Working Paper Series

Firms as Clubs in Walrasian Markets with Private Information: Technical Appendix

WP 05-11

Edward Simpson Prescott
Federal Reserve Bank of Richmond

Robert M. Townsend
University of Chicago
Federal Reserve Bank of Richmond

THE FEDERAL RESERVE BANK OF RICHMOND
RICHMOND ▪ BALTIMORE ▪ CHARLOTTE

Firms as Clubs in Walrasian Markets with Private Information: Technical Appendix

Edward Simpson Prescott*
Federal Reserve Bank of Richmond
Robert M. Townsend*
University of Chicago
Federal Reserve Bank of Chicago

Working Paper 05-11

November 1, 2005

Abstract

This paper proves the Welfare Theorems and the existence of a competitive equilibrium for the club economies with private information in Prescott and Townsend (2005). The proofs cover lottery economies with a finite number of goods and without free disposal. A mapping based on Negishi (1960) is used.

JEL Classification: D51, D71, D82, L11

Keywords: competitive equilibrium, clubs, private information

*Townsend would also like to thank the NSF and the NIH for financial support. The views expressed in this paper are solely those of the authors and do not necessarily reflect the views of the Federal Reserve Banks of Chicago and Richmond or the Federal Reserve System. Contact information: Prescott, Federal Reserve Bank of Richmond, P.O. Box 27622, Richmond, VA 23261, Edward.Prescott@rich.frb.org; Townsend, Dept. of Economics, University of Chicago, 1126 E. 59th St., Chicago, IL 60637, rtownsen@midway.uchicago.edu.

1 Introduction

This paper is the technical appendix to Prescott and Townsend (2005). It contains proofs of the Welfare theorems and of the existence of competitive equilibria for their model. The proofs are based on Negishi (1960) who proved existence by finding a fixed point in the space of Pareto weights that is a competitive equilibrium with zero transfers. While the linear structure of Prescott and Townsend (2005) simplifies the problem relative to his, there are two other differences that complicates the problem. Because of these differences, some modifications to his proof are required.

The first difference is that we do not assume free disposal. Free disposal makes prices positive and in our economies, with its hedonic properties, this is entirely inappropriate. Negishi's proof, as it stands, cannot handle negative prices. In his mapping, the n prices are mapped into an $n-1$ dimensional simplex. We modify his approach to incorporate negative prices by requiring prices to lie in the closed unit ball. The unit ball includes the zero price point. In proofs based on the excess demand correspondence this can cause problems because that mapping is not upper hemi-continuous. As we shall see, this is not a concern for Negishi's mapping in our context.

The second difference between Negishi's economy and ours is that Negishi assumes that endowments are interior to consumption sets. There are two reasons he makes this assumption. First, it satisfies Slater's condition, a sufficient condition for the application of the Kuhn-Tucker theorem. Assuming an interior endowment is clearly inappropriate in our economies. Instead, we obtain the existence of Kuhn-Tucker vectors directly from the linear structure of our economy. A Kuhn-Tucker vector, the dual variables in a linear program, will exist if a solution to the linear program is finite.

The second reason that Negishi assumes that endowments are interior to consumption sets is that when combined with positive prices (implied by free disposal and a non-satiation assumption) there exists a cheaper consumption point for each agent. The existence of a cheaper consumption point implies that the Pareto weights at the fixed point are strictly positive. Strictly positive Pareto weights are important because they imply that the fixed point of Negishi's mapping is a competitive equilibrium rather than just a compensated equilibrium. In our economies, it is natural to endow agents with zero units of almost every commodity. Consequently, we cannot make the interiority assumption. Instead, we will guarantee the existence of a cheaper point by making reasonable assumptions on the production technology in our economy.

2 A Short Review of Prescott and Townsend (2005)

Prescott and Townsend (2005) study a model in which agents purchase memberships and jobs in a variety of firms. Each type of a firm is defined by its contractual terms. These include the inputs it uses, how hard participants work, and the output-contingent compensation of all members of the firm. The joint consumption and joint production features of these multi-agent contracts are what make them club goods.

Private information of agents' efforts is a critical feature of the environment. We assume that these contracts are exclusive in the sense that an agent cannot side trade away from the contractual terms. This assumption is important because an agent who is not fully insured by a contract has an incentive to purchase additional insurance, which undoes the original contract.

Participating in a firm, or club or contract, gives an agent utility, and creation of a firm uses resources. In this appendix we do not specify the details of these contracts; they are described in the main paper. Instead, we define utility and resource usage directly in terms of the contracts and jobs because that is all that matters for the decentralization theory.

There is a continuum of agents but only a finite number of types. Each type i is a positive fraction α_i of the population. In most of Prescott and Townsend (2005) the only difference between agents is their wealth. For consistency, we lay out the model with this assumption. For the proofs, however, the notation is expanded to be applicable to the case where agents differ in their abilities or preferences.

Type-i agents are endowed with capital κ_i. Agents sell their capital for income that they use to purchase membership in single-agent or multi-agent firms. An incentive compatible self-employment firm is a $b_1 \in B_1$. An agent's utility from participation in a b_1 contract is $u(b_1)$. Each contract or firm produces output and uses consumption. The net amount of such consumption is $r_{(c-q)}(b_1)$. Similarly, each contract uses $r_k(b_1)$ of the capital input.

Multi-agent firms are similar. The main paper distinguishes among several different classes of firms: a supervisor and a worker, relative performance, and team production. In this section, to simplify the exposition we only consider the supervisor-worker firms. Let $b_2 \in B_2$ be the contract describing one such firm. The utility an agent receives from participation in a b_2 firm also depends on his job within the firm. We write his utility as $u(b_2, j)$, $j = w, s$, where $j = w$ indicates that the agent is the worker and $j = s$ indicates that the agent is the supervisor. Resource usages for a two-person firm are $r_{(c-q)}(b_2)$ and $r_k(b_2)$.

We assume that there is a finite number of elements in B_1 and B_2 so the commodity space is $L = \Re^{\#B_1+2(\#B_2)+1}$, where $\#B_i$ denotes the number of elements in set B_i. The commodity space reflects a choice among the $\#B_1$ of self-employment firms, the $\#B_2$ of worker-supervisor firms along with a position in that firm, and the level of the capital endowment sold. Let $p(b_1)$ denote the price of a b_1 firm and let $p(b_2, j)$ be the price of a b_2 firm joint with position $j = w, s$ in that firm. We let p_k denote the price of a unit of the capital endowment.

Agents purchase membership in one of several types of firms plus, if appropriate, a job in that firm. Let $x_i(b_1)$ denote the measure of type-b_1 firms purchased by a type-i agent, and let $x_i(b_2, j)$ denote the measure of job-j in type-b_2 firms purchased by a type-i agent. In equilibrium, these measures will indicate the fraction of type-i agents assigned to a particular type of firm, and, if appropriate, to a particular job within a firm. A type-i's consumption set X_i is

$$X_i = \{x_i(b_1) \geq 0, x_i(b_2, j) \geq 0, j = w, s | \sum_{b_1} x_i(b_1) + \sum_{b_2,j} x_i(b_2, j) = 1\}.$$

This set guarantees that $(x_i(b_1), x(b_2, j))$ is a probability measure.[1]

The problem of a type-i consumer is

$$\max \sum_{b_1} x_i(b_1)u(b_1) + \sum_{b_2,j} x_i(b_2, j)u(b_2, j)$$

subject to the budget constraint

$$\sum_{b_1} x_i(b_1)p(b_1) + \sum_{b_2,j} x_i(b_2, j)p(b_2, j) \leq p_k\kappa_i,$$

[1] Note that $X_i \subset L$ and that, in particular, there is no choice corresponding to the last component, the capital endowment, as that is supplied inelastically.

and $x_i \in X_i$.

The production sector forms firms. Its input is capital and its outputs are memberships in firms, or clubs. To do this, it converts the capital stock into the capital input and it creates firms by supplying the capital input, state-contingent consumption, and people to staff the positions. Each multi-agent firm needs an agent to purchase the worker position and an agent to purchase the manager position. Because there is constant returns to scale in setting up firms, only one profit-maximizing entity is needed. To distinguish it from the firms of our theory, we refer to it as the production sector.

Let $\delta(b_1)$ be the number of b_1 firms produced by the sector and let $\delta(b_2)$ be the number of b_2 firms produced. The sector purchases capital y_κ from consumers and then uses it to create the capital input used by the firms that it creates. As is conventional, the input y_k is negative in sign, and outputs, the number of firms of each type, are positive in sign. The capital constraint is

$$\sum_{b_1} \delta(b_1) r_k(b_1) + \sum_{b_2} \delta(b_2) r_k(b_2) + y_\kappa \leq 0, \tag{1}$$

so capital sold as part of firms is less than or equal to capital purchased.

The sector also transfers contingent consumption between agents. It collects premia and distributes indemnities to firms experiencing low output, and these premia and indemnities need to balance in order to sum to zero. With the resource requirements of the consumption good $r_{c-q}(b_i)$, written in expectation, the consumption resource constraint is

$$\sum_{b_1} \delta(b_1) r_{(c-q)}(b_1) + \sum_{b_2} \delta(b_2) r_{(c-q)}(b_2) \leq 0. \tag{2}$$

With a continuum of agents the left-hand side is a fixed number.

In selling supervisor-worker firms, b_2, the production sector also staffs positions in them. Each one of these b_2 firms requires two members, so for each level of b_2 produced there needs to be one worker for each supervisor. We write this club or matching constraint by creating notation $y(b_2, w)$ for the number of multi-agent firms with a worker and $y(b_2, s)$ for the number of multi-agent firms with a supervisor, then requiring

$$\forall b_2,\ \delta(b_2) = y(b_2, w) = y(b_2, s). \tag{3}$$

There is also a trivial constraint for the self-employment firm:

$$\forall b_1,\ \delta(b_1) = y(b_1). \tag{4}$$

The production set is

$$Y = \{y \in L | \exists \delta(b_1) \in \Re_+^{\#B_1}, \delta(b_2) \in \Re_+^{\#B_2} \ni (1), (2), (3), (4) \text{ hold}\}.$$

Y is the aggregate production set.

Given prices, the production sector maximizes

$$\max_{y \in Y, \delta(b_1), \delta(b_2)} \sum_{b_1} p(b_1) y(b_1) + \sum_{b_2, j} p(b_2, j) y(b_2, j) + p_k y_\kappa.$$

Market clearing requires that the number of firms of each possible type purchased by agents be equal in number to the those sold by the production sector, and similarly that the capital good sold by agents equal that bought by the production sector:

$$\forall b_1, \ \sum_i \alpha_i x_i(b_1) = y(b_1), \tag{5}$$

$$\forall b_2, j, \ \sum_i \alpha_i x_i(b_2, j) = y(b_2, j), \tag{6}$$

$$\sum_i \alpha_i \kappa_i + y_\kappa = 0. \tag{7}$$

Let x_i denote the vector of purchases $x_i(b_1)$ and $x_i(b_2, j)$ for type i and let x be the vector of consumptions across the I types of agents. Similarly, let y denote the vector of output vectors $y(b_1)$, $y(b_2, j)$, and y_k, and let p denote the vector of prices $p(b_1)$ and $p(b_2, j)$.

Definition 1 *A competitive equilibrium for this economy is an (x^*, y^*, p^*) such that*

1. *$\forall i$, x_i^* maximizes $\sum_{b_1} x_i(b_1) u(b_1) + \sum_{b_2, j} x_i(b_2, j) u(b_2, j)$ subject to $x_i \in X_i$ and $\sum_{b_1} x_i(b_1) p^*(b_1) + \sum_{b_2, j} x_i(b_2, j) p^*(b_2, j) \leq p_k^* \kappa_i$.*

2. *y^* maximizes $p^* y$ subject to $y \in Y$.*

3. *Markets clear, that is, (5), (6), and (7) hold.*

Pareto optimum can be found by solving the Pareto program. Let λ_i denote the Pareto weight on type-i agents and λ the corresponding vector across types. The Pareto program is:

$$\max_{x, y, \delta, k} \sum_i \lambda_i \alpha_i \left(\sum_{b_1} x_i(b_1) u(b_1) + \sum_{b_2, j} x_i(b_2, j) u(b_2, j) \right)$$

subject to $x_i \in X_i$, $y \in Y$, (5), (6), and (7). Note that combining market clearing with the production set gives the more traditional representation of the resource constraints. Also note that this program is a linear program.

3 Proofs

In this section we provide proofs for a linear economy with lotteries that cover the economy described above. The proofs also cover the important case where agents are heterogeneous in ability and preferences. Because of the additional generality, we will modify the above notation and make it more abstract. First, we define the consumption set by a finite number of linear inequalities, that is,

$$X_i = \{x_i \in \Re^n_+ | g_i x_i - b_i \leq 0\}.$$

As before x_i is a vector. There may be any finite number of constraints so the object g_i is a matrix. These constraints may hold as an equality or as an inequality. (There is no need to worry about $\geq$ constraints. Just multiply these by negative one to put them in the above form.) For expositional convenience, we will use the above notation to reflect that either case is possible and then examine each case in the proofs as needed. In the economy laid out above, the only consumption set constraint is to guarantee that the agent chooses a probability measure. In that case, g_i is a vector of ones, $b_i = 1$, and the constraint holds at equality. If agents are heterogeneous in their abilities or preferences, as in the extension of the above model sketched out in Prescott and Townsend (2005), then there are constraints that require $x_i = 0$ for allocations that assign an agent to a contract that is infeasible for his type. In general, however, the only requirement for the proofs below is that there be a finite number of linear constraints, $x_i \geq 0$, and that X_i be bounded.

In the above model agents inelastically supply the capital endowment. Consequently, X_i would be in a lower-dimensional space then Y. Rather than distinguishing between the endowment and consumption goods we simply expand the dimension of X_i, here, by the number of inelastically supplied endowment goods. Consumption of these goods are then constrained to equal zero. Of course, agents get no direct utility from supplying these goods so the utility coefficient on them is zero.

Production is defined similarly. Again, we represent the production set by a finite number of linear inequalities. The production set is

$$Y = \{y \in \Re^n | fy \leq 0\},$$

where f is a matrix. As with the consumption set, the notation represents the possibility that the constraints are either inequalities or equalities. In the model economy, the club constraints hold at equality while the production constraints on consumption and capital are inequalities. The production set is a convex cone.

Finally, we let ξ_i denote the vector endowment of a type i. In the economy described above, the natural endowment is for ξ_i to be zero everywhere except for a κ_i in the last component. We do not impose that restriction here for two reasons. First, there is no need to limit the applicability of the proofs below. Second, in the above economy there are Pareto optima that cannot be supported as competitive equilibria with a natural endowment distribution, that is, where agents' endowments only differ in their holding of the capital (and they cannot be endowed with a negative quantity of capital). These optima can be supported if income transfers are allowed, and by considering the full set of endowments we can study these income transfer cases as well.

Using dot-production notation we have the following definitions:

Definition 2 *A compensated equilibrium in this economy is an* (x^*, y^*, p^*) *such that*

1. $\forall i$, x_i^* *minimizes* $p^* x_i$ *subject to* $x_i \in X_i$ *and* $u_i x_i \geq u_i x_i^*$.

2. y^* *maximizes* $p^* y$ *subject to* $y \in Y$.

3. $\sum_i \alpha_i (x_i^* - \xi_i) = y^*$.

Definition 3 *A competitive equilibrium in this economy is the same as a compensated equilibrium except that 1. above is replaced by*

1. $\forall i$, x_i^* *maximizes* $u_i x_i$ *subject to* $x_i \in X$ *and* $p^* x_i \leq p^* \xi_i$.

Our proofs are based on solving the Pareto problem so we rewrite it in the general notation. Goods are indexed by j. Consequently, the jth coefficient of the vector x_i will be denoted $x_{i,j}$. For matrices like f or g_i, let f_j and $g_{i,j}$ denote the jth column of that matrix. The Pareto program is

$$\max_{\{x_i\} \geq 0, y} \sum_i \lambda_i \alpha_i u_i x_i$$

$$
\begin{aligned}
\text{s.t.} \quad \sum_i \alpha_i(x_i - \xi_i) - y &= 0, \\
fy &\leq 0, \\
\forall i, \;\; g_i x_i &\leq b_i.
\end{aligned}
$$

4 Proofs

The economy described above is linear. In this section, we provide proofs of the Welfare Theorems and of the existence of a competitive equilibrium for linear economies with a finite number of goods and without free disposal.

Our commodity space is Euclidean and consumption is bounded from below by zero. The only additional assumptions we make are

Assumption 1 *$\forall i$, X_i is bounded,*

Assumption 2 $\sum_i \alpha_i(X_i + \xi_i) \cap Y \neq \emptyset$,

Assumption 3 *$\forall i$, for any x_i component of a feasible allocation there exists $x_i' \in X_i$ such that $u_i x_i' > u_i x_i$.*

The first assumption is satisfied when the consumption set has a probability measure constraint. Because the consumption set is represented by a finite set of linear inequalities, it is convex and closed. The boundedness assumption then makes this set compact. No additional assumptions are needed on the production set because it is a closed, convex cone. The second assumption just assumes that a feasible allocation exists. The third assumption is a no-satiation assumption and is easy to satisfy by simple assumptions on the underlying economy. Furthermore, this assumption plus the linearity in the economy implies that there are no local satiation points in the consumption set. Finally, note that the linear structure of this economy means that the utility function and constraints are all differentiable.

We will use Kakutani's fixed point theorem to prove the existence of a competitive equilibrium. We state it without proof.

Theorem 1 *Let Z be a compact convex set in $\Re^n$ and let $h(z)$ be a non-empty, compact-valued, convex-valued, upper hemi-continuous correspondence mapping Z into Z. Then, there is a fixed point $\widehat{z}$ such that $\widehat{z} \in f(\widehat{z})$.*

There are no features of our model that require modification of the standard proofs of the First Welfare Theorem so we state it without proof:

Theorem 2 (First Welfare Theorem) *If (x, y, p) is a competitive equilibrium and no agents are satiated at x, then the allocation (x, y) is Pareto optimal.*

The Second Welfare Theorem as well as the existence proof use properties of solutions to the Pareto program.

Theorem 3 *Assuming that a feasible solution to the Pareto program exists, then for any set of weights $\lambda \in S^{I-1}$, an optimal feasible solution to the Pareto program exists.*

Proof: The constraint set is closed. It is also bounded because for each i, X_i is bounded. Furthermore, the set of feasible y is bounded by the resource constraint in conjunction with the bounds on X_i. Therefore, x and y are chosen from compact sets. The objective function is continuous. Therefore, a maximum exists. **Q.E.D.**

We now characterize a solution to the Pareto program with the first-order conditions. Let $\widehat{p}_j$ be the dual variable (or multiplier) on the market clearing constraint for the jth good, let $\widehat{\mu}$ be the vector of dual variables on the production set constraints, and let $\widehat{\gamma}_i$ be the dual variables on agent i's consumption set constraints. Dual variables for equality constraints can be of any sign while those for less-than-or-equal constraints are restricted to be non-negative. When the notation is designed to capture constraints of either form, the sign restrictions on the dual variables will not be explicitly written out.

Theorem 4 *An allocation (x, y) with $x \geq 0$ is a solution to the Pareto program if and only if it and the dual variables $(\widehat{p}, \widehat{\mu}, \widehat{\gamma} \geq 0)$ satisfy*

$$\forall i, j, \;\; \lambda_i \alpha_i u_{i,j} - \widehat{p}_j \alpha_i - \widehat{\gamma}_i g_{i,j} = 0, \;\; (\leq 0 \textit{ if } x_{i,j} = 0) \tag{8}$$

$$\forall j, \;\; \widehat{p}_j - \widehat{\mu} f_j = 0, \tag{9}$$

$$\widehat{p}(\sum_i \alpha_i(x_i - \xi_i) - y) \;=\; 0, \;\; \sum_i \alpha_i(x_i - \xi_i) - y = 0, \tag{10}$$

$$\widehat{\mu} f y = 0, \qquad f y \leq 0, \tag{11}$$

$$\forall i, \;\; \widehat{\gamma}_i(g_i x_i - b_i) = 0, \qquad g_i x_i \leq b_i. \tag{12}$$

Proof: The Pareto program is a linear program so the Kuhn-Tucker conditions are necessary and sufficient. **Q.E.D.**

At this point, it is worth proving a preliminary result.

Lemma 1 *For $(\widehat{p}, y)$ to be part of a solution to the Pareto program, $\widehat{p}y = 0$.*

Proof: Multiplying (9) by y_j and summing over j gives $\widehat{p}y = \widehat{\mu}fy$. By (11), $\widehat{\mu}fy = 0$ so $\widehat{p}y = 0$. **Q.E.D.**

The other lemma we need is that with non-satiation $\widehat{p} \neq 0$.

Lemma 2 *With non-satiation, there does not exist a solution to the Pareto program such that $\widehat{p} = 0$.*

Proof: By non-satiation there exists for each i, $x'_i \in X_i$ such that $u_i x'_i > u_i x_i$. If $\widehat{p} = 0$, then by (8) and (12), $\lambda_i \alpha_i u_i x_i = \widehat{\gamma}_i g_i x_i = \widehat{\gamma}_i b_i$. By (12), $\widehat{\gamma}_i b_i \geq \widehat{\gamma}_i g_i x'_i$. (If $\widehat{\gamma}_i = 0$ it holds trivially, while if $\widehat{\gamma}_i \geq 0$ it holds because $g_i x'_i \leq b_i$.) Therefore, $\lambda_i \alpha_i u_i x'_i > \widehat{\gamma}_i g_i x'_i$. Then because $x'_i \geq 0$, for some j, $\lambda_i \alpha_i u_{i,j} > \widehat{\gamma}_i g_{i,j}$. But that contradicts (8) so $\widehat{p} \neq 0$. **Q.E.D.**

We express the compensated equilibrium conditions in terms of the necessary and sufficient conditions. For the consumer's minimization problem let $\beta_i \geq 0$ be the dual variable on the constraint $u_i x_i \geq u_i x_i^*$, and let υ_i be the vector of dual variables on the consumption set constraints. For the production sector, let μ be the vector of dual variables on the production constraints.

Lemma 3 *Conditions 1 and 2 in the definition of a compensated equilibrium can be rewritten in the following form:*

1. *$\forall i$, $x_i^* \geq 0$ and dual variables $(\beta_i \geq 0, \upsilon_i)$ satisfy condition 1 of a compensated equilibrium if and only if they satisfy*

$$\forall j, \;\; \beta_i u_{i,j} - p_j - \upsilon_i g_{i,j} \;=\; 0, (\leq 0 \text{ if } x_{i,j} = 0) \tag{13}$$

$$\beta_i(u_i x_i - u_i x_i^*) \;=\; 0, \;\; u_i x_i - u_i x_i^* \geq 0, \tag{14}$$

$$\upsilon_i(g_i x_i - b_i) \;=\; 0, \;\; g_i x_i - b_i \leq 0. \tag{15}$$

2. y and dual variable μ satisfy condition 2 of a compensated equilibrium if and only if

$$\begin{aligned} \forall j,\ \ p_j - \mu f_j &= 0 \qquad (16) \\ \mu f y &= 0, \ \ fy \leq 0. \qquad (17) \end{aligned}$$

Proof: Both problems are linear programs. **Q.E.D.**

Theorem 5 (Second Welfare Theorem) *Any solution to the Pareto program for some Pareto weights $\lambda \geq 0$ can be supported as a compensated equilibrium.*

Proof: Given a solution to the Pareto program, necessary and sufficient condition (10) implies that condition three of a compensated equilibrium, market clearing, holds. Let $p = \widehat{p}$, $\beta_i = \lambda_i$, $\nu_i = \widehat{\gamma}_i/\alpha_i$, and $\mu = \widehat{\mu}$. Comparison of necessary and sufficient conditions (9) and (11) with the necessary and sufficient conditions (16) and (17) shows that the condition two of a compensated equilibrium holds. Comparison of (8) and (12) with (13) and (15) shows that these two necessary and sufficient conditions of a compensated equilibrium holds. The remaining part of condition one, (14), holds trivially because $x_i = x_i^*$. **Q.E.D.**

The conditions for a competitive equilibrium are nearly identical. The only change is the necessary and sufficient conditions for the Consumer's problem. Let $\theta_i \geq 0$ be the dual variable on agent i's budget constraint and let ω_i be the dual variable on agent i's consumption set constraints. Then, $x_i \geq 0$ and dual variables $(\theta_i \geq 0, \omega_i)$ satisfy condition one of a competitive equilibrium if and only if they satisfy

$$\begin{aligned} \forall j,\ \ u_{i,j} - \theta_i p_j - \omega_i g_{i,j} &= 0, \ \ (\leq 0 \text{ if } x_i = 0) \qquad (18) \\ \theta_i p(x_i - \xi_i) &= 0, \ \ p(x_i - \xi_i) \leq 0, \qquad (19) \\ \omega_i(g_i x_i - b_i) &= 0, \ \ g_i x_i - b_i \leq 0. \qquad (20) \end{aligned}$$

Theorem 6 *Any solution to the Pareto program for some Pareto weights $\lambda > 0$ can be supported as a competitive equilibrium.*

Proof: As in the proof of Theorem 5, set $p = \widehat{p}$, and $\mu = \widehat{\mu}$. In addition, set $\forall i$, $\xi_i = x_i^*$, $\theta_i = 1/\lambda_i$ and $\omega_i = \widehat{\gamma}_i/(\lambda_i \alpha_i)$. Market clearing and the Production sector's problem are satisfied by the same arguments used in the proof to Theorem 5. The Consumer's problem condition (19) holds trivially because $\xi_i = x_i^*$. Substituting into (8) gives us that condition (18) holds. Finally, substituting into (12) and dividing by the scalar $(\lambda_i \alpha_i)$ demonstrates that condition (20) holds. **Q.E.D.**

In Debreu (1954), the existence of a cheaper point is used to show that a compensated equilibrium is a competitive equilibrium. This argument is commonly referred to as Arrow's Remark. Here, we provide a similar result by demonstrating that the existence of a cheaper point means that the Pareto weights are strictly positive. Later, this result will be used to employ Theorem 6 when proving the existence of a competitive equilibrium.

Theorem 7 *Take a solution to the Pareto program for $\lambda \geq 0$. If, at the corresponding compensated equilibrium, there exists for all i, a cheaper point satisfying $x_i \in X_i$, then $\lambda > 0$.*

Proof: Let x be the solution to the Pareto program. From (8), for each i, $\lambda_i \alpha_i u_i x_i - \alpha_i \widehat{p} x_i - \widehat{\gamma}_i g_i x_i = 0$. Take any $x_i' \in X_i$. Also by (8), $\lambda_i \alpha_i u_i x_i' - \alpha_i \widehat{p} x_i' - \widehat{\gamma}_i g_i x_i' \leq 0$. Therefore,

$$\lambda_i \alpha_i u_i x_i - \alpha_i \widehat{p} x_i - \widehat{\gamma}_i g_i x_i \geq \lambda_i \alpha_i u_i x_i' - \alpha_i \widehat{p} x_i' - \widehat{\gamma}_i g_i x_i', \ \forall x_i' \in X_i.$$

Assume there exists a cheaper point $x_i' \in X_i$ and consider the $\lambda_i = 0$ case. Then,

$$\widehat{p} x_i' \geq \widehat{p} x_i + \widehat{\gamma}_i g_i x_i / \alpha_i - \widehat{\gamma}_i g_i x_i' / \alpha_i, \ \forall x_i' \in X_i.$$

By (12), $\widehat{\gamma}_i (g_i x_i - b_i) = 0$ and $\widehat{\gamma}_i (g_i x_i' - b_i) \leq 0$, $\forall x_i' \in X_i$. Therefore, $\widehat{p} x_i' \geq \widehat{p} x_i$, $\forall x_i' \in X_i$, which contradicts the assumption that a cheaper point exists. **Q.E.D.**

4.1 Existence Theorem

A Negishi-style proof works by showing that at a fixed point of Negishi's mapping, consumers' budget constraints are satisfied by a solution to the Pareto program. As discussed above, we do not have free disposal so prices may be negative. To allow for negative prices, we restrict prices to the closed unit ball, that is,

$$P = \{p \in \Re^n | \sqrt{p \cdot p} \leq 1\}.$$

We can not use as our set the unit sphere because $\{p \in \Re^n | \sqrt{p \cdot p} = 1\}$ is not convex. However, we will use this normalization in our mapping. In proofs based on excess demand correspondences, a complication with using the unit ball is that the excess demand correspondence is not upper hemi-continuous at $p = 0$. As we will see, this is not a problem for Negishi's mapping here.

To show that a competitive equilibrium exists, we use the following mapping of $(\lambda, x, y, p) \rightarrow (\lambda, x, y, p)$. The mapping consists of the following two parts

$$\begin{aligned} \lambda &\rightarrow (x', y', p') \\ (\lambda, x, y, p) &\rightarrow \lambda'. \end{aligned}$$

Each variable is restricted to lie in a compact, convex set. Remember that I is the number of types so we restrict $\lambda \in S^{I-1}$ (the unit simplex), $x \in X$, $y \in \Gamma$, $p \in P$, where X is the cross-product over I of the X_i, and $\Gamma = \{y \in Y | y \text{ is feasible}\}$. The set P is compact and convex. Market clearing and the compactness of the X_i guarantee that Γ is a compact set. The set Γ is also convex.

For any $\lambda \in S^{I-1}$ there is a solution to the Pareto program $(x^*, y^*, p^*, \mu^*, \gamma^*)$. Then we normalize p to lie in the unit circle with

$$\widetilde{p} = \left\{ \frac{p^*}{\sqrt{p^* \cdot p^*}} \right. .$$

This normalization will not work if prices are zero but by Lemma 1, $p^* \neq 0$. Finally, we add one more step to the mapping by including in the correspondence the convex hull of prices calculated from the normalization, that is,

$$p' \in co\widetilde{P},$$

where $\widetilde{P}$ is the set of normalized prices. This step is important because it preserves convexity of the mapping while keeping prices in P. Furthermore, the set of $\widetilde{p}$ is convex so normalizing it to lie on the unit circle and then taking the convex hull does not add any new set of relative prices to the mapping.

The first part of the mapping gives us (x', y', p'). This mapping is non-empty, compact-valued, convex-valued, and upper hemi-continuous. Non-emptiness, boundedness, and convex valued are trivial. Closed valued and upper hemi-continuity are not too difficult to show. For example, the normalization of prices is a continuous function, which preserves upper hemi-continuity. Furthermore, the convex hull of an upper hemi-continuous correspondence is itself upper hemi-continuous. (See, for example, Proposition 11.29(a) of Border (1985).)

The second part of the mapping calculates the new Pareto weights, λ', as a function of the transfers needed to support an allocation x from prices p. Because the sets P and X are bounded, there exists a positive number A such that

$$\sum_i |p(\xi_i - x_i)| < A,$$

for any $(x, p) \in X \times P$. Then, for any $\lambda \in S^{I-1}$, let

$$\hat{\lambda}_i = \max\{0, \lambda_i + \frac{p(\xi_i - x_i)}{A}\}, \text{ and } \lambda_i' = \frac{\hat{\lambda}_i}{\sum_i \hat{\lambda}_i}. \tag{21}$$

This mapping is a continuous function so it is trivially a non-empty, compact-valued, convex valued, and upper hemi-continuous correspondence.[2]

To summarize, we have defined a mapping from $S^{I-1} \times X \times \Gamma \times P \to S^{I-1} \times X \times \Gamma \times P$. Each of these sets is convex and compact and cross-products of convex and compact sets are convex and compact. Each part of the mapping is non-empty, convex-valued, compact-valued, and upper hemi-continuous. Cross-products of correspondences preserve these properties. Therefore, we have proven the following theorem.

Theorem 8 *A fixed point (λ, x, y, p) of the mapping exists.*

Proof: Apply the conditions of Kakutani's fixed point theorem.

The fixed point is a solution to the Pareto program so by Theorem 1 it is also a compensated equilibrium. (Note that prices can be scaled by a constant without affecting the solution to compensated or competitive equilibria.) The second part of the mapping, subject to one condition, will provide us with a proof that the compensated equilibrium is a competitive equilibrium.

Theorem 9 (Existence) *For any given distribution of endowments, if the Pareto weights at a fixed point of the mapping are non-zero, then a competitive equilibrium exists.*

[2]Note that y is not explicitly used in this portion of the mapping. It is not needed because by Lemma 1 any solution to the Pareto program satsifies $py = 0$ and scaling p does not change this.

Proof: By Theorem 8 a fixed point to the mapping exists. The first part of the mapping is a solution to the Pareto program so by Theorem 5 the solution can be supported as a compensated equilibrium. Now assume that $\lambda > 0$. From the second part of the mapping, (21), at a fixed point $p(\xi_i - x_i)$ must be the same sign for all i. From the necessary and sufficient conditions to the Pareto program, $p(\sum_i \alpha_i(x_i - \xi_i) - y) = 0$ and from Lemma 1 $py = 0$. Therefore, $p(\xi_i - x_i) = 0, \forall i$.

This satisfies one part of the necessary and sufficient conditions of the Consumer's problem. The remaining conditions simply require setting $\theta_i = 1/\lambda_i$ and $\omega_i = \gamma_i/(\lambda_i\alpha_i)$. Since $\lambda_i > 0$ we can make this substitution. **Q.E.D.**

The term $\theta_i = 1/\lambda_i$ is consumer i's marginal utility of income.

4.2 Cheaper point

The existence proof only guarantees the existence of a competitive equilibrium if $\lambda > 0$. Because our fixed point is a compensated equilibrium, Theorem 7 implies that if there is a cheaper point then $\lambda > 0$. Consequently, if a cheaper point exists then our fixed point is a competitive equilibrium.

In Prescott and Townsend (2005) a self-employment contract $b_1 \in B_1$ is an output dependent consumption schedule $c(q)$, an effort level a, and a capital input level k for which effort is incentive compatible. Production is $f(q|a,k)$, which is the probability distribution of output q given the effort level and capital input. The price of one of these contracts is

$$p(b_1) = (E(c) - E(q))\mu_{(c-q)} + kp_k,$$

where $\mu_{(c-q)} > 0$ is the shadow price on consumption and the expectation is taken with respect to the a and k in the contract.

Fortunately, only weak conditions are needed in this environment to guarantee the existence of a cheaper point. For example, when each agent is endowed with a positive amount of capital, that is, $\forall i,\ \kappa_i > 0$, and capital is scarce, that is, $p_k > 0$, and each agent has positive income. Because of the non-satiation assumption the shadow price of consumption is positive ($\mu_{(c-q)} > 0$), so if $E(q|a = 0, k = 0) \geq 0$, then a contract where the agent gets zero consumption, works zero, and uses no capital for production is cheaper than the agent's equilibrium allocation.

In the important case where there is no capital input into production, each agent's income is zero. In this case, $p(b_1) = (E(c) - E(q))\mu_{(c-q)}$ with $\mu_{(c-q)} > 0$ because of the non-satiation assumption. Then, as long there exists $(c(q), a) \in B_1$ such that $E(c) \leq E(q)$, there will be a cheaper point than each agent's equilibrium allocation. Since the moral-hazard literature is usually concerned with situations where the agent produces a surplus for the principal, this condition seems mild indeed.

References

[1] Border, Kim. *Fixed Point Theorems with Applications to Economics and Game Theory.* Cambridge: Cambridge University Press, 1985.

[2] Debreu, Gerard. "Valuation Equilibrium and Pareto Optimum." *Proceedings of the National Academy of Sciences* 40 (1954): 588-92.

[3] Negishi, Takashi. "Welfare Economics and the Existence of an Equilibrium for a Competitive Economy." *Metroeconomica* 12 (1960): 3-8.

[4] Prescott, Edward Simpson and Robert M. Townsend. "Firms as Clubs in Walrasian Markets with Private Information." Manuscript, October 2005.