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MEASURING THE DEFAULT RISK OF BONDS USING YIELDS TO MATURITY

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I. INTRODUCTION

In both the theoretical and empirical literature of finance the relative riskiness of two debt instruments identical in all respects save the likelihood of default on payments of principal and/or interest has generally been measured by the difference between the yields to maturity of the two debt instruments.[1] In a recent paper Benson and Rogowski [1] argue that the relative yield spread, defined as the yield spread divided by the less risky (or riskless) yield, is a better measure of default risk, because the value of the expected loss due to default risk should be "greater the higher are interest rates."[2] Unfortunately, those using the yield spread or the relative yield spread as default risk measures have not discussed the relationship between their default risk measure and the way in which investors adjust future promised payments for default risk.[3] The purpose of this paper is to examine this relationship. In Section II the relationship is examined in the context of a simple model where investors are risk-neutral and where debt instruments differ only in the probability of default on future promised payments, and an alternative measure of default risk is proposed. Section III uses the results of Section II to explain previous empirical findings concerning the behavior of yield spreads over time, especially the relationship found between default risk and the level of interest rates.

II. ADJUSTING FUTURE PROMISED PAYMENTS FOR RISK

To simplify the analysis of investors' adjustment of future cash payments for default risk, the following assumptions are made:[4]

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[1] E.g., see Fisher [4], Jaffee [5], Cook and Hendershott [3], and Van Horne [7]

[2] [1], p. 352.

[3] An exception is Cook and Hendershott [3, p. 1180].

[4] These assumptions are not critical, but they do make the exposition easier. For example, if investors were risk-averse, then p_t would incorporate both the expected value of the future payment and some premium demand for risk-bearing. In either case, p_t represents the dollar amount of a certain future payment in t years that investors see as equivalent to a risky future payment in t years of $1.

a) investors are risk-neutral;

b) default is an all-or-nothing event--i.e., if default occurs, there is no payment;

c) the probability of default on a future risky payment in t years is d_t, and the probability of payment is $p_t = 1-d_t$, where $0 < d_t < 1$.

A. The Single Payment Case

The present value of a future risky payment of \$1 in t years is given by

$$(1) \quad V = \frac{1}{(1+r)^t},$$

where r is the yield to maturity. Risk-neutral investors will be indifferent between a risky payment in t years of \$1 and a certain future payment of $\$p_t$. Consequently, the present value of the future risky \$1 payment in t years can also be written

$$(2) \quad V = \frac{p_t}{(1+i)^t},$$

where i is the default-free yield, with $r>i$, because $p_t<1$. Equating (1) and (2),

$$(3) \quad p_t = \left[\frac{(1+i)}{(1+r)}\right]^t .$$

It seems clear that any measure of "default risk" should be dependent only on the probability of payment p_t (or the probability of default d_t). It is easily demonstrated that neither the yield spread $(r-i)$ nor the relative yield spread $(\frac{r-i}{i})$ are such measures. Solving (3) for the yield spread and the relative yield spread, one finds that

$$(4) \quad r-i = \frac{(1+i)[1-(p_t)^{1/t}]}{(p_t)^{1/t}}$$

and

$$(5) \quad \frac{r-i}{i} = \frac{(1+i)}{i}\,\frac{[1-(p_t)^{1/t}]}{(p_t)^{1/t}} .$$ [5]

Treating p_t as a constant, it is obvious from (4) and (5) that the yield spread $(r-i)$ is positively related to the level of interest rates, and the relative yield spread $\frac{(r-i)}{i}$ is negatively related to the level of interest rates. That is,

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[5] These relationships are also derived in Bierman and Hass [2] and Yawitz [8].

$$(6)\quad \frac{\partial(r-i)}{\partial i} \quad \frac{[1-(p_t)^{1/t}]}{(p_t)^{1/t}} > 0 \text{ since } 0<p_t<1;$$

and

$$(7)\quad \frac{\partial[\frac{r-i}{i}]}{\partial i} = \frac{-[1-(p_t)^{1/t}]}{i^2(p_t)^{1/t}} < 0 \quad \text{since } 0<p_t<1.$$

Thus neither the yield spread nor the relative yield spread is an adequate measure of default risk alone because each is dependent on the level of interest rates.

To illustrate, suppose that initially the yield on a riskless one-year no coupon discount bond is 5 percent, and that the probability of default on a risky one-year no-coupon discount bond is 0.03 (or p_t=0.97). From (4) and (5) it can be seen that $r-i = \frac{(1.05)(1-.97)}{.97} = 3.25$ percentage points, and $\frac{r-i}{i} = \frac{(1.05)}{.05}\frac{(1-.97)}{.97} = 0.65$. Now suppose the riskless yield i increases to 10 percent, while the probability of default remains 0.03. The yield spread will rise slightly to $\frac{(1.10)(1-.97)}{.97} = 3.40$ percentage points, while the relative yield spread is now $\frac{r-i}{i} = \frac{(1.10)}{.1}\frac{(1-.97)}{.97} = 0.34$, a drop of almost half.

Figure 1 further illustrates the relationship between these two risk measures and the level of interest rates, by showing how the yield spread and relative yield spread vary with the level of rates for three different values of p_t, 0.96, 0.98, and 0.99. The figure shows that the yield spread is linearly related to the level of rates when the probability of default is constant,[6] while the relative yield spread is linearly related to the <u>inverse</u> of the riskless rate. Thus the relative yield spread is an especially poor measure of default risk, fluctuating greatly when the level of rates changes and the probability of default remains constant.

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[6] Thus the inclusion of the riskless interest rate as an independent variable in an equation where the dependent variable is the spread between risky and riskless rates is theoretically correct ([3, p. 1180]). However, because the level of rates moves cyclically, its inclusion as a regressor may lead to spurious estimates when examining the cyclical variability of default risk.

It can be seen from (4) that an adequate measure of default risk, at least for no-coupon discount bonds, is $\frac{r-i}{(1+i)}$, i.e., the yield spread divided by one plus the riskless yield (not simply the riskless yield), because it is a function of the probability of default only.[7]

B. The Multiperiod Payments Case

Deriving a single measure of default risk for bonds with multiperiod payments is more difficult because there may be a different probability of default associated with each future payment. For example, the probability that next year's coupon payment will be defaulted may differ from, and may have no relation to, the probability that the coupon payment due in 10 years will be defaulted.

If there exist bonds in identical risk classes but of maturities of 1, 2, 3, . . . N years, as well as riskless bonds corresponding to each risky bond maturity, then it would be possible to solve for all of the p_t's (since there would be N equations and N unknowns), and thus obtain a default risk measure for each future promised payment. Unfortunately, such bonds seldom if ever exist, so the use of this procedure is limited.[8]

One way to analyze the problem is to make a plausible assumption regarding the functional relationship between p_t and t. In the following discussion the relationship between a risky and a riskless bond, both with coupon payments of C, face value at maturity of \$100, and maturity of N years is examined under the assumption that the conditional probability of payment on any future promised payment is a constant p, and that when default occurs no future promised payments will be paid. In other words, the probability that

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[7]Another adequate default risk measure for no-coupon discount bonds would be the spread between continuously compounded rates of return. For example, if a one-year risky no-coupon bond with a par value of \$100 sells for \$X, then its continuously compounded rate of return is $r = \ln(100)-\ln(X)$, the continuously compounded riskless rate of return will be $i = \ln(100p_1)-\ln(X) = \ln(100)+\ln(p_1)-\ln(X)$, and $r-i = -\ln(p_1)>0$ since $0<p_1<1$.

[8]Silvers [6] uses the prices of bonds in similar risk classes but of maturities of 5, 15, and 30 years to estimate the "certainty equivalent coefficients" (the p_t discussed in footnote 4 above) for these years.

a promised payment in t years will be paid is conditional on the payment of all previous promised payments. This implies that the actual probability that a payment due in t years will be paid (p_t) will equal p^t--i.e., the probability of payment falls exponentially over time.[9] An assumption of a declining p_t over time is intuitively appealing, at least for most risky bonds, and empirical evidence suggests that p_t does indeed fall in an approximately exponential fashion over time for rated bonds.[10]

The present value of the risky bond described above is given by

$$(8) \quad V = C\sum_{t=1}^{N} \frac{1}{(1+r)^t} + \frac{100}{(1+r)^N} .$$

Because risk-neutral investors are indifferent between a risky payment in t years of \$1 and a certain payment in t years of $\$p^t$, (8) can also be written

$$(9) \quad V = C\sum_{t=1}^{N} \frac{p^t}{(1+i)^t} + \frac{100}{(1+i)^N} .$$

Rearranging (9) yields

$$(10) \quad V = C\sum_{t=1}^{N} \frac{1}{[1+\frac{(1+i-p)}{p}]^t} + \frac{100}{[1+\frac{(1+i-p)}{p}]^N} .$$

Setting (10) equal to (8), one finds that

$$(11a) \quad r-i = \frac{(1+i)(1-p)}{p}$$

and

$$(11b) \quad \frac{r-i}{i} = \frac{(1+i)(1-p)}{i \quad p} .$$

Thus one finds that if the probability of payment falls exponentially over time, then the yield spread is positively (and linearly) related to the level of interest rates, and the relative yield spread is negatively related to the level of rates. The relative yield spread continues to be an especially poor measure of default

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[9] Bierman and Hass [2] and Yawitz [8] also use this assumption to examine the relationship between risky and riskless yields.

[10] See Silvers [6], pp. 947-949.

risk, and $\frac{r-i}{1+i}$ is again an acceptable measure.[11]

III. PREVIOUS EMPIRICAL FINDINGS

The results in the previous section suggest that if default risk remains constant, then an increase in the level of interest rates will a) raise the spread between a risky bond yield and a riskless bond yield, and b) lower the relative yield spread of these two bonds. Consequently, in empirical work relating yield spreads and relative yield spreads to the level of interest rates, our *a priori* expectation is for a positive correlation between yield spreads and the level of rates and a negative correlation between relative yield spreads and the level of rates.

These expectations are realized in the empirical work of others. In one of the most widely cited articles on the cyclical variability of bond yield spreads, Jaffee [5] found that the spread between lower-rated and higher-rated bond yields (e.g., BAA vs. AAA) was positively related to the level of interest rates. Benson and Rogowski [1] found that the spread between the yields on risky and riskless (AAA) municipal bonds was positively related to the level of municipal bond rates. They also found that this municipal bond yield spread divided by the riskless yield--i.e., the relative yield spread--was negatively (and very strongly) related to the level of rates.

If, as was argued in the previous section, neither the yield spread nor the relative yield spread is an adequate default risk measure, then using these measures as proxies for default risk may lead to unwarranted conclusions. For example, Benson and Rogowski argue that the negative relationship between the relative municipal

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[11]Of course, it should be realized that (11a) and (11b) will hold only for the case where $p_t = p^t$, and may not hold for all risky bonds. However, if one wishes to use the yields to maturity to measure default risk some functional relationship between p_t and t must be assumed, and the above relationship seems quite plausible.

bond yield spread and the level of bond rates suggests that "Commercial bank withdrawal from the market during tight credit conditions causes yields on high-grade bonds to rise faster than low-grade bonds, resulting in narrower yield spreads."[12] The results of this paper, however, suggest that this negative relationship is to be expected simply because of the way in which investors adjust future promised payments for risk. Consequently, Benson and Rogowski's results say nothing concerning the effect of bank portfolio behavior on yield spreads.

One final comment should be made concerning the use of default risk proxies. If the default risk measure employed is sensitive not only to default risk but also to the level of interest rates, then it may be difficult to separate the cyclical variation in default risk from the cyclical variation in interest rates. Thus any finding that yield spreads or relative yield spreads have a cyclical component may be the result of a misspecified dependent variable, and not of increased riskiness.

IV. CONCLUSION

This paper has examined the relationship between previously used measures of default risk and the way in which investors adjust future promised payments for default risk. The paper argues that if investors are risk-neutral then any default risk measure should depend solely on the probability of default. The paper then demonstrated that neither the yield spread nor the relative yield spread satisfy this criterion, with the spread between risky and riskless yields being positively related to the level of interest rates and the ratio of this spread to the riskless yield being negatively related to the level of rates when the probability of default remains constant. In contrast, the ratio of the yield spread to one plus the riskless yield was found to be dependent solely on the probability of default, and was

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[12] [1], p. 355. Actually, this statement should read ". . . in narrower <u>relative</u> yield spreads."

thus judged to be an adequate measure of default risk. Finally, the paper indicates that recent empirical work supports the findings of this paper.

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Figure 1. Yield Spread and Relative Yield Spread vs. Riskless Interest Rate (i) For Different Probabilities of Payment