probably the only way it could exist would be if individual notes had built-in sensors and transmitters. Without such data, we need to estimate the parameters of π. We use two data sets for this purpose. One data set describes the marginal quality distributions only and has extremely broad coverage. The other data set is at the level of individual notes, and contains age of notes as well as quality. It has more limited coverage.

Large Data Set Describing Marginal Distributions

The large data set comprises fitness data for the entire Federal Reserve System for the months of January 2004 and May 2004, provided by the Currency Technology Office (CTO) at the Federal Reserve Bank of Richmond. This data characterizes the marginal quality distributions for more than two-and-

Figure 2 Marginal Quality Distributions, $10 Notes



a-half billion notes. The data are at the level of office location, date, shift, supervisor, and denomination. For a particular denomination, we assume that summing these data over all dates, supervisors, and shifts generates a precise estimate of the steady-state marginal distribution over each quality level. Figures 1 and 2 plot the marginal distributions over soil level, ink wear and graffiti for the combined January and May 2004 data, for $5 and $10 notes (solid lines).[15]

The raw data have 26 quality levels for each category. However, for many quality levels there are very few notes, and for speed of computation it is advantageous to decrease the number of quality levels. For each denomination

[15] The same data set covers $20 notes, but as described in the Conclusion, our limited analysis of the 20s has not used this data.

and each category (e.g., SLF) we have, therefore, combined multiple quality levels into one. For example, our new soil level zero for the \$10 notes includes all notes with soil levels zero through 2 in the data. Table 1 contains comprehensive information about how we combine quality levels. Boxes around multiple quality levels indicate that we have combined them, and the columns labeled "q" contain the quality level numbers corresponding to our smaller set of quality levels. After combining in this way, we are left with between 7 and 13 quality levels for each denomination and category. For each denomination and each dimension, there are three unfit quality levels. For example, for the \$5 notes SLF, quality levels 9, 10, and 11 are unfit.

Per-Note Data

In addition to the comprehensive data set describing marginal distributions, we use per-note data sets covering approximately 45,000 notes each of \$5 notes and \$10 notes. These data were gathered at nine Federal Reserve offices in February and March 2006. For each note, there is information on the date of issue, as well as quality level in at least 21 categories, including SLF, GWF, and IWWF. The dotted lines in Figures 1 and 2 are the marginal quality distributions for SLF, IWWF, and GWF from the per-note data for the \$5 and \$10 notes. There are minor differences relative to the marginal distributions from the large data set, but the broad patterns are the same. This gives us some confidence that the per-note data are representative samples.

Because the note data contain date of issue for each note, we are able to get an estimate of the age distribution of notes. In Figure 3, the jagged dotted line is a smoothed version of the age distribution of unfit notes from the note data. The smoothing method involves taking a three-month moving average. Without smoothing, the age distributions would be extremely choppy. Note that in Figure 3, we plot the age distribution of *unfit notes*. It is the unfit notes with which we are most concerned for this study, and whose age distribution we care most about matching with the model. Unfit notes are those notes whose quality is worse than the shred threshold in any dimension—major or minor.

4. CHOOSING THE MODEL'S PARAMETERS

There are $Q^2 + 3Q + 1$ parameters in each model; they comprise the Q^2 elements of π, the $3Q$ elements of α, ρ, and g, and the single parameter γ.[16] Since Q is between 7 and 13, the number of parameters is between 71 and 209. We select the model's parameters in several stages.[17]

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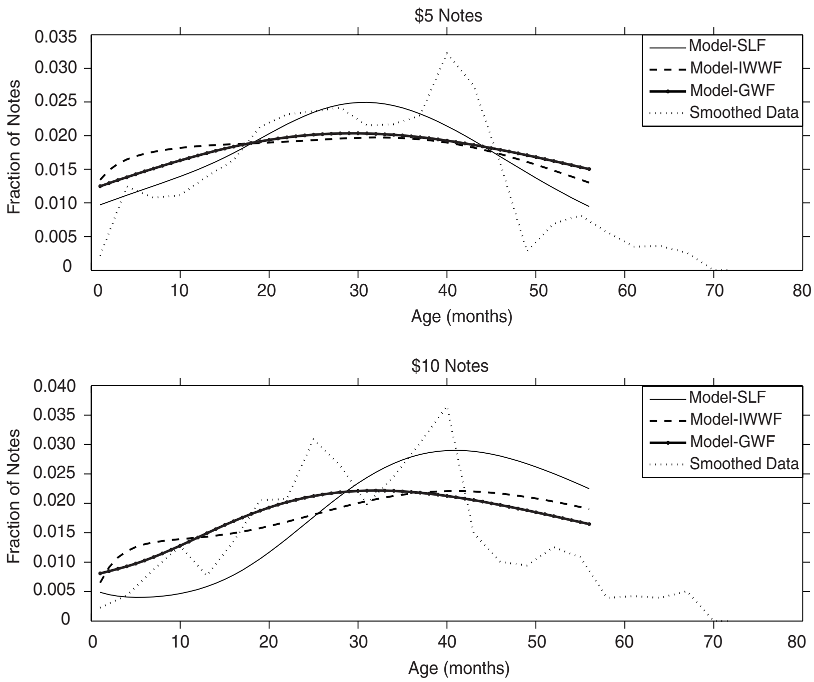
[16] Recall that Q is either n_s, n_i, or n_g, depending on the version of the model.

[17] Because our approach to selecting parameters is ad hoc, we hesitate to talk about "estimating the model." However, in effect that is what we are doing.

Table 1 Marginal Quality Distributions and Combined Quality Levels

$5 Notes								$10 Notes						
Quality Level	SLF	q	IWWF	q	GWF	q		Quality Level	SLF	q	IWWF	q	GWF	q
0	0.000		0.059	0	0.000	0		0	0.000		0.040	0	0.001	0
1	0.000		0.342	1	0.333			1	0.000	0	0.300	1	0.602	
2	0.000	0	0.121	2	0.454	1		2	0.001		0.106	2	0.257	1
3	0.004		0.086	3	0.138	2		3	0.014	1	0.084	3	0.078	2
4	0.039		0.067	4	0.036	3		4	0.068		0.075	4	0.030	3
5	0.094	1	0.054	5	0.015			5	0.122	2	0.069	5	0.012	4
6	0.116	2	0.044	6	0.008	4		6	0.143	3	0.063	6	0.006	
7	0.128	3	0.037		0.005			7	0.157	4	0.056		0.004	5
8	0.140	4	0.031		0.003	5		8	0.153	5	0.049	7	0.003	
9	0.139	5	0.027	7	0.002			9	0.130	6	0.041		0.002	
10	0.122	6	0.023		0.002	6		10	0.096	7	0.033		0.001	
11	0.093	7	0.019		0.001			11	0.058	8	0.026	8	0.001	
12	0.056	8	0.017		0.001			12	0.027		0.019		0.001	
13	0.027		0.014	8	0.001			13	0.012	9	0.013	9	0.001	
14	0.013	9	0.011		0.001			14	0.006		0.009		0.000	
15	0.007		0.009		0.000			15	0.004		0.006		0.000	
16	0.005		0.007	9	0.000			16	0.003	10	0.003	10	0.000	
17	0.004	10	0.005		0.000			17	0.002		0.002		0.000	6
18	0.003		0.004		0.000	7		18	0.002		0.001		0.000	
19	0.002		0.004	10	0.000			19	0.001		0.001	11	0.000	
20	0.002		0.003		0.000			20	0.001		0.000		0.000	
21	0.002		0.002		0.000			21	0.000	11	0.000		0.000	
22	0.001	11	0.002		0.000			22	0.000		0.000		0.000	
23	0.001		0.002	11	0.000			23	0.000		0.000	12	0.000	
24	0.001		0.001		0.000			24	0.000		0.000		0.000	
25	0.000		0.008		0.000			25	0.000		0.001		0.000	

Figure 3 Age Distribution of Unfit Notes



First, we make some *a priori* assumptions on the transition matrix π that decrease the number of free parameters. Next, we pin down g, α, γ, and ρ based on information from the Federal Reserve System's Currency Technology Office, the Federal Reserve Board, and preliminary analysis of the data. We select the remaining parameters so that the model's steady-state distribution matches the quality and age distributions in Figures 1–3.

At this point, it may be useful to remind the reader where we are: we have specified a model of the evolution of currency quality, and we will now use data from the period before implementation of the currency recirculation policy in order to choose parameters of the model. Once the parameters have been chosen, we will simulate the model under particular assumptions about how DI behavior will change in response to the recirculation policy. The recirculation policy itself is "outside the model"; the model does not address pricing of currency processing by the Fed, and the model does not address (intraweek) cross-shipping because it is specified at a monthly frequency.

A Priori Restrictions on π

We reduce the number of parameters determining π by imposing the restriction that notes never improve in quality, except that soil level may "improve" to zero if a note is laundered (i.e., the note has gone through a washing machine). This restriction means that almost half the elements of π are zeros. For the ink wear and graffiti model, all elements above the main diagonal are zero. For the soil level model, the elements above the main diagonal are zero except in the first row, which may contain nonzero elements in every column to account for the possibility of laundered notes; in the first column, the first row contains a one and all other rows contain zeros, because a laundered note always remains laundered. The numbers of nonzero elements in π are thus $\left(\frac{n_s(n_s+1)}{2} + n_s - 1\right)$, $\frac{n_i(n_i+1)}{2}$ and $\frac{n_g(n_g+1)}{2}$ for the three models. The last restriction we impose on π is an inherent feature of the model: the columns of π must sum to one, and π is a stochastic matrix with each element weakly between zero and one. This adds Q restrictions, subtracting an equal number of parameters.

Choosing α, g, ρ, and γ

The Federal Reserve chooses the definition of fit notes, so there would be no difficulty determining α if the model were specified in terms of all quality dimensions simultaneously; α_q would be one for fit notes and zero for unfit notes. However, since we specify the model in terms of only one dimension, we need to adjust the shred parameter α to reflect the fact that notes may be unfit even though they are fit according to the dimension in which the model is specified. For example, if the model is specified in terms of soil level, a note that is very clean may nonetheless be unfit because of its level of ink wear. We adjust for this possibility as follows, using the soil level example: for each *fit* degree of soil level q, calculate the fraction of notes with soil level q that are unfit according to other dimensions and subtract that fraction from α_q. That calculation is necessarily based on the per-note data, as it requires going beyond marginal distributions. The corrections we make to α are shown in Table 2.

The vector g represents the quality distribution of newly printed notes. Our estimates of g are from the Federal Reserve System's Currency Technology Office (unpublished data), and these are presented in Table 3. Sorting behavior by DIs is captured by the vector ρ.

Table 2 Corrections to α Vector

$5 Notes						$10 Notes					
q	**SLF**	**q**	**IWWF**	**q**	**GWF**	**q**	**SLF**	**q**	**IWWF**	**q**	**GWF**
0	0	0	0.0850	0	0.0624	0	0	0	0.0255	0	0.0374
1	0.0375	1	0.1080	1	0.1215	1	0.0215	1	0.0545	1	0.1142
2	0.0640	2	0.1445	2	0.2795	2	0.0106	2	0.0654	2	0.2294
3	0.0867	3	0.1754	3	0.5783	3	0.0120	3	0.0868	3	0.3392
4	0.1150	4	0.1801	4	0	4	0.0272	4	0.0844	4	0
5	0.1417	5	0.2048	5	0	5	0.0389	5	0.0857	5	0
6	0.1852	6	0.2109	6	0	6	0.0622	6	0.0878	6	0
7	0.2546	7	0.2461	7	0	7	0.1132	7	0.1036		
8	0.3658	8	0.2913			8	0.1890	8	0.1251		
9	0	9	0			9	0	9	0.1496		
10	0	10	0			10	0	10	0		
11	0	11	0			11	0	11	0		
								12	0		

Table 3 Quality Distribution of New Notes

$5 Notes						$10 Notes					
q	**SLF**	**q**	**IWWF**	**q**	**GWF**	**q**	**SLF**	**q**	**IWWF**	**q**	**GWF**
0	0	0	1	0	0.935	0	0	0	1	0	1
1	0.010	1	0	1	0.065	1	0.965	1	0	1	0
2	0.695	2	0	2	0	2	0.035	2	0	2	0
3	0.295	3	0	3	0	3	0	3	0	3	0
4	0	4	0	4	0	4	0	4	0	4	0
5	0	5	0	5	0	5	0	5	0	5	0
6	0	6	0	6	0	6	0	6	0	6	0
7	0	7	0	7	0	7	0	7	0		
8	0	8	0			8	0	8	0		
9	0	9	0			9	0	9	0		
10	0	10	0			10	0	10	0		
11	0	11	0			11	0	11	0		
								12	0		

We assume that DIs do not sort, which implies that all elements of ρ are identical and are equal to the fraction of notes that DIs deposit each period.[18] We set each element of ρ to 0.1165 for the $5 notes and 0.1322 for the $10 notes. These numbers are based on data from the Federal Reserve Board (S. Ferrari, pers. comm.). Finally, γ is the growth rate of the stock of currency. We have set the annual growth rate at 1.78 percent for the $5 notes, and 0.38 percent for the $10 notes, again based on data from the Federal Reserve Board (S. Ferrari, pers. comm.).

Matching the Quality and Age Data

We select the remaining parameters of the matrix π—for each specification of the model—so that the model's steady-state distribution matches as closely as possible two features of the data. First, we want to match the marginal quality distribution from the 2004 comprehensive data (Figures 1 and 2, solid line). Second, we want to match the age distribution of unfit notes from the 2006 per-note data (Figure 3). Concretely, we select the parameters of π to minimize a weighted average of (i) the sum of squared deviations between the marginal quality distribution and that predicted by the model, and (ii) the sum of squared deviations between the unfit age distributions and that predicted by

[18] A recent internal Federal Reserve study confirmed that DIs have not been sorting to any appreciable extent, as the quality distribution of currency that the Federal Reserve receives from DIs is close to the quality distribution of currency in circulation (Board of Governors of the Federal Reserve System 2007). However, the recirculation policy—in particular, the fee for cross-shipping *fit* currency—gives DIs an incentive to sort. We address this issue in the Conclusion.

Table 4 π Matrix for \$5 Notes According to GWF

q	0	1	2	3	4	5	6	7
0	0.9469	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.0531	0.9755	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0000	0.0224	0.9647	0.0000	0.0000	0.0000	0.0000	0.0000
3	0.0000	0.0000	0.0353	0.9945	0.0000	0.0000	0.0000	0.0000
4	0.0000	0.0022	0.0000	0.0000	0.8828	0.0000	0.0000	0.0000
5	0.0000	0.0000	0.0000	0.0054	0.1148	0.8294	0.0000	0.0000
6	0.0000	0.0000	0.0000	0.0001	0.0024	0.1706	0.9995	0.0000
7	0.0000	0.0000	0.0000	0.0000	0.0001	0.0000	0.0005	1.0000

Notes: The row r, column c element of this matrix is the probability that a note will become quality r next period, conditional on it being of quality c in this period. For example, the probability that a note will be of quality 4 in the next period, given that it is quality 1 in this period is 0.0022, the element in row 4, column 1.

the model.[19] Table 4 contains one example of the π matrix; it is for the GWF model of \$5 notes.

With respect to the marginal quality distributions, we have no trouble matching the data. In all of the model specifications, we match the marginal quality distributions nearly perfectly. The age distributions are a different matter, which perhaps is not surprising given their choppiness in the data—the model wants to make the age distribution of unfit notes smooth. Figure 3 plots the age distributions implied by each specification of the model, along with the age distributions from the data.[20] With the exception of the SLF model for \$5 notes, the age distributions implied by the model involve too many unfit notes more than approximately four years old.

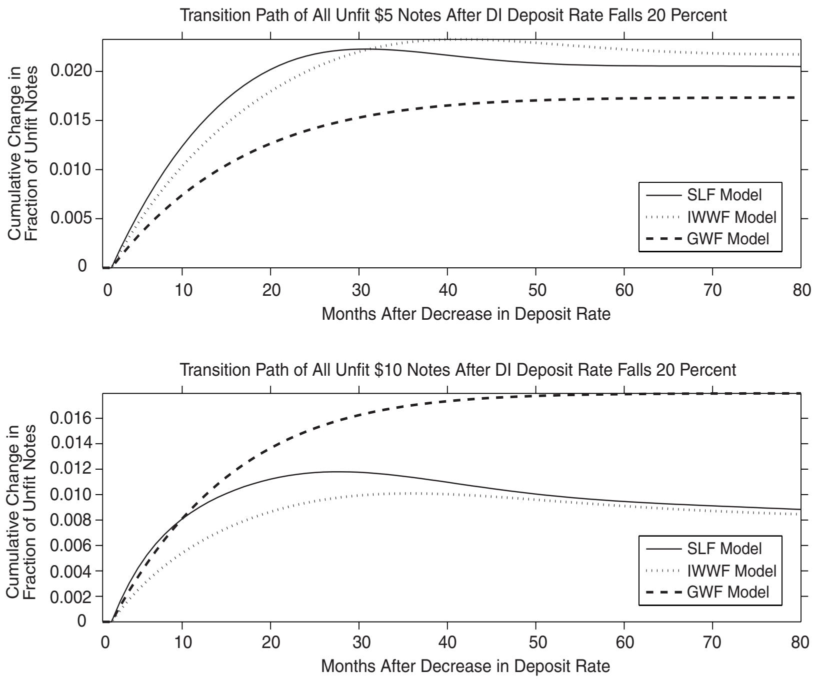
5. SIMULATING A CHANGE IN DI BEHAVIOR

Because the response of the quality distribution to a decrease in deposit rates depends on the transition matrix π, the fact that we have multiple models means that we generate a range of responses to a decrease in deposit rates. Figures 4 and 5 plot the time series for the fraction of unfit notes, in response to 20 and 40 percent decreases in DIs' deposit rates, respectively. According to final Currency Recirculation Policy (Board of Governors of the Federal Reserve System 2006b), of the \$10 and \$20 notes processed by the Fed in

[19] We have also experimented with adding to our estimation criterion the fraction of age k notes that are unfit, for $k = 1, 2, \ldots$ For moderate weights on this component the results are not materially affected.

[20] In Figure 3, the lines associated with the model stop at 56 months because we did not attempt to match the age distribution beyond 56 months.

Figure 4 Response to 20 Percent Decrease in Deposit Rate

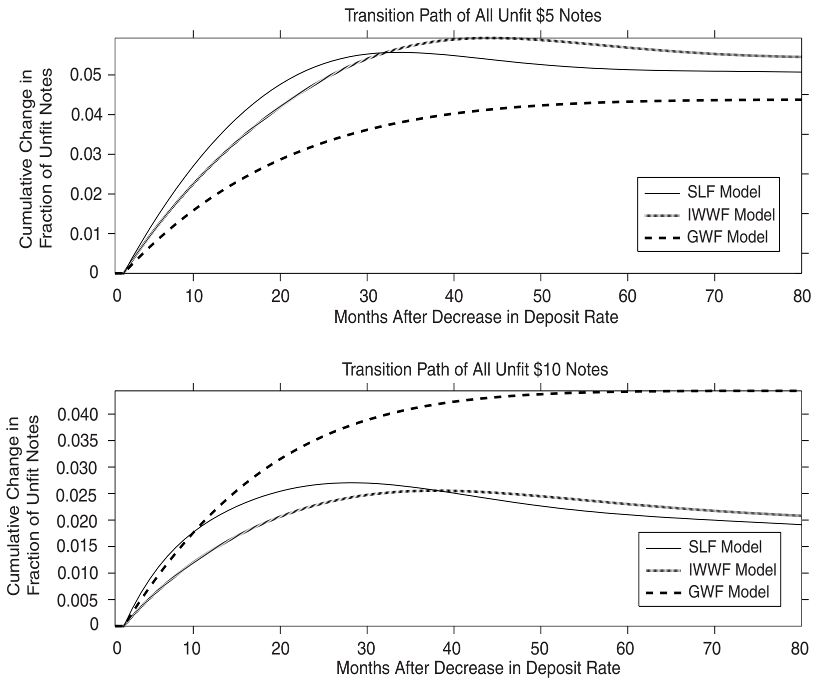


2004, 40.4 percent were cross-shipped. Thus, a 40 percent decrease in deposits corresponds to DIs ceasing entirely to cross-ship. This seems unlikely, so we view the 40 percent number as an upper bound on the effect of the recirculation policy. In addition, cross-shipping is likely more important for $20 notes than $10 notes, because of the necessity of having crisp (fit) $20 notes in ATM machines. Since the DIs always receive fit notes from the Federal Reserve System, a larger volume of $20 notes are cross-shipped than any other denomination.[21] Thus, the 40.4 percent upper bound for $10 notes and $20 notes combined is higher than the upper bound for the $10 notes or $5 notes.

Each line in Figures 4 and 5 represents the transition path for the fraction of unfit notes for a different major dimension model (soil level, ink wear, graffiti). In response to a 20 percent decrease in the deposit rate, the models predict a long-run increase in the *fraction* of unfit notes of between 0.017 and 0.025 for the $5 notes (i.e., around two percentage points), and between 0.008 and 0.018 for the $10 notes. In our large data sets, the total fractions

[21] In 2005, the volume of $5, $10, and $20 notes that were cross-shipped were 12.7 percent, 9.0 percent, and 78.3 percent, respectively.

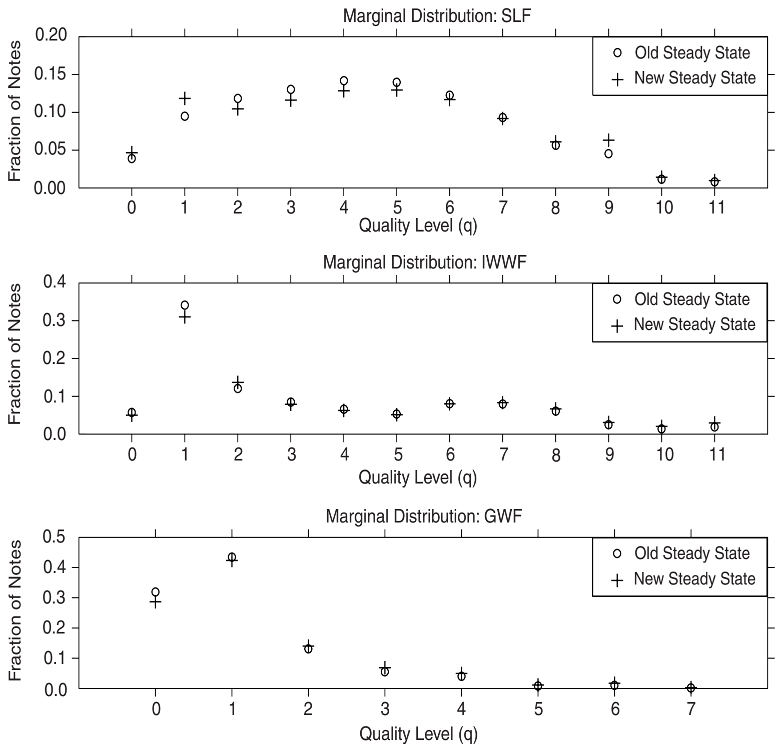
Figure 5 Response to 40 Percent Decrease in Deposit Rate



of unfit notes are 0.173 for the $5 notes and 0.150 for the $10 notes. Note that the model that provides the best fit to the age distribution ($5 SLF) is also the model that predicts the largest increase in the fraction of unfit notes, 0.025. Not surprisingly, a 40 percent decrease in deposit rates generates a larger increase in the fraction of unfit notes—between 0.044 and 0.055 for the $5 notes and between 0.019 and 0.044 for the $10 notes.

Figures 6, 7, 8, and 9 provide a different perspective on the effects of a decrease in deposit rates. These figures plot on the same panel the initial steady-state quality distribution (prior to the drop in deposit rates) and the new steady-state quality distribution corresponding to the lower deposit rate. For the 20 percent experiment (Figures 6 and 7), the long-run effects on quality are generally small, reinforcing the message of Figure 4. There are, however, certain quality levels that are strongly affected. For example, the fraction of $10 notes at soil level 6 (in Figure 7) eventually rises from 0.13 to 0.1832 in response to the 20 percent drop in deposits. For the 40 percent experiment, things look somewhat more dramatic: for example, the fraction of $10 notes at soil level 6 increases from 0.13 to 0.27 (in Figure 9). To put this change in perspective though, Table 2 tells us that only 6.2 percent of the level 6 SLF

Figure 6 Effect of 20 Percent Deposit Rate Decrease on Quality Distributions of $5 Notes



$10 notes are unfit, so the big increase in notes at that level (which is still fit according to SLF) brings with it an increase of less than one percentage point in unfit notes. Recall that the change in total fraction of unfit notes is shown in Figures 4 and 5.

If the Fed wished to offset the quality deterioration caused by a decrease in deposit rates, a natural policy would be to shred notes of higher quality. Table 5 displays scenarios for fraction of notes to shred at each quality level in order to maintain the fraction of unfit notes at its old steady-state level. For example, if deposit rates fall 20 percent, our SLF model for $5 notes implies that shredding all notes in the worst-fit category and shredding 35 percent of notes in the second worst-fit category would counteract the deposit decrease, leaving the fraction of notes unchanged. The columns in this table should be read independently, as they each apply to distinct models. In other words, the column labeled $5 SLF provides a policy change for SLF that is predicted to bring about a stable fraction of unfit notes; no changes are made

Table 5 Policy Response to Offset Effect of Deposit Rate Decrease

20 Percent Decrease in Deposits: Fraction of Notes to Shred

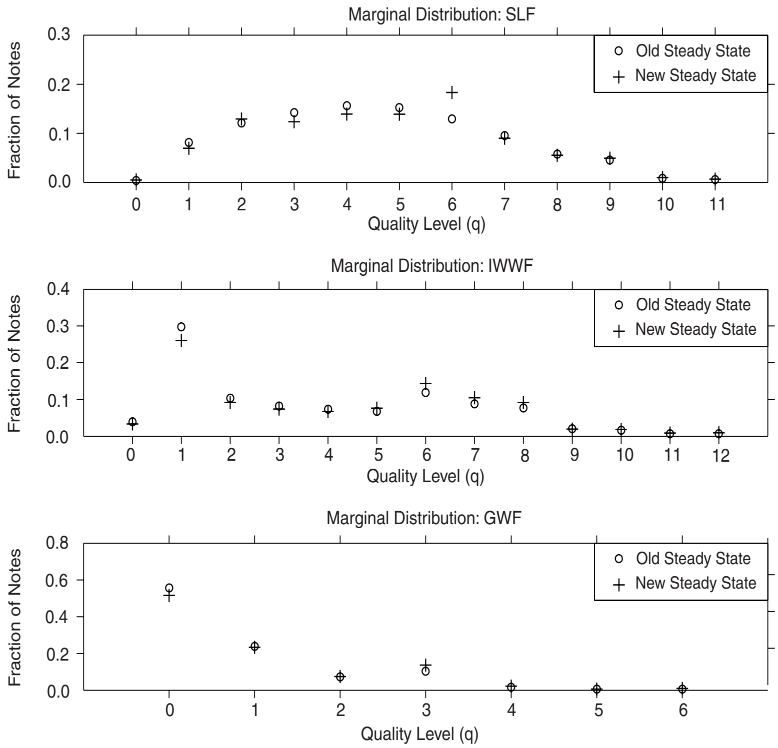
$5 Notes				$10 Notes			
q	**SLF**	**q**	**IWWF**	**q**	**SLF**	**q**	**IWWF**
0	1	0	0	0	1	0	0
1	0	1	0	1	0	1	0
2	0	2	0	2	0	2	0
3	0	3	0	3	0	3	0
4	0	4	0	4	0	4	0
5	0	5	0	5	0	5	0
6	0	6	0	6	0	6	0
7	0.3512	7	0.0255	7	0	7	0
8	1	8	1	8	0.545	8	0
9	1	9	1	9	1	9	0.6845
10	1	10	1	10	1	10	1
11	1	11	1	11	1	11	1
						12	1

40 Percent Decrease in Deposits: Fraction of Notes to Shred

$5 Notes				$10 Notes			
q	**SLF**	**q**	**IWWF**	**q**	**SLF**	**q**	**IWWF**
0	1	0	0	0	1	0	0
1	0	1	0	1	0	1	0
2	0	2	0	2	0	2	0
3	0	3	0	3	0	3	0.22
4	0	4	0	4	0	4	1
5	0.198	5	0	5	0	5	1
6	1	6	0.48	6	0	6	1
7	1	7	1	7	0.125	7	1
8	1	8	1	8	1	8	1
9	1	9	1	9	1	9	1
10	1	10	1	10	1	10	1
11	1	11	1	11	1	11	1
						12	1

to shred thresholds for other dimensions. Note that we have omitted GWF from the analysis in Table 1; we were not successful in finding policies that counteracted the quality decline by changing the shred policy for GWF. In order to counteract the effects of a 40 percent decrease in deposits, Reserve Banks would have to shred currency at significantly higher quality levels, depending on the particular model specification. In the most extreme case, which is the IWWF model for $10 notes, the worst six levels of fit notes would have to be shredded (quality levels four through nine), and 22 percent of notes at quality level 3 would have to be shredded to prevent overall quality from deteriorating. Recall, however, that the 40 percent decrease in deposit rates

Figure 7 Effect of 20 Percent Deposit Rate Decrease on Quality Distributions of $10 Notes

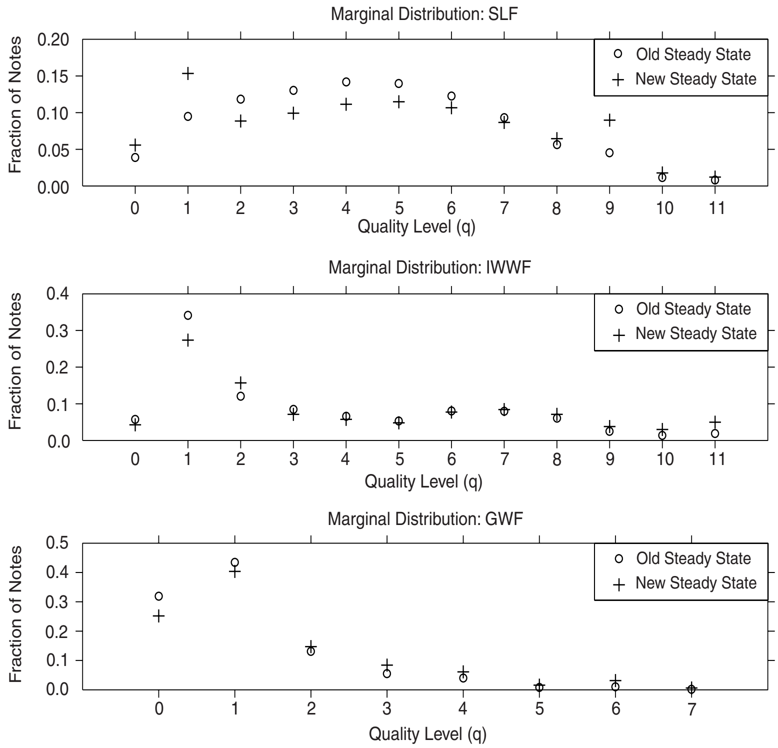


represents an upper bound on how we expect DIs to change their behavior in response to the recirculation policy.

6. CONCLUSION

The quality of currency in circulation is an important policy objective for the Federal Reserve. Changes in the behavior of depository institutions, whether caused by Fed policy or by independent factors, can have implications for the evolution of currency quality. Currently the Fed is implementing a recirculation policy, which is expected to cause changes in the behavior of DIs and, therefore, affect currency quality. The mechanical model of currency quality in this article can be used to study the effects of changes in DI behavior and changes in Fed policy on the quality distribution of currency. In general, the model predicts relatively modest responses of currency quality to decreases in DI deposit rates that are anticipated to occur as a consequence of the recircu-

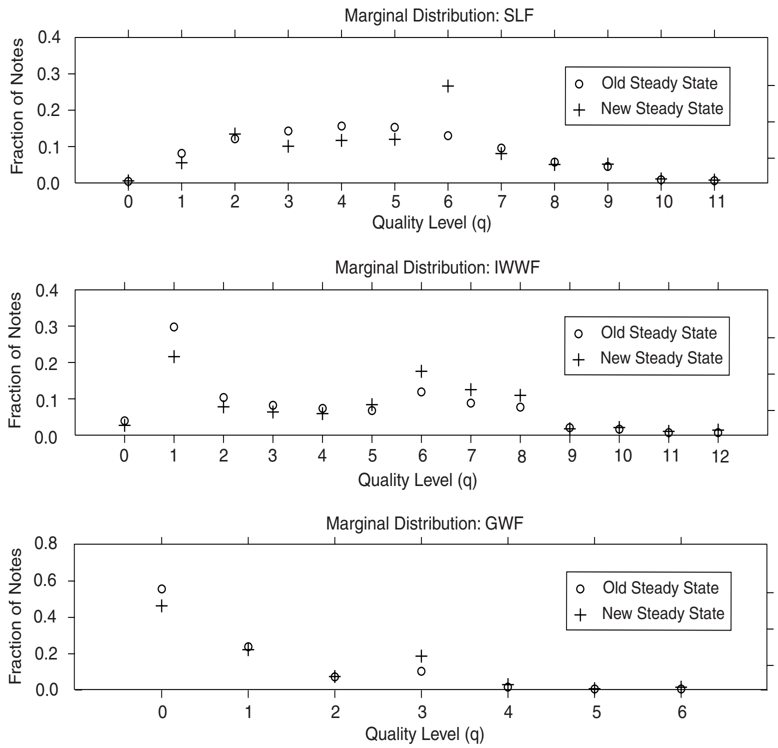
Figure 8 Effect of 40 Percent Deposit Rate Decrease on Quality Distributions of $5 Notes



lation policy. For $5 and $10 notes, our model is able to match the marginal quality distributions perfectly, and the age distributions of unfit notes reasonably well. Thus, we have some confidence in the range of predictions that the different model specifications make for the effects on currency quality of a decrease in deposit rates. In what follows, we discuss potential extensions to the current analysis.

Although our framework allows for sorting by DIs, the quantitative analysis has assumed no sorting occurs. If DIs do sort, then the researcher must take into account that the distribution of currency in circulation is not the same as the distribution of currency that visits the Fed. The derivations in this report do not differentiate between the two distributions, but it is straightforward to do so. If DIs were to sort using the same criteria as the Federal Reserve, then it is likely that the results presented here would overstate the decline in currency quality following implementation of the recirculation policy; by depositing with the Fed only low-quality notes, DIs would offset the deleterious effect of

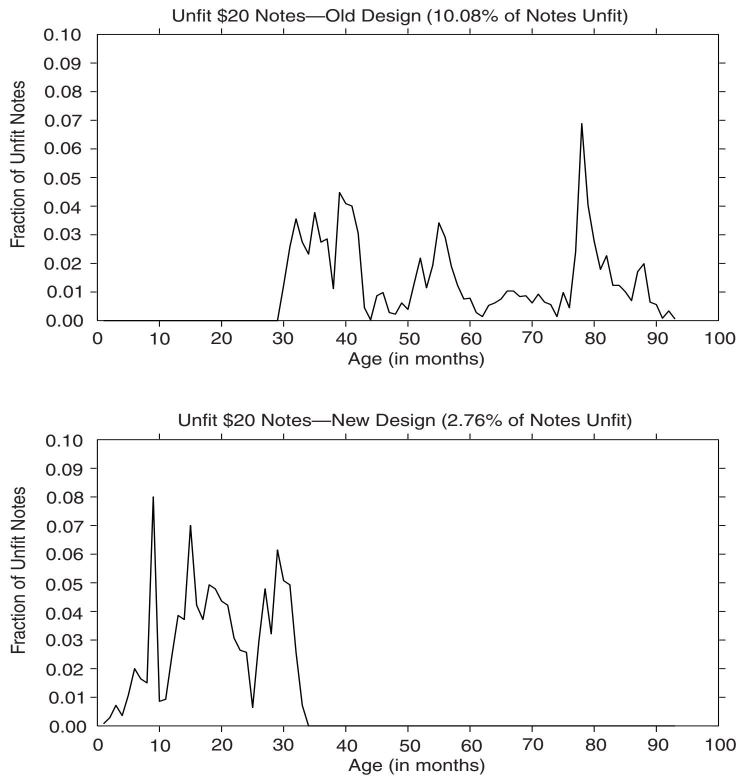
Figure 9 Effect of 40 Percent Deposit Rate Decrease on Quality Distributions of \$10 Notes



depositing fewer notes. The recirculation policy clearly provides an incentive for at least some DIs to sort because it imposes fees for cross-shipment of fit currency only.

Our analysis has not addressed \$20 notes. Figure 10 illustrates the difficulty they present: they are not in a steady state but are transiting from the old to the new design. Of the old notes, more than 10 percent are unfit, whereas of the new notes, less than 3 percent are unfit. All the old notes are more than two years old, whereas all the new notes are less than three years old. Our model is not inherently restricted to steady state situations. To apply it to the 20s, one would want to use the form of the model in (7) and also allow for γ (the growth rate of currency) to be time-varying or at least allow γ to vary across designs. The non-steady-state form of the model (7) also could be useful more generally, in providing a check on our estimates. If there is good data on marginal quality distributions available monthly, then that data can be used to generate forecast errors for the model on a real-time basis.

Figure 10 Age Distributions of Unfit $20 Notes



One reason to question the steady-state assumption is the possibility that the payments system is in the midst of a transition away from the use of currency and toward electronic forms of payment. Although it is difficult to distinguish a change in the trend from a transitory shock, data on the stock of currency does give some credence to this concern: from 2002 to 2007 the growth rate of currency has declined steadily, and at 2 percent for the 12 months ending in June 2007 it is currently growing more slowly than most measures of nominal spending. A decreasing currency growth rate means that there is a decreasing rate of new notes introduced into circulation. This would likely require stronger measures by the Federal Reserve to maintain currency quality in response to a decrease in deposit rates.

The version of the model estimated here is very small and easy to estimate. Expanding the model so that it describes the joint distribution of all three quality dimensions studied here leads to an unmanageably large system. A middle ground that might be worth pursuing would be to specify the model in terms of two dimensions, say graffiti and soil level, and include information about unfitness in other dimensions, as we have done here.

Finally, it would be useful to embed the currency quality model of this article in an economic model of DIs and households. The DIs' deposit rate and sorting policy (both summarized by ρ) would then be endogenously determined. Such a model could be used to predict the effects of a change in the Federal Reserve's pricing policy on DI behavior. It could also be used to conduct welfare analysis of different pricing and shredding policies. The model in Lacker (1993) is a natural starting point.

APPENDIX: DETAILS OF CALCULATING AGE DISTRIBUTION

It is straightforward to compute the age distribution of notes for any quality level and the quality distribution at any age. Begin by defining the fraction of notes at quality level q and age k to be $h_{q,k}$. These fractions satisfy

$$1 = \sum_{k=0}^{\infty} \sum_{q=1}^{Q} h_{q,k}. \tag{13}$$

For convenience, define h_k to be the Q-vector containing in element q, the fraction of notes that are k-periods old, and in quality level q :

$$\underset{Qx1}{h_k} = \begin{bmatrix} h_{1,k} \\ h_{2,k} \\ \vdots \\ h_{Q,k} \end{bmatrix}. \tag{14}$$

We also have that $h_{q,k} = \left(e_q^Q\right)' h_k$, where e_q^Q is a $Qx1$ selection vector with a 1 in the q^{th} element and zeros elsewhere.

The fraction of brand-new notes is

$$\sum_{q=1}^{Q} h_{q,0} = \frac{\gamma}{1+\gamma} + \sum_{j=1}^{N} \left(1-\alpha_j\right) \rho_j f_j^*, \tag{15}$$

and since the quality distribution of new notes is g, the fractions of notes that are new and in each quality level q are

$$h_0 = \left(\frac{\gamma}{1+\gamma} + \sum_{j=1}^{N} \left(1-\alpha_j\right) \rho_j f_j^* \right) \cdot g. \tag{16}$$

For one-period old notes, the fractions are

$$h_1 = \left[\frac{\pi \cdot diag\,(1-\rho) + diag\,(\alpha \odot \rho)}{1+\gamma} \right] \cdot h_0. \tag{17}$$

Likewise, we have

$$h_{k+1} = \left[\frac{\pi \cdot diag\,(1-\rho) + diag\,(\alpha \odot \rho)}{1+\gamma} \right]^{k+1} \cdot h_0, \text{ for } k = 0, 1, ..., \tag{18}$$

with h_0 determined by (16). Thus, we can calculate the fraction of notes at any age-quality combination as

$$h_{q,k} = \left(e_q^Q\right)' \left[\frac{\pi \cdot diag\,(1-\rho) + diag\,(\alpha \odot \rho)}{1+\gamma} \right]^{k} \cdot h_0. \tag{19}$$

The age distribution of quality-q notes is

$$\frac{1}{\sum_{k=0}^{\infty} h_{q,k}} \begin{bmatrix} h_{q,0} \\ h_{q,1} \\ \vdots \\ h_{q,\infty} \end{bmatrix}. \tag{20}$$

and the quality distribution of age-k notes is

$$\frac{1}{\sum_{q=0}^{Q} h_{q,k}} \begin{bmatrix} h_{1,k} \\ h_{2,k} \\ \vdots \\ h_{Q,k} \end{bmatrix}. \tag{21}$$

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