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ABSTRACT: In a two-agent, two-good, private information, risk-sharing environment, conditions are described under which a collateralized debt contract is the optimal allocation within the set of all resource and incentive feasible allocations, even allowing for extraneous randomization. In a collateralized debt contract, the borrower usually pays a fixed amount of one good and none of the second good, but if the first good is insufficient, all of the first good and some of the second good are paid. The second good thus serves as collateral, and is essential in nontrivial contracts in order to satisfy incentive constraints. The critical conditions for optimality are that preferences are heterogeneous in a particular way, and that the absolute risk aversion of the borrower is nonincreasing. The amount of the collateral good available to the borrower can sharply constrain contracts in the sense that it imposes an upper limit on the compensation available to the lender in feasible contracts. The results suggest applications to a wide range of financial arrangements.

Why is there debt? In so many observed contractual arrangements an agent's payment is noncontingent over a wide range of circumstances, while occassionally, as in a "default," little or no payment is made. This is puzzling because the treatment of uncertain, publicly observed events in general equilibrium theory suggests that contractual payments will in general be fully contingent [1, 11]. If instead some relevant information is private to the agent, then contractual payments will be completely noncontingent [2], but this seems inconsistent with the possibility of payments occasionally smaller than otherwise, as in a "default." Efforts to resolve this paradox and find environments in which the optimal arrangement is a noncontingent payment with occasional default have met with only limited success. The answer proposed in this paper builds directly on the insight of Arrow's. The key innovation is to examine a *two-good* risk sharing environment. Conditions are found under which the optimal arrangment is a "collateralized debt contract." The second good serves as collateral for insufficient payment of the first good, and this feature is derived endogenously.

In the environment studied here, the endowment of one agent (the "borrower") of one of the goods is random but the realized value is private and nonverifiable; for simplicity all other endowments are nonrandom. A "contract" in this setting is a pair of arbitrary payment schedules, one for each good, agreed upon as a *quid pro quo* for some consideration given earlier, such as a loan advance. An optimal contract is the solution to an "Arrow-Debreu program," as in Townsend [37], maximizing the weighted average of agents' expected utilities constrained only by resource feasibility and the incentive conditions implied by private information. Contracts of maximum generality are allowed, possibly involving extraneous lotteries as in Prescott and Townsend [28].

In a collateralized debt contract the borrower pays a fixed amount, R, of the random good whenever the realized endowment is at least R, and pays all of the random good whenever it is less than R. The nonrandom good is transfered only when the random endowment is less than R; the exact schedule is determined by the incentive constraints. Thus the nonrandom good functions as collateral in that transfers of it "make up for" the gap between the actual payment of the random good and the fixed payment R. This feature follows quite directly from the incentive constraints. An interesting property of collateral here is that its role is not to compensate the lender but to ensure the honesty of the borrower.

The main result of this paper is a set of conditions under which the collateralized debt contract is the optimal allocation in this environment. First, and most critically, a certain type of diversity of preferences is needed. The lender must like the nonrandom (collateral) good less, relative to the random good, than does the borrower, in the sense that their marginal rates of substitution are everywhere bounded apart. Essentially, the indifference curves of the two agents must never be tangent inside the Edgeworth Boxes drawn for each realized state (see Figure 3 below). Two other assumptions that play minor roles are that marginal utilities are finite, and that the borrower's utility displays nonincreasing absolute risk aversion with respect to the random good. The results rely on optimal control theory, and allow for contracts which are functions of bounded variation. *Ex post* verification plays no role.

The persuasiveness of the explanation of debt contracts proposed here depends on the plausibility of the required conditions on preferences, in particular the way in which preferences are heterogeneous. The difference in prefences implies that the optimal contract attempts, *ceteris paribus*, to minimize the amount of the collateral good that the lender receives. It would seem that something like this has to be involved in any collateralized debt contract; otherwise why wouldn't the contract have the borrower hand over some of the collateral in every state?[1] Imagine a loan to a farmer who will repay out of the (private) proceeds from the next harvest. The collateral good corresponds to the chattels of the farmer: durable, portable, personal property. The lender possesses property as well, but if chattels are unique and specially suited to an individual, the borrower's chattels might be of only limited direct utility to the lender. So the optimal contract has the farmer repay a fixed amount out of the harvest, with any shortfall made up by the surrender of some of the farmer's chattels.

It is easy to imagine related economic environments that might also give rise to a similar "reduced form" structure of preferences, and so might give rise to collateralized debt as the optimal contract. The collateral good might be a plot of land or a durable capital good with which the borrower is more productive than anyone else, either because of a stock of knowledge previously acquired by using it, or because of a learning or setup cost to transferring it to some other agent's use. Nobody else will be willing to pay as much as the machine is worth to the borrower, and so the optimal contract naturally calls for the borrower retaining the machine as often as possible.

The collateral good is essential to the optimal contract in the sense that without it the only contracts that are resource and incentive feasible are the trivial constant contracts. In fact, the amount of the collateral good held by the borrower can sharply constrain attainable allocations, in that it imposes an upper bound on the expected utility that the lender can obtain under feasible contracts. As a result, the borrower might not be able to obtain a desired loan. In a sense this paper provides a generalization of the phenomenon of "borrowing constraints" to situations in which collateral is available but perhaps insufficient. Briefly, examples of the implications of collateral constraints include the failure to equalize marginal intertemporal rates of substitution across agents, associated distortions in the intertemporal allocation of borrowers' consumption, an upper limit on the loan amount a borrower can obtain, distortions in the rate of capital formation because otherwise efficient projects are not fully funded [41], and distortions in the choice of capital inputs because some types, such as human capital, serve poorly as collateral [15].

The results reported here, along with the potential extensions noted above, suggest that *perhaps all debt contracts are implicitly collateralized*. After all, it is rarely the case that a borrower has literally no resources other than those of the exact type promised in payment; income streams at other future dates are available, for example, and these often serve as implicit collateral for "unsecured" lending. Indeed, a creditor's rights in legal bankruptcy proceedings can be viewed as a contingent claim, distinct from the promised repayment, that implicitly collateralizes an unsecured debt. Although the complex phenomena associated with bankruptcy are far beyond the simple models we have, so far, of financial contracts, including the present one, it is worth considering that the distinction between unsecured and (explicitly) secured debt may be a subtle one.

The economic environment is described in Section 1. Section 2 describes contracts. Section 3 defines optimal contracts and displays the programming problem that finds them. Section 4 contains the main results; conditions under which the optimal contract is a collateralized debt contract. A discussion of collateral constrained contracts is contained in Section 5. The remainder of the introduction discusses previous attempts to derive optimal debt contracts. (A fuller discussion appears in the working paper version [22].)

Probably the best known models of optimal debt contracts are those based on Townsend's costly state verification [36, 40, 14], but it is well known that randomized verification policies and randomized payment schedules can dominate deterministic schedules and do not in general give rise to debt contracts [38, 24, 21, 28, 38]. In contrast, arbitrary randomized contracts are allowed in the present paper, and a simple condition is found under which randomization is unnecessary.

In the context of a model of "credit rationing," Bester and Hellwig [8] assume that "default states" cannot be faked while the return to the borrower in all other states is unobserved. This asymmetry seems implausible in most contexts, and it evades the crucial question of why borrowers do not exploit the possibility of simulating "default." In the present paper the borrower can hide any amount the return in any state.

Hart and Moore [16] and Kahn and Huberman [19], in models focused on renegotiation, assume that the borrower's resources are observed by both borrower and lender, but are not verifiable by a third party such as a court, and thus "enforceable" contracts cannot be made contingent. But courts ascertain litigants' wealth quite regularly, and often enforce contracts that are *ex ante* contingent. A longstanding legal principle is that an admissible obligation be a "sum certain," meaning an amount that is calculable *at the time the suit is brought* [34, pp. 59-70], and besides, partnership arrangements, in which settling the claim requires reckoning profits, have been enforceable since at least the 12th century [29, 26]. It seems difficult to reconcile such an imperfection in the legal system with the widespread use of contracts, other than debt, that are genuinely contingent.

Diamond [13], in a risk-neutral, one-good, investment loan model, presents an optimal debt contract that relies on the lender committing to impose "nonpecuniary penalties" on the borrower in the event of default. If these penalties are interpreted as a second good to be forfeited by the borrower, his environment is virtually a special case of the one presented here. Thus the present paper unifies the treatment of collateral and penalties in loan contracts, and highlights their essential similarity.

Innes [17, 18] recently described *ex ante* moral hazard and *ex ante* assymetric information environments in which uncollateralized debt is an optimal contract. The optimality of the debt contract requires risk neutrality and restrictions on probability distributions and utilities such as the monotone likelihood ratio property and uniqueness of effort choice. In addition, attention must be restricted to contracts in which the payment is nondecreasing. Innes provides two alternative technological assumptions that justify this restriction: one is that the borrower cannot hide the realized return but can costlessly produce evidence of larger than actual return; and the other is that the lender can costlessly hide the return.

These results seem somewhat limited. First, risk neutrality is quite essential, because without it, as is well known, some general risk sharing arrangement is optimal in his setup and the debt contract is not, even with the restriction to monotone contracts. Second, one might question the plausibility of assuming that a borrower can produce evidence of nonexistent resources but cannot hide existing resources, or of assuming that the lender is able to hide existing resources from the borrower.

Vast literatures examine the effects of debt contracts in various settings, but except for the works cited above and others based on them, the form of contracts available to agents are generally taken as given. It is worth noting that in a number of papers that have investigated the role of collateral in loan contracts, it is generally assumed that the collateral is "worth less" to the lender than to the borrower, consistent with the present paper (for example, [3, 4, 10]). Collateral is also important in the large literature on "credit rationing" due to adverse selection (private *ex ante* information of the borrower concerning future returns) [5, 6, 7, 8, 35, 39], but again, a collateralized debt contract is imposed. In general, the adverse selection "credit rationing" results depend on the use of contracts with collateral that is insufficient in the sense that, if it were possible, the borrower would in every state prefer to hand over all of the promised collateral [5, 6, 7, 8]. Such contracts would not be incentive compatible under the assumptions of the present paper. Thus the possibility of credit rationing due to adverse selection might be quite sensitive to the way in which debt contracts are assumed to arise.

1. Environment

There are two agents, named *a* and *b*, and indexed by *h*. There are two goods, called *1* and *2*, and indexed by *i*. Agent a will occasionally be referred to as "the borrower," and agent *b* as "the lender." The endowment of agent *h* of good *i* is e_{hi}, $h=a,b$, $i=1,2$. Endowments e_{a2}, e_{b1}, and e_{b2}, are known nonrandom, nonnegative constants. The endowment, e_{a1}, of agent a of good *1* is random, and will be denoted by θ. Good 2 will turn out to function as collateral, so it will be refered to as the "collateral good." Attention will be restricted to the case in which $e_{b2} = 0$, and the lender has no endowment of the collateral good.[2]

Agent a is to make transfers (possibly contingent, possibly negative) of y_1 of good *1* and y_2 of good *2* to agent *b* after the endowments are received. After the transfers take place the goods are consumed. Consumptions are given by

$$
\begin{aligned}
c_{a1} &= \theta - y_1, \qquad & c_{a2} &= e_{a2} - y_2, \\
c_{b1} &= e_{b1} + y_1, \qquad & c_{b2} &= e_{b2} + y_2.
\end{aligned}
\tag{1.1}
$$

In a previous period, before the endowments are realized, agent *b* gives agent *a* some consideration--a loan advance, for example. The transfers from *a* to *b* then compensate *b* for the earlier consideration.

Agents derive utility from consumptions according to the functions $u_a(c_{a1},c_{a2})$ and $u_b(c_{b1},c_{b2})$ for agents *a* and *b* respectively. Both agents' utility functions are assumed to be additively separable, and so can be written $u_h(c_{h1},c_{h2}) = u_{h1}(c_{h1}) + u_{h2}(c_{h2})$, for $h=a,b$.[3]

The realized endowment of the random good is essentially private information to agent *a*. To be specific, it is costless for agent *a* to hide the good and pretend that the realized amount is smaller than it actually is. In contrast, it is assumed to be prohibitively costly to falsify units of the good that do not exist, pretending that the realized amount is larger than it actually is. The implication of this assumption is that incentive constraints need only be checked against the alternative report that the state was lower than actually realized. The usual private information assumption is that falsification costs are symmetric and zero for both hiding and faking, and thus incentive constraints must be checked against both lower and higher reports. The constraint that one cannot produce evidence of goods that do not exist does not bind for the equilibrium collateralized debt contract. See Lacker and Weinberg [23] for an analysis of costly falsification.

The assumptions about the primitive elements of our environment are summarized as follows:

Assumption 1: (a) Endowments: $e_{a2} > 0$, $e_{b1} \geq 0$, $e_{b2} = 0$, $e_{a1} = \theta$, θ random. (b) The distribution of θ is absolutely continuous with density f( ) that is strictly positive on Ω, the support of θ, where $\Omega = [\theta_0, \theta_1]$, $0 \leq \theta_0 < \theta_1 < +\infty$. (c) For all feasible, nonnegative consumptions $u_{hi}(\ )$, $h=a,b$, $i=1,2$, are continuous, concave, twice continuously differentiable; u_{a1}, u_{a2}, and u_{b1} are strictly increasing; and u_{a1}, u_{a2}, and u_{b2} have finite derivitives. (d) It is perfectly costless for agent *a* to hide good *1* and make it seem that θ is smaller than actually realized, but it is prohibitively costly to make it seem that θ is larger than actually realized.

Note that u_{b2} need not be increasing, so that "acquisition" of the collateral good might leave agent *b* indifferent or even worse off. Thus having agent *a* surrender some of good 2 might be a pure punishment that provides no gain to agent *b*, or may in fact be costly.[4]

2. Contracts

Agents meet at an initial period and, in exchange for a loan advance, agree to a repayment contract to be described in more detail below. The loan advance will not be treated explicitly, so that we may focus on the characteristics of the contract governing repayment. Then some time later the random endowment θ is realized, along with the other nonrandom endowments. Agent *a* observes the realized value of θ and either allows the true value to be seen by agent *b*, or makes it seem that the realized value was some smaller value θ'. The displayed value, θ', is not necessarily equal to the true value, θ. Given the amount displayed by agent *a*, transfers are to take place according to a schedule agreed to in the contract. For maxiumum generality, following Prescott and Townsend [28] and Townsend [37], these tranfers are allowed to be random. For a given display, θ', the result is a measure, $\pi(dy_1, dy_2|\theta')$, that specifies a probability distribution over transfers (y_1, y_2).[5] A contract is a family of probability measures, $\pi(\ ,\ |\theta)$ for each $\theta \in \Omega$, on B, the Borel sets of the set of feasible tranfers $[-e_{b1}, \theta] \times [0, e_{a2}]$.

An application of the well-known Revelation Principle (see Myerson [25], Townsend [38]), allows us to restrict attention to allocations that satisfy a self-selection constraint. Specifically, for a given contract π, and a given state θ, agent *a* chooses an announcement $\theta' \in [\theta_0, \theta]$ to maximize expected utility, given by

$$\iint [u_{a1}(\theta - y_1) + u_{a2}(e_{a2} - y_2)] \, \pi(dy_1, dy_2 | \theta') \ . \tag{2.1}$$

Define $\theta^*(\theta)$ as the announced state chosen by agent *a* when the true state is θ. Since both agents are aware of the choice problem facing agent *a* for any given contract, both agents can calculate $\theta^*(\theta)$. Therefore, for a contract $\bar{\pi}(dy_1, dy_2 | \theta)$, both agents know that the actual schedule relating transfers and the realized state will be $\pi(dy_1, dy_2 | \theta) \equiv \bar{\pi}(dy_1, dy_2 | \theta^*(\theta))$. This effective schedule has the property that

$$\iint [u_{a1}(\theta - y_1) + u_{a2}(e_{a2} - y_2)] \, \pi(dy_1, dy_2 | \theta)$$
$$\geq \iint [u_{a1}(\theta - y_1) + u_{a2}(e_{a2} - y_2)] \, \pi(dy_1, dy_2 | \theta') \tag{IF'}$$
$$\forall (\theta, \theta') \in \Omega \times \Omega, \text{ s.t. } \theta' < \theta.$$

A contract is <u>incentive feasible</u> if it satisfies (IF'). (IF') follows from the fact that $\theta^*(\theta)$ maximizes (2.1). Under a contract π that satisfies (IF'), agent *a* always truthfully reveals the actual state.[6] For any given contract, there exists a contract which satisfies (IF') and which results in an identical allocation. Thus there is no loss in generality in restricting attention to contracts which satisfy the incentive feasibility condition (IF').

A contract π is <u>resource feasible</u> if it is a probability measure over feasible transfers, and thus satisfies

$$\pi: B \times B \times \Omega \rightarrow [0,1],$$

$$\iint \pi(dy_1, dy_2 | \theta) \quad = \quad 1, \quad \forall\theta\in\Omega. \tag{RF'}$$

A <u>deterministic contract</u> is one in which transfers are deterministic functions of the state, $y_1(\theta)$ and $y_2(\theta)$. A deterministic contract is <u>resource feasible</u> if it satisfies

$$(y_1(\theta)), y_2(\theta)) \in [-e_{b1}, \theta] \times [0, e_{a2}], \ \forall\theta\in\Omega \tag{RF}$$

A deterministic contract is <u>incentive feasible</u> if it satisfies the following deterministic version of (IF')

$$u_{a1}(\theta - y_1(\theta)) + u_{a2}(e_{a2} - y_2(\theta)) \quad \geq \quad u_{a1}(\theta - y_1(\theta')) + u_{a2}(e_{a2} - y_2(\theta'))$$
$$\forall(\theta,\theta')\in\Omega\times\Omega, \text{ s.t. } \theta'<\theta. \tag{IF}$$

A <u>collateralized debt contract</u>, or <u>debt contract</u> for short, is a resource and incentive feasible contract, $(y_1^*(\theta,R), y_2^*(\theta,R))$, satisfying:

$$y_1^*(\theta,R) \quad = \quad \text{MIN}[\theta,R], \qquad \forall\theta\in\Omega, \tag{2.2}$$

$$y_2^*(\theta,R) \quad = \quad 0 \qquad \forall\theta\in[R,\theta_1], \tag{2.3}$$

$$y_2^{*\prime}(\theta,R) \quad = \quad -u'_{a1}(0)/u'_{a2}(e_{a2} - y_2^*(\theta,R)) \qquad \forall\theta\in[\theta_0,R). \tag{2.4}$$

where R is an arbitrary constant in (θ_0,θ_1). An example of a debt contract is plotted in Figure 1. The corresponding consumption schedules are shown in Figure 2. For realizations of θ that are large enough, agent a transfers a constant amount, R, of good *1*, and none of good *2*. If the realization of θ is not sufficient to allow the payment R, then agent a transfers all of good *1*, and transfers some of good *2*. Condition (2.4), along with the endpoint condition $y_2^*(\theta,R) = 0$ for $\theta=R$, determines $y_2^*(\theta,R)$ for θ below R, and will ensure incentive feasibility. Condition (2.4) makes $y_2^*(\theta,R)$ strictly decreasing for $\theta\in[\theta_0,R]$. It can be readily verified that the debt contract satisfies resource and incentive feasibility.

The largest amount of the collateral good ever transferred under the debt contract is $y_2^*(\theta_0,R)$, and this can be termed the collateral associated with the contract R. There may be some value of R below θ_1, call it $\bar{R}$, such that $y_2^*(\theta_0,\bar{R}) = e_{a2}$, and the debt contract corresponding to $\bar{R}$ requires transfer of all of the collateral good in the lowest state. For $R > \bar{R}$, a collateralized debt contract cannot be constructed since (2.4) is undefined for small θ; the contract would require more collateral than agent a has available. *$\bar{R}$ is the value of R for which the collateral constraint $y_2(\theta) \leq e_{a2}$ just binds, and no debt contract with $R > \bar{R}$ is feasible.*

3. Optimal Contracts

An optimal contract is one that is resource and incentive feasible, and for which there is no alternative resource and incentive feasible contract that makes one agent better off (in the sense of *ex ante* expected utility) without making the other agent worse off. Optimal contracts can be found as solutions to a particular private information "Arrow-Debreu program" that is appropriate for this environment, as in Townsend [37]. Proof of this would merely be an extension of a result of Prescott and Townsend [28] to a continuous state space, and is omitted. The program is to choose probability measures over transfers $\pi(\ ,\ |\theta)$ for each $\theta\in\Omega$, to

$$\text{MAX} \quad \lambda_a \iiint [u_{a1}(\theta-y_1) + u_{a2}(e_{a2}-y_2)]\pi(dy_1,dy_2|\theta)f(\theta)d\theta \qquad \text{(P1)}$$

$$+ \quad \lambda_b \iiint [u_{b1}(e_{b1}+y_1) + u_{b2}(y_2)]\pi(dy_1,dy_2|\theta)f(\theta)d\theta$$

s.t. (RF′) and (IF′)

where λ_a and λ_b are arbitrary nonnegative Pareto weights. In P1 the contract is chosen to maximize the weighted sum of the two agents' expected utilities, subject to feasibility and incentive-compatibility constraints. The program P1 is as fully general as possible, as in Prescott and Townsend [28] and Townsend [37]. Because the choice variables enter linearly in both the objective function and the constraints, the constraint set is convex. If the constraint set is nonempty, as Assumptions 1(a) and 1(b) guarantee, then we know that a solution exists.

4. Optimality of Collateralized Debt Contracts

Assumption 1 is not sufficient for the optimality of the debt contract. In this section three further conditions are described which, together with Assumption 1, imply that the debt contract is optimal. The following assumption permits restricting attention to deterministic contracts.

Assumption 2: $-u_{a1}''(c_{a1})/u_{a1}'(c_{a1})$ is nonincreasing.

Proposition 1: Let Assumptions 1 and 2 hold. Then a deterministic contract solves P1.

Proofs are in the Appendix. $-u''_{a1}(c_{a1})/u'_{a1}(c_{a1})$ is the absolute risk aversion of agent *a* with respect to good *1*, the random good. Nonincreasing absolute risk aversion allows a wide of range utility functions (see Pratt [27]), but keep in mind that marginal utilities must be finite. Thus $u_{a1}(c_{a1}) = (1-\alpha)^{-1}(\delta+c_{a1})^{1-\alpha}$ is allowed only if $\delta > 0$.

For some insight into Proposition 1, consider a given deterministic contract that is resource and incentive feasible. Fix $\hat{\theta}$, and add randomness to the allocation for that state, $\pi(\ ,\ |\hat{\theta})$, in such a way that agent *a*'s expected utility for that state (assuming truthtelling) is unchanged. Could this relax the incentive feasibility constraints? The left side of (IF′) is unchanged for $\theta = \hat{\theta}$, so the incentive constraints are still satisfied for $\theta = \hat{\theta}$, and $\theta' < \hat{\theta}$. For $\theta > \hat{\theta}$ and $\theta' = \hat{\theta}$ the right side of (IF′) may be different. Because absolute risk aversion is nonincreasing for u_{a1}, the utility function $u_{a1}(\theta-\theta'+\theta'-y_1) + u_{a2}(e_{a2}-y_2)$ is no more risk averse with regard to the distribution of y_1 and y_2 than is $u_{a1}(\theta'-y_1) + u_{a2}(e_{a2}-y_2)$, and so the right side of (IF′) is certainly no smaller than before. Thus adding randomness does not introduce "slack" into any constraints, and may in fact cause violations of incentive feasibility. If instead u_{a1} displayed increasing absolute risk aversion, the right side of (IF′) would be made smaller with the addition of randomness for $\theta > \hat{\theta}$ and $\theta' = \hat{\theta}$, and the slack obtained might be enough to compensate both agents for the extraneous risk.

Assumption 1(d) concerning falsification costs plays a role in Proposition 1. If faking nonexistant goods is possible and is costless, as is implicit in the standard private information assumption, then additional incentive constraints are required for $\theta < \theta'$. In this case the proof outlined above would fail because for $\theta < \hat{\theta}$ and $\theta' = \hat{\theta}$ the right side of

(IF′) is no smaller, and is strictly larger if risk aversion is strictly decreasing. However, the proof of Proposition 1 makes no use of the fact that in private information risk-sharing environments of this type incentive constraints for $\theta < \theta'$ generally do not bind for the optimal contract. A reasonable conjecture is that one could relax Assumption 1(d) but use some of the implications of optimality to prove a version Proposition 1.

Proposition 1 allows us to rewrite problem P1 with attention restricted to deterministic contracts. The problem is now to choose payment schedules y_1, and y_2 to

$$\begin{aligned} \text{MAX} \quad & \lambda_a \int [u_{a1}(\theta - y_1(\theta)) + u_{a2}(e_{a2} - y_2(\theta))]f(\theta)d\theta \\ + \quad & \lambda_b \int [u_{b1}(e_{b1} + y_1(\theta)) + u_{b2}(y_2(\theta))]f(\theta)d\theta \\ & \text{s.t.} \quad (\text{RF}) \quad \text{and} \quad (\text{IF}) \end{aligned} \tag{P2}$$

In P2 the contract is chosen to maximize the weighted sum of the two agents' expected utilities, subject to the simpler feasibility and incentive-compatibility constraints relevant to deterministic contracts, (RF) and (IF).

The program P2 still presents some difficulties. The central problem is that for each θ, (IF) involves a continuum of constraints, one for each alternative $\theta' < \theta$. In order to make the problem tractable these constraints need to be reduced to a managable form. The approach taken here (in fact the approach almost always taken when the state space is continuous) is to replace (IF) with a weaker first order condition. The problem then takes the form of a control problem, because the constraints are in terms of functions y_1 and y_2 and their derivatives at each point in the state space. The control theory requires that we restrict attention to functions of bounded variation.[7] The derivative of a function of bounded variation exists almost everywhere. This fact can be used to derive a convenient property of incentive-compatible contracts.

Proposition 2: Let Assumption 1 hold. If the contract (y_1, y_2) satisfies (IF) and is of bounded variation, then

$$- y_1'(\theta)u_{a1}'(\theta - y_1(\theta)) \quad - \quad y_2'(\theta)u_{a2}'(e_{a2} - y_2(\theta)) \quad \geq \quad 0 \qquad \text{a. e.} \qquad (4.1)$$

Condition (4.1) states that for any given θ, agent a's utility is a nondecreasing function of the announcement, θ', for θ' very close to θ. Condition (4.1) does not by itself imply incentive feasibility. If the contract payment schedules are absolutely continuous functions, then condition (4.1) is necessary and sufficient for a contract to satisfy (IF) because of the concavity of utilities, but this is not true for functions of bounded variation. Thus (4.1) is weaker than (IF). Because the debt contract satisfies (IF) with equality by construction, if (4.1) is substituted for (IF) and the debt contract is optimal under this weaker condition, then the debt contract is optimal under the stronger condition (IF).

An immediate implication of (4.1) is that there is a limit to the risk that agent *a* can shed via an incentive feasible contract. Define $v_a(\theta)$ as the *ex ante* utility of agent a, $v_a(\theta) = u_{a1}(\theta - y_1(\theta)) + u_{a2}(e_{a2} - y_2(\theta))$. Then Proposition 2 implies

<u>Corollary 1</u>: Let Assumption 1 hold. If the contract (y_1,y_2) satisfies (IF) and is of bounded variation, then

$$v_a'(\theta) \geq u_{a1}'(\theta-y_1(\theta)) \qquad \text{a.e.}$$

Thus, in a sense the borrower bears "most" of the risk, in that the minimum slope of the borrower's *ex post* utility is the partial derivitive of $u_{a1}(\theta-y_1(\theta')) + u_{a2}(e_{a2}-y_2(\theta'))$ with respect to θ. Note that if u_{a1} is strictly concave, a smaller payment $y_1(\theta)$ in a given state θ implies a smaller value of $u_{a1}'(\theta-y_1(\theta))$ in that state. This affords an indirect opportunity for risk sharing by reducing $v_a'(\theta)$, the slope of agent a's *ex post* utility.

Some additional notation will be helpful. Define $v_a(\theta,R)$ and $v_b(\theta,R)$ by:

$$v_a(\theta,R) = u_{a1}(\theta-y_1^*(\theta,R)) + u_{a2}(e_{a2}-y_2^*(\theta,R))$$

$$v_b(\theta,R) = u_{b1}(e_{b1}+y_1^*(\theta,R)) + u_{b2}(y_2^*(\theta,R))$$

These are the *ex post* utilities of the two agents in state θ, under the debt contract R.

We can now describe conditions under which the collateralized debt contract R is the unique optimal contract. The first condition is that the debt contract satisfy

<u>**Condition 1:**</u>

$$\lambda_a E\left[\frac{\delta v_a(\theta,R)}{\delta R}\right] + \lambda_b E\left[\frac{\delta v_b(\theta,R)}{\delta R}\right] = \mu$$

$$\mu \geq 0, \qquad \mu(\bar{R} - R) = 0.$$

(Derivations appear in the Appendix.) μ is the multiplier associated with the collateral constraint $y_2(\theta_0) \leq e_{a2}$. (Recall that $y_2^*(\theta_0,\bar{R}) = e_{a2}$ by definition.) If the collateral constraint does <u>not</u> bind, so that $R < \bar{R}$, then $\mu = 0$, and Condition 1 determines the value of λ_a/λ_b for which R is optimal. If $\mu > 0$, then the collateral constraint <u>does</u> bind, $R = \bar{R}$, and Condition 1 determines μ given λ_a and λ_b.

The second, and crucial condition for the optimality of debt contracts is that the preferences of the two agents are sufficiently different in the sense that their marginal rates of substitution are bounded apart:

<u>**Condition 2:**</u> For all θ in the interior of Ω,

$$\left[u'_{b1}(e_{b1}+y_1^*(\theta,R)) - u'_{a1}(\theta-y_1^*(\theta,R))\frac{u'_{b2}(y_2^*(\theta,R))}{u'_{a2}(e_{a2}-y_2^*(\theta,R))}\right]\frac{f(\theta)}{F(\theta)}$$

$$> \quad -\rho(\theta,R)\left\{\left(\frac{\lambda_a}{\lambda_b}\right)E\left[\frac{\delta v_a(\hat{\theta},R)}{\delta R}\middle|\hat{\theta}\leq\theta\right] + E\left[\frac{\delta v_b(\hat{\theta},R)}{\delta R}\middle|\hat{\theta}\leq\theta\right]\right\}$$

where $\rho(\theta,R) = -u''_{a1}(\theta-y_1^*(\theta,R))/u'_{a1}(\theta-y_1^*(\theta,R))$ is the coefficient of absolute risk aversion of agent a with respect to good *1*. The term in braces on the right side of Condition 2 is always negative, and will be discussed below. The main result can now be stated:

Proposition 3: Let Assumptions 1 and 2 hold, and suppose that a collateralized debt contract R satisfies Conditions 1 and 2 with $\mu = 0$, so that the collateral constraint does not bind. Then R is the unique optimal contract.

It is easiest to understand this result in two steps. First, abstract from risk-sharing considerations and assume for now that u_{a1} is linear. In this case $\rho(\theta,R) = 0$ and Condition 2 is equivalent to

Condition 3: For all $\theta \in \Omega$,

$$\frac{u'_{b1}(e_{b1}+y_1^*(\theta,R))}{u'_{a1}(\theta-y_1^*(\theta,R))} > \frac{u'_{b2}(y_2^*(\theta,R))}{u'_{a2}(e_{a2}-y_2^*(\theta,R))}.$$

Condition 3 is simply that for any given state θ, the two agents' indifference curves are never tangent, and the *borrower* values the collateral good more highly than does the lender. Indifference curves that satisfy Condition 3 are shown in the Edgeworth Box for an arbitrary state in Figure 3. In this case a move towards the northwest, giving the borrower more collateral good and the lender more corn, can, *ceteris paribus*, make both agents better off. The collateralized debt contract is the incentive feasible contract that lies entirely on the boundary of the Edgeworth Boxes; the northern edge for $\theta > R$, and the western edge for $\theta < R$. This contract minimizes the lender's expected consumption of the collateral good.

A very simple example illustrates the principle at work here. Suppose that all utility functions are linear, and that $u_{a1}(c) = u_{a2}(c) = u_{b1}(c) = c$, and $u_{b2}(c) = qc$. Condition 3 is equivalent to $q < 1$. Incentive feasibility is $y_1'(\theta) + y_2'(\theta) \geq 0$; in other words, the total payment (evaluated using agent *a*'s valuation of the collateral good) is nondecreasing. Consider the following problem, equivalent to P2:

$$\begin{aligned} \text{MAX} \quad & \int [\theta - y_1(\theta) + e_{a2} - y_2(\theta)]f(\theta)d\theta \qquad \text{(P2')} \\ \text{s.t.} \quad & \int [e_{b1} + y_1(\theta) + qy_2(\theta)]f(\theta)d\theta \quad \geq \quad \bar{v}_b \\ & \text{(RF) and (IF)} \end{aligned}$$

where $\bar{v}_b$ is some arbitrary reservation utility for the lender. It is easy to show that P2′ is equivalent to minimizing $E[(1-q)y_2(\theta)]$, the expected loss due to the transfer of collateral from the higher value user (the borrower) to the lower value user (the lender), subject to (RF), (IF), and the $\bar{v}_b$ constraint. The collateralized debt contract is the unique feasible contract that minimizes this loss.

When u_{a1} is <u>not</u> linear, the borrower's consumption of corn has a direct effect on the incentive constraints. Alternative resource feasible contracts can affect the slope of the borrower's *ex post* utility via Corollary 1, and thus can affect the risk born by agent *a*. Imagine a marginal decrease in $y_1(\theta)$ for some given state θ, accompanied by a marginal increase in $y_2(\theta)$ just large enough to maintain incentive feasibility. $u_{a1}'(\theta-y_1(\theta))$ is now lower, and so $v_a'(\theta)$ can be lower, allowing a reduction in the total variation in the borrower's *ex post* utility. The borrower's consumption schedule is now marginally less risky, and this allows a gain in *ex ante* expected utility that can be shared by the two agents. The right side of Condition 2 is a measure of the value of this improvement in risk-sharing. The gain is compared to the loss, measured by the left side of Condition 2, associated with the wedge between agents' indifference curves; the marginal changes in $y_1(\theta)$ and $y_2(\theta)$ are movements toward the interior of the Edgeworth Box. Condition 2 states that the value of the improvement in risk-sharing is less than the disutility of giving more of the collateral good to the lender.

For a bit more insight into the right side of Condition 2, note that

$$\lambda_a \frac{\delta v_a(\theta,R)}{\delta R} + \lambda_b \frac{\delta v_b(\theta,R)}{\delta R}$$

$$= \left[-\lambda_a u'_{a1}(\theta - y_1^*(\theta,R)) + \lambda_b u'_{b1}(e_{b1}+y_1^*(\theta,R)) \right] \frac{\delta y_1^*(\theta,R)}{\delta R}$$

$$+ \left[-\lambda_a u'_{a2}(e_{a2} - y_2^*(\theta,R)) + \lambda_b u'_{b2}(y_2^*(\theta,R)) \right] \frac{\delta y_2^*(\theta,R)}{\delta R},$$

for all θ. In the same environment *without* incentive constraints, the expression above would be zero for every state, since weighted marginal utilities would be equated state-by-state. Instead, *average* marginal utilities are equated in Condition 1. As a result, the bracketed term on the right side of Condition 2 is always negative, and is a measure of the utility cost in states $\theta' \le \theta$ of imperfect risk-sharing, due, of course, to the informational imperfection. The entire term is scaled by the coefficient of absolute risk aversion, $\rho(\theta,R)$, which here measures the effect on $u'_{a1}(\theta - y_1(\theta))$ of a perturbation in $v_a(\theta)$.

It bears emphasizing that the right side of Condition 2 is quite literally a "second-order effect." By giving the borrower less collateral and more corn the marginal utility of corn for the borrower is reduced, and this relaxes the incentive constraint and allows the borrower to bear less risk. If $\rho(\theta,R)$ is small the change in u'_{a1} is small and little improvement in risk-sharing is acheived. Thus *Condition 2 can be thought of as an upper bound on* $\rho(\theta,R)$. Alternatively, the value of indirect risk-sharing measured by the right side of Condition 2 implies a minimum value for the wedge between the indifference curves of the two agents, as measured by the left side of Condition 2. Thus *Condition 2 can be thought of as an upper bound on the lender's valuation of the collateral good.*

Some interesting aspects of debt contracts in this setting are worth commenting on. In the collateralized debt contract the lender takes a "loss," in terms of good 1, of $R - \theta$ for $\theta < R$. The lender is compensated (if $u'_{b2} \geq 0$) by the transfer of some collateral in these states, but because the collateral is not as valuable to the lender as it is to the borrower, the lender's *ex post* utility is always lower for $\theta < R$; in fact it is strictly increasing in θ over this range. From the lender's point of view the debt is undercollateralized, though the contract is fully collateralized when evaluated from the borrower's point of view. Note also that as long as the collateral good has a positive value to the lender ($u'_{b2} > 0$), it will be in the lender's interest to take possession of the collateral for nonpayment, as called for in the original contract. Thus the contract is fully time consistent and renegotiation-proof in this case, unlike the costly verification setup (see [21] for a discussion of time consistency in the costly verification environment).

In one important respect optimal collateralized debt contracts in this setting differ significantly from debt contracts in some other settings, such as the costly verification environment. It is easy to establish that when

$u'_{b2} \geq 0$, $E[v_b(\theta,R)]$ is strictly increasing in R, because for any two feasible debt contracts, R and $R' > R$, the *ex post* utility of agent b is greater in every state under the contract R'. Because of this fact, the lender's expected utility has no interior maximum with respect to R, and the lender always prefers a larger R to a smaller one. This implies that in this environment there will be no "Williamson credit rationing" (see Williamson [40] and Lacker [20]), which depends on just such an interior maximum. However, as the next section explores, the availability of collateral to the borrower can constrain contracts in a way that might be thought of as "credit rationing."

5. Collateral Constrained Contracts

The borrower's endowment of the collateral good can impose a sharp constraint on contracts. As mentioned earlier, there may be a value $\bar{R}$ (less than θ_1) for which the collateralized debt contract utilizes all of the borrower's endowment of collateral. It is impossible to construct a collateralized debt contract for values of R greater than $\bar{R}$ without violating resource or incentive feasibility.

Proposition 3 covers the case in which the collateral constraint is not binding so that the multiplier μ is zero. When $\mu > 0$, Condition 2 is no longer sufficient, and so we need a more general result to cover this case.

<u>**Condition 4:**</u> For all θ in the interior of Ω,

$$\left[u'_{b1}(e_{b1}+y_1^*(\theta,R)) \quad - \quad u'_{a1}(\theta-y_1^*(\theta,R))\frac{u'_{b2}(y_2^*(\theta,R))}{u'_{a2}(e_{a2}-y_2^*(\theta,R))}\right]\frac{f(\theta)}{F(\theta)}$$

$$> \quad -\rho(\theta,R)\left\{\left(\frac{\lambda_a}{\lambda_b}\right)E\left[\frac{\delta v_a(\hat{\theta},R)}{\delta R}\middle|\hat{\theta}\leq\theta\right] \quad + \quad E\left[\frac{\delta v_b(\hat{\theta},R)}{\delta R}\middle|\hat{\theta}\leq\theta\right]\right\}$$

$$+ \quad \mu\frac{\rho(\theta,R)}{\lambda_b F(\theta)}.$$

<u>**Proposition 4:**</u> Let Assumptions 1 and 2 hold, and suppose that the collateral constraint binds (so that $R = \bar{R}$), and that $\bar{R}$ satisfies Conditions 1 and 4 with $\mu \geq 0$. Then $\bar{R}$ is the unique optimal contract.

Condition 4 differs from Condition 2 by the presence of a positive term containing μ on the right side. A binding collateral constraint gives rise to a further hurdle for the optimality of the debt contract. As was noted above, it is impossible to constuct a debt contract that provides greater expected utility to the lender than does $\bar{R}$, the collateral constrained debt contract. However, interior contracts that do not resemble debt can provide more expected utility for the lender via the indirect risk-sharing effect described earlier. By giving the borrower more corn and less collateral, the incentive constraint in Corollary 1 can be relaxed slightly (at the rate $\rho(\theta,R)$), reducing the risk born by the borrower. The reduction in the borrower's "risk premium" can be used to increase the lender's expected utility. If λ_b is large enough relative to λ_a, so that $R = \bar{R}$ and μ is large enough, then the value of providing agent b with more expected utility will exceed the cost associated with the lender's lower valuation of the collateral good. Once again, this is essentially either an upper bound on $\rho(\theta,R)$, or an upper bound on the lender's valuation of the collateral good.

The collateral constraint could be quite severe. Notice, for example, that the constraint is independent of the expected utility of agent a under the contract. If agent a is an entreprenuer considering an investment project, he may be unable to obtain financing due to the collateral constraint. This could occur despite the project "having a positive net present value" in the sense that agent a could obtain financing in a perfect information environment.

More generally, the collateral constraint will drive a wedge between the intertemporal marginal rates of substitution of the borrower and the lender. Such a wedge, derived directly from the primitives of the environment here, could have quite far-reaching implications, since a key property of many dynamic models is that intertemporal rates of substitution are equated across agents.

6. Concluding Remarks

An explanation of the ubiquity of debt contracts has been proposed. The analysis necessarily has been carried out in the simplest possible environment. Whether this explanation is plausible depends on how attractive one finds the "match" between the Assumptions and Conditions and the situations in which people actually find themselves. The plausibility also depends on whether more "realistic" environments can be found which deliver, as suggested in the Introduction, analogous results. Consequently, this should perhaps be viewed as only a small step towards an improved understanding of financial contracts.

Appendix

Proposition 1: Suppose a contract π solves P1. Replace it with the certainty equivalent contract $(y_1(\theta), y_2(\theta))$ defined by

$$u_{a1}(\theta - y_1(\theta)) = \iint u_{a1}(\theta - y_1)\pi(dy_1, dy_2|\theta),$$

$$u_{a2}(e_{a2} - y_2(\theta)) = \iint u_{a2}(e_{a2} - y_2)\pi(dy_1, dy_2|\theta).$$

Because u_{a1} and u_{a2} are concave, the risk premium, $y_i(\theta) - E_\pi[y_i|\theta]$, is nonnegative. Because u_{b1} and u_{b2} are concave, the value of the objective function is not reduced by this substitution. Feasibility is obviously satisfied, so we only need to check incentive compatibility. Define a variable $y_1(\theta,\theta')$ by

$$u_{a1}(\theta - y_1(\theta,\theta')) = \iint u_{a1}(\theta - y_1)\pi(dy_1, dy_2|\theta');$$

$y_1(\theta,\theta')$ is the certainty equivalent of the random variable y_1 if the announced state is θ' but the true state is θ. Note that $y_1(\theta,\theta') = y_1(\theta)$. By Assumption 2, and Pratt [27], Theorem 2, $y_1(\theta,\theta')$ is nonincreasing in θ for any given θ'. Using this and (IF'):

$$
\begin{aligned}
u_{a1}(\theta-y_1(\theta)) &+ u_{a2}(e_{a2}-y_2(\theta)) \\
&= \iint [u_{a1}(\theta-y_1) + u_{a2}(e_{a2}-y_2)]\pi(dy_1,dy_2|\theta) \\
&\geq \iint [u_{a1}(\theta-y_1) + u_{a2}(e_{a2}-y_2)]\pi(dy_1,dy_2|\theta') \\
&= u_{a1}(\theta-y_1(\theta,\theta')) + u_{a2}(e_{a2}-y_2(\theta')) \\
&\geq u_{a1}(\theta-y_1(\theta')) + u_{a2}(e_{a2}-y_2(\theta')),
\end{aligned}
$$

and thus (y_1,y_2) satisfies (IC).#

<u>**Proposition 2**</u>: Suppose (y_1,y_2) is of bounded variation and satisfies (IC). Then

$$
\begin{aligned}
u_{a2}(e_{a2}-y_2(\theta)) &- u_{a2}(e_{a2}-y_2(\theta')) \\
&\geq u_{a1}(\theta-y_1(\theta)) - u_{a1}(\theta-y_1(\theta')), \\
&\qquad \forall(\theta',\theta)\in\Omega\times\Omega, \text{ s.t. } \theta'<\theta.
\end{aligned}
$$

This implies that where both left derivatives $D^-y_1(\theta)$ and $D^-y_2(\theta)$ exist, $D^-y_1(\theta)u'_{a2}(e_{a2}-y_2(\theta)) \leq D^-y_1(\theta)u'_{a1}(\theta-y_1(\theta))$. Because the derivatives of y_1 and y_2 both exist almost everywhere, where they do exist they must satisfy (4.1).#

<u>**Propositions 3 and 4**</u>: Define $x_1(\theta) = u_{a1}(\theta-y_1(\theta))$ and $x_2(\theta) = u_{a2}(e_{a2}-y_2(\theta))$, and define the vector $x(\theta) = (x_1(\theta),x_2(\theta))'$. A contract is now an <u>arc</u> $x:\Omega\to\Re^2$. Let **A** be the set of absolutely continuous arcs from Ω to $\Re^2$, and let **B** be the set of arcs of bounded variation from Ω to $\Re^2$. Define ϕ_1 as the inverse of u_{a1} and ϕ_2 as the inverse of u_{a2}. Given x, we can recover $y_1(\theta) = \theta - \phi_1(x_1(\theta))$, and $y_2(\theta) = e_{a2} - \phi_2(x_2(\theta))$. P2 can now rewritten as follows:

Choose an arc $x \in \mathbf{B}$ to (P3)

$$\text{MAX} \quad \lambda_a \int [x_1(\theta) + x_2(\theta)]f(\theta)d\theta + \lambda_b \int [\psi_1(\theta, x_1(\theta)) + \psi_2(x_2(\theta))]f(\theta)d\theta$$

$$\text{s.t.} \quad x'(\theta) \in \mathbf{Z}(\theta, x(\theta)) \quad \text{a.e.}$$

$$x(\theta) \in \mathbf{X}(\theta) \quad \forall \theta \in \omega,$$

where $\mathbf{X}(\theta) = [\underline{x}_1, \overline{x}_1(\theta)] \times [\underline{x}_2, \overline{x}_2]$, $\underline{x}_1 = u_{a1}(0)$, $\overline{x}_1(\theta) = u_{a1}(\theta + e_{b1})$, $\underline{x}_2 = u_{a2}(0)$, $\overline{x}_2 = u_{a2}(e_{a1} + e_{a2})$, $\mathbf{Z}(\theta, x) = \{z \in \mathbb{R}^2 | z_1 + z_2 \geq u'_{a1}(\phi_1(x_1))\}$, $\psi_1(\theta, x_1) = u_{b1}(e_{b1} + \theta - \phi_1(x_1))$, and $\psi_2(x_2) = u_{b2}(e_{b2} + e_{a2} - \phi_2(x_2))$. Note that $\{(x,z) \in \mathbb{R}^4 | z \in \mathbf{Z}(\theta, x), x \in \mathbf{X}(\theta)\}$ is convex, using Assumption 2.

An arc $x: \Omega \rightarrow \mathbb{R}^2$ of bounded variation gives rise to a $\mathbb{R}^2$-valued regular Borel measure dx on Ω. For each $\theta \in \text{int}\Omega$, the left and right limits of x unambiguously exist; call them $x_-(\theta)$ and $x_+(\theta)$ respectively. At the endpoints, define $x_-(\theta_0)$ as $x(\theta_0)$ and $x_+(\theta_1)$ as $x(\theta_1)$. The atoms of dx occur only at discontinuities of x. If there is a discontinuity at θ and $x_-(\theta) = x_+(\theta)$, the discontinuity is said to be <u>removable</u>. Discontinuities at endpoints are not removable, and they play a crucial role below. Two arcs x and $\hat{x}$ that differ only by removable discontinuities are said to be <u>equivalent</u>. An arc of bounded variation is thus an equivalence class, but arcs will be refered to as functions when no ambiguity would result.

The <u>jump</u> in x at θ is $\Delta x(\theta) = x_+(\theta) - x_-(\theta)$. The derivitive $x'(\theta)$ exists almost everywhere. The measure dx can be decomposed into an absolutely continuous measure $x'(\theta)d\theta$ and a measure $\xi(\theta)d\tau(\theta)$, where $\xi(\theta)$ is a Borel measurable function and $d\tau(\theta)$ is a measure that is singular with respect to the Lebesgue measure.

Define the Lagrangian function as

$$L(\theta,x,z) = -\lambda_a(x_1 + x_2)f(\theta) - \lambda_b[\psi_1(\theta,x_1) + \psi_2(x_2)]f(\theta) \quad \text{if } x\in\mathbf{X}(\theta) \text{ and } x'\in\mathbf{Z}(\theta,x)$$

$$= +\infty \quad \text{otherwise.}$$

Define $r_L(\theta,z)$ as the *recession function* of L (see [30, sec. 8]):

$$r_L(\theta,z) = \lim_{\alpha\to+\infty} [L(\theta,x_0,z_0+\alpha z) - L(\theta,x_0,z_0)]/\alpha$$

r_L is well-defined and independent of (x_0,z_0) as long as $L(\theta,x_0,z_0) < +\infty$. For our problem $r_L(\theta,z) = 0$ if $z_1 + z_2 \geq 0$, and $r_L(\theta,z) = +\infty$ if $z_1 + z_2 < 0$. Define the functional

$$J_L(x) = \int_{\theta_0}^{\theta_1} L(\theta,x(\theta),x'(\theta))d\theta + \int_{\theta_0}^{\theta_1} r_L(\theta,\xi(\theta))d\tau(\theta)$$

for $x\in\mathbf{B}$, where $\xi(\theta)d\tau(\theta)$ is any representation of the singular measure $dx - x'(\theta)d\theta$. If $x\in\mathbf{A}$ then the second term in J_L vanishes. Our problem can be restated:

$$\text{Choose } x\in\mathbf{B} \text{ to MIN } J_L(x) \tag{P4}$$

We will apply results of Rockafellar's [32] for Lagrange problems; that is, problems with fixed endpoints. Consider the following Problem of Lagrange:

Choose $x \in \mathbf{B}$ to MIN $J_L(x)$ (P5)

$$\text{s.t.} \quad x(\theta_0) = x^0 \quad \text{and} \quad x(\theta_1) = x^1$$

where $x^0 \in \mathbf{X}(\theta_0)$ and $x^1 \in \mathbf{X}(\theta_1)$ are arbitrary endpoints. Define $\underline{x}^0 = (\underline{x}_1, \underline{x}_2)$ and $\bar{x}^1 = (\bar{x}_1(\theta_1), \bar{x}_2)$, the smallest feasible left endpoint and the largest feasible right endpoint, respectively.

Lemma 1: For any arc $x \in \mathbf{B}$ with $x(\theta_0) \in \mathbf{X}(\theta_0)$, $x(\theta_1) \in \mathbf{X}(\theta_1)$, and $J_L(x) < +\infty$, there exists an arc $\hat{x} \in \mathbf{B}$, equivalent to x on $\mathrm{int}\Omega$, with $\hat{x}(\theta_0) = \underline{x}^0$, $\hat{x}(\theta_1) = \bar{x}^1$, and $J_L(\hat{x}) = J_L(x)$.

Proof: For all such x and $\hat{x}$, we have:

$$J_L(\hat{x}) \quad = \quad J_L(x) \quad + \quad r_L(\theta_0, x_+(\theta_0) - \underline{x}^0) \quad + \quad r_L(\theta_1, \bar{x}^1 - x_+(\theta_1)),$$

but $r_L(\theta_0, x_+(\theta_0) - \underline{x}^0) = r_L(\theta_1, \bar{x}^1 - x_-(\theta_1)) = 0$, for all $x(\theta_0) \in \mathbf{X}(\theta_0)$ and $x(\theta_1) \in \mathbf{X}(\theta_1)$, so $J_L(\hat{x}) = J_L(x)$.#

This implies that any feasible arc $x \in \mathbf{B}$ is endpoint-equivalent to any other arc $\hat{x} \in \mathbf{B}$ such that $\hat{x}(\theta) = x(\theta)$ $\forall \theta \in \mathrm{int}\Omega$, and $r_L(\theta_0, x_+(\theta_0) - \hat{x}(\theta_0)) = r_L(\theta_1, \hat{x}(\theta_1) - x_+(\theta_1)) = 0$. For any arc that solves P4, there is an endpoint-equivalent arc that solves P5 for endpoints $\underline{x}^0$ and $\bar{x}^1$. Therefore P4 is equivalent to the fixed endpoint problem P5, with endpoints $\underline{x}^0$ and $\bar{x}^1$.

Define the Hamiltonian

$$H(\theta, x, p) \quad = \quad \sup_z \{p \cdot z - L(\theta, x, z) \mid z \in \mathbb{R}^2\},$$

where $p \in \mathbb{R}^2$ are multipliers. Define $\mathbf{P}(\theta)$ as $\{p \in \mathbb{R}^2 | H(\theta,x,p)<+\infty\}$. Then for our problem $\mathbf{P}(\theta) = \{p \in \mathbb{R}^2 | p_1 \leq 0, p_2=p_1\}$, i.e. the normal cone to $\mathbf{Z}(\theta,x)$. Then we have:

$$H(\theta,x,p) = \begin{cases} +\infty & \text{if } p \notin \mathbf{P}(\theta) \\ p_1(\theta)u'_{a1}(\phi_1(x_1)) - L(\theta,x,z) & \text{if } p \in \mathbf{P}(\theta) \end{cases}$$

where we have used the fact that $p_1(\theta) = p_2(\theta)$ for $p \in \mathbf{P}(\theta)$. For arcs $x \in \mathbf{A}$ and $p \in \mathbf{A}$, the <u>Hamiltonian conditions</u> can be written:

$$(-p'(\theta),x'(\theta)) \in \partial H(\theta,x(\theta),p(\theta)) \quad \text{a.e.} \tag{HC}$$

where $\partial H(\theta,x(\theta),p(\theta))$ is the set of subgradients of $H(\theta,\cdot,\cdot)$ at $(x(\theta),p(\theta))$. Arcs $x \in \mathbf{B}$ and $p \in \mathbf{B}$ are said to satisfy <u>Extended Hamiltonian conditions</u> (EHC), if (HC) holds along with the following: (a) $\forall \theta \in \Omega$, $x_-(\theta) \in \mathbf{X}(\theta)$, $x_+(\theta) \in \mathbf{X}(\theta)$, $p_-(\theta) \in \mathbf{P}(\theta)$, and $p_+(\theta) \in \mathbf{P}(\theta)$; (b) for any representation

$$\xi(\theta)d\tau(\theta) = dx - x'(\theta)d\theta, \qquad \pi(\theta)d\tau(\theta) = dp - p'(\theta)d\theta$$

(where ξ and π are Borel measurable and $d\tau(\theta)$ is nonnegative) it is true that (i) $\pi(\theta)$ is normal to $\mathbf{X}(\theta)$ at $x_-(\theta)$ and $x_+(\theta)$ $[\tau]$-a.e., and (ii) $\xi(\theta)$ is normal to $\mathbf{P}(\theta)$ at $p_-(\theta)$ and $p_+(\theta)$ $[\tau]$-a.e.

<u>Lemma 2</u>: Any pair of arcs $x \in \mathbf{A}$ and $p \in \mathbf{A}$ that satisfy (HC) also satisfy (EHC).

<u>Proof</u>: See Rockafellar [32], Proposition 3, p. 168.#

Now define

$$M(\theta,p,s) \quad = \quad \sup_{x,z} \{p\cdot z + s\cdot x - L(\theta,x,z) \,|\, x\in\mathbb{R}^2, z\in\mathbb{R}^2\}.$$

Define the functional

$$J_M(p) \quad = \quad \int_{\theta_0}^{\theta_1} M(\theta,p(\theta),p'(\theta))d\theta \quad + \quad \int_{\theta_0}^{\theta_1} r_M(\theta,\pi(\theta))d\tau(\theta)$$

where $r_M(\theta,s)$ is the recession function of $M(\theta,\cdot,\cdot)$.

It is readily verified that L is a <u>Lebesgue-normal integrand</u>; that is, the epigraph

$$\text{epi } L(\theta,\cdot,\cdot) \quad = \quad \{(x,z,\alpha)\in\mathbb{R}^5 \,|\, \alpha\geq L(\theta,x,z)\}$$

is closed and depends Lebesgue-measurably on θ. Also, L is <u>convex</u> and is <u>proper</u> in the sense that L is not equal to $+\infty$ everywhere. Furthermore, the correspondences $\mathbf{X}(\theta)$ and $\mathbf{P}(\theta)$ are closed-valued and upper semicontinuous. We can then apply the following theorem:

> **<u>Theorem</u>**: Suppose L is a proper, convex, Lesbegue-normal integrand, and $\mathbf{X}(\theta)$ and $\mathbf{P}(\theta)$ are closed-valued and upper semicontinuous correspondences. Let $x\in\mathbf{B}$ and $p\in\mathbf{B}$ be a pair of arcs that satisfy (EHC) and such that $J_L(x)$ and $J_M(p)$ are not oppositely infinite. Then x is optimal for P5 and $J_L(x)$ and $J_M(p)$ are both finite.

Proof: This is a simplified version of Rockafellar's Theorem 2 [32, p. 171].

We will now show that a collateralized debt *contract satisfies* the conditions of the Theorem and thus is optimal for P5 (with endpoints $\underline{x}^0$ and $\bar{x}^1$) and thus is optimal for P4. First, take an arbitrary feasible value for R and construct the corresponding debt contract; call this x_R^*. Since we are seeking a solution to P5, we set the endpoints of x_R^* equal to $\underline{x}^0$ and $\bar{x}^1$ instead of the relevant limits. Thus x_R^* is absolutely continuous on intΩ, but discontinuous at the endpoints. Next we construct an arc $p_R \in \mathbf{B}$ that, along with x_R^*, satisfies the (EHC). Here we treat λ_a/λ_b as a free parameter, and derive an expression for the value of λ_a/λ_b that is consistent with the posited debt contract R. A value, $\underline{\lambda}$, is found that λ_a/λ_b must exceed for the collateral constraint not to bind, as assumed in Proposition 3. For convenience we will suppress the dependence of x_R^* and p_R on R where no ambiguity results, and write x^* and p. We will restrict attention to $R \in (-e_{b1}, \theta_1)$. Note that if $R = \bar{R}$, then $x_2^*(\theta_0) = \bar{x}_2$.

The Hamiltonian conditions (HC) can be written:

$$p'(\theta) \in \rho(\theta)p(\theta) + L^*_{x_1}(\theta)$$

(A.1)

$$p'(\theta) \in L^*_{x_2}(\theta)$$

where $\rho(\theta) = -u''_{a1}(\phi_1(x^*_1(\theta)))/u'_{a1}(\phi_1(x^*_1(\theta)))$,

$$L^*_{x_1}(\theta) = \begin{cases} (-\lambda_a + \lambda_b\nu_1(\theta))f(\theta) & \text{if } x^*_1(\theta)\in(\underline{x}_1,\bar{x}_1(\theta)), \\ [(-\lambda_a + \lambda_b\nu_1(\theta))f(\theta),+\infty) & \text{if } x^*_1(\theta)=\bar{x}_1(\theta), \\ (-\infty,(-\lambda_a + \lambda_b\nu_1(\theta))f(\theta)] & \text{if } x^*_1(\theta)=\underline{x}_1, \end{cases}$$

$$L^*_{x_2}(\theta) = \begin{cases} (-\lambda_a + \lambda_b\nu_2(\theta))f(\theta) & \text{if } x^*_2(\theta)\in(\underline{x}_2,\bar{x}_2), \\ [(-\lambda_a + \lambda_b\nu_2(\theta))f(\theta),+\infty) & \text{if } x^*_2(\theta)=\bar{x}_2, \\ (-\infty,(-\lambda_a + \lambda_b\nu_2(\theta))f(\theta)] & \text{if } x^*_2(\theta)=\underline{x}_2, \end{cases}$$

$$\nu_1(\theta) = u'_{b1}(e_{b1}+\theta-\phi_1(x^*_1(\theta)))/u'_{a1}(\phi_1(x^*_1(\theta)))$$

$$\nu_2(\theta) = u'_{b2}(e_{b2}+e_{a2}-\phi_2(x^*_2(\theta)))/u'_{a2}(\phi_2(x^*_2(\theta)))$$

In the collateralized debt contract, $x^*_1(\theta) = \underline{x}_1$ for $\theta\in[\theta_0,R]$, and $x^*_1(\theta)\in(\underline{x}_1,\bar{x}_1(\theta))$ for $\theta\in(R,\theta_1]$, while $x^*_2(\theta)\in(\underline{x}_2,\bar{x}_2)$ for $\theta\in(\theta_0,R)$ and $x^*_2(\theta) = \bar{x}_2$ for $\theta\in[R,\theta_1]$. If $x^*_2(\theta_0) = \underline{x}_2$, the contract is "collateral constrained." Using these facts we have:

$$p'(\theta) = \begin{cases} (-\lambda_a + \lambda_b\nu_2(\theta))f(\theta) & \text{if } \theta\in(\theta_0,R) \\ \rho(\theta)p(\theta) + (-\lambda_a + \lambda_b\nu_1(\theta))f(\theta) & \text{if } \theta\in(R,\theta_1). \end{cases} \qquad \text{(A.2)}$$

We will demonstrate later that

$$\rho(\theta)p(\theta) \quad + \quad (-\lambda_a + \lambda_b\nu_1(\theta))f(\theta) \quad \geq \quad (-\lambda_a + \lambda_b\nu_2(\theta))f(\theta), \tag{A.3}$$

so that the formula for $p'(\theta)$ given in (A.2) is consistent with (A.1).

The normality of $\pi(\theta)$ in (EHC) implies that if $\Delta p(\theta) \neq 0$ then $\sup\{\Delta p(\theta)(x_1+x_2) \mid (x_1,x_2) \in X(\theta)\}$ is achieved by $(x_1^*(\theta),x_2^*(\theta))$, $(x_{1-}^*(\theta),x_{2-}^*(\theta))$, and $(x_{1+}^*(\theta),x_{2+}^*(\theta))$. See [32]. The normality of $\xi(\theta)$ implies that if $\Delta x^*(\theta) \neq 0$, then

$$\begin{aligned} r_L(\theta,\Delta x^*(\theta)) \quad &= \quad (\Delta x_1^*(\theta) + \Delta x_2^*(\theta))p(\theta) \\ &= \quad (\Delta x_1^*(\theta) + \Delta x_2^*(\theta))p_-(\theta) \\ &= \quad (\Delta x_1^*(\theta) + \Delta x_2^*(\theta))p_+(\theta) \end{aligned}$$

where r_L is the recession function defined above. Together, these two conditions can be shown to imply the following endpoint relations:

$$\begin{aligned} &R > -e_{b1} \;\Rightarrow\; p(\theta_1) = \Delta p(\theta_1) = p_-(\theta_1) = 0 \\ &R < \bar{R} \;\Rightarrow\; p(\theta_0) = \Delta p(\theta_0) = p_+(\theta_0) = 0 \\ &p_+(\theta_0) < 0 \;\Rightarrow\; R = \bar{R} \\ &\theta \in \text{int}\Omega \;\Rightarrow\; \Delta p(\theta) \;=\; 0 \end{aligned}$$

Solving the differential equation (A.2), with endpoints $p_+(\theta_0) \leq 0$ and $p_-(\theta_1) = 0$, yields:

$$\mu = \lambda_a \int_{\theta_0}^{\theta_1} -u'_{a1}(\theta - y_1^*(\theta,R))f(\theta)d\theta \; +$$

$$\lambda_b \int_{\theta_0}^{R} u'_{b2}(y_2^*(\theta,R))\frac{u'_{a1}(\theta - y_1^*(\theta,R))}{u'_{a2}(e_{a2}-y_2^*(\theta,R))}f(\theta)d\theta$$

$$+ \quad \lambda_b \int_{R}^{\theta_1} u'_{b1}(e_{b1}+y_1^*(\theta,R))f(\theta)d\theta$$

$$= \quad \lambda_a E\left[\frac{\delta v_a(\theta,R)}{\delta R}\right] + \lambda_b E\left[\frac{\delta v_b(\theta,R)}{\delta R}\right], \qquad (A.4)$$

where $\mu = -p_+(\theta_0)u'_{a1}(0)$. The first term on the right side is negative and increasing in absolute value in R, while the second term is positive and decreasing in R. Define $\underline{\lambda}$ by

$$\underline{\lambda} E\left[\frac{\delta v_a(\theta,\bar{R})}{\delta R}\right] \quad = \quad - E\left[\frac{\delta v_b(\theta,\bar{R})}{\delta R}\right] .$$

For $\lambda_a/\lambda_b \geq \underline{\lambda}$, R is determined by (A.4) with $\mu = 0$. If $\lambda_a/\lambda_b < \underline{\lambda}$, then $R = \bar{R}$ and (A.4) determines μ.

Solving for $p(\theta)$, $\theta \in (\theta_0, \theta_1)$, yields:

$$p(\theta)u'_{a1}(\theta - y_1^*(\theta,R)) \quad = \quad p_+(\theta_0)u'_{a1}(0)$$

$$+ \quad \lambda_a E\left[\frac{\delta v_a(\hat{\theta},R)}{\delta R}\middle|\hat{\theta}\leq\theta\right]F(\theta) + \lambda_b E\left[\frac{\delta v_b(\hat{\theta},R)}{\delta R}\middle|\hat{\theta}\leq\theta\right]F(\theta) \qquad (A.5)$$

(A.4) and the fact that $\lambda_a \delta v_a(\theta,R)/\delta R + \lambda_b \delta v_b(\theta,R)/\delta R$ is increasing in θ implies that $p(\theta) < 0$ in (A.5). It remains to show that (A.3) holds. Using (A.5) to substitute for $p(\theta)$ in (A.3), we obtain

$$\left[u'_{b1}(e_{b1}+y_1^*(\theta,R)) \quad - \quad u'_{a1}(\theta-y_1^*(\theta,R))\frac{u'_{b2}(y_2^*(\theta,R))}{u'_{a2}(e_{a2}-y_2^*(\theta,R))}\right]\frac{f(\theta)}{F(\theta)}$$

$$\geq \quad - \frac{\rho(\theta,R)p(\theta)u'_{a1}(\theta-y_1^*(\theta,R))}{\lambda_b F(\theta)}$$

$$= \quad - \rho(\theta,R)\left\{\left(\frac{\lambda_a}{\lambda_b}\right)E\left[\frac{\delta v_a(\hat{\theta},R)}{\delta R}\middle|\hat{\theta}\leq\theta\right] + E\left[\frac{\delta v_b(\hat{\theta},R)}{\delta R}\middle|\hat{\theta}\leq\theta\right]\right\}$$

$$- \quad \rho(\theta,R)p_+(\theta_0)\frac{u'_{a1}(0)}{\lambda_b F(\theta)} \tag{A.6}$$

When $R < \bar{R}$, $p_+(\theta_0) = 0$ and (A.6) is implied by Condition 2. When $p_+(\theta_0) < 0$, then (A.6) is implied by Condition 4. Since x^* is feasible by construction, $J_L(x^*)$ is finite, and thus $J_L(x^*)$ and $J_M(p)$ are not oppositely infinite. Thus the conditions of the Theorem are fulfilled and x^* is optimal for P4 and P5.

To establish the uniqueness of the optimal contract, suppose that x^* is an optimal debt contract as above, and that $\hat{x}$ is some other optimal contract. Obviously $\hat{x}$ must be feasible, and so $J_L(\hat{x}) < +\infty$. Without loss of generality, we can assume that $\hat{x}$ and x^* differ on a set of positive Lebesgue measure, for otherwise they would belong to the same equivalence class. Then we have

$$
\begin{aligned}
J_L(\hat{x}) \quad - \quad J_L(x^*) & \\
= \quad & \int_{\theta_0}^{\theta_1} [L(\theta,\hat{x}(\theta)) \quad - \quad L(\theta,x^*(\theta))]d\theta \\
\geq \quad & \int_{\theta_0}^{\theta_1} [L^*_{x_1}(\theta,x^*(\theta))(\hat{x}_1(\theta) - x_1^*(\theta)) \quad - \quad L^*_{x_2}(\theta,x^*(\theta))(\hat{x}_2(\theta) - x_2^*(\theta))]d\theta \\
\geq \quad & \int_{\theta_0}^{\theta_1} [p'(\theta) - \rho(\theta)p(\theta)](\hat{x}_1(\theta) - x_1^*(\theta)) \quad + \quad p'(\theta)(\hat{x}_2(\theta) - x_2^*(\theta))]d\theta \\
= \quad & -\int_{\theta_0}^{\theta_1} p(\theta)[\hat{x}_1'(\theta) + \hat{x}_2'(\theta) + \rho(\theta)(\hat{x}_1(\theta) - x_1^*(\theta)) - x_1^{*\prime}(\theta) - x_2^{*\prime}(\theta)]d\theta \\
& - \int_{(\theta_0,\theta_1)} p_-(\theta)[\hat{\xi}_1(\theta) + \hat{\xi}_2(\theta)]d\theta \\
> \quad & 0.
\end{aligned}
$$

The first two inequalities follow from the convexity of L, the Hamiltonian conditions, and the fact that $L^*_{x_1} > L^*_{x_2}$ from Assumption 3. The subsequent equality uses the "integration-by-parts" formula in Rockafellar [32], Proposition 1, p. 161, and the fact that $p(\theta_0) = \Delta p(\theta_0) = p(\theta_1) = \Delta p(\theta_1) = 0$. The final inequality uses: $p(\theta) \leq 0, \forall\theta$; $\hat{\xi}_1(\theta) + \hat{\xi}_2(\theta) \geq 0 \ \forall\theta$ (otherwise $J_L(\hat{x}) = +\infty$); and the convexity of $u'_{a1}(\phi_1(x_1))$ in x_1 (from Assumption 2). The latter implies that

$$
\begin{aligned}
\hat{x}_1'(\theta) \quad + \quad \hat{x}_2'(\theta) \quad \geq \quad & u'_{a1}(\phi_1(\hat{x}_1(\theta))) \\
\geq \quad & u'_{a1}(\phi_1(x_1^*(\theta))) \quad - \quad \rho(\theta)(\hat{x}_1(\theta) \quad - \quad x_1^*(\theta)) \\
= \quad & x_1^{*\prime}(\theta) - x_2^{*\prime}(\theta) \quad - \quad \rho(\theta)(\hat{x}_1(\theta) \quad - \quad x_1^*(\theta)).
\end{aligned}
$$

Because $-p(\theta) > 0$ $\forall\theta\in\text{int}\Omega$, the last inequality is strict unless $\hat{x}'(\theta) = x^{*\prime}(\theta)$ a.e. and $\hat{\xi}(\theta) = 0$. But if this is true $\hat{x}$ and x^* are equivalent. Thus $J_L(\hat{x}) > J_L(x^*)$ and x^* is the unique optimal contract.

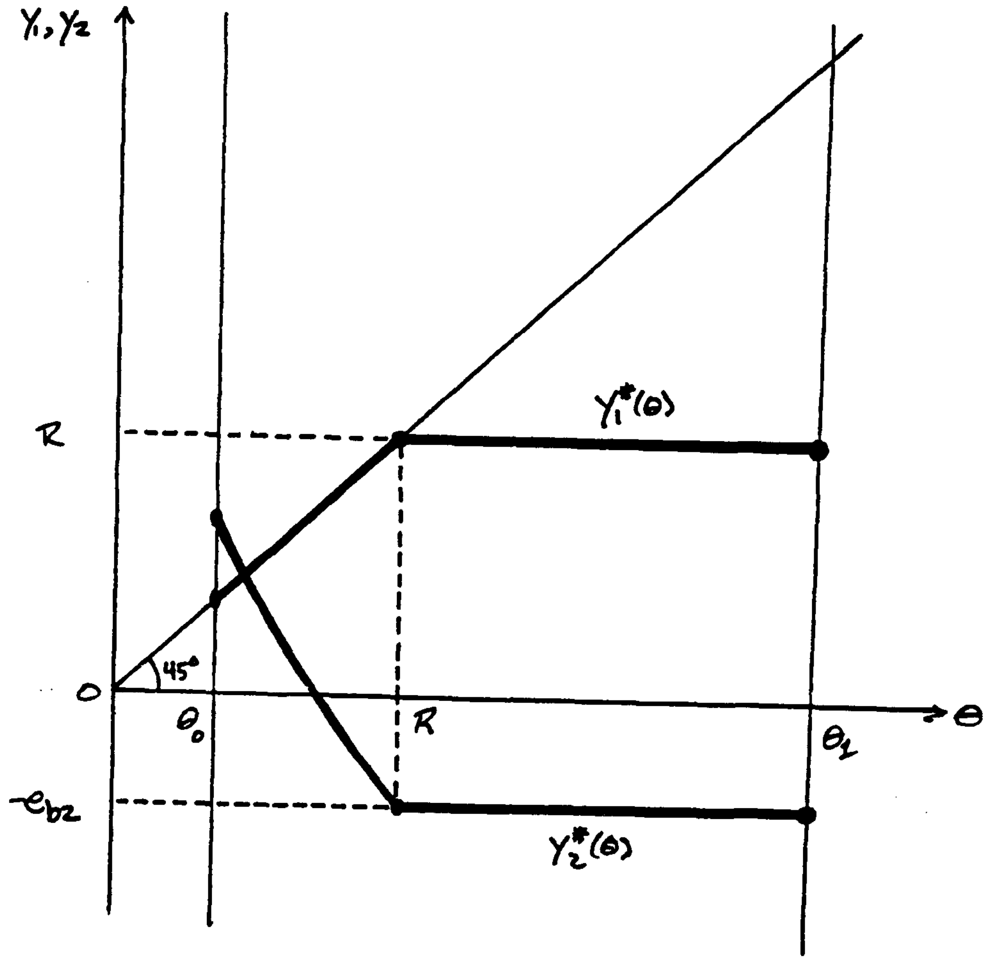
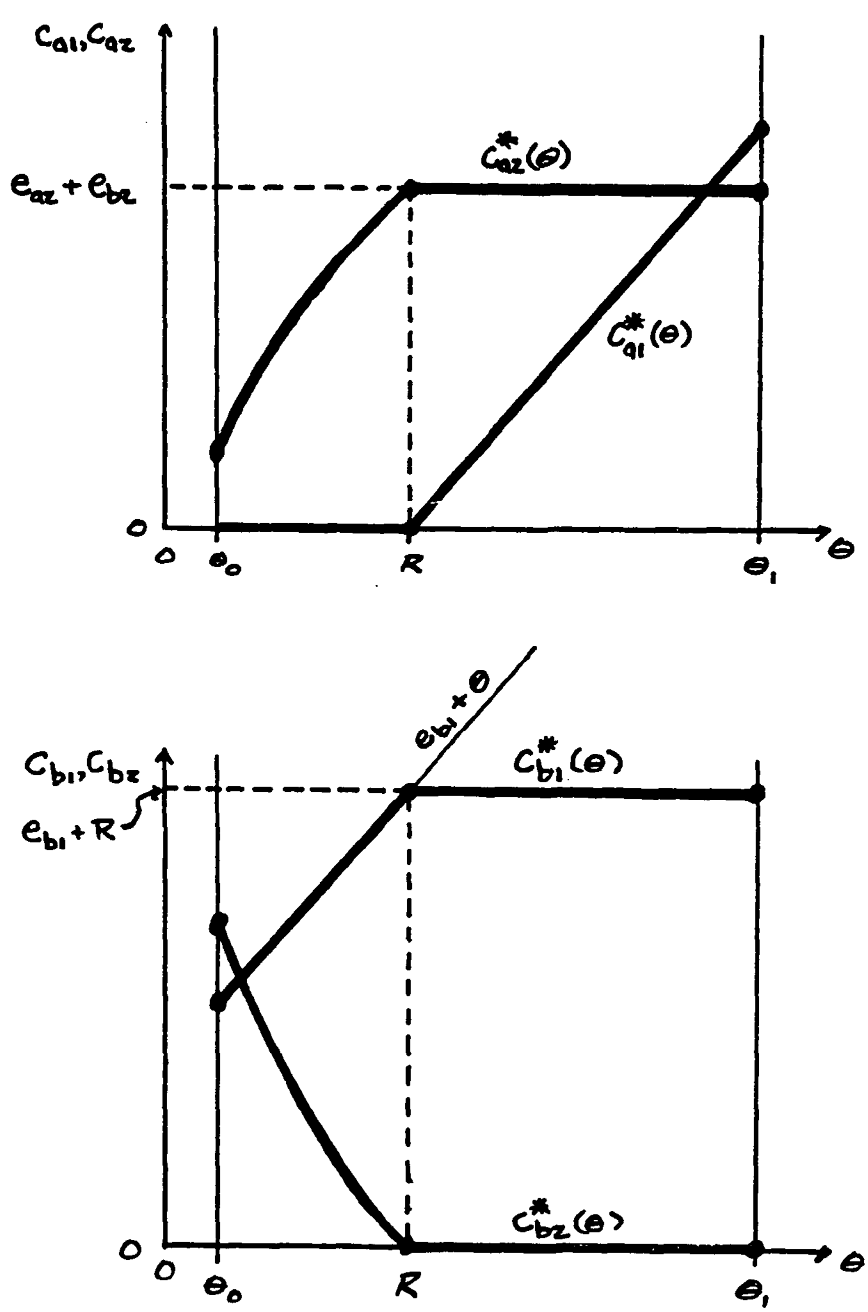


Figure 1. A Collaterallized Debt Contract.

Figure 2. Consumption Schedules under

A Collateralized Debt Contract.



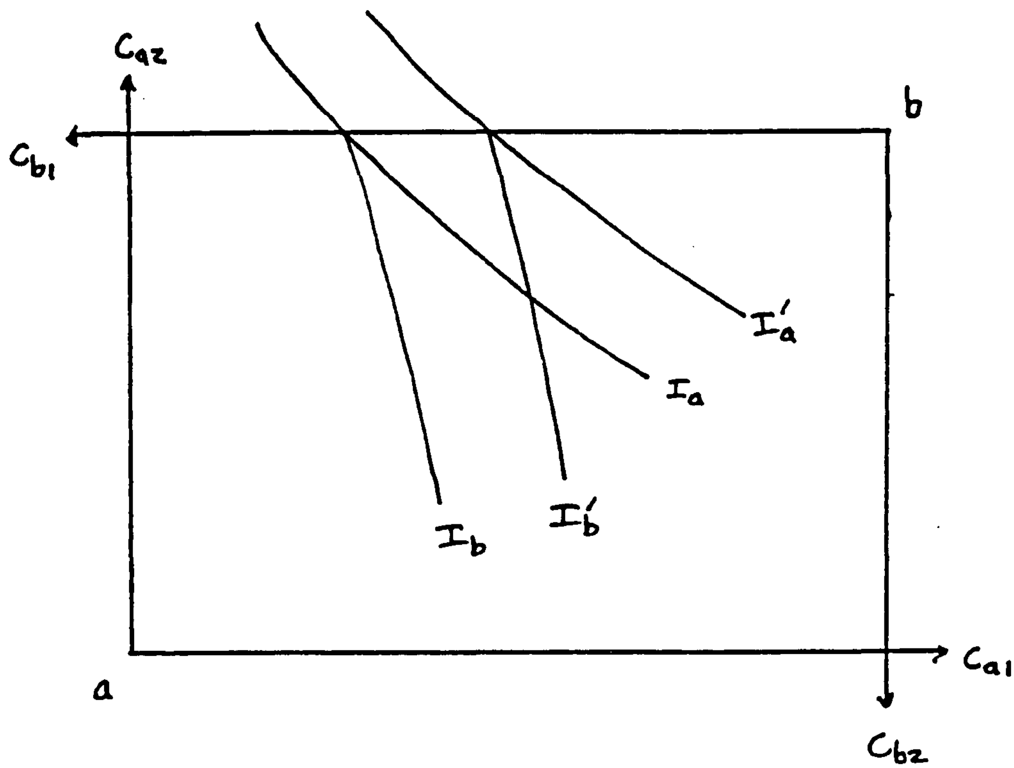


Figure 3. Indifference Curves That Satisfy

Condition 2, for a fixed θ

NOTES

1. Indeed, it is often difficult to find environments in which loan contracts are distinguished from the closely related arrangements of partnership and pawnbroking (see Lacker and Weinberg [23] and the citations there concerning the former, and de Roover [12] and Caskey [9] concerning the latter). With two alterations, the model presented here delivers pawnbroking as an optimal arrangement: first, the collateral good is durable, and exists at the initial contracting date; and there is a possibility of the borrower absconding.

2. It might be fruitful to embed the environment specified here in a setting where agents take actions *ex ante* which determine these endowment patterns. One might conjecture that the lender would choose $e_{b2} = 0$.

3. Additive separability of u_b can easily be dispensed with, and it seems as if nothing in principle prevents dispensing with it for u_a as well, although the analysis would be considerably more cumbersome.

4. Diamond's [13] setup can be obtained by setting $e_{b1} = e_{b2} = 0$, $\theta_0 = 0$, $u'_{b2}(c_{b2}) \equiv 0$, $u_{a1}(c_{a1}) = c_{a1}$, $u_{a2}(c_{a2}) = c_{a2}$, and then relaxing Assumption 1(d) to allow costless faking of goods that do not exist. The latter is inconsequential, so his environment is effectively a special case of Assumption 1. The "nonpecuniary penalties" are $u_{a2}(e_{a2}) - u_{a2}(e_{a2}-y_2)$.

5. This presumes that agents have access to an enforcement facility of some sort that can punish agent a for withholding stipulated transfers, but

is not capable of overcoming the informational imperfection of costless falsification.

6. (IF′) holds even if $\theta^*(\theta)$ is not unique or if agent a randomizes.

7. A real-valued function on an interval is of bounded variation if it is the difference of two monotone real-valued functions on the interval. A function of bounded variation can have a countable number of discontinuities. See Royden [33, pp. 98-100].

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