Working Paper Series

Optimal Personal Bankruptcy Design: A Mirrlees Approach

WP 08-05

Borys Grochulski
Federal Reserve Bank of Richmond

THE FEDERAL RESERVE BANK OF RICHMOND
RICHMOND ▪ BALTIMORE ▪ CHARLOTTE

Optimal Personal Bankruptcy Design: A Mirrlees Approach*

Borys Grochulski
Federal Reserve Bank of Richmond Working Paper 08-05

September 2008

Abstract

In this paper, we develop a normative theory of unsecured consumer credit and personal bankruptcy based on the optimal trade-off between incentives and insurance. First, in order to characterize this trade-off, we solve a dynamic moral hazard problem in which agents' private effort decisions influence the life-cycle profiles of their earnings. We then show how the optimal allocation of individual effort and consumption can be implemented in a market equilibrium in which (i) agents and intermediaries repeatedly trade in secured and unsecured debt instruments, and (ii) agents obtain (restricted) discharge of their unsecured debts in bankruptcy. The structure of this equilibrium and the associated restrictions on debt discharge closely match the main qualitative features of personal credit markets and bankruptcy law that actually exist in the United States.

Keywords: Bankruptcy, unsecured credit, moral hazard.

1 Introduction

Provision of debt relief and "a fresh start" to "the honest but unfortunate debtor" is recognized in the legal literature as the main role for the institution of personal bankruptcy.[1] In the language of economics, this role amounts to the provision of insurance and has been recognized as such in the literature on the economics of personal bankruptcy.[2] In this paper, we take this role as given and ask the following normative question: How should the institution of personal bankruptcy be designed to fulfill its role efficiently?

We approach this question in two steps. In the first step, we propose an economic environment that precisely determines what efficient provision of insurance means. Specifically, we consider a dynamic moral hazard environment in which agents' private effort decisions influence the life-cycle

*I wish to thank Huberto Ennis, Mikhail Golosov, John Karaken, Erzo Luttmer, Leonardo Martinez, Christopher Phelan, B. Ravikumar, Pierre Sarte, and Jan Werner for their helpful comments on this paper. Any remaining errors are mine. The views expressed here are mine and do not necessarily reflect those of the Federal Reserve Bank of Richmond or the Federal Reserve System. Email address: borys.grochulski@rich.frb.org

[1] This role is expressed, e.g., in the 1934 Supreme Court decision *Local Loan Co. v. Hunt*, 292 U.S. 234, 244 (1934). See also Jackson (1985) and references therein.

[2] See, e.g., Athreya (2002), White (2007).

profiles of their income. High effort mitigates the income risk but cannot eliminate it completely. In this environment, the (constrained) efficient allocation of consumption and effort recommends high effort and does not provide full insurance against the income risk, as incentives for high effort must be provided through a positive correlation between income and consumption. This correlation reflects the optimal trade-off between incentives and insurance.

In the second step, we demonstrate how this solution to the moral hazard problem can be implemented as a competitive equilibrium outcome in a market economy in which agents (consumers) repeatedly trade with free-entry financial intermediaries in a set of secured and unsecured debt instruments. Unsecured consumer debt is subject to discharge under specific rules of a personal bankruptcy law, which we characterize. Since this bankruptcy law implements the efficient amount of income risk insurance, it efficiently fulfills the role assigned to the institution of bankruptcy.

The outcome of our normative analysis provides a theory of unsecured credit and personal bankruptcy. We proceed then by taking a first step toward confronting this theory with the data. In Section 7, we compare qualitatively the bankruptcy law and the structure of the unsecured credit markets that emerge in our model with the main features of the bankruptcy law and personal credit markets that actually exist in the United States. The basic structures of the two sets of institutions turn out to match closely.

First, the efficient bankruptcy law of the model consists of (1) an income-tested bankruptcy eligibility condition; (2) a discharge provision, which frees the bankrupt agent from all unsecured debt obligations; and (3) a liquidation rule with an exemption provision. Liquidation means that the bankrupt agent's assets in excess of a given exemption level are seized from the agent and used to (at least partially) repay the creditors. The exemption provision sets the asset exemption level as well as frees all current and future labor income of the agent from any further creditors' claims. These three properties emerge endogenously as efficient personal bankruptcy rules in our normative model. In Section 7, we document that the same three properties characterize actual law that regulates personal bankruptcy in the United States. In particular, properties (1)-(3) are central features of the so-called U.S. chapter 7 personal bankruptcy procedure.

Second, the structure of the unsecured credit markets in our model is very similar to the structure of the unsecured credit markets in the U.S. economy. In the model, competitive intermediaries offer unsecured credit to the consumers in the form of loans characterized by an interest rate and a credit limit. In equilibrium, these interest rates and credit limits depend on consumers' observable characteristics that include income, debt, and assets. Intermediaries do have information about their prospective borrowers' unsecured debts outstanding with all other intermediaries, i.e., consumers cannot borrow anonymously. In Section 7, we document that all these features obtained in our normative model also characterize the actual structure of the unsecured consumer credit markets in the United States.

The hypothesis adopted in this paper is that (i) social insurance is provided through unsecured credit and bankruptcy discharge, and (ii) the trade-off between insurance and incentives that arises from moral hazard is important for credit market and bankruptcy arrangements. Our theory of unsecured consumer credit and personal bankruptcy is built by deriving the implications of this hypothesis under the requirement of efficiency. The broad consistency of these implications with the observed institutions ought to be viewed as evidence validating our hypothesis.

Agents' private effort is the sole friction in the primitives of the economic environment we study in this paper. Consistently, therefore, in the market economy implementing the optimal allocation, we assume that unobservable effort is the only friction. In particular, full enforcement of private promises to repay debt is assumed to be available and, thus, consumers can borrow and lend at a risk-free rate. As well, we assume that all trades are publicly observable. Any relaxation of these assumptions would introduce an additional friction into the underlying economic environment, which would be inconsistent with our objective of isolating the implications of moral hazard for unsecured credit and personal bankruptcy.

In the life-cycle model we consider, there are only two possible realizations of agents' income in each period and the income shocks that agents experience are persistent. This formulation is suitable for studying the provision of insurance through personal bankruptcy. It is generally understood that insuring the frequent, small, and transitory shocks that households routinely experience over the life-cycle is not a role for the institution of personal bankruptcy. Such granular shocks are probably best insured through other means, or—possibly because of moral hazard—may have to go uninsured altogether. To reflect this, we assume in our model a two-point support for the income in shock each period and interpret the first low income realization in the life-cycle as a shock sufficiently large to trigger bankruptcy. Our model with persistence admits a large class of low-frequency income processes, which makes our formulation both empirically plausible and suitable for studying optimal personal bankruptcy design.

Relation to the literature Methodologically, this paper is closely related to the Mirrleesian dynamic optimal taxation literature (e.g., Kocherlakota 2005, Albanesi and Sleet 2006, Golosov and Tsyvinski 2006). We follow the same approach to the question of optimal design of the bankruptcy code as that literature uses with regard to the question of optimal design of the tax code. In this approach, following the seminal work of Mirrlees (1971), optimal institutions emerge as mechanisms that implement optimal allocations derived directly from the primitives of preferences, technology, and information. Incomplete public information is a key friction shaping the optimal allocations and the institutions that attain them.[3] In our model, private information takes the form of the lack of public observability of effort. Unlike most papers in the literature on dynamic moral hazard, our stochastic structure is not *iid*.[4] We consider a finite-horizon, life-cycle model with a stochastic structure allowing for income persistence and age effects.[5] The optimal allocation obtained in our model is recursive in an agent's continuation utility and, due to the persistence of income, the most recent realization of individual income.

[3] Among other topics, dynamic models with private information have also been used to study optimal unemployment insurance (e.g., Atkeson and Lucas 1995, Hopenhayn and Nicollini 1997) and optimal financial structure for a firm (e.g., Clementi and Hopenhayn 2006, DeMarzo and Sannikov 2006).

[4] Contributions to the repeated moral hazard literature include Rogerson (1985), Spear and Srivastava (1987), Phelan and Townsend (1991).

[5] Note that the persistent variable, i.e., income, is public. This is unlike in Fernandes and Phelan (2000) where the persistent variable is private.

At the technical level, our implementation with bankruptcy has features common with several tax implementations studied in the dynamic optimal taxation literature. Similar to the tax system of Albanesi and Sleet (2006), which is recursive in wealth, our implementation with bankruptcy has a recursive structure. In our model, the state vector characterizing an agent consists of two variables: wealth and the most recent realization of individual income. Similar to the asset-tested disability insurance system of Golosov and Tsyvinski (2006), the optimal bankruptcy rules of our model introduce a kink (a point of non-differentiability) in the budget set faced by the agents. Most papers in the dynamic optimal taxation literature study implementations in which the government is the sole provider of social insurance.[6] In our implementation with bankruptcy, the role of the government is restricted to designing a bankruptcy law that allows agents to optimally self-insure by trading (repeatedly) with profit-maximizing private intermediaries.

Obviously, the implementation mechanism we propose is not unique in the environment we study. Prescott and Townsend (1984) and Atkeson and Lucas (1992), among others, provide examples of market-like implementations of solutions to optimal allocation problems with private information. These examples can be easily adapted to our life-cycle environment with moral hazard. What differentiates our implementation is its similarity to the U.S. unsecured credit markets and personal bankruptcy laws. The realism of our implementation mechanism makes it useful for thinking about the connection between real-world personal bankruptcy regulations and efficient solutions to normative optimal allocation problems with private information.

This paper is primarily related to the theoretical literature on default and personal bankruptcy. Papers in this literature can be divided into three groups. First, there are papers that study default in economies with exogenously incomplete markets. Second, there are papers that study default and bankruptcy in economies with limited enforcement. In the third group are papers that, as we do herein, study default and bankruptcy in environments with private information.

Dubey, Geanakoplos and Shubik (2005) is a seminal paper in the literature on default (as opposed to bankruptcy) with exogenously incomplete markets.[7] That paper makes two important contributions. First, it extends the classic Arrow-Debreu model of general equilibrium to allow for defaultable assets and competitive asset pools. Second, it demonstrates that such assets and pools may improve efficiency of the equilibrium outcome when asset markets are incomplete. In our paper, we use the competitive equilibrium construct of Dubey, Geanakoplos and Shubik (2005) to model the unsecured consumer credit market. Unlike Dubey, Geanakoplos and Shubik (2005), however, we do not assume an exogenously incomplete asset market structure. Rather, our model's asset market structure is endogenously incomplete, with a set of traded contracts emerging as a mechanism implementing the (constrained) optimal allocation under moral hazard. It is important to note that the fact that our model is built without exogenous contract-space restrictions allows us to characterize an optimal—not merely an efficiency-improving—unsecured credit market structure

[6] Golosov and Tsyvinski (2007) study the provision of social insurance by competitive insurance firms and show that the competitive outcome may be inefficient when agents have access to hidden re-trade markets. In this paper, we study optimal social insurance in an environment in which moral hazard is the only friction, i.e., all trades are observable and the results of Prescott and Townsend (1984) imply that the competitive outcome is efficient.

[7] Other contributions to this literature include Zame (1993), Araujo and Pascoa (2002).

and bankruptcy arrangement. Also, the abstract model of Dubey, Geanakoplos and Shubik (2005) introduces default but does not explicitly define an institution of personal bankruptcy, which makes this model difficult to compare with the observed institutions. The unsecured credit markets and the bankruptcy code of our model, in contrast, have clear counterparts in the institutions observed in the U.S. economy.

The papers that study default and bankruptcy in environments with limited enforcement include Kehoe and Levine (1993, 2001, 2006), Alvarez and Jermann (2000), among others. In this literature, enforcement of individual promises is restricted by the agents' ability to leave the economy and consume their individual endowment (i.e., their labor income). The loss of the ability to trade with others is the only penalty faced by the agents who leave. Most papers in this literature interpret leaving the economy as default. Under this interpretation, the possibility of default restricts feasible risk sharing, but default never actually occurs in equilibrium. Under this interpretation, thus, limited enforcement does not deliver a theory of default or bankruptcy.

In a recent paper, Kehoe and Levine (2006) abandon this interpretation of default in a limited enforcement environment. They demonstrate how the optimal allocation can be implemented as an equilibrium of an economy with defaultable assets in which default and bankruptcy do occur along the equilibrium path. This implementation mechanism is similar to the one we use in that the event of default/bankruptcy is identified with the provision of an implicit insurance payment to agents hit by an adverse income shock. The optimal institution of bankruptcy obtained in Kehoe and Levine (2006), however, differs from the one that we obtain in our private information model. In their model, bankrupt agents are allowed to keep the returns on the loans they make to other agents but lose their holdings of all other assets. In our model, finite but non-zero asset exemptions emerge as a key element of the optimal bankruptcy arrangement. Also, the structure of the unsecured credit markets we obtain in our model differs significantly from the mutual credit arrangement studied in Kehoe and Levine (2006). The results we obtain in this paper suggest that moral hazard is an important force shaping the observed bankruptcy institutions. The results of Kehoe and Levine (2006) indicate that limited enforcement may be important as well.

The third strand of the theoretical literature on default and bankruptcy includes the papers that study environments with private information. Typically, papers in this literature study private information contracting problems in which agents' ability to declare bankruptcy is taken as an exogenous constraint on the set of feasible contracts (see, e.g., Bizer and DeMarzo 1999, Bisin and Rampini 2006). In this paper, in contrast, bankruptcy is an element of a mechanism implementing the optimal allocation obtained in an environment in which private information is the sole friction. In our model, thus, bankruptcy is an endogenous outcome rather than an exogenous constraint.

In a recent paper, Rampini (2005) studies an optimal risk sharing problem in a static private information environment and interprets the net transfers to agents hit by an adverse idiosyncratic income shock as default. That paper characterizes the size of the net transfers as a function of the realization of an aggregate income shock, which is observable. Net transfers are interpreted as default but actual borrowing and lending is left implicit. Rampini (2005) does not formally define an institution of bankruptcy and does not consider the question of implementation of the optimal allocation in a market economy with default/bankruptcy. In our paper, in contrast, we not only characterize the optimal allocation but also demonstrate how it can be implemented in a market economy with unsecured credit and bankruptcy. Also, we study a dynamic moral hazard environment with no aggregate risk, whereas Rampini (2005) studies a static hidden income environment with an aggregate shock.

Indirectly, this paper is also related to the quantitative literature on consumer credit, default and personal bankruptcy.[8] This literature builds on the theoretical foundation of the incomplete markets literature, in which, as in Dubey, Geanakoplos and Shubik (2005), the role for default and bankruptcy stems from exogenous restrictions on the set of traded assets. The choice of these restrictions is important because the quantitative results obtained in this literature depend on the exact structure of these restrictions.[9] In our paper, analogous restrictions emerge endogenously as a mechanism implementing an optimal allocation. Therefore, the structure of the unsecured credit markets and the bankruptcy institution obtained in our model may be useful in guiding the choices of the credit market and bankruptcy structures studied in the quantitative literature. In particular, in the concluding Section 8 we briefly discuss two key features of the optimal market arrangement obtained in our model that have not been incorporated in the market arrangements studied in the quantitative literature.

Organization Section 2 lays out the environment and defines efficiency. Section 3 provides a characterization of the optimal allocation. Section 4 lays out the market economy with unsecured credit and an institution of bankruptcy. It also formally defines and proves implementation, and provides a partial characterization of an optimal market arrangement and bankruptcy code. Section 5 provides further characterization by showing how optimal unsecured credit limits and asset exemption levels change with wealth. Section 6 isolates the effect that moral hazard has on the structure of the optimal market arrangement and bankruptcy code of Section 4 by comparing it with a credit-and-bankruptcy system that would be optimal in our environment had moral hazard been absent. Section 7 discusses the similarities and dissimilarities between the optimal arrangement obtained in the model and the structure of unsecured consumer credit contracts, markets for consumer credit, and bankruptcy law currently functioning in the United States. Section 8 concludes.

2 Environment and efficiency

The time horizon is finite with $T+1$ periods indexed by $t = 1, ..., T+1$. There is a single consumption good in every period. The model economy is populated by a continuum of agents. All agents are ex ante identical with respect the their income-earning abilities and preferences over consumption and effort.

[8] Papers in this literature include Athreya (2002), Chatterjee et al. (2007), Li and Sarte (2006), and Livshits, MacGee and Tertilt (2007).

[9] See Townsend (1988) for a discussion of the limitations of policy analysis with exogenous restrictions on the set of contracts that agents can enter.

2.1 Individual income, preferences, and information

Agents consume in all $T+1$ periods. Consumption in period t is denoted by c_t. Individual income of an agent in period $t = 1, ..., T$, denoted by y_t, takes on values from the set $\Theta_t = \{\theta_t^L, \theta_t^H\}$, with $\theta_t^L < \theta_t^H$ for all $t \leq T$. Agents earn no income in the final time period $T+1$, i.e., $y_{T+1} \equiv 0$. Individual effort in period $t \leq T$, denoted by x_t, takes on values from $\{0,1\}$ for $t = 1, .., T$. There is no effort in period $T+1$.

The distribution of individual income in period t depends on the current effort and the previous period's income level. The probability that individual income y_t is realized at the value θ_t^H, conditional on effort x_t and income y_{t-1}, is denoted by $\pi_{t,y_{t-1}}(\theta_t^H|x_t)$ for $t \leq T$.[10] We assume that effort is productive:

$$\pi_{t,y_{t-1}}(\theta_t^H|1) > \pi_{t,y_{t-1}}(\theta_t^H|0) \tag{1}$$

for any y_{t-1}. We allow for persistence in the income process:

$$\pi_{t,\theta_{t-1}^H}(\theta_t^H|x_t) \geq \pi_{t,\theta_{t-1}^L}(\theta_t^H|x_t) \tag{2}$$

for any x_t. Note that $\Theta_t \neq \Theta_s$ for $t \neq s$ allows for life-cycle effects.

Timing within a period is a follows. First, agents consume and expend effort. Then individual income is realized. This means that consumption c_t cannot be conditioned on contemporaneous income y_t.[11]

Let $y^t = (y_1, ..., y_t)$ denote the partial history of realized income up to period t. The set of all income histories of length t is given by $\Theta^t \equiv \Theta_1 \times ... \times \Theta_t$.[12] An individual consumption plan is $c = (c_1, ..., c_{T+1})$, where $c_t : \Theta^{t-1} \to \mathbb{R}_+$. Here, $c_t(y^{t-1})$ represents the consumption assigned in period t to an agent whose individual income history coming into period t is y^{t-1}. An individual effort plan is $x = (x_1, ..., x_T)$, where $x_t : \Theta^{t-1} \to \{0,1\}$ represents the effort recommended in period t to an agent whose individual income history is y^{t-1}.

Let E_x denote the expectation operator over the paths $y^T \in \Theta^T$ conditional on an effort plan x. Agents' preferences over pairs (c, x) are represented by the expected utility function

$$\mathcal{U}(c,x) \equiv E_x[\sum_{t=1}^{T} \beta^{t-1} \{V_t(x_t) + U(c_t)\} + \beta^T U_{T+1}(c_{T+1})],$$

where period utility functions $U : \mathbb{R}_+ \to \mathbb{R}$, $U_{T+1} : \mathbb{R}_+ \to \mathbb{R}$, and $V_t : \{0,1\} \to \mathbb{R}$ satisfy $U' > 0$, $U'' < 0$, $U'_{T+1} > 0$, $U''_{T+1} < 0$, and $V_t(1) < V_t(0)$ for all $t \leq T$.

Throughout the paper, we assume that effort is private information of the agents. All other variables are publicly observable.

[10] Here, y_0 denotes the initial empty income history, the same for all agents.
[11] This timing assumption is not essential.
[12] Also, Θ^0 will denote the initial empty history y_0.

2.2 Allocations and efficiency

Agents are ex ante heterogeneous with respect to their initial promised utility ω. Let Ω denote the distribution of agents with respect to the promised utility value ω, and let $S(\Omega) \subset \mathbb{R}$ denote the support of this distribution.

An *allocation* is an assignment of a pair (c, x) to each promised utility value ω in $S(\Omega)$.[13] We will denote an allocation by $A = (c(\omega), x(\omega))_{\omega \in S(\Omega)}$. Since effort is private information of the agents, we restrict attention to incentive compatible allocations.[14] Allocation A is incentive compatible (IC) if

$$x(\omega) \in \arg\max_{\tilde{x} \in \mathcal{E}} \mathcal{U}(c(\omega), \tilde{x})$$

for all $\omega \in S(\Omega)$, where $\mathcal{E}$ is the set of all individual effort strategies, i.e., the (finite) set of all mappings $\tilde{x}_t : \Theta^{t-1} \to \{0, 1\}$ for $t \leq T$.

An IC allocation $A = (c(\omega), x(\omega))_{\omega \in S(\Omega)}$ delivers the promised utility distribution Ω if

$$\mathcal{U}(c(\omega), x(\omega)) = \omega$$

for all $\omega \in S(\Omega)$.

Let $\{q_t\}_{t=1}^{T}$ be an exogenous sequence of one-period discount rates at which resources can be transferred across time in this economy.[15] Given these prices, an IC allocation $A = (c(\omega), x(\omega))_{\omega \in S(\Omega)}$ that delivers the promised utility distribution Ω generates a net cost $C_1^A(\Omega)$ given by

$$C_1^A(\Omega) \equiv \int_{S(\Omega)} E_{x(\omega)}[\sum_{t=1}^{T+1} \left(\Pi_{s=1}^{t-1} q_s\right) \{c_t(\omega) - y_t(\omega)\}]\Omega(d\omega), \tag{3}$$

where $y_t(\omega)$ denotes the income process induced by the effort assignment $x(\omega)$, and $\Pi_{s=1}^{0} q_s \equiv 1$.

Allocation A is *efficient* if it is IC, if it delivers the initial distribution of promised utility Ω, and if it minimizes, among all IC allocations that deliver Ω, the net cost $C_1^A(\Omega)$.

2.3 Recursive component planning problem

Let $C \equiv U^{-1}$, $C_{T+1} \equiv U_{T+1}^{-1}$, and $X_t \equiv V_t^{-1}$ for $t \leq T$. For any $t \leq T$ and $y_{t-1} \in \Theta_{t-1}$, the *component planning problem* is to find the cost function $B_{t,y_{t-1}} : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$B_{t,y_{t-1}}(w_t) = \min_{u,v,w'(y_t)} C(u) + \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t | X_t(v)) \left\{-y_t + q_t B_{t+1,y_t}(w'(y_t))\right\},$$

[13] Note that under an allocation, all agents with the same ω receive the same treatment. Such allocations are often called type-identical.

[14] By the Revelation Principle, this is without loss of generality.

[15] These outside markets do not have to be interpreted as international credit markets. They can be domestic markets in which the interest rate is determined by the marginal productivity of capital in the business sector. Production and capital accumulation processes are outside of the model, i.e., our economy represents the consumer sector for which the intertemporal resource prices are exogenous.

where minimization is subject to the temporary incentive compatibility (TIC) constraint

$$v + \beta \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|X_t(v))w'(y_t) \geq \tilde{v} + \beta \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|X_t(\tilde{v}))w'(y_t), \tag{4}$$

where $\tilde{v} = V_t(1 - X_t(v))$, and the promise keeping (PK) constraint

$$v + u + \beta \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|X_t(v))w'(y_t) = w_t, \tag{5}$$

and where the function B_{t+1,y_t} solves the component planning problem at $(t+1, y_t)$. At $(T+1, y_T)$, the component planning problem is to find the cost function $B_{T+1,y_T} : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$B_{T+1,y_T}(w_{T+1}) = \min_u C_{T+1}(u),$$

where minimization is subject to the PK constraint

$$u = w_{T+1}.$$

In these recursively defined minimization problems, $B_{t,y_{t-1}}(w_t)$ represents the minimum resource cost at t to provide continuation utility w_t to an agent whose previous period's income is y_{t-1}. For any $t \leq T$, $y_{t-1} \in \Theta_{t-1}$ and any number w_t, let $u^*_{t,y_{t-1}}(w_t)$, $v^*_{t,y_{t-1}}(w_t)$, and $w^*_{t+1,y_{t-1}}(w_t, y_t)$ denote policies that attain $B_{t,y_{t-1}}(w_t)$. Also, let $u^*_{T+1,y_T}(w_{T+1})$ denote a policy that attains $B_{T+1,y_T}(w_{T+1})$.

An initial distribution of promised utility Ω and policies $\{(v^*_{t,y_{t-1}}, u^*_{t,y_{t-1}}, w^*_{t+1,y_{t-1}})^{t=1...T}_{y_{t-1}\in\Theta_{t-1}}, u^*_{T+1,y_T}\}$ define an allocation $A = (c(\omega), x(\omega))_{\omega \in S(\Omega)}$ as follows. Let $w^* = (w^*_1, ..., w^*_{T+1})$, where $w^*_t : \mathbb{R} \times \Theta^{t-1} \to \mathbb{R}$, be an *optimal continuation utility process* defined as a solution to the difference equations $w_{t+1} = w^*_{t+1,y_{t-1}}(w_t, y_t)$ with the initial value $w^*_1 = \omega$.[16] For any $\omega \in S(\Omega)$, the individual consumption plan $c(\omega)$ is given by

$$c_t(\omega, y^{t-1}) = C(u^*_{t,y_{t-1}}(w^*_t(\omega, y^{t-1}))), \tag{6}$$

for all t and $y^{t-1} \in \Theta^{t-1}$, and the effort plan $x(\omega)$ is given by

$$x_t(\omega, y^{t-1}) = X_t(v^*_{t,y_{t-1}}(w^*_t(\omega, y^{t-1}))),$$

for $t \leq T$ and $y^{t-1} \in \Theta^{t-1}$. It is a straightforward modification of the results of Atkeson and Lucas (1992) to show that such defined allocation $A = (c(\omega), x(\omega))_{\omega \in S(\Omega)}$ is efficient. We will refer to this allocation as the optimum, and denote it by $A^* = (c^*(\omega), x^*(\omega))_{\omega \in S(\Omega)}$.

[16] To clarify the notation: w_{t+1} represents a generic continuation utility level in period $t+1$, $w^*_{t+1,y_{t-1}}(w_t, y_t)$ is an optimal policy function in the component planning problem, and $w^*_{t+1}(\omega, y^t)$ is the value of the optimal continuation utility process generated from the initial promised utility ω and the sequential application of policy functions $w^*_{t+1,y_{t-1}}(w_t, y_t)$ along the history y^t.

3 Properties of the optimum

To avoid dealing with trivial cases, we assume that high effort is efficient at all dates and states.[17]

Assumption 1 *The parameters of the environment are such that high effort is efficient at all dates and states, i.e.,*

$$x_t^*(\omega, y^{t-1}) = 1$$

for all $t \leq T$ and $y^{t-1} \in \Theta^{t-1}$.

Given the flexibility of the specification of preferences, technology and the support of the initial distribution Ω, it is clear that such parameters actually exist. Note that this assumption implies that $v^*_{t,y_{t-1}}(w_t^*(\omega, y^{t-1})) = V_t(1)$ for all $\omega \in S(\Omega)$, $t \leq T$, and $y^{t-1} \in \Theta^{t-1}$. We maintain Assumption 1 throughout.

The following lemma establishes, as a consequence of Assumption 1, some properties of the solutions to the component planning problems.

Lemma 1 *For any $t \leq T+1$, $y_{t-1} \in \Theta_{t-1}$, the cost functions $B_{t,y_{t-1}}$ are strictly increasing, strictly convex, and differentiable with*

$$B_{t,\theta^H_{t-1}} \leq B_{t,\theta^L_{t-1}}. \tag{7}$$

For any $t \leq T$, $y_{t-1} \in \Theta_{t-1}$ and w_t, the solution to the component planning problem has the following properties:

$$w^*_{t+1,y_{t-1}}(w_t, \theta^H_t) > w^*_{t+1,y_{t-1}}(w_t, \theta^L_t), \tag{8}$$

$$-\theta^H_t + q_t B_{t+1,\theta^H_t}(w^*_{t+1,y_{t-1}}(w_t, \theta^H_t)) \leq -\theta^L_t + q_t B_{t+1,\theta^L_t}(w^*_{t+1,y_{t-1}}(w_t, \theta^L_t)), \tag{9}$$

$$B'_{t,y_{t-1}}(w_t) = C'(u^*_{t,y_{t-1}}(w_t)), \tag{10}$$

$$B'_{t+1,\theta^L_t}(w^*_{t+1,y_{t-1}}(w_t, \theta^L_t))q_t\beta^{-1} < B'_{t,y_{t-1}}(w_t) < B'_{t+1,\theta^H_t}(w^*_{t+1,y_{t-1}}(w_t, \theta^H_t))q_t\beta^{-1}, \tag{11}$$

$$q_t\beta^{-1} \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|x^*_t) B'_{t+1,y_t}(w^*_{t+1,y_{t-1}}(w_t, y_t)) = B'_{t,y_{t-1}}(w_t). \tag{12}$$

Also, the component planner policy functions $u^*_{t,y_{t-1}}(w_t)$ and $w^*_{t+1,y_{t-1}}(w_t, y_t)$, $y_t \in \Theta_t$, are strictly increasing in w_t.

Proof In Appendix. ■

Inequality (7) follows from the persistence of income. Intuitively, when income is persistent, delivering a given amount of utility w_t to an agent whose past income is high is less costly than delivering the same w_t to an agent whose past income is low. Properties (8)-(12) are standard in dynamic moral hazard models. In particular, inequality (8) means that agents continuation value increases with realized income. This property follows from the need to reward high effort. If it did not hold, high effort would not be incentive compatible for the agents. Inequality (9) means that

[17] Our results can be easily extended to the environments in which the optimal effort recommendation is zero in some states. In these states, the incentive problem vanishes and characterization of the optimum and implementation are straightforward.

the component planner provides a net payment to the low-income agents and receives a net payment from the high income agents. If this were not true, the high effort recommendation would not be optimal for the planner.

We now demonstrate two important properties of the optimum.

Proposition 1 *At the optimum A^*, all agents are*

1. *insurance-constrained:*

$$\beta U'(c^*_{t+1}(\omega,(y^{t-1},\theta^H_t))) \quad < \quad q_t U'(c^*_t(\omega,y^{t-1})) \tag{13}$$

$$< \quad \beta U'(c^*_{t+1}(\omega,(y^{t-1},\theta^L_t))) \tag{14}$$

for any $\omega \in S(\Omega)$, $t \leq T$, and $y^{t-1} \in \Theta^{t-1}$; and

2. *savings-constrained:*

$$\frac{U'(c^*_t(\omega,y^{t-1}))}{\sum_{y_t\in\Theta_t}\pi_{t,y_{t-1}}(y_t|1)\beta U'(c^*_{t+1}(\omega,(y^{t-1},y_t)))} < \frac{1}{q_t} \tag{15}$$

for any $\omega \in S(\Omega)$, $t \leq T$, and $y^{t-1} \in \Theta^{t-1}$.

Proof Inequalities (13) and (14) follow from the two inequalities in (11) after substituting from (10), (6), and using the inverse function theorem. Inequality (15) follows from (12), (10), (6), the inverse function theorem, and the Jensen inequality. ■

Inequalities (13) and (14) mean that the optimal amount of insurance provided to the agents is less-than-full. At the optimal allocation, if an agent had an opportunity to take out insurance against the consumption risk remaining in the optimal consumption allocation, she would be willing to pay more than the fair-odds premium for it.

Inequality (15) means that the optimal amount of intertemporal consumption-smoothing provided to the agents is less-than-full. At the optimal allocation, if an agent had an opportunity to borrow or save, she would be willing to save at a gross rate of interest smaller than $1/q_t$, i.e., pay a premium relative to the intertemporal cost of resources q_t.

4 Market equilibrium implementation

We proceed now to showing how the optimum can be implemented as an equilibrium outcome of a market economy in which agents sequentially trade with zero-profit intermediaries in secured and unsecured debt instruments subject to debt discharge regulated by an institution similar to the U.S. bankruptcy law.

4.1 Inefficiency of the riskless claims equilibrium and advantages of unsecured lending

As a point of departure we take a result of Atkeson and Lucas (1992), which demonstrates that the standard riskless claims market equilibrium is incapable of the implementation of the private information optimum. Consider, in the context of our environment, a market mechanism consisting simply a set of riskless claims markets.[18] With free entry into the riskless borrowing and lending, the presence of the outside markets for riskless claims with prices $\{q_t\}_{t=1}^{T}$ implies (by arbitrage) that the equilibrium claims prices must be identically equal to $\{q_t\}_{t=1}^{T}$. An equilibrium allocation of consumption under such a set of markets, denoted by $\hat{c}$, must satisfy the standard Euler equation

$$U'(\hat{c}_t)q_t = \beta E_t\left[U'(\hat{c}_{t+1})\right].$$

Therefore, $\hat{c}$ cannot coincide with c^*, as c^* satisfies the strict inequality (15), which can be rewritten as

$$U'(c_t^*)q_t < \beta E_t\left[U'(c_{t+1}^*)\right].$$

This, as pointed out in Atkeson and Lucas (1992), means that a simple set of riskless claims markets does not implement the private-information optimum.[19]

Intuitively, two factors contribute to the riskless claims markets' failure to implement the optimum. First, riskless claims' payoffs are uncontingent, i.e., they are not contingent on individual agents' income realizations. Thus, riskless claims markets do not allow the agents to sufficiently insure their individual income risk. Second, riskless claims markets provide unrestricted access to self-insurance via savings. In the presence of the first failure, agents over-self-insure (i.e., over-save) in the riskless claims equilibrium.

How can these two failures be avoided with unsecured lending? Suppose that the riskless claims markets are supplemented with unsecured debt, and that agents can discharge their unsecured debt obligations if their individual income realizations are low. Such an expanded set of markets, clearly, can provide better insurance against individual-specific income shocks. For an equilibrium of such a set of markets to be consistent with the optimal allocation c^* at which agents are insurance- and savings-constrained, mechanisms must exist to discourage over-insurance and over-saving.

In the market arrangement that we formally define in the next subsection, competitive intermediaries extend unsecured credit to the agents. Dischargeability of unsecured credit is regulated by rules akin to bankruptcy law. Under these rules, only low-income agents are eligible to receive discharge of their unsecured loans. The discharged loans have to be written off by the intermediaries as losses. This makes for an implicit transfer from the intermediaries to the low-income agents. High-income agents, however, by design of the bankruptcy rules, are ineligible for discharge. They must repay the unsecured obligations with interest. Interest paid by the borrowers whose loans

[18] The promise to repay embedded in a riskless claim is secured by an external enforcement mechanism. In this paper, we focus on private information as the only friction in the environment and thus assume that such an enforcement mechanism is available. We identify riskless claims with secured debt.

[19] In the above, E_t denotes the expectation conditional on information available at the beginning of period t, i.e., $E_t[U'(c_{t+1})]$ is y^{t-1}-measurable.

are not discharged is the intermediaries' profit, i.e., it makes for a transfer from the high-income agents to the intermediaries. The equilibrium interest rate on the unsecured loans (the default premium) is determined at the level at which the intermediaries break even (make zero profit). Thus, this pattern of unsecured borrowing and income-contingent discharge implements a transfer from the high-income agents to the low-income agents, i.e., provides insurance payments contingent on individual income realizations.

In order for the intermediaries to break even, the probability of default on the unsecured loans has to be priced correctly. This probability, however, depends on the agents' effort, which is private information and, thus, cannot be written into the unsecured loan contract. The intermediaries can break even and provide inexpensive unsecured credit (which maximizes the agents' welfare) only if the amount of unsecured credit available to each agent is restricted sufficiently to avoid giving the agent an incentive to over-insure and expend low effort. Thus, in equilibrium each agent can obtain unsecured credit only up to a limit. This limit is determined at the maximum level consistent with the agent's expending high effort. Under this limit, agents remain insurance-constrained in equilibrium, as they are at the optimal allocation c^*.

The second failure of the riskless claims equilibrium (over-self-insurance) is resolved by designing the bankruptcy law in such a way that the benefit of unsecured credit discharge is tied to not over-saving. Agents can freely save, i.e., they can accumulate wealth by buying riskless claims in any quantity they want and can afford. Excessive amounts of wealth, however, cannot be retained by agents who seek discharge of their unsecured debt obligations in bankruptcy. This, effectively, makes dischargeability of the nominally unsecured debt conditional on the debtor's wealth in a way that reduces the benefit of the bankruptcy option for over-savers. Agents are free to save, but they value the option of discharge. This mechanism, which is absent when only the riskless claims are traded, discourages over-saving. What exactly constitutes over-saving follows from the optimal amount of "savings" implicit in the optimal allocation c^*.

4.2 Unsecured credit markets and bankruptcy discharge conditions

In this subsection, we lay out a market economy with unsecured credit and a formal institution of bankruptcy. The timing of interaction is as follows.

4.2.1 Market interaction in periods $t \leq T$

The sequence of events within each period $t \leq T$ is divided into three stages.

Stage 1 Agents enter period t with bonds b_t. Intermediaries offer unsecured credit to the agents. Agents make three decisions. They decide how much unsecured credit to take out with the intermediaries, and how to split the resources available to them between current consumption and savings, which they take into the second stage of interaction. Let $h_t \geq 0$ denote the amount of unsecured credit taken out. Resources $b_t + h_t$ are split between consumption c_t and savings s_t. The third decision agents take in stage 1 is their effort decision $x_t \in \{0, 1\}$.

Unsecured credit is available to the agents as a loan offer extended by the intermediaries. This loan is short-term: it matures within the period, at stage 2, after agents produce period income y_t. The amount h_t of the unsecured loan that each agent takes out is publicly observable[20]. A loan consists of an interest rate and a credit limit. The terms of the loan depend on the agent's observable characteristics. The gross interest rate in period t, denoted by $R_{t,y_{t-1}}(b_t)$, depends on last-period's income y_{t-1} and wealth b_t. Similarly, the unsecured credit limit in period t, denoted by $\bar{h}_{t,y_{t-1}}(b_t)$, depends on agent's y_{t-1} and b_t. At the end of stage 1, an agent initially characterized by wealth b_t and past income data y_{t-1} holds assets $s_t = b_t + h_t - c_t$, owes $h_t R_{t,y_{t-1}}(b_t)$ to the intermediaries, and has expended privately effort $x_t \in \{0, 1\}$. Note that each agent simultaneously holds assets, s_t, and has liabilities, $h_t R_{t,y_{t-1}}$. Hence, assets s_t are leveraged by debt $h_t R_{t,y_{t-1}}$ (in contrast to beginning-of-period wealth b_t).

Stage 2 In the second stage within the period, individual income y_t is realized and the unsecured loans h_t are due repayment. At this stage, each agent chooses how to settle their unsecured debt obligation. There are, potentially, two options: to pay back, or to default and seek discharge in bankruptcy. Let $d_t \in \{0, 1\}$ be the indicator of the decision to default in period t.

What happens after default is regulated by a bankruptcy law, which is specified as follows. First, there is a discharge eligibility condition f_t, which specifies that only low-income agents are eligible for discharge of debt in bankruptcy.[21] By this condition, the repayment of high-income agents' unsecured loans will be enforced.[22] Those with the low income realization $y_t = \theta_t^L$ meet the eligibility condition f_t. If they choose to not default, which is denoted by $d_t = 0$, they repay the loan h_t with interest (just like the high-income agents do), i.e., they pay $h_t R_{t,y_{t-1}}$ to the intermediaries. If they choose to default, which is denoted by $d_t = 1$, the settlement of their obligations is handled (by a bankruptcy court) according to the rules specified in the bankruptcy law. These rules are as follows.

1. The unsecured debt obligations of the bankrupt agent, $h_t R_{t,y_{t-1}}$, are discharged.
2. All current and future income of the bankrupt agent is out of reach of the unsecured creditors (i.e., is *exempt*).
3. The assets held by the bankrupt agent, s_t, are exempt as well, up to a maximum $\bar{s}_{t,y-1}(b_t)$. Any assets in excess of the exemption level $\bar{s}_{t,y-1}(b_t)$ are seized from the bankrupt agent and used to (at least partially) repay the unsecured creditors.

Under these rules, the discharged loans have to be written off by the intermediaries as losses. Creditors exit stage 2 with income from the repaid loans and losses on the loans that were discharged. (In equilibrium, these profits and losses will add up to zero). Agents who did not obtain discharge

[20] Throughout the analysis, we assume that effort is the only piece of information that is private.

[21] More generally, discharge eligibility could be contingent on the whole history of the observable characteristics of an agent, which in our environment means everything but the history of effort. We restrict attention to a simple current-income-based test for discharge eligibility because this test turns out to be sufficient in our environment.

[22] From the outset, we have assumed that full enforcement of contracts is possible in the our moral hazard environment.

exit stage 2 with their current income y_t and their savings s_t minus the amount $h_t R_{t,y_{t-1}}$, which they must repay to the creditors. Those who obtained discharge exit stage 2 with their current income y_t and their exempt assets given by the smaller of s_t and $\bar{s}_{t,y-1}$, and with no unsecured debt obligations.

Stage 3 At the third stage and final stage, agents use their post-settlement resources to purchase claims b_{t+1}, which will be their wealth entering period $t+1$.

4.2.2 Market interaction in period $T+1$

In the final period $T+1$, the sequence of events is much simpler. Agents enter with wealth b_{T+1}. There is no effort decision or income risk in this period. Claims b_{T+1} pay off and agents consume.

4.2.3 Individual optimization problems

Bankruptcy Code formalism In the model, the discharge eligibility condition is represented by the functions $f_t : \Theta_t \to \{0, 1\}$. The bankruptcy asset exemption level is formally represented by the functions $\bar{s}_{t,y_{t-1}} : \mathbb{R} \to \mathbb{R}$. In this notation, $f_t(y_t)$ is the bankruptcy eligibility indicator for an agent whose income in period t is y_t. The value $\bar{s}_{t,y_{t-1}}(b_t)$ represents the exemption level, i.e., the amount of assets s_t that an agent can shield from his creditors in bankruptcy. Note that the exemption level depends on beginning-of-period wealth b_t, as well as on income from the previous period, y_{t-1}.[23] We will refer to $\bar{s}_{t,y_{t-1}}(b_t)$ as the exemption level for type (y_{t-1}, b_t).

Agents' problem Agents take as given the riskless bond prices $\{q_t\}_{t=1}^T$, the unsecured loans pricing and credit limit schedules $\{\{R_{t,y_{t-1}}, \bar{h}_{t,y_{t-1}}\}_{y_{t-1}\in\Theta_{t-1}}\}_{t=1}^T$, and the rules of bankruptcy $\{f_t, \{\bar{s}_{t,y_{t-1}}\}_{y_{t-1}\in\Theta_{t-1}}\}_{t=1}^T$. Given an initial wealth b_1, an agent solves the following recursive maximization problem:

$$W_{t,y_{t-1}}(b_t) = \max_{\substack{x,c,h,s,\\ d(y_t),b'(y_t)}} V_t(x) + U(c) + \beta \sum_{y_t\in\Theta_t} \pi_{t,y_{t-1}}(y_t|x) W_{t+1,y_t}(b'(y_t))$$

subject to the budget constraints

$$\begin{aligned}
0 &\leq h \leq \bar{h}_{t,y_{t-1}}(b_t), && (16)\\
c + s &= b_t + h, && (17)\\
d(y_t) &\in \{0, 1\}, && (18)\\
d(y_t) &\leq f_t(y_t), && (19)\\
q_t b'(y_t) &= y_t + s - (1 - d(y_t))hR_{t,y_{t-1}}(b_t) - d(y_t)\max\{s - \bar{s}_{t,y_{t-1}}(b_t), 0\}, && (20)
\end{aligned}$$

[23] Similar to discharge eligibility, the exemption level could be in our model a function of the whole history of agents' observable characteristics. We restrict attention to the dependence of the exemption level in period t on y_{t-1} and b_t.

for $t \leq T$; with

$$W_{T+1,y_T}(b_{T+1}) = \max_{c \leq b_{T+1}} U_{T+1}(c).$$

In the above problem, $d(y_t)$ is the indicator of the agent's decision to go bankrupt in income state y_t. The budget constraint (20) incorporates the consequences of this decision. If $d(y_t) = 1$, which by (19) is only feasible if $f_t(y_t) = 1$, then $hR_{t,y_{t-1}}$ is discharged and non-exempt assets $s - \bar{s}_{t,y_{t-1}}$ are seized (if $s \leq \bar{s}_{t,y_{t-1}}$, the amount seized is zero). If the agent does not go bankrupt, i.e., if $d(y_t) = 0$, which by (19) is always feasible because $f_t \geq 0$, then (20) reduces to the standard budget constraint $q_t b'(y_t) = y_t + s - hR_{t,y_{t-1}}(b_t)$.

Importantly, agents cannot conceal income y_t or wealth b_t or s_t. This assumption is consistent with the moral hazard environment we study in this paper, in which agents' effort is the only piece of private information, and all other variables and parameters of the environment are publicly observable.

For any $t \leq T$ and $y_{t-1} \in \Theta_{t-1}$ and wealth b_t, we will denote the agents' individually optimal policies for effort, consumption, unsecured borrowing, intra-period savings, default, and next period wealth, i.e., the policies that attain the utility value $W_{t,y_{t-1}}(b_t)$ by, respectively, $x_{t,y_{t-1}}(b_t)$, $c_{t,y_{t-1}}(b_t)$, $h_{t,y_{t-1}}(b_t)$, $s_{t,y_{t-1}}(b_t)$, $d_{t,y_{t-1}}(b_t, y_t)$, $b_{t+1,y_{t-1}}(b_t, y_t)$. Also, by $c_{T+1,y_T}(b_{T+1})$ we will denote the consumption policy that attains $W_{T+1,y_T}(b_{T+1})$.

Unsecured lenders' problem Following Dubey, Geanakoplos and Shubik (2005), we model unsecured credit markets as perfectly competitive. In this model, lenders take the terms of the unsecured loan contacts as given. An unsecured loan offer extended to an agent whose last period's income is y_{t-1} and whose wealth is b_t will be referred to as loans of type (y_{t-1}, b_t). The lenders take as given the following characteristics of an unsecured loan of type (y_{t-1}, b_t): the gross interest rate $R_{t,y_{t-1}}(b_t)$, the credit limit $\bar{h}_{t,y_{t-1}}(b_t)$, the expected loan demand $h^e_{t,y_{t-1}}(b_t)$, the expected default rate $D^e_{t,y_{t-1}}(b_t)$, and the expected principal recovery rate $\gamma^e_{t,y_{t-1}}(b_t)$ on the loans in default.

The expected profit on a loan of type (y_{t-1}, b_t) is given by

$$\Pi^e_{y_{t-1},b_t} = \left(-1 + (1 - D^e_{t,y_{t-1}}(b_t))R_{t,y_{t-1}}(b_t) + D^e_{t,y_{t-1}}(b_t)\gamma^e_{t,y_{t-1}}(b_t)\right) h^e_{t,y_{t-1}}(b_t).$$

In equilibrium, lenders' expectations are correct and thus the (ex ante) expected profit equals the actually realized (ex post) profit on a fully diversified portfolio of unsecured loans of type (y_{t-1}, b_t). We assume that intermediaries diversify, i.e., each lender holds a portfolio of loans made out to a non-zero mass of consumers of type (y_{t-1}, b_t). Investing in a fully diversified portfolio of loans of a type (y_{t-1}, b_t) is a constant returns to scale activity. With constant returns to scale, free entry into unsecured lending implies that equilibrium profits must be zero. Since in equilibrium the intermediaries make zero profits on each type of loan in every period, the number of intermediaries operating in equilibrium is indeterminate. It is important to note that the credit limit $\bar{h}_{t,y_{t-1}}(b_t)$ applies to the total amount of unsecured credit that an agent of type (y_{t-1}, b_t) can take out with the whole unsecured lending industry. Here, as in Dubey, Geanakoplos and Shubik (2005), we assume here that the intermediaries can take the enforcement of this credit limit as a given.

In our formulation of the lenders' problem, the lenders take the terms of the unsecured loan contracts as given, i.e., market-determined. We do not model explicitly the process in which these terms are derived. It is worth pointing out, however, that our results do not depend on the assumption of perfect competition in unsecured lending. Khan and Mookherjee (1995) study a strategic contracting game in which financial intermediaries offer (non-exclusive) insurance contracts to an agent whose income is subject to risk influenced by his private effort. They show that the (constraint) optimal allocation emerges as an equilibrium outcome of this interaction. This result can be easily adapted to the moral hazard environment with observable trades that we study in this paper.[24] Therefore, our implementation of the optimal allocation in a set of competitive unsecured credit markets and bankruptcy does not depend on the assumption of perfect competition in unsecured lending.

4.3 Equilibrium

Definition 1 *Given an initial distribution of wealth Ψ, the riskless bond prices $\{q_t\}_{t=1}^T$, and the rules of bankruptcy $(f,\bar{s}) = \{f_t, \{\bar{s}_{t,y_{t-1}}\}_{y_{t-1}\in\Theta_{t-1}}\}_{t=1}^T$; **recursive competitive equilibrium with bankruptcy** consists of the consumers' value functions $W_{t,y_{t-1}}$ and individual policies $\hat{x}_{t,y_{t-1}}$, $\hat{h}_{t,y_{t-1}}$, $\hat{c}_{t,y_{t-1}}$, $\hat{s}_{t,y_{t-1}}$, $\hat{d}_{t,y_{t-1}}$, $\hat{b}_{t+1,y_{t-1}}$, one for every $t \le T$ and $y_{t-1} \in \Theta_{t-1}$, the value functions W_{T+1,y_T} and policies $\hat{c}_{T+1,y_T}$ for $y_T \in \Theta_T$, interest rates and credit limits $\{\{R_{t,y_{t-1}}, \bar{h}_{t,y_{t-1}}\}_{y_{t-1}\in\Theta_{t-1}}\}_{t=1}^T$ on the unsecured loans, expected loan demand functions $h^e_{t,y_{t-1}}$,expected default rate functions $D^e_{t,y_{t-1}}$, and recovery rate functions $\gamma^e_{t,y_{t-1}}$ for every $t \le T$ and $y_{t-1} \in \Theta_{t-1}$, such that*

1. *the value functions and individual policies solve the agents' problem;*
2. *intermediaries' profits on every loan type are zero, i.e.,*

$$\Pi^e_{y_{t-1},b_t} = 0$$

for all t, y_{t-1}, b_t;

3. *expectations are correct, i.e.,*

$$\begin{aligned}
h^e_{t,y_{t-1}}(b_t) &= \hat{h}_{t,y_{t-1}}(b_t), \\
D^e_{t,y_{t-1}}(b_t) &= \sum_{y_t\in\Theta_t} \pi_{t,y_{t-1}}(y_t|\hat{x}_{t,y_{t-1}}(b_t))\hat{d}_{t,y_{t-1}}(b_t), \\
\gamma^e_{t,y_{t-1}}(b_t) &= \frac{\max\{\hat{s}_{t,y_{t-1}}(b_t) - \bar{s}_{t,y_{t-1}}(b_t), 0\}}{\hat{h}_{t,y_{t-1}}(b_t)} \quad \textit{if} \quad \hat{h}_{t,y_{t-1}}(b_t) > 0,
\end{aligned}$$

for all t, y_{t-1}, b_t.

Let $\mathcal{E}(f,\bar{s},\Psi)$ denote the set of objects that constitute a recursive equilibrium under the bankruptcy rules $(f,\bar{s})$ and the initial wealth distribution Ψ. An *equilibrium allocation* is an assignment of an individual consumption plan c and an individual effort plan x to each initial wealth value

[24] In such an extensive-form decentralization of the competitive equilibrium concept that we use herein, the terms of the unsecured loan contracts would be determined explicitly as a subgame perfect Nash equilibrium outcome.

$\psi \in S(\Psi)$. Equilibrium allocations will be denoted by $\hat{A}(\Psi) = (\hat{c}(\psi), \hat{x}(\psi))_{\psi \in S(\Psi)}$. A given recursive equilibrium $\mathcal{E}(f, \bar{s}, \Psi)$ defines an equilibrium allocation $\hat{A}(\Psi)$ as follows. Let $\hat{b} = (\hat{b}_1, ..., \hat{b}_{T+1})$, where $\hat{b}_t : \mathbb{R} \times \Theta^{t-1} \to \mathbb{R}$, denote the equilibrium wealth process given by the solution to the difference equations $b_{t+1} = \hat{b}_{t+1,y_{t-1}}(b_t, y_t)$ with the initial value $\hat{b}_1 = \psi$.[25] For any $\psi \in S(\Psi)$, the individual consumption plan $\hat{c}(\psi)$ is given by

$$\hat{c}_t(\psi, y^{t-1}) = \hat{c}_{t,y_{t-1}}(b_t(\psi, y^{t-1}))$$

for all t and $y^{t-1} \in \Theta^{t-1}$, and the individual effort plan $\hat{x}(\psi)$ is given by

$$\hat{x}_t(\psi, y^{t-1}) = \hat{x}_{t,y_{t-1}}(b_t(\psi, y^{t-1}))$$

for all $t \leq T$ and $y^{t-1} \in \Theta^{t-1}$.

4.4 Implementation

We now formally define implementation. Our definition is similar to the one used in Albanesi and Sleet (2006). Recall from Section 2 that $A^*(\Omega) = (c^*(\omega), x^*(\omega))_{\omega \in S(\Omega)}$ denotes an efficient allocation that attains the minimum cost of provision of the initial distribution of promised utility Ω, $C_1^*(\Omega)$.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable function assigning initial wealth ψ to each initial promised utility ω, i.e., for all $\omega \in S(\Omega)$ we have $\phi(\omega) = \psi$. For a given distribution of promised utility Ω, ϕ induces a distribution of wealth Ψ that we will denote by $\phi(\Omega)$.[26]

Definition 2 *We say that a bankruptcy code $(f, \bar{s})$ and an initial wealth assignment function ϕ **implement** an efficient allocation $A^*(\Omega)$ if there exists a recursive competitive equilibrium with bankruptcy $\mathcal{E}(f, \bar{s}, \phi(\Omega))$ such that the equilibrium allocation $\hat{A}(\phi(\Omega))$ coincides with the optimal allocation $A^*(\Omega)$, i.e., for all $\omega \in S(\Omega)$*

$$\hat{x}(\phi(\omega)) = x^*(\omega), \tag{21}$$

$$\hat{c}(\phi(\omega)) = c^*(\omega). \tag{22}$$

Note that the conditions (21) and (22) imply that

$$\int_{S(\Omega)} E_{\hat{x}(\phi(\omega))}[\sum_{t=1}^{T+1} \left(\Pi_{s=1}^{t-1} q_s\right) \{\hat{c}_t(\phi(\omega)) - y_t(\omega)\}]\Omega(d\omega) = C_1^*(\Omega),$$

i.e., the amount of resources used by the equilibrium allocation $\hat{A}(\phi(\Omega))$ is equal to the minimum $C_1^*(\Omega)$.

[25] To clarify the notation, b_{t+1} is generic notation for wealth at the beginning of period $t+1$, $\hat{b}_{t+1,y_{t-1}}(b_t, y_t)$ is an optimal policy function in the consumer utility maximization problem, and $\hat{b}_{t+1}(\omega, y^t)$ is the value of the equilibrium wealth process generated from the initial wealth ψ and the sequential application of policy functions $\hat{b}_{t+1,y_{t-1}}(b_t, y_t)$ along the history y^t.

[26] By definition, for any measurable set of wealth levels S, $\phi(\Omega)(S) = \Omega(\phi^{-1}(S))$.

Theorem 1 *There exists a bankruptcy code* $(f, \bar{s})$ *and a wealth assignment* ϕ *that implement the efficient allocation* $A^*(\Omega)$ *under any distribution of initial utility* Ω*. In particular, implementation is attained by the code* $(f, \bar{s})$ *and the function* ϕ *given as follows. The code* $(f, \bar{s})$ *is given by*

$$f_t(y_t) = \begin{cases} 1 & \text{if} \quad y_t = \theta_t^L \\ 0 & \text{if} \quad y_t = \theta_t^H \end{cases}, \tag{23}$$

for $t \leq T$*, and*

$$\bar{s}_{t,y_{t-1}}(\cdot) = q_t B_{t+1,\theta_t^L}(w^*_{t+1,y_{t-1}}(B^{-1}_{t,y_{t-1}}(\cdot), \theta_t^L)) - \theta_t^L, \tag{24}$$

for $t \leq T$ *and* $y_{t-1} \in \Theta_{t-1}$*, where* B_{t+1,y_t} *is the component planning cost function at* (t, y_{t-1})*, and* $w^*_{t+1,y_{t-1}}$ *is the continuation utility policy function from this component planning problem. The wealth assignment function* ϕ *is given by*

$$\phi = B_{1,y_0}. \tag{25}$$

The optimal bankruptcy eligibility condition (23) states simply that only low-income agents are eligible for bankruptcy discharge. The optimal asset exemption level, given in (24), is determined by the solutions to the component planning problems.[27] As in the implementation of Albanesi and Sleet (2006), the optimal wealth assignment function ϕ endows each agent of type ω with initial wealth $b_1 = B_{1,y_0}(\omega)$, i.e., with initial resources equal to the minimum cost of the component planner at date $t = 1$ to deliver promised utility ω.

4.5 Proof of Theorem 1

We prove this theorem using backward induction. The first step is the following lemma.

Lemma 2 *For both* $y_T \in \Theta_T$*,*

$$W_{T+1,y_T} = B^{-1}_{T+1,y_T}.$$

Moreover, the equilibrium individual policy $\hat{c}_{T+1,y_T}$ *satisfies*

$$\hat{c}_{T+1,y_T}(\cdot) = C_{T+1}(u^*_{T+1,y_T}(B^{-1}_{T+1,y_T}(\cdot)))$$

for $y_T \in \Theta_T$*.*

Proof In Appendix. ■

The inductive step is given by the following proposition.

Proposition 2 *Fix* $t \leq T$ *and* $y_{t-1} \in \Theta_{t-1}$*. Under bankruptcy rules* $(f_t, \bar{s}_{t,y_{t-1}})$ *given in (23)-(24), if*

$$W_{t+1,y_t} = B^{-1}_{t+1,y_t}, \tag{26}$$

[27] In the next section, we provide a closer characterization of functions $\bar{s}_{t,y_{t-1}}$.

for both $y_t \in \Theta_t$*, then*

$$W_{t,y_{t-1}} = B^{-1}_{t,y_{t-1}}.$$

Moreover, equilibrium individual policies $\hat{x}_{t,y_{t-1}}$*,* $\hat{c}_{t,y_{t-1}}$*,* $\hat{b}_{t+1,y_{t-1}}$ *satisfy*

$$\begin{aligned}
\hat{x}_{t,y_{t-1}}(\cdot) &= X_t(v^*_{t,y_{t-1}}(B^{-1}_{t,y_{t-1}}(\cdot))), && (27)\\
\hat{c}_{t,y_{t-1}}(\cdot) &= C(u^*_{t,y_{t-1}}(B^{-1}_{t,y_{t-1}}(\cdot))), && (28)\\
\hat{b}_{t+1,y_{t-1}}(\cdot, y_t) &= B_{t+1,y_t}(w^*_{t+1,y_{t-1}}(B^{-1}_{t,y_{t-1}}(\cdot), y_t)), && (29)
\end{aligned}$$

for $y_t \in \Theta_t$*.*

Proof of Theorem 1 follows from Lemma 2 and Proposition 2. First, Lemma 2 and Proposition 2 imply that

$$W_{t,y_{t-1}} = B^{-1}_{t,y_{t-1}}$$

for all $t = 1, ..., T+1$ and $y_{t-1} \in \Theta_{t-1}$. Note now that applying Proposition 2 at $t = 1$, we have that equilibrium allocation $\hat{A}(\phi(\Omega))$ satisfies

$$\begin{aligned}
\hat{x}_1(\phi(\omega), y_0) &= x^*_1(\omega, y_0),\\
\hat{c}_1(\phi(\omega), y_0) &= c^*_1(\omega, y_0),
\end{aligned}$$

and that the equilibrium wealth process $\hat{b}$ satisfies $\hat{b}_1 = \phi(\omega)$ and, for any $y_1 \in \Theta_1$,

$$\begin{aligned}
\hat{b}_2(\phi(\omega), y^1) &= \hat{b}_{2,y_0}(\hat{b}_1, y_1)\\
&= B_{2,y_1}(w^*_{2,y_0}(\phi^{-1}(\hat{b}_1), y_1))\\
&= B_{2,y_1}(w^*_2(\omega, y^1))
\end{aligned}$$

for all $\omega \in S(\Omega)$, where the last line follows from definition of the continuation utility process w. Similarly, applying Proposition 2 repeatedly at dates $t = 2, ..., T$ we get that the equilibrium allocation coincides with the optimum

$$\begin{aligned}
\hat{x}_t(\phi(\omega), y^{t-1}) &= x^*_t(\omega, y^{t-1}),\\
\hat{c}_t(\phi(\omega), y^{t-1}) &= c^*_t(\omega, y^{t-1}),
\end{aligned}$$

at these dates, and the equilibrium wealth process $\hat{b}$ coincides with the component planner cost functions evaluated at the continuation value process w^*, i.e., for any $\omega \in S(\Omega)$, $t \leq T$ and $y^t \in \Theta^t$

$$\begin{aligned}
\hat{b}_{t+1}(\phi(\omega), y^t) &= \hat{b}_{t+1,y_{t-1}}(\hat{b}_t(\phi(\omega), y^{t-1}), y_t)\\
&= B_{t+1,y_t}(w^*_{t+1,y_{t-1}}(B^{-1}_{t,y_{t-1}}(\hat{b}_t(\phi(\omega), y^{t-1})), y_t))\\
&= B_{t+1,y_t}(w^*_{t+1,y_{t-1}}(w^*_t(\omega, y^{t-1}), y_t))\\
&= B_{t+1,y_{t-1}}(w^*_{t+1}(\omega, y^t)).
\end{aligned}$$

In particular, at $t = T$, we have

$$\hat{b}_{T+1}(\phi(\omega), y^T) = B_{T+1,y_T}(w^*_{T+1}(\omega, y^T))$$

for all $\omega \in S(\Omega)$ and $y^T \in \Theta^T$. Lemma 2 implies now that

$$\hat{c}_T(\phi(\omega), y^T) = c^*_T(\omega, y^T)$$

for all $\omega \in S(\Omega)$ and $y^T \in \Theta^T$. Thus, allocations $\hat{A}(\phi(\Omega))$ and $A^*(\Omega)$ coincide.

It order to complete the proof of Theorem 1, we need to prove Proposition 2, which provides the key argument of our implementation result.

Proof of Proposition 2 The proof is constructive. We specify a set of prices, credit limits, expected credit usage, default, and recovery rates for unsecured loans, as well as consumer borrowing, saving, effort, consumption and default policies and verify that these objects are consistent with equilibrium under the assumed consumers' continuation value functions W_{t+1,y_t} given in (26).

For a loan type (y_{t-1}, b_t), the gross interest rate is given by

$$R_{t,y_{t-1}}(b_t) = 1/\pi_{t,y_{t-1}}(\theta^H_t|1), \tag{30}$$

and the credit limit is

$$\bar{h}_{t,y_{t-1}}(b_t) = C(u^*_{t,y_{t-1}}(B^{-1}_{t,y-1}(b_t))) + \bar{s}_{t,y_{t-1}}(b_t) - b_t. \tag{31}$$

The expected credit usage, default, and recovery rates are given by, respectively,

$$\begin{aligned} h^e_{t,y_{t-1}}(b_t) &= \bar{h}_{t,y_{t-1}}(b_t), \\ D^e_{t,y_{t-1}}(b_t) &= \pi_{t,y_{t-1}}(\theta^L_t|1), \\ \gamma^e_{t,y_{t-1}}(b_t) &= 0. \end{aligned}$$

The proposed equilibrium consumer policies are as follows: effort $\hat{x}_{t,y_{t-1}}$ as in (27), consumption $\hat{c}_{t,y_{t-1}}$ as in (28), the unsecured borrowing and saving policies given by, respectively,

$$\begin{aligned} \hat{h}_{t,y_{t-1}}(b_t) &= \hat{c}_{t,y_{t-1}}(b_t) + \bar{s}_{t,y_{t-1}}(b_t) - b_t, \\ \hat{s}_{t,y_{t-1}}(b_t) &= \bar{s}_{t,y_{t-1}}(b_t), \end{aligned}$$

for any b_t. The consumer default policy given by

$$\begin{aligned} \hat{d}_{t,y_{t-1}}(b_t, \theta^H_t) &= 0, \\ \hat{d}_{t,y_{t-1}}(b_t, \theta^L_t) &= 1, \end{aligned}$$

and claims purchases $\hat{b}_{t+1,y_{t-1}}(b_t, y_t)$ as given in (29).

We need to show that the proposed equilibrium loan terms and consumer policies, and expectations are consistent with the three equilibrium conditions of Definition 1.

We start with the expectation consistency conditions. Substituting the proposed equilibrium consumers' policies, we directly obtain, for any b_t,

$$\begin{aligned} h^e_{t,y_{t-1}}(b_t) &= C(u^*_{t,y_{t-1}}(B^{-1}_{t,y-1}(b_t))) + \bar{s}_{t,y_{t-1}}(b_t) - b_t \\ &= \hat{c}_{t,y_{t-1}}(b_t) + \bar{s}_{t,y_{t-1}}(b_t) - b_t \\ &= \hat{h}_{t,y_{t-1}}(b_t), \end{aligned}$$

and

$$\begin{aligned} D^e_{t,y_{t-1}}(b_t) &= \pi_{t,y_{t-1}}(\theta^L_t\,|1) \\ &= \pi_{t,y_{t-1}}(\theta^L_t\,|1)1 + \pi_{t,y_{t-1}}(\theta^H_t\,|1)0 \\ &= \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|\hat{x}_{t,y_{t-1}}(b_t))\hat{d}_{t,y_{t-1}}(b_t), \end{aligned}$$

which means that the loan demand and default rate expectations are consistent with the proposed equilibrium agent behavior. Also, under the exemption rule $\bar{s}_{t,y_{t-1}}$ given in (24), the amount of assets seized in equilibrium in bankruptcy from debtors of type y_{t-1}, b_t is

$$\begin{aligned} \max\left\{\hat{s}_{t,y_{t-1}}(b_t) - \bar{s}_{t,y_{t-1}}(b_t), 0\right\} &= \max\left\{\bar{s}_{t,y_{t-1}}(b_t) - \bar{s}_{t,y_{t-1}}(b_t), 0\right\} \\ &= 0, \end{aligned}$$

which means that the recovery rate expectation $\gamma^e_{t,y_{t-1}}(b_t) = 0$ is consistent with agents' equilibrium behavior, as well.

Checking the zero-profit condition for loans of type (y_{t-1}, b_t), we get

$$\begin{aligned} \Pi^e_{y_{t-1},b_t} &= \left(-1 + (1 - D^e_{t,y_{t-1}}(b_t))R_{t,y_{t-1}}(b_t) + D^e_{t,y_{t-1}}(b_t)\gamma^e_{t,y_{t-1}}(b_t)\right) h^e_{t,y_{t-1}}(b_t) \\ &= \left(-1 + (1 - \pi_{t,y_{t-1}}(\theta^L_t\,|1))/\pi_{t,y_{t-1}}(\theta^H_t\,|1) + 0\right) h^e_{t,y_{t-1}}(b_t) \\ &= (-1 + 1 + 0)\, h^e_{t,y_{t-1}}(b_t) \\ &= 0. \end{aligned}$$

What remains to be shown is that the proposed agents' behavior policies are consistent with agents' utility maximization under the continuation value functions (26), unsecured debt pricing schedules (30) and bankruptcy rules $(f_t, \bar{s}_t)$ given in (23)-(24).

We first demonstrate that the proposed behavior is budget-feasible. Straightforward substitution of the proposed equilibrium policies to the budget constraints (16)-(20) shows this directly, except for the requirement $\hat{h}_{t,y_{t-1}}(b_t) \geq 0$. Under the proposed equilibrium behavior, we have

$$\hat{h}_{t,y_{t-1}}(b_t) = C(u^*_{t,y_{t-1}}(B^{-1}_{t,y-1}(b_t))) - \theta^L_t + q_t B_{t+1,\theta^L_t}(w^*_{t+1,y_{t-1}}(B^{-1}_{t,y_{t-1}}(b_t), \theta^L_t)) - b_t$$

for any b_t. That the expression on the right-hand side of this equality is positive follows from the fact that, at the solution to the component planning problem, a positive amount of resources is delivered to the low-income agents in period t. To see this, note that since (by definition)

$$C(u^*_{t,y_{t-1}}(w_t)) + \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|1)\{-y_t + q_t B_{t+1,y_t}(w^*_{t+1,y_{t-1}}(w_t, y_t))\} - B_{t,y_{t-1}}(w_t) = 0, \qquad (32)$$

inequality (9) implies

$$C(u^*_{t,y_{t-1}}(w_t)) - \theta^L_t + q_t B_{t+1,\theta^L_t}(w^*_{t+1,y_{t-1}}(w_t, \theta^L_t)) - B_{t,y_{t-1}}(w_t) \geq 0,$$

for all w_t, i.e., $\hat{h}_{t,y_{t-1}} \geq 0$.

We must show that the proposed choices, in addition to being budget-feasible, are in fact optimal in the utility maximization problem, i.e., that there does not exist a budget feasible plan that yields more utility than the proposed equilibrium policies. The standard argument for showing that would examine the first-order conditions (FOC) of this problem. The maximization problem at hand, however, has a non-convex budget set. We handle the non-convexity in the following way: we divide the budget set into several subsets, each of which is given by linear inequalities. The problem of overall utility maximization is split into the several problems of maximization over the convex subsets. In each of these subsets, we use the standard FOC-based argument. The proposed equilibrium behavior dominates the solutions to each of the sub-problems, and thus it is an overall solution to the utility maximization problem.

We start out by examining the set of default plans available to the agents. Under the eligibility rule (23) the choice $d(\theta^H_t) = 1$ is not budget-feasible. Thus, there are only two budget-feasible default plans for agents of all wealth levels b_t:

$$(d(\theta^H_t), d(\theta^L_t)) = (0, 0) \qquad (33)$$

and

$$(d(\theta^H_t), d(\theta^L_t)) = (0, 1). \qquad (34)$$

The default premium, i.e., the fact that $R_{t,y_{t-1}}(b_t) > 1$, implies that if $d(\theta^L_t) = 0$, then $h = 0$, i.e., if an agent plans to never default, borrowing in the unsecured instrument is suboptimal. To see this, note that under the no-default plan (33), the budget constraints (16)-(20) reduce to

$$\begin{aligned} c + s &= b_t + h, \\ q_t b'(y_t) &= y_t + s - h R_{t,y_{t-1}}(b_t), \end{aligned}$$

for $y_t \in \Theta_t$. Given that agents' continuation value W_{t+1,y_t} is strictly increasing in $b'(y_t)$, any choice of c, s, h and $b'(y_t)$, such that $h > 0$ is strictly dominated by the feasible choice $\tilde{c} = c$, $\tilde{s} = s - h$,

$\tilde{h} = 0$, and

$$\begin{aligned} \tilde{b}'(y_t) &= q_t^{-1}(y_t + \tilde{s}) \\ &= q_t^{-1}(y_t + s - h) \\ &> q_t^{-1}(y_t + s - R_{t,y_{t-1}}(h, b_t)h) \\ &= b'(y_t) \end{aligned}$$

for $y_t \in \Theta_t$. We thus have that, under the proposed equilibrium pricing, the best allocation that agents can individually obtain using the no-default plan (33) coincides with the best allocation they obtain under self-insurance. Given that self-insurance is inefficient in the environment at hand, the proposed equilibrium allocation dominates any individual plan that does not involve default in the low income state. Thus, no such plan can deliver higher individual utility than that delivered at the optimum.

We now consider deviations from the proposed equilibrium behavior under the other feasible default plan (34). We prove the following lemma.

Lemma 3 *At any $t \leq T$, for any b_t and $y_{t-1} \in \Theta_{t-1}$, under the proposed equilibrium prices, credit limits and bankruptcy rules, conditional on the default plan (34) and high effort $x_t = 1$, the proposed equilibrium policies $\hat{c}_{t,y_{t-1}}, \hat{h}_{t,y_{t-1}}, \hat{s}_{t,y_{t-1}}, \hat{b}_{t+1,y_{t-1}}$ solve the consumer utility maximization problem.*

Proof In Appendix. ■

The next lemma shows that the same conclusion is true under low effort $x_t = 0$.

Lemma 4 *At any $t \leq T$, for any b_t and $y_{t-1} \in \Theta_{t-1}$, under the proposed equilibrium prices, credit limits and bankruptcy rules, conditional on the default plan (34) and low effort $x_t = 0$, the proposed equilibrium policies $\hat{c}_{t,y_{t-1}}, \hat{h}_{t,y_{t-1}}, \hat{s}_{t,y_{t-1}}, \hat{b}_{t+1,y_{t-1}}$ solve the consumer utility maximization problem.*

Proof In Appendix. ■

By Lemma 3 and Lemma 4, we have that agents find it optimal to follow the same policies $\hat{c}_{t,y_{t-1}}, \hat{h}_{t,y_{t-1}}, \hat{s}_{t,y_{t-1}}, \hat{b}_{t+1,y_{t-1}}$ under either effort choice $x_t \in \{0, 1\}$, where $\hat{c}_{t,y_{t-1}}$ satisfies (28) and $\hat{b}_{t+1,y_{t-1}}$ satisfies (29). Thus, the value $W_{t,y_{t-1}}(b_t)$ of the utility maximization problem is given by

$$W_{t,y_{t-1}}(b_t) = \max_{x_t \in \{0,1\}} V_t(x_t) + U(\hat{c}_{t,y_{t-1}}(b_t)) + \beta \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|x_t) W_{t+1,y_t}(\hat{b}_{t+1,y_t}(b_t, y_t)). \quad (36)$$

From (26) and (29), we have that $W_{t+1,y_t}(\hat{b}_{t+1,y_t}(b_t, y_t)) = w^*_{t+1,y_{t-1}}(B^{-1}_{t,y_{t-1}}(b_t), y_t)$. From (28) and the definition of C, we have $U(\hat{c}_{t,y_{t-1}}(b_t)) = u^*_{t,y_{t-1}}(B^{-1}_{t,y_{t-1}}(b_t))$. Substituting into (36), we have

$$\begin{aligned} W_{t,y_{t-1}}(b_t) &= \max_{v_t \in \{V_t(0), V_t(1)\}} v_t + u^*_{t,y_{t-1}}(B^{-1}_{t,y_{t-1}}(b_t)) + \beta \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|X_t(v_t)) w^*_{t+1,y_{t-1}}(B^{-1}_{t,y_{t-1}}(b_t)) \\ &= V_t(1) + u^*_{t,y_{t-1}}(B^{-1}_{t,y_{t-1}}(b_t)) + \beta \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|1) w^*_{t+1,y_{t-1}}(B^{-1}_{t,y_{t-1}}(b_t), y_t), \quad (37) \end{aligned}$$

where the last line follows from the fact that the recommendation $v^*_{t,y_{t-1}}(w_t) = V_t(1)$ is an optimal policy in the component planning problem, i.e., the recommendation $v^*_{t,y_{t-1}}$ and the continuation utility policy $w^*_{t+1,y_{t-1}}$ satisfy the TIC constraint (4) of this problem. Finally, that $W_{t,y_{t-1}}(b_t) = B^{-1}_{t,y_{t-1}}(b_t)$ for every b_t follows from (37) and the fact that the solution to the component planning problem, $v^*_{t,y_{t-1}}(w_t) = V_t(1)$, $u^*_{t,y_{t-1}}(w_t)$, $w^*_{t+1,y_{t-1}}(w_t, y_t)$, satisfies the PK constraint (5) of this problem for every w_t. ■

A key step in the proof of this implementation result is checking that the socially optimal allocation A^* is consistent with agents' individual utility maximization in the market economy. The optimal allocation is determined by the solutions to component planning problems, in which the planner directly controls everything but agent's private effort. By Proposition 1, at the socially optimal allocation agents are insurance- and savings-constrained. In our market economy, agents have much freedom in choosing their insurance and savings levels through their trades in the riskless claims markets, their unsecured borrowing, and bankruptcy decisions. Why do the agents not find it individually optimal to trade away from the socially optimal allocation?

First, consider the insurance wedge given in (13) and (14). In the market economy with unsecured credit and bankruptcy, agents' access to insurance is restricted by the unsecured credit limit $\bar{h}_{t,y_{t-1}}$ and the bankruptcy asset exemption level $\bar{s}_{t,y_{t-1}}$. Inequality (13) implies that agents would like to deviate from the optimal allocation by selling contingent claims to their income in the high state $y_t = \theta^H_t$. The credit limit $\bar{h}_{t,y_{t-1}}$ makes such a sale infeasible, i.e., outside the budget set faced by the agents the market economy. Inequality (14) implies that agents would like to trade away from the optimal allocation by buying a claim that would pay off in their low income state $y_t = \theta^L_t$. Conditional on an agent's plan to obtain bankruptcy discharge in the low income state, however, this trade is not in the budget set, either. Saving more than the asset exemption level $\bar{s}_{t,y_{t-1}}$ and going into bankruptcy in the low income state does not increase the agent's post-settlement resources in this state because savings in excess of the exemption level $\bar{s}_{t,y_{t-1}}$ are seized from the agent.[28] The same mechanisms makes the savings wedge (15) consistent with agents' individual optimization in the market economy. The optimal bankruptcy rules provide a disincentive to save via the exemption cap $\bar{s}_{t,y_{t-1}}$. Note that agents do have the option to not subject themselves to the savings restriction $\bar{s}_{t,y_{t-1}}$ by never going into bankruptcy. This plan of action, however, is suboptimal, as it eliminates the benefit of insurance that bankruptcy provides to the agents.

As Lemma 4 demonstrates, the unsecured credit limit $\bar{h}_{t,y_{t-1}}$ and the bankruptcy asset exemption level $\bar{s}_{t,y_{t-1}}$ not only discourage deviations from the optimum in asset market trades but also the so-called joint deviations in which agents simultaneously adjust their asset market trades and private effort. In fact, an agent's deviation to low effort $x_t = 0$ increases his demand for insurance against the low income shock $y_t = \theta^L_t$. It is thus intuitive that if the credit limit $\bar{h}_{t,y_{t-1}}$ and the exemption cap $\bar{s}_{t,y_{t-1}}$ prevent a deviation in asset trades under high effort, the more so they do under low effort.

[28] Note that the exemption cap $\bar{s}_{t,y_{t-1}}$ can be imposed because agents cannot conceal their assets in our environment, in which all parameters and variables but effort are publicly observable.

5 History dependence through wealth

In this section, we examine how the optimal exemption level $\bar{s}_{t,y_{t-1}}(b_t)$ and the unsecured credit limit $\bar{h}_{t,y_{t-1}}(b_t)$ depend on wealth b_t.

Proposition 3 *For any $t \leq T$ and $y_{t-1} \in \Theta_{t-1}$, the optimal exemption level $\bar{s}_{t,y_{t-1}}(b_t)$ is strictly increasing in wealth b_t at a rate strictly smaller that one. The equilibrium unsecured credit limit $\bar{h}_{t,y_{t-1}}(b_t)$ is strictly decreasing in b_t.*

Proof The optimal exemption level is defined in (24) as

$$\bar{s}_{t,y_{t-1}}(b_t) = q_t B_{t+1,\theta_t^L}(w^*_{t+1,y_{t-1}}(B^{-1}_{t,y_{t-1}}(b_t), \theta_t^L)) - \theta_t^L.$$

Since $B_{t,y_{t-1}}$ is strictly increasing, so is $B^{-1}_{t,y_{t-1}}$. By Lemma 1, $w^*_{t+1,y_{t-1}}(w_t, \theta_t^L)$ is strictly increasing in w_t. Thus, since B_{t+1,θ_t^L} is strictly increasing, we have that $B_{t+1,\theta_t^L}(w^*_{t+1,y_{t-1}}(B^{-1}_{t,y_{t-1}}(b_t), \theta_t^L))$ is strictly increasing in b_t.

Using (24), the expression for the equilibrium credit limit (31) can be rewritten as follows

$$\bar{h}_{t,y_{t-1}}(b_t) = C(u^*_{t,y_{t-1}}(B^{-1}_{t,y-1}(b_t))) + q_t B_{t+1,\theta_t^L}(w^*_{t+1,y_{t-1}}(B^{-1}_{t,y_{t-1}}(b_t), \theta_t^L)) - \theta_t^L - b_t.$$

From (32), we have

$$C(u^*_{t,y_{t-1}}(B^{-1}_{t,y-1}(b_t))) = b_t - \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|1)\{-y_t + q_t B_{t+1,y_t}(w^*_{t+1,y_{t-1}}(B^{-1}_{t,y-1}(b_t), y_t))\},$$

thus

$$\begin{aligned}
\bar{h}_{t,y_{t-1}}(b_t) &= -\theta_t^L + q_t B_{t+1,\theta_t^L}(w^*_{t+1,y_{t-1}}(B^{-1}_{t,y_{t-1}}(b_t), \theta_t^L)) \\
&\quad - \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|1)\{-y_t + q_t B_{t+1,y_t}(w^*_{t+1,y_{t-1}}(B^{-1}_{t,y-1}(b_t), y_t))\} \\
&= \pi_{t,y_{t-1}}(\theta_t^H|1) \times \\
&\quad \left(\theta_t^H - \theta_t^L + q_t[B_{t+1,\theta_t^L}(w^*_{t+1,y_{t-1}}(B^{-1}_{t,y_{t-1}}(b_t), \theta_t^L)) - B_{t+1,\theta_t^H}(w^*_{t+1,y_{t-1}}(B^{-1}_{t,y_{t-1}}(b_t), \theta_t^H))]\right).
\end{aligned}$$

That $\bar{h}_{t,y_{t-1}}(b_t)$ is strictly decreasing follows now from the fact that $B^{-1}_{t,y_{t-1}}$ is strictly increasing, (44) and (11).

To show that the rate at which $\bar{s}_{t,y_{t-1}}(b_t)$ increases in b_t is strictly less than one, we write (31) as

$$\bar{h}_{t,y_{t-1}}(b_t) - C(u^*_{t,y_{t-1}}(B^{-1}_{t,y-1}(b_t))) = \bar{s}_{t,y_{t-1}}(b_t) - b_t$$

for all b_t. Since $C(u^*_{t,y_{t-1}}(B^{-1}_{t,y-1}(b_t)))$ is increasing in b_t, the left-hand side of this identity is strictly decreasing in b_t. So must be the right-hand side. ■

The dependence of the optimal exemption level and equilibrium credit limit on wealth follows from the underlying incentive problem quite intuitively. The exception level $\bar{s}_{t,y_{t-1}}(b_t)$ restricts, and in equilibrium determines, the amount of wealth that an agent of type (y_{t-1}, b_t) can take into period $t+1$ following the low income realization $y_t = \theta_t^L$. Since it is efficient in the component planning problem to intertemporally smooth the provision of the continuation value to the agent, the optimal continuation value process is persistent. In the market economy, wealth b_t tracts each agent's continuation value. Thus, wealth must be persistent in equilibrium, i.e., for each realization of current income y_t, wealth taken into period $t+1$ is increasing with wealth held at the beginning of period t. In particular, it is increasing for $y_t = \theta_t^L$, and thus $\bar{s}_{t,y_{t-1}}(b_t)$ must be increasing in b_t.

That $\bar{h}_{t,y_{t-1}}(b_t)$ is strictly decreasing in wealth b_t follows from the decreasing marginal utility of wealth and the need to provide an incentive to exert effort. High effort at date t carries the same disutility $V_t(1) - V_t(0)$ for agents of all wealth levels b_t. Because agents' value functions are strictly concave, richer agents have lower marginal utility of wealth. For richer agents, therefore, it takes more spreading in the equilibrium wealth process to provide a given amount of spreading in their continuation value process. Thus, in order to provide enough incentives for the agents to choose to incur the high effort disutility $V_t(1) - V_t(0)$, less insurance against the individual income shock in period t is optimally provided to agents with higher wealth b_t. In equilibrium, agents obtain insurance through unsecured borrowing and discharge. Thus, the amount of unsecured credit available to richer agents is decreasing with wealth b_t.

6 Isolating the effects of moral hazard

In this section, we isolate those features of the optimal bankruptcy code and the unsecured credit arrangement that are due to moral hazard. To this end, we consider a full-information version of our environment. The efficient allocation of the full-information environment, similar to the optimum of the moral hazard environment, can be implemented as an equilibrium outcome of an unsecured credit market economy with debt discharge regulated by a bankruptcy code. In particular, the full-information optimum can be implemented in a market arrangement in which agents face no unsecured credit limit, and bankruptcy provides unlimited asset exemption to all bankruptcy-eligible agents. The associated eligibility condition, however, is more stringent, as eligibility is determined on the basis of both income and effort.

6.1 Efficiency with full information

Under full information, an allocation $A = (c(\omega), x(\omega))_{\omega \in \Omega}$ is efficient if it delivers the initial distribution of promised utility Ω, and if it minimizes, among all allocations that deliver Ω, the net cost $C_1^A(\Omega)$ given in (3). Note that the incentive compatibility requirement is absent from this definition. Let $A^{**}(\Omega) = (c^{**}(\omega), x^{**}(\omega))_{\omega \in \Omega}$ denote the full-information optimum, and let $C_1^{**}(\omega)$ denote the ω-component of the minimized cost objective. It is a standard result that A^{**} satisfies the following optimality condition

$$\beta U'(c_{t+1}^{**}(\omega, (y^{t-1}, y_t))) = q_t U'(c_t^{**}(\omega, y^{t-1}))$$

for any $\omega \in S(\Omega)$, $t \leq T$, and $y^{t-1} \in \Theta^{t-1}$ and $y_t \in \Theta$. At A^{**}, therefore, agents are fully insured and their intertemporal consumption profiles are fully smoothed, i.e., agents are neither savings- nor borrowing-constrained. (Compare this with Proposition 1 in the moral hazard case.) As before, we will assume that high effort is optimal at all dates and states.

6.2 Unsecured credit market implementation

Consider now the market economy defined in Section 4.2, in which agents trade secured (riskless) bonds and take out unsecured loans with competitive financial intermediaries. In this economy, let us specify the bankruptcy rules $(f, \bar{s})$ as follows. Let the eligibility condition f_t be given by

$$f_t(x_t, y_t) = \begin{cases} 1 & \text{if } y_t = \theta_t^L \text{ and } x_t = 1, \\ 0 & \text{otherwise} \end{cases} \tag{38}$$

for $t \leq T$. Note that effort x_t, which now is publicly observable, is an argument of f_t. Also, set the asset exemption level $\bar{s}_{t,y_{t-1}}(b_t)$ equal to plus infinity for all t, y_{t-1} and b_t (i.e., eliminate the possibility of seizure of assets). Under these bankruptcy rules, the unsecured debts of all low-income, high-effort agents are dischargeable, and all assets held by a bankrupt agent, in addition to his current and future income, are exempt from creditors' claims.

Following the steps of the proof of Theorem 1, it is straightforward to verify that these bankruptcy rules constitute an optimal bankruptcy code in the full-information economy. In particular, under the above bankruptcy rules and with the initial assignment of wealth $\phi = C_1^{**}$, an equilibrium exists in which the unsecured credit limit $\bar{h}_{t.y_{t-1}}(b_t)$ is plus infinity for all all t, y_{t-1} and b_t, and the gross interest rate on the unsecured loans available to the consumers is given by, as in (30), $R_{t,y_{t-1}} = 1/\pi_{t,y_{t-1}}(\theta_t^H|1)$. In equilibrium, the size of the unsecured loan h taken out by an agent who faces the rate $R_{t,y_{t-1}}$ is $\hat{h}_{t,y_{t-1}} = (\theta_t^H - \theta_t^L)/R_{t,y_{t-1}}$, independently of wealth b_t. The discharge eligibility condition (38) is sufficient to induce high effort. If an agent decides to exert low effort, it is optimal for him to borrow $h = 0$ because he will not be able to discharge h and $R_{t,y_{t-1}} > 1$. With low effort, thus, the best an agent can do is to self-insure. This strategy, however, is dominated by the equilibrium strategy, because self-insurance is inefficient.

6.3 Comparing moral hazard and full information

Unsecured credit markets and bankruptcy can be used to implement optimal social insurance in a full information economy as well as in the economy with moral hazard. The results of the last subsection demonstrate that unsecured credit limits and bounded bankruptcy asset exemptions are inessential under full information. Both of these are essential, however, in the moral hazard economy, as follows from our proof of Theorem 1. Also, it is easy to show that conditioning on effort is essential in implementation of the full-information optimum A^{**}. Theorem 1 shows that such conditioning is not needed in market implementation of the constrained optimum A^* under moral hazard. In our model, therefore, moral hazard necessitates credit limits and bankruptcy asset exemption caps. This result suggests that moral hazard may explain why credit limits and bankruptcy exemption caps are observed in real-life unsecured consumer credit markets and bankruptcy arrangements.

7 Discussion of positive properties

In this paper, we use an abstract, stylized environment to study normative implications of dynamic moral hazard for the structure of unsecured credit markets and personal bankruptcy. Thus, we develop a normative, efficiency-based theory of unsecured consumer credit and bankruptcy. It must be emphasized that our model was not designed to replicate any particular set of facts about the structure of actual credit markets or bankruptcy laws functioning in any given country. Nevertheless, it is useful to examine the outcome of our normative analysis from a positive perspective. The objective of this section is to discuss the similarities and dissimilarities between the normative prescriptions of the model and the data in order to explore both the limitations of the model and the possibilities for identifying potential inefficiencies in the design of the observed institutions.

In the first subsection, we compare the basic structure of the credit market and bankruptcy arrangement obtained in our normative model with the basic structure of the institutions currently functioning in the United States. We conclude that these structures are remarkably similar. We take this similarity as evidence in support of the main hypothesis of this paper: the institution of personal bankruptcy is an insurance mechanism (against the risk of idiosyncratic income loss) and moral hazard is an important force shaping this institution.

In the second subsection, we look beyond the basic structure of the model. Several of the additional qualitative features of the model highlight dimensions on which our analysis could be extended as well as suggest directions that can be taken in thinking about potential policy improvements.

7.1 Basic structure

The structure of the unsecured credit market obtained in our model has the following four basic characteristics:

1. Default and bankruptcy discharge of unsecured debt occur in equilibrium.
2. Consumers borrow unsecurely at high interest rates and, simultaneously, hold low-yielding liquid assets, which could be used to reduce or eliminate their unsecured debt. Consumers face limits on the amount of unsecured credit they can obtain.
3. Lenders have information about all unsecured loans that their borrowers obtain from other lenders. Lenders write off loans discharged in bankruptcy as losses.
4. The unsecured lending sector is competitive.

These four properties are well documented in the U.S. data. At a general level, they are broadly consistent with the stylized facts provided in Chatterjee et al. (2007). More particularly, Gropp, Scholz and White (1997) and Sullivan, Warren and Westbrook (2000) present data on the prevalence of unsecured consumer credit and personal bankruptcy in the United States. The fact that a large number of credit card borrowers hold liquid assets in significant quantities is sometimes referred to as the credit card puzzle. This fact is documented in, e.g., Gross and Souleles (2001).[29] Pagano and

[29] See also Telyukova (2006) and Telyukova and Wright (2008).

Jappelli (1993) and Hunt (2003) describe the structure of consumer credit reporting in the United States. Clearly, U.S. households do not borrow anonymously. When making credit award decisions, lenders have a great deal of knowledge about the prospective borrower's debts outstanding with other lenders. Evans and Schmalensee (2005) document competition in credit card lending in the United States.

The optimal bankruptcy rules obtained in our model have the following three basic properties:

1. Bankruptcy discharges unsecured debt obligations.

2. Bankruptcy rules provide limited asset exemptions for debtors obtaining discharge. In addition to the exempt assets, all current and future individual income is exempt.

3. Eligibility for bankruptcy debt discharge is means-tested on the basis of current income.

These three properties make the bankruptcy institution derived in our model strikingly similar to the liquidation procedure of the U.S. bankruptcy law, i.e., the so-called chapter 7 bankruptcy. Chapter 7 bankruptcy in fact discharges all unsecured debt obligations, provides asset exemptions for debtors, and frees all current and future income of the debtor from claims of the creditors.[30] Means-testing for debtor eligibility to obtain discharge in chapter 7 bankruptcy was introduced by the Bankruptcy Abuse Prevention and Consumer Protection Act of 2005 (BAPCPA). The BAPCPA test is based primarily on the debtor's current income.[31] Similar to the eligibility rule of our model, all low-income agents are eligible for chapter 7 discharge under BAPCPA, where low income is defined as below state median. However, unlike in our model, high-income agents (i.e., those with income above state median) are not automatically excluded under the BAPCPA test. If the amount of unsecured debt of a high-income individual is large enough (relative to his disposable income, which is defined as income less allowable expenses), the debtor is eligible for discharge on grounds of being unable to repay his unsecured debt. Clearly, this feature of the BAPCPA test may provide an above-median income household with an incentive to increase borrowing in order to qualify for chapter 7 discharge.

7.2 Additional implications

Beyond the basic structure, a number of additional features of our model can be confronted with the corresponding features of the market arrangement in the U.S. economy. In the model, seizure of non-exempt assets is an off-equilibrium event, i.e., the recovery rate on loans in bankruptcy is zero. Sullivan, Warren and Westbrook (2000) cite evidence showing that gross (96 percent in a large sample) of chapter 7 bankruptcy cases are the so-called no-asset cases, in which debtors do not hold any non-exempt assets. Consequently, conditional on bankruptcy, lenders' recovery rates are close to zero.

[30] See Administrative Office of the United States Courts (2008) for a more detailed description of the chapter 7 bankruptcy liquidation proceeding.

[31] See Wedoff (2007) for a detailed description of the BAPCPA and its means test.

By Proposition 3, the level of the asset exemption provided by the optimal bankruptcy rule of our model is strictly increasing in the agent's wealth. As a fraction of wealth, however, the exemption is decreasing. In the model, therefore, the exemption is regressive (i.e., increasing at a decreasing rate).

It is not immediately clear how the asset exemption provided by the U.S. chapter 7 bankruptcy law depends on the level of debtor's wealth. Chapter 7 bankruptcy provides not one but many exemptions, one for each of the several asset classes distinguished in the law.[32] The total wealth exemption, therefore, will depend on how a debtor's wealth is allocated among these different asset classes. Within each asset class, the exemption is given by a simple ceiling level specified in absolute terms.[33] The overall exemption level, however, can increase in debtors' total wealth. This increase will be observed if wealthier debtors' wealth is spread among a larger number of asset classes recognized by the law. Thus, the fact that the within-class exemption caps are absolute is not inconsistent with the overall exemption level being increasing in debtors' wealth.

It is clear, however, that the observed structure of bounded within-class chapter 7 asset exemptions makes the overall exemption level a regressive function of wealth. For debtors with higher overall wealth, within-class exemptions granted in the U.S. chapter 7 procedure will bind within a larger number of asset classes. Eventually, the within-class exemptions will be exhausted, which makes the overall exemption level regressive, as it is in our model (see Proposition 3). In the model, however, the exemption level is smoother than the one provided by chapter 7. Also, even at very high levels of wealth, the exemption level of our model is slightly increasing. This means that wealthy agents in our model receive more insurance than what chapter 7 appears to be providing to wealthy households in the United States.

Closely related to the asset exemption level is the unsecured credit limit faced by an agent. In our model, as shown in Proposition 3, the unsecured credit limit $\bar{h}_{t,y_{t-1}}$ is decreasing in individual wealth. It is important to point out, however, that it would be incorrect to identify the credit limit $\bar{h}_{t,y_{t-1}}$ with the nominally unsecured credit limit of a household in the data (measured by the sum of limits on credit card accounts and other unsecured loans held by the household). In the model, agents borrow unsecurely purely for insurance purposes and do not hold any non-exempt wealth in equilibrium. Thus, 100% of the unsecured credit extended to the agents in our model is dischargeable in personal bankruptcy. If we define *de facto unsecured credit* as precisely what can be discharged in bankruptcy, all of the unsecured credit used in our model is de facto unsecured. In reality, households hold non-exempt assets and use unsecured credit not exclusively for insurance purposes. Non-exempt wealth held by a household in the U.S. implicitly collateralizes the *nominally unsecured credit* available to the household.[34] Therefore, credit card limits and the unsecured credit

[32] Different exemption levels apply to such assets as primary residence equity, other real estate, one motor vehicle, additional motor vehicles, funds in retirement accounts, value of life insurance policies, cash, household goods.

[33] There are some exceptions. For example, the unused portion of the primary residence exemption can be used to exempt other property. For details, see Administrative Office of the United States Courts (2008) and references therein.

[34] For example, an agent whose nominal credit limit on a credit card is $50,000 faces a de facto unsecured credit limit of zero if his non-exempt assets stand in excess of $50,000. An agent with a $5,000 credit limit on a credit card and no non-exempt assets has access to de facto unsecured credit in the amount of $5,000.

limit of our model are two different objects.

The amount of de facto unsecured credit available to a household in the United States, which would be comparable with our limit $\bar{h}_{t,y_{t-1}}$, is hard to measure directly. This amount depends on the overall wealth exemption level facing the household in the chapter 7 procedure, which in turn depends on the allocation of wealth among the various asset classes distinguished in the law.

It is clear, however, that the bounded within-class asset exemptions of chapter 7 constitute a mechanism that decreases the de facto unsecured credit limit available to a household whose wealth increases. In fact, as wealth of a debtor increases, so does the non-exempt portion of it, as the allowed exemptions are not unbounded. The non-exempt wealth reduces dollar-for-dollar the de facto unsecured portion of a given nominally unsecured credit limit faced by the debtor.[35] A similar mechanism functions in our model.

In this paper, all kinds of real and financial wealth held by households are represented by a single, abstract riskless bond. Our model, therefore, is not suitable to address the interesting question of why different asset classes receive different exemption levels in the U.S. bankruptcy law. In the normative exercise we perform, individual income shocks are the only risk that agents face over the life-cycle, and unsecured borrowing and bankruptcy are, by construction, the only means for the agents to obtain insurance against these shocks. This modeling approach, commonly used in the normative literature (including the literature on optimal taxation), puts limits on how realistic the model prescriptions can be. Given these obvious limitations, it should be rather surprising how well the outcome of our normative analysis corresponds to the actual institutions observed in the U.S. economy.

Both in the data and in the model, the level of bankruptcy asset exemption provided to households is regressive in household wealth (i.e., is increasing at a decreasing rate). It appears, however, that the exemption provided in the U.S. chapter 7 law is too regressive relative to the prescriptions of our model. In effect, the insurance opportunities that chapter 7 procedure provides to wealthy households appear to be less than what theoretically could be provided through this channel if the exemption level schedule were chosen to be less regressive.

8 Conclusions

In this paper, we pose and answer the normative question of how unsecured credit markets and the institution of personal bankruptcy should be organized in order to implement an efficient allocation of effort and consumption in a stylized life-cycle moral hazard economy. In equilibrium, agents borrow unsecurely and, simultaneously, save in the riskless asset. If hit by an adverse idiosyncratic income shock, agents use personal bankruptcy to obtain discharge of their unsecured debt obligations. In bankruptcy, agents use the asset exemption allowed by the optimal bankruptcy rules to shield their savings from the creditors. In effect, they obtain insurance against the adverse income shock. The

[35] In the example of the previous footnote, if the agent with de facto unsecured credit line of $5,000 suddenly receives a wealth injection of $3,000 in cash (a non-exempt asset), his de facto unsecured credit line drops automatically to $2,000.

optimal bankruptcy rules are designed to ensure incentive compatibility of this insurance mechanism. Eligibility for discharge is means-tested and asset exemptions are bounded. Access to unsecured credit, as well, is limited. The unsecured credit limits are tight just enough to not give agents an incentive to shirk. Unsecured credit is priced at actuarially fair odds and lenders break even in equilibrium.

Viewed from a positive perspective, our model formalizes the notion that unsecured credit and personal bankruptcy constitute a mechanism for the provision of social insurance under asymmetric information. In our analysis, from this notion we derive an efficiency-based theory on how unsecured credit markets and personal bankruptcy should be organized. The institutions obtained in our analysis turn out to be qualitatively similar to those actually observed in the United States. Our study, therefore, supports the notion that bankruptcy is an insurance mechanism whose functioning is constrained by private information.

Agents' unobservable effort is the only friction in the otherwise frictionless environment we study. This assumption allows us to isolate the implications of moral hazard for unsecured lending and personal bankruptcy. By confronting these implications with the main features of the institutions functioning in the U.S. economy, we gain insight into what other frictions, in addition to moral hazard, may be important in shaping the observed institutions. In particular, given the important role collateral plays in actual bankruptcy laws, the availability of costless enforcement of private contracts stands out as a strong assumption in our analysis. It seems that examining the implications of private information jointly with costly contract enforcement may provide further insights into the functioning of consumer credit markets and optimal design of the institution of bankruptcy.

Beyond these general lessons, our results can be useful for quantitative studies of personal bankruptcy in the United States. An essential feature of the optimal arrangement of our model is that unsecured debt is distinct from negative wealth. Also, our model demonstrates that bankruptcy asset exemptions are important in determining what portion of nominally unsecured credit is de facto unsecured, i.e., dischargeable in bankruptcy. These features of our model suggest that disentangling unsecured credit from negative net worth and accounting for asset exemptions may be a productive next step to take in quantitative work.

Appendix

Proof of Lemma 1

Note first that the solution to the component planning problem at $T+1$ is immediate:

$$B_{T+1,y_T}(w_{T+1}) = C_{T+1}(w_{T+1}),$$

independently of y_T. Because U_{T+1} is twice differentiable, strictly increasing and strictly concave, B_{T+1,y_T} is twice differentiable, strictly increasing and strictly convex.

Consider now the cost function $B_{T,y_{T-1}}$. Since B_{T+1,y_T} is strictly convex, $B_{T,y_{T-1}}$ is the value of a minimization problem with a strictly convex objective and linear constraints. By strict convexity, this problem has a unique solution and $B_{T,y_{T-1}}$ is also strictly increasing and strictly convex. Proceeding backwards, we have that all functions $B_{t,y_{t-1}}$ are strictly increasing and strictly convex.

We now show that (8) holds. Fix $t \leq T$, $y_{t-1} \in \Theta_{t-1}$, and w_t. The solution to the component planning problem must satisfy the temporary incentive compatibility (TIC) constraint (4), which can be written as

$$\begin{aligned} &V_t(1) + \beta \pi_{t,y_{t-1}}(\theta_t^H|1)\left(w^*_{t+1,y_{t-1}}(w_t,\theta_t^H) - w^*_{t+1,y_{t-1}}(w_t,\theta_t^L)\right) \geq \\ &V_t(0) + \beta \pi_{t,y_{t-1}}(\theta_t^H|0)\left(w^*_{t+1,y_{t-1}}(w_t,\theta_t^H) - w^*_{t+1,y_{t-1}}(w_t,\theta_t^L)\right). \end{aligned}$$

Thus,

$$\begin{aligned} w^*_{t+1,y_{t-1}}(w_t,\theta_t^H) - w^*_{t+1,y_{t-1}}(w_t,\theta_t^L) &\geq \frac{V_t(0)-V_t(1)}{\pi_{t,y_{t-1}}(\theta_t^H|1) - \pi_{t,y_{t-1}}(\theta_t^H|0)}\beta^{-1} \\ &> 0, \end{aligned}$$

where the strict inequality follows from $V_t(0) > V_t(1)$ and (1).

From (8) and the strict convexity of B_{t+1,y_t} it follows that the TIC constraint (4) binds. If it did not, it would be possible to decrease the difference between $w^*_{t+1,y_{t-1}}(w_t,\theta_t^H)$ and $w^*_{t+1,y_{t-1}}(w_t,\theta_t^L)$ without changing the expected value of $w^*_{t+1,y_{t-1}}$ and obtain in this way a feasible component planner policy that would generate a lower cost than the optimum (by the strict concavity of B_{t+1,y_t}), a contradiction.

Since the TIC binds, it must hold with equality, i.e.,

$$V_t(1) + \beta \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|1) w^*_{t+1,y_{t-1}}(w_t,y_t) = V_t(0) + \beta \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|0) w^*_{t+1,y_{t-1}}(w_t,y_t).$$

Directly from this equality we have that the recommendation of low effort $x = 0$ satisfies the TIC in the minimization problem defining $B_{t,y_{t-1}}(w_t)$. Under this recommendation, the promise keeping (PK) constraint (5) is satisfied as well, as either side of the above TIC constraint equals $w_t - u^*_{t,y_{t-1}}(w_t)$. Thus, the recommendation of low effort $x = 0$, the consumption utility assignment $u^*_{t,y_{t-1}}(w_t)$ and the continuation utility assignment $w^*_{t+1,y_{t-1}}(w_t,y_t)$ for $y_t \in \Theta_t$ are a feasible choice for the component planner in the minimization problem defining $B_{t,y_{t-1}}(w_t)$. We now use this fact to show inequality (9).

If (9) is violated, (1) implies that the low effort recommendation $x = 0$, the consumption utility assignment $u^*_{t,y_{t-1}}(w_t)$, and the continuation utility assignment $w^*_{t+1,y_{t-1}}(w_t, y_t)$ for $y_t \in \Theta_t$ is a policy choice that achieves a cost strictly lower than $B_{t,y_{t-1}}(w_t)$ as

$$\begin{aligned} B_{t,y_{t-1}}(w_t) &= C(u^*_{t,y_{t-1}}(w_t)) + \sum_{y_t\in\Theta_t} \pi_{t,y_{t-1}}(y_t|1)\{-y_t + q_t B_{t+1,y_t}(w^*_{t+1,y_{t-1}}(w_t, y_t))\} \\ &> C(u^*_{t,y_{t-1}}(w_t)) + \sum_{y_t\in\Theta_t} \pi_{t,y_{t-1}}(y_t|0)\{-y_t + q_t B_{t+1,y_t}(w^*_{t+1,y_{t-1}}(w_t, y_t))\}. \end{aligned}$$

This contradicts Assumption 1. Thus, inequality (9) holds.

To show inequality (7), note that the value $\tilde{u}$ defined as

$$\tilde{u} = u^*_{t,\theta^L_{t-1}}(w_t) - \beta \left(w^*_{t+1,\theta^L_{t-1}}(w_t, \theta^H_t) - w^*_{t+1,\theta^L_{t-1}}(w_t, \theta^L_t)\right) \left(\pi_{t,\theta^H_{t-1}}(\theta^H_t|1) - \pi_{t,\theta^L_{t-1}}(\theta^H_t|1)\right)$$

satisfies $\tilde{u} \leq u^*_{t,\theta^L_{t-1}}(w_t)$ because the terms in parentheses are positive by (8) and (2), respectively. Also, it is easy to check (using (2) again) that the utility assignment $\tilde{u}$ and the continuation value assignments $w^*_{t+1,\theta^L_{t-1}}(w_t, y_t)$ for $y_t \in \Theta_t$, are feasible in the constraint set of the minimization problem defining $B_{t,\theta^H_{t-1}}(w_t)$. Thus,

$$\begin{aligned} B_{t,\theta^H_{t-1}}(w_t) &\leq C(\tilde{u}) + \sum_{y_t\in\Theta_t} \pi_{t,\theta^H_{t-1}}(y_t|1)\{-y_t + q_t B_{t+1,y_t}(w^*_{t+1,\theta^L_{t-1}}(w_t, y_t))\} \\ &\leq C(u^*_{t,\theta^L_{t-1}}(w_t)) + \sum_{y_t\in\Theta_t} \pi_{t,\theta^L_{t-1}}(y_t|1)\{-y_t + q_t B_{t+1,y_t}(w^*_{t+1,\theta^L_{t-1}}(w_t, y_t))\} \\ &= B_{t,\theta^L_{t-1}}(w_t), \end{aligned}$$

where the second inequality follows from $\tilde{u} \leq u^*_{t,\theta^L_{t-1}}(w_t)$, (9) and (2).

Cost function differentiability: Fix $t \leq T$, $y_{t-1} \in \Theta_{t-1}$ and consider the difference $B_{t,y_{t-1}}(w_t) - B_{t,y_{t-1}}(w_t - \varepsilon)$ for some w_t and $\varepsilon \neq 0$, with ε small in absolute value. Because the policies $u^*_{t,y_{t-1}}(w_t - \varepsilon) + \varepsilon$ and $w^*_{t+1,y_{t-1}}(w_t - \varepsilon, y_t)$ for $y_t \in \Theta_t$ are feasible in the minimization problem defining $B_{t,y_{t-1}}(w_t)$, we get

$$B_{t,y_{t-1}}(w_t) - B_{t,y_{t-1}}(w_t - \varepsilon) \leq C(u^*_{t,y_{t-1}}(w_t - \varepsilon) + \varepsilon) - C(u^*_{t,y_{t-1}}(w_t - \varepsilon)).$$

Also, since the policies $u^*_{t,y_{t-1}}(w_t) - \varepsilon$ and $w^*_{t+1,y_{t-1}}(w_t, y_t)$ for $y_t \in \Theta_t$ are feasible in the minimization problem defining $B_{t,y_{t-1}}(w_t - \varepsilon)$, we get

$$B_{t,y_{t-1}}(w_t) - B_{t,y_{t-1}}(w_t - \varepsilon) \geq C(u^*_{t,y_{t-1}}(w_t)) - C(u^*_{t,y_{t-1}}(w_t) - \varepsilon).$$

Dividing by ε, we get

$$\begin{aligned}\frac{C(u^*_{t,y_{t-1}}(w_t)) - C(u^*_{t,y_{t-1}}(w_t) - \varepsilon)}{\varepsilon} &\leq \frac{B_{t,y_{t-1}}(w_t) - B_{t,y_{t-1}}(w_t - \varepsilon)}{\varepsilon} \\ &\leq \frac{C(u^*_{t,y_{t-1}}(w_t - \varepsilon) + \varepsilon) - C(u^*_{t,y_{t-1}}(w_t - \varepsilon))}{\varepsilon}.\end{aligned}$$

Taking the (left and right) limit as $\varepsilon \to 0$, we get that $B'_{t,y_{t-1}}(w_t)$ exists with

$$B'_{t,y_{t-1}}(w_t) = C'(u^*_{t,y_{t-1}}(w_t)).$$

We can now further characterize the cost functions $B_{t,y_{t-1}}$ by using the first-order and envelope conditions. For every $(t, y_{t-1}) \in \{1, ..., T\} \times \Theta_{t-1}$ and w_t, the first-order conditions are as follows

$$C'(u^*_{t,y_{t-1}}(w_t)) = \mu_{t,y_{t-1},w_t}, \tag{39}$$

$$B'_{t+1,y_t}(w^*_{t+1,y_{t-1}}(w_t, y_t))q_t\beta^{-1} = \lambda_{t,y_{t-1},w_t}(1 - LR_{t,y_{t-1}}(y_t)) + \mu_{t,y_{t-1},w_t}, \tag{40}$$

for $y_t \in \Theta_t$, where $\lambda_{t,y_{t-1},w_t} > 0$ is the Lagrange multiplier on the TIC constraint, $\mu_{t,y_{t-1},w_t} > 0$ is the Lagrange multiplier on the PK constraint, and $LR_{t,y_{t-1}}(y_t)$ is the likelihood ratio, given by

$$LR_{t,y_{t-1}}(y_t) = \frac{\pi_{t,y_{t-1}}(y_t|0)}{\pi_{t,y_{t-1}}(y_t|1)}.$$

The envelope condition is

$$B'_{t,y_{t-1}}(w_t) = \mu_{t,y_{t-1},w_t}. \tag{41}$$

Conditional on y_{t-1}, the expectation of the likelihood ratio, under optimal effort $x^*_{t,y_{t-1}}(w_t) = 1$ is one:

$$\begin{aligned}\sum_{y_t\in\Theta_t} \pi_{t,y_{t-1}}(y_t|1) LR_{t,y_{t-1}}(y_t) &= \sum_{y_t\in\Theta_t} \pi_{t,y_{t-1}}(y_t|1) \frac{\pi_{t,y_{t-1}}(y_t|0)}{\pi_{t,y_{t-1}}(y_t|1)} \\ &= 1. \end{aligned} \tag{42}$$

Since effort is productive, i.e., (1), we thus have

$$LR_{t,y_{t-1}}(\theta^H_t) < 1 < LR_{t,y_{t-1}}(\theta^L_t) \tag{43}$$

for any $y_{t-1} \in \Theta_{t-1}$.

Condition (10) follows directly from (39) and (41). Inequalities (11) follow from (40) with (41) and the inequalities (43). Condition (12) follows by adding up equations (40) over $y_t \in \Theta_t$ and using (42).

We now show that the policy functions are strictly increasing in the utility value delivered.

Writing the binding TIC constraint as follows

$$w^*_{t+1,y_{t-1}}(w_t,\theta^H_t) = w^*_{t+1,y_{t-1}}(w_t,\theta^L_t) + \frac{V_t(0)-V_t(1)}{\pi_{t,y_{t-1}}(\theta^H_t|1)-\pi_{t,y_{t-1}}(\theta^H_t|0)}\beta^{-1}, \tag{44}$$

we get that $w^*_{t+1,y_{t-1}}(w_t,\theta^H_t)$ and $w^*_{t+1,y_{t-1}}(w_t,\theta^L_t)$ are co-monotone in w_t, as the changes in $w^*_{t+1,y_{t-1}}(w_t,\theta^H_t)$ and $w^*_{t+1,y_{t-1}}(w_t,\theta^L_t)$ associated with any change in w_t must be exactly equal to each other. From (12) and (10), we have

$$C'(u^*_{t,y_{t-1}}(w_t)) = q_t\beta^{-1}\sum_{y_t\in\Theta_t}\pi_{t,y_{t-1}}(y_t|1)B'_{t+1,y_t}(w^*_{t+1,y_{t-1}}(w_t,y_t)). \tag{45}$$

Since B'_{t+1,y_t} is increasing for both $y_t\in\Theta_t$, the right-hand side of this equation is co-monotone with the continuation values $w^*_{t+1,y_{t-1}}(w_t,\theta^H_t)$ and $w^*_{t+1,y_{t-1}}(w_t,\theta^L_t)$. Thus, so must be the left-hand side. By concavity of C, we get that $u^*_{t,y_{t-1}}(w_t)$, as well, is co-monotone with the continuation values $w^*_{t+1,y_{t-1}}(w_t,\theta^H_t)$ and $w^*_{t+1,y_{t-1}}(w_t,\theta^L_t)$. Thus, as w_t varies, $u^*_{t,y_{t-1}}(w_t)$ and $w^*_{t+1,y_{t-1}}(w_t,\theta^H_t)$ and $w^*_{t+1,y_{t-1}}(w_t,\theta^L_t)$ all change in the same direction. The promise-keeping constraint (5) implies that these three values must be increasing in w_t with at least one value strictly increasing. That all three are strictly increasing follows from (45) and (44). ■

Proof of Lemma 2

Solving for W_{T+1,y_T} from definition, for any b_{T+1}, we get

$$\begin{aligned} W_{T+1,y_T}(b_{T+1}) &= \max_{c_{T+1}\le b_{T+1}} U_{T+1}(c_{T+1}) \\ &= U_{T+1}(b_{T+1}), \end{aligned}$$

with optimal consumption

$$\hat{c}_{T+1,y_T}(b_{T+1}) = b_{T+1}. \tag{46}$$

Similarly, from definitions of B_{T+1,y_T} and C_{T+1} we have for any w_{T+1}

$$\begin{aligned} B_{T+1,y_T}(w) &= \min_{u_{T+1}=w_{T+1}} C_{T+1}(u_{T+1}) \\ &= C_{T+1}(w_{T+1}) \\ &= U^{-1}_{T+1}(w_{T+1}), \end{aligned} \tag{47}$$

with

$$u^*_{T+1,y_T}(w_{T+1}) = w_{T+1}. \tag{48}$$

Thus,

$$B^{-1}_{T+1,y_T} = (U^{-1}_{T+1})^{-1} = U_{T+1} = W_{T+1,y_T},$$

and, for both $y_T \in \Theta_T$,

$$\begin{aligned} C_{T+1}(u^*_{T+1,y_T}(B^{-1}_{T+1,y_T}(b_{T+1}))) &= C_{T+1}(B^{-1}_{T+1,y_T}(b_{T+1})) \\ &= b_{T+1} \\ &= \hat{c}_{T+1,y_T}(b_{T+1}), \end{aligned}$$

where the first equality follows from (48), the second from (47), and the third from (46). ■

Proof of Lemma 3

Under the assumptions of the lemma, the problem of a consumer of type (y_{t-1}, b_t) reduces to

$$\max_{c,h,s,b'(y_t)} V_t(1) + U(c) + \beta \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|1) W_{t+1,y_t}(b'(y_t)) \tag{49}$$

subject to the budget constraints

$$\begin{aligned} 0 &\leq h \leq \bar{h}_{t,y_{t-1}}(b_t), && (50) \\ c + s &= b_t + h, && (51) \\ q_t b'(\theta^H_t) &= \theta^H_t + s - \frac{1}{\pi_{t,y_{t-1}}(\theta^H_t|1)} h, && (52) \\ q_t b'(\theta^L_t) &= \theta^L_t + s - \max\{s - \bar{s}_{t,y_{t-1}}(b_t), 0\} \\ &= \theta^L_t + \min\{s, \bar{s}_{t,y_{t-1}}(b_t)\}. && (53) \end{aligned}$$

Let Γ denote the budget set given by (50)-(53). Γ is not convex, due to (53). We will now divide Γ into two convex subsets as follows. Let Γ_1 be given by

$$\begin{aligned} s &\leq \bar{s}_{t,y_{t-1}}(b_t), \\ 0 &\leq h \leq \bar{h}_{t,y_{t-1}}(b_t), \\ c + s &= b_t + h, \\ q_t b'(\theta^H_t) &= \theta^H_t + s - \frac{1}{\pi_{t,y_{t-1}}(\theta^H_t|1)} h, \\ q_t b'(\theta^L_t) &= \theta^L_t + s. \end{aligned}$$

Let Γ_2 be given by

$$\begin{aligned} s &\geq \bar{s}_{t,y_{t-1}}(b_t), \\ 0 &\leq h \leq \bar{h}_{t,y_{t-1}}(b_t), \\ c + s &= b_t + h, \\ q_t b'(\theta^H_t) &= \theta^H_t + s - \frac{1}{\pi_{t,y_{t-1}}(\theta^H_t|1)} h, \\ q_t b'(\theta^L_t) &= \theta^L_t + \bar{s}_{t,y_{t-1}}(b_t). \end{aligned}$$

Both Γ_1 and Γ_2 are convex, as they are given by linear equality and inequality constraints. Also, the union of Γ_1 and Γ_2 is Γ. The vector of choices for $h, s, c, b'(\theta_t^H), b'(\theta_t^L)$ prescribed by the proposed equilibrium policies evaluated at b_t, denote this vector by z^e, is feasible in both Γ_1 and Γ_2. Define now two auxiliary utility maximization problems as follows: 1) maximize (49) over Γ_1, and 2) maximize (49) over Γ_2. In order to show that z^e maximizes (49) over the whole Γ, it is sufficient to show that z^e solves both of the two auxiliary maximization problems. Indeed, if there exists in Γ a vector $\tilde{z}$ that dominates z^e (with respect to the objective (49)), then z^e cannot solve both of the two auxiliary problems, as $\tilde{z}$ must be feasible in at least one of them.

Since the objective is strict concavity and the constraint set is convex, it is sufficient to show that z^e satisfies the FOCs in each of the two auxiliary problems. In the first sub-problem, they are (ignoring the non-negativity constraint on h):

$$U'(c) \geq \beta q_t^{-1} W'_{t+1,\theta_t^H}(b'(\theta_t^H)), \tag{54}$$

with equality if $h < \bar{h}_{t,y_{t-1}}(b_t)$ and

$$U'(c) \leq \beta q_t^{-1} \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|1) W'_{t+1,y_t}(b'(y_t)), \tag{55}$$

with equality if $s < \bar{s}_{t,y_{t-1}}(b_t)$. Using the inductive assumption (26), the inverse function theorem, and substituting in the proposed equilibrium policies for the consumer choice variables, these sufficient conditions read, respectively,

$$\frac{1}{C'(u^*_{t,y_{t-1}}(w_t))} \geq \beta q_t^{-1} \frac{1}{B'_{t+1,\theta_t^H}(w^*_{t+1,y_{t-1}}(w_t, \theta_t^H))},$$

and

$$\frac{1}{C'(u^*_{t,y_{t-1}}(w_t))} \leq \beta q_t^{-1} \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|1) \frac{1}{B'_{t+1,\theta_t^H}(w^*_{t+1,y_{t-1}}(w_t, y_t))}$$

with $w_t = B^{-1}_{t,y_{t-1}}(b_t)$. In this form, these conditions are expressed purely in terms of objects determined in the solution to the component planning problems. That the first of these conditions holds follows from equality (10) and the right inequality in (11) of Lemma 1. In fact, this condition is satisfied with strict inequality. That the second of these conditions is true follows from (10) and the application of Jensen inequality to equation (12) of Lemma 1. This condition, as well, is satisfied with strict inequality. Thus, the proposed equilibrium behavior vector z^e does solve the first auxiliary problem.

The sufficient FOC of the second problem are as follows (ignoring again the non-negativity constraint on h):

$$U'(c) \geq \beta q_t^{-1} W'_{t+1,\theta_t^H}(b'(\theta_t^H)), \tag{56}$$

with equality if $h < \bar{h}_{t,y_{t-1}}(b_t)$ and

$$U'(c) \geq \beta q_t^{-1} \pi_{t,y_{t-1}}(\theta_t^H|1) W'_{t+1,\theta_t^H}(b'(\theta_t^H)), \tag{57}$$

with equality if $s > \bar{s}_{t,y_{t-1}}(b_t)$. Note that (56) coincides with (54), thus, is satisfied by the proposed equilibrium vector z^e. The second FOC (57) follows directly from (56), as $\pi_{t,y_{t-1}}(\theta_t^H|1) \leq 1$. Thus, the proof of the lemma is complete. ■

Proof of Lemma 4

Conditional on the choice of low effort $x_t = 0$, the consumer chooses $c, h, s, b'(y_t)$ so as to maximize the objective

$$V_t(0) + U(c) + \beta \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|0) W_{t+1,y_t}(b'(y_t)) \tag{58}$$

subject to the budget set Γ. The difference between this problem and the problem consumers solve under effort $x_t = 1$ (Lemma 3) is the additive component $V_t(x_t)$ in the objective and the conditional probability distribution $\pi_{t,y_{t-1}}(y_t|x_t)$. We proceed as in Lemma 3. We divide the consumer problem into two sub-problems. The first sub-problem is defined as maximization of the objective (58) on the budget set Γ_1. Sufficient FO conditions of this problem are:

$$U'(c) \geq \beta q_t^{-1} \frac{\pi_{t,y_{t-1}}(\theta_t^H|0)}{\pi_{t,y_{t-1}}(\theta_t^H|1)} W'_{t+1,\theta_t^H}(b'(\theta_t^H)), \tag{59}$$

with equality if $h < \bar{h}_{t,y_{t-1}}(b_t)$ and

$$U'(c) \leq \beta q_t^{-1} \sum_{y_t \in \Theta_t} \pi_{t,y_{t-1}}(y_t|0) W'_{t+1,y_t}(b'(y_t)), \tag{60}$$

with equality if $s < \bar{s}_{t,y_{t-1}}(b_t)$. Condition (59) follows from the fact that (54) is satisfied by the proposed equilibrium policies, and the left inequality in (43). Similarly, that the proposed equilibrium policies satisfy (60) follows from (55), (1), and (11).

The sufficient FO conditions of the second sub-problem (maximization of (58) on Γ_2) are:

$$U'(c) \geq \beta q_t^{-1} \frac{\pi_{t,y_{t-1}}(\theta_t^H|0)}{\pi_{t,y_{t-1}}(\theta_t^H|1)} W'_{t+1,\theta_t^H}(b'(\theta_t^H)), \tag{61}$$

with equality if $h < \bar{h}_{t,y_{t-1}}(b_t)$ and

$$U'(c) \geq \beta q_t^{-1} \pi_{t,y_{t-1}}(\theta_t^H|0) W'_{t+1,\theta_t^H}(b'(\theta_t^H)), \tag{62}$$

with equality if $s > \bar{s}_{t,y_{t-1}}(b_t)$. Condition (61) is the same as (59), thus, is satisfied by the proposed equilibrium. Finally, (62) follows from (57) and (1). ■

References

Administrative Office of the United States Courts (2008), "Bankruptcy Basics," http://www.uscourts.gov/bankruptcycourts/bankruptcybasics.html

Albanesi, S., and C. Sleet (2006), "Dynamic Optimal Taxation with Private Information," *Review of Economic Studies,* 73, 1-30.

Alvarez, F. and U. Jermann (2000), "Efficiency, Equilibrium and Asset Pricing with Risk of Default" *Econometrica,* 68(4), 775-798.

Araujo, A.P. and M.R. Pascoa (2002), "Bankruptcy in a model of unsecured claims," *Economic Theory* 20, 455–481.

Athreya, K. (2002), "Welfare Implications of the Bankruptcy Reform Act of 1999," *Journal of Monetary Economics,* 49, 1567-1595.

Atkeson, A. and R.E. Lucas, Jr. (1992), "On efficient distribution with private information." *Review of Economic Studies,* 59, 427–453.

Atkeson, A. and R.E. Lucas, Jr. (1995), "Efficiency and Equality in a Simple Model of Efficient Unemployment Insurance," *Journal of Economic Theory,* 66, 64-88.

Bisin, A. and A. Rampini (2006), "Exclusive contracts and the institution of bankruptcy" *Economic Theory,* 27, 277-304.

Bizer, D.S. and P.M. DeMarzo (1999), "Optimal Incentive Contracts when Agents Can Save, Borrow, and Default" *Journal of Financial Intermediation,* 8(4), 241–269.

Chatterjee, S., D. Corbae, M. Nakajima, J.V. Rios-Rull (2007), "A quantitative theory of unsecured consumer credit with risk of default." *Econometrica,* 75(6), 1525-1589.

Clementi, G.L. and H.A. Hopenhayn (2006), "A Theory of Financing Constraints and Firm Dynamics" *Quarterly Journal of Economics,* 121(1), 229-265.

DeMarzo, P.M. and Y. Sannikov (2006), "Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model," *Journal of Finance,* 66(6), 2681-2724.

Dubey, P., J. Geanakoplos and M. Shubik (2005), "Default and Punishment in General Equilibrium," *Econometrica,* 73(1), 1-37.

Fernandes, A. and C. Phelan, (2000), "A recursive formulation for repeated agency with history dependence," *Journal of Economic Theory,* 91, 223-247.

Golosov, M. and A. Tsyvinski (2006), "Designing Optimal Disability Insurance: A Case for Asset Testing," *Journal of Political Economy* 114(2), 257-279.

Golosov, M. and A. Tsyvinski (2007), "Optimal Taxation with Endogenous Insurance Markets," *Quarterly Journal of Economics,* 122(2), 487-534.

Gropp, R., J.K Scholz and M.J. White (1997), "Personal Bankruptcy and Credit Supply and Demand" *Quarterly Journal of Economics,* 112(1), 217-251.

Gross, D. and N. Souleles (2002), "Do Liquidity Constraints and Interest Rates Matter for Consumer Behavior? Evidence from Credit Card Data." *Quarterly Journal of Economics,* 117(1), 149-185.

Hopenhayn, H.A. and J.P. Nicollini (1997), "Optimal Unemployment Insurance," *Journal of Political Economy,* 105(2), 412-438.

Hunt, R.M. (2006), "The Development and Regulation of Consumer Credit Reporting in America," in G. Bertola and C. Grant, eds., The Economics of Consumer Credit: European Experience and Lessons from the US. Boston: MIT Press.

Jackson, T.H. (1985), "The fresh start policy in bankruptcy law." *Harvard Law Review,* 98, 1393–1449.

Kehoe, T.J. and D.K. Levine (1993), "Debt Constrained Asset Markets," *Review of Economic Studies,* 60(4), 865-888.

Kehoe, T.J. and D.K. Levine (2001), "Liquidity Constrained Markets versus Debt Constrained Markets," *Econometrica,* 69(3), 575-598.

Kehoe, T.J. and D.K. Levine (2006), "Bankruptcy and Collateral in Debt Constrained Markets," FRBM Research Department Staff Report 380.

Khan, C.M. and D. Mookherjee (1995), "Market failure with moral hazard and side trading" *Journal of Public Economics*, 159-184.

Kocherlakota, N. (2005), "Zero expected wealth taxes: A Mirrlees approach to dynamic optimal taxation," *Econometrica,* 73(5), 1587-1621.

Li, W. and P.D. Sarte (2006), "U.S. Consumer Bankruptcy Choice: The Importance of General Equilibrium Effects," *Journal of Monetary Economics,* 53, 613-631.

Livshits, I., J. MacGee, and M. Tertilt (2007), "Consumer Bankruptcy: A Fresh Start," *American Economic Review,* 97(1), 402-418.

Mirrlees, J.A. (1971), "An Exploration in the theory of optimum income taxation," *Review of Economic Studies,* 38, 175-208.

Pagano, M. and T. Jappelli (1993), "Information Sharing in Credit Markets" *Journal of Finance,* 48(5), 1693-1718.

Phelan, C. and R.M. Townsend (1991), "Computing Multi-Period, Information-Constrained Optima" *Review of Economic Studies,* 58, 853-881.

Prescott, E.C. and R.M. Townsend (1984), "Pareto Optima and Competitive Equilibria with Adverse Selection and Moral Hazard," *Econometrica,* 52(1), 21-45.

Rampini, A. (2005), "Default and Aggregate Income" *Journal of Economic Theory,* 122, 225-253.

Rogerson, W. (1985), "Repeated Moral Hazard" *Econometrica,* 53(1), 69-76.

Spear, S.E. and S. Srivastava (1987), "On Repeated Moral Hazard with Discounting" *Review of Economic Studies,* 54, 599-617.

Telyukova, I. (2006), "Household Need for Liquidity and the Credit Card Debt Puzzle" UCSD Working Paper 2008-01.

Telyukova, I and R. Wright (2008), "A Model of Money and Credit, with Application to the Credit Card Debt Puzzle," *Review of Economic Studies,* forthcoming.

Townsend, R. (1988), "Information Constrained Insurance. The Revelation Principle Extendend." *Journal of Monetary Economics,* 21, 411-450.

Wedoff, E.R. (2007), "Major Consumer Bankruptcy Effects of BAPCPA," *University of Illinois Law Review,* 2007(1), 31-66.

White, M.J. (2007), "Bankruptcy Reform and Credit Cards, " *Journal of Economic Perspectives,* 21(4), 175-199.

Zame, W. (1993), "Efficiency and the Role for Default when Markets are Incomplete" *American Economic Review,* 83(5), 1142-1164.