Online Appendix: Should platforms be allowed to charge ad valorem fees?

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Abstract

In this online appendix, we provide the Proof of Proposition 2 (from the main text), and the analysis for the case in which $d > 0$ and in which there are a continuum of markets, which are noted in the main text. We also provide the figures decomposing the change in welfare from allowing ad valorem fees into changes in platform profit and consumer surplus for our calibrated model.

1 Proof of Proposition 2

Recall demand is given by

$$Q_c\left(T_c\right) = \left(1 + \frac{\lambda\left(\sigma - 1\right)T_c}{c}\right)^{\frac{1}{1-\sigma}}, \tag{1}$$

where $\lambda > 0, \sigma < 2$. We consider three cases.

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(i) Demand is log-concave: Suppose demand is log-concave so $\sigma < 1$. Then there is a choke price $T_c' = c/\left(\lambda\left(1-\sigma\right)\right)$ at which demand becomes zero for market c. Let c_L be fixed and consider increasing k and so c_H. Let $z = \left(\frac{1}{2-\sigma}\right)\left(1-\frac{1-\sigma}{2-\sigma}\right)^{\frac{1}{1-\sigma}}$ where $0 < z < e^{-1}$ given $\sigma < 1$. Under price discrimination, the profit from the high-demand market is $c_H z/\lambda \to \infty$ as $k \to \infty$. The profit from the low-demand market is fixed at $c_L z/\lambda$. Total profit is unbounded as k increases. On the other hand, with a uniform price the profit is bounded if both markets continue to operate since the price cannot exceed the choke price for market c_L, which is $c_L/\left(\lambda\left(1-\sigma\right)\right)$. Therefore, there exists a high enough k such that the monopolist will give up on the low-demand market if it is forced to set a single price. The threshold k_0 such that the monopolist will no longer keep the low-demand market open whenever $k \geq k_0$ is determined by

$$\left(\frac{1}{2-\sigma}\right)^{\frac{2-\sigma}{1-\sigma}} = \frac{1}{(k_0+1)}\left[\left(\frac{k_0+\sigma}{k_0+1}\right)^{\frac{1}{1-\sigma}} + \left(\frac{k_0\sigma+1}{k_0+1}\right)^{\frac{1}{1-\sigma}}\right], \tag{2}$$

which is obtained by comparing the monopolist's profit with and without shutting down the low-demand market under uniform pricing. Note k_0 only depends on σ. For example, in the case of linear demand, solving (2) with $\sigma = 0$ implies $k_0 = 3$. With price discrimination, the monopolist will set the same price for the high-demand market as it would under uniform pricing, and set a lower price for the low-demand market to ensure it operates, thereby generating additional profit, consumer surplus and welfare.

(ii) Demand is exponential: Suppose demand is exponential (i.e. $Q_c(T_c) = e^{-\frac{\lambda T_c}{c}}$) which corresponds to the limit case of (1) when $\sigma \to 1$. Then there is no choke price at which demand becomes zero. We compare welfare directly. Welfare from market c is

$$W_c = \int_0^{Q_c} T_c\left(Q\right)dQ = \int_0^{Q_c} \left(-\frac{c}{\lambda}\ln Q\right)dQ,$$

so that $W_c\left(T_c^*\right) = 2ce^{-1}/\lambda$ under price discrimination given $Q_c\left(T_c\right) = e^{-\frac{\lambda T_c}{c}}$ and

$$T_c^* = \frac{\lambda d + c}{\lambda\left(2-\sigma\right)} \tag{3}$$

with $d = 0$ and $\sigma = 1$. Therefore, welfare from both markets under price discrimination is $W_{PD} = 2c_L(1+k)e^{-1}/\lambda$.

Now consider welfare without price discrimination. The monopolist will set the uniform price T to maximize

$$\Pi = T\left(e^{-\frac{\lambda T}{c_L}} + e^{-\frac{\lambda T}{c_H}}\right).$$

The optimal uniform price $\widehat{T}$ solves the first-order condition

$$e^{-\frac{\lambda \widehat{T}}{c_L}}\left(1 - \frac{\lambda \widehat{T}}{c_L}\right) + e^{-\frac{\lambda \widehat{T}}{c_H}}\left(1 - \frac{\lambda \widehat{T}}{c_H}\right) = 0.$$

The solution can be written as $\widehat{T} = \rho c_L/\lambda$, where ρ solves

$$(1-\rho)e^{-\rho} + \left(1 - \frac{\rho}{k}\right)e^{-\frac{\rho}{k}} = 0$$

and ρ is just a function of k.

Welfare under uniform pricing is

$$\begin{aligned} W_U &= \widehat{T}\left(e^{-\frac{\lambda \widehat{T}}{c_L}} + e^{-\frac{\lambda \widehat{T}}{c_H}}\right) + c_L\left(\frac{e^{-\frac{\lambda \widehat{T}}{c_L}}}{\lambda}\right) + c_H\left(\frac{e^{-\frac{\lambda \widehat{T}}{c_H}}}{\lambda}\right) \\ &= \frac{c_L}{\lambda}\left((k+\rho)e^{-\frac{\rho}{k}} + (1+\rho)e^{-\rho}\right). \end{aligned}$$

Therefore,

$$W_{PD} - W_U = \frac{c_L}{\lambda}\left(2(1+k)e^{-1} - (1+\rho)e^{-\rho} - (k+\rho)e^{-\frac{\rho}{k}}\right).$$

Since ρ is just a function of k, and the term in brackets in $W_{PD} - W_U$ is just a function of ρ and k, the sign of $W_{PD} - W_U$ just depends on k. Evaluating this confirms $W_{PD} - W_U > 0$ for all $k > 1$, and so welfare is higher under price discrimination.

The limit case as $k \to \infty$ provides some insight into what happens as demands become more dispersed across markets. In the limit as $k \to \infty$, it can be shown that $\frac{\rho}{k} \to 1$. Accordingly, $\frac{\widehat{T}}{T^*_{c_H}} = \frac{\rho}{k} \to 1$ and $W_{PD} - W_U \to 2c_L e^{-1}/\lambda$. In other words, for large k, the uniform price converges to the optimal discriminatory price that the monopolist would set for the high-demand market, and the welfare gain of price discrimination converges to the welfare generated from the low-demand market under monopoly.

(iii) Demand is log-convex: Suppose demand is log-convex so $1 < \sigma < 2$. Welfare from market c is

$$W_c = \int_0^{Q_c} T_c(Q)\, dQ = \int_0^{Q_c} \frac{c\left(1 - Q_c^{1-\sigma}\right)}{\lambda(1-\sigma)} dQ, \tag{4}$$

so that

$$W_c(T_c^*) = \frac{c}{\lambda(\sigma-1)}\left[\left(\frac{1}{2-\sigma}\right)^{2+\frac{1}{1-\sigma}} - \left(\frac{1}{2-\sigma}\right)^{\frac{1}{1-\sigma}}\right]$$

under price discrimination where demand given in (1) and price T_c^* given in (3) have been substituted into (4). Therefore, welfare from both markets under price discrimination is

$$W_{PD} = \frac{c_L(1+k)}{\lambda(\sigma-1)}\left[\left(\frac{1}{2-\sigma}\right)^{2+\frac{1}{1-\sigma}} - \left(\frac{1}{2-\sigma}\right)^{\frac{1}{1-\sigma}}\right]. \tag{5}$$

Now consider welfare without price discrimination. The monopolist will set the uniform price T to maximize

$$\underset{T}{Max}\ \Pi = T\left[\left(1 + \frac{\lambda(\sigma-1)T}{c_L}\right)^{\frac{1}{1-\sigma}} + \left(1 + \frac{\lambda(\sigma-1)T}{c_H}\right)^{\frac{1}{1-\sigma}}\right].$$

The optimal uniform price $\widehat{T}$ solves the first-order condition

$$\begin{aligned} &\left(1 + \frac{\lambda(\sigma-1)\widehat{T}}{c_L}\right)^{\frac{1}{1-\sigma}} + \left(1 + \frac{\lambda(\sigma-1)\widehat{T}}{c_H}\right)^{\frac{1}{1-\sigma}} \\ = &\ \frac{\lambda\widehat{T}}{c_L}\left(1 + \frac{\lambda(\sigma-1)\widehat{T}}{c_L}\right)^{\frac{\sigma}{1-\sigma}} + \frac{\lambda\widehat{T}}{c_H}\left(1 + \frac{\lambda(\sigma-1)\widehat{T}}{c_H}\right)^{\frac{\sigma}{1-\sigma}}. \end{aligned}$$

The solution can be written as $\widehat{T} = \rho c_L / (\lambda(\sigma-1))$, where for any given σ, the term ρ is just a function of k which solves

$$(1+\rho)^{\frac{1}{1-\sigma}} - \frac{\rho}{\sigma-1}(1+\rho)^{\frac{\sigma}{1-\sigma}} + \left(1 + \frac{\rho}{k}\right)^{\frac{1}{1-\sigma}} - \frac{\rho}{k(\sigma-1)}\left(1 + \frac{\rho}{k}\right)^{\frac{\sigma}{1-\sigma}} = 0.$$

Welfare under uniform pricing is

$$W_U = \frac{c_L}{\lambda(\sigma-1)}\left[\frac{(1+\rho)^{\frac{2-\sigma}{1-\sigma}}}{2-\sigma} - (1+\rho)^{\frac{1}{1-\sigma}} + \frac{k\left(1+\frac{\rho}{k}\right)^{\frac{2-\sigma}{1-\sigma}}}{2-\sigma} - k\left(1+\frac{\rho}{k}\right)^{\frac{1}{1-\sigma}}\right]. \tag{6}$$

Since ρ is just a function of k and the term in brackets in W_U is just a function of ρ and k for any given σ, the sign of $W_{PD} - W_U$ just depends on k for any particular σ. Evaluating (5) and (6) confirms $W_{PD} - W_U > 0$ for all $k > 1$ for any $1 < \sigma < 2$, so that welfare is higher under price discrimination.

Again, the limit case as $k \to \infty$ provides some insight into what happens as demands become more dispersed across markets. In the limit as $k \to \infty$, it can be shown that $\frac{\rho}{k} \to \frac{\sigma-1}{2-\sigma}$. Accordingly, $\frac{\widehat{T}}{T^*_{c_H}} = (\frac{\rho}{k})/(\frac{\sigma-1}{2-\sigma}) \to 1$ and $W_{PD} - W_U \to \frac{c_L}{\lambda(\sigma-1)}\left[\left(\frac{1}{2-\sigma}\right)^{2+\frac{1}{1-\sigma}} - \left(\frac{1}{2-\sigma}\right)^{\frac{1}{1-\sigma}}\right]$. In other words, for large k, the uniform price converges to the optimal discriminatory price that the monopolist would set for the high-demand market, and the welfare gain of price discrimination converges to the welfare generated from the low-demand market under monopoly.

2 Positive platform costs ($d > 0$)

In this section we illustrate the robustness of our price discrimination results with two markets (Proposition 2) to the possibility that $d > 0$, so that the platform faces some positive (albeit small) marginal cost. To illustrate that the same mechanism works with $d > 0$ in the case in which the low-demand market shuts down under uniform pricing, consider the case $\sigma = 0$ so that the monopolist faces linear demand in market c given by

$$Q_c\left(T_c\right) = \left(1 - \frac{\lambda T_c}{c}\right). \tag{7}$$

Figure 1 illustrates the monopolist's fees under price discrimination and under uniform pricing.[1] It shows that as d increases, the critical value of k above which the low-demand market shuts down under uniform pricing actually decreases as d increases. Thus, the range of k over which price discrimination increases welfare would be greater when we allow for modest levels of positive d.

Figure 2 illustrates a similar logic works, even though the low-demand market is always served, when demand is exponential. It shows the monopolist's optimal prices with and without price discrimination as k varies.

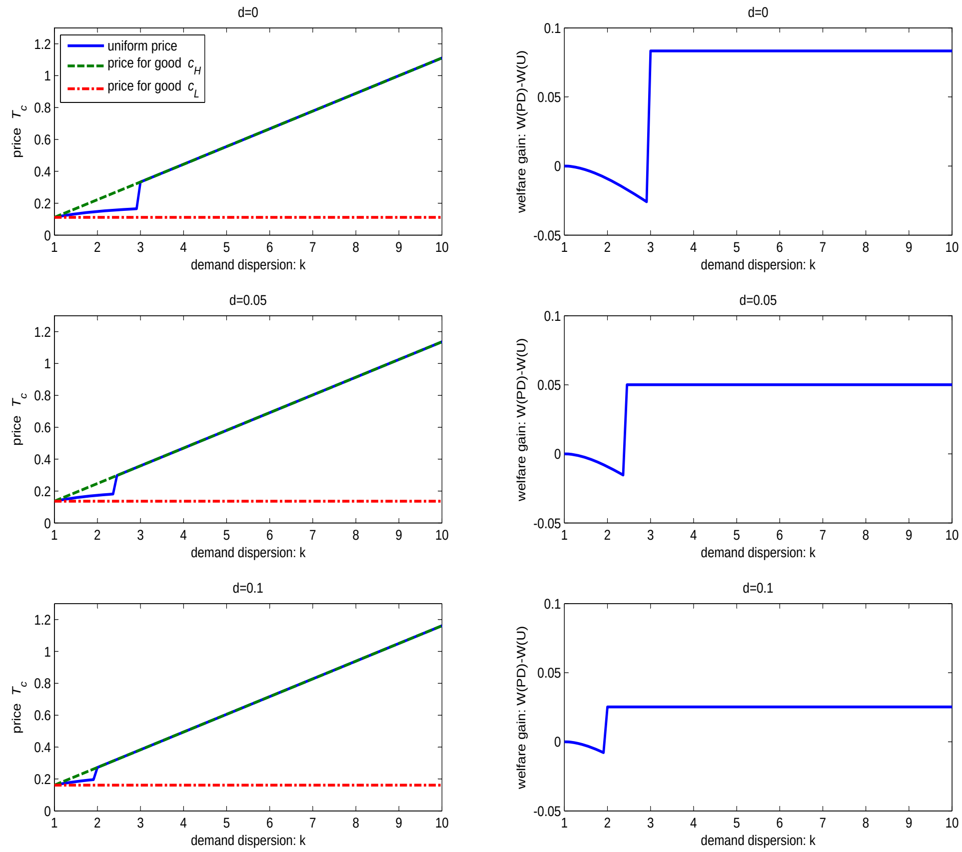
As can be seen in the figure, the uniform price is very close to the monopoly price in the high-demand market either when k is close to 1 and when k is large, and for intermediate levels of k, it is still closer to the high-demand monopoly price than the low-demand monopoly price. The figure shows this pattern is replicated even when $d > 0$.

The welfare effects of allowing price discrimination for the different cases captured by Proposition 2 are summarized in Figure 3.

The figure considers three different values of d and a range of values of k and σ. The dark blue area in the figure indicates a welfare loss due to price discrimination, while the light orange area indicates a welfare gain. When $\sigma < 1$, there is a discrete jump between these two areas when k becomes sufficiently large, reflecting that the low-demand market gets shut down if price discrimination is not allowed. In the log-concave case with $d = 0$, the critical level of k

[1] To plot the figure, we normalize $\lambda = 4.5$ and $c_L = 1$. Note that the values of λ and c_L just scale the results, but do not affect welfare findings in any of our exercises in this section.

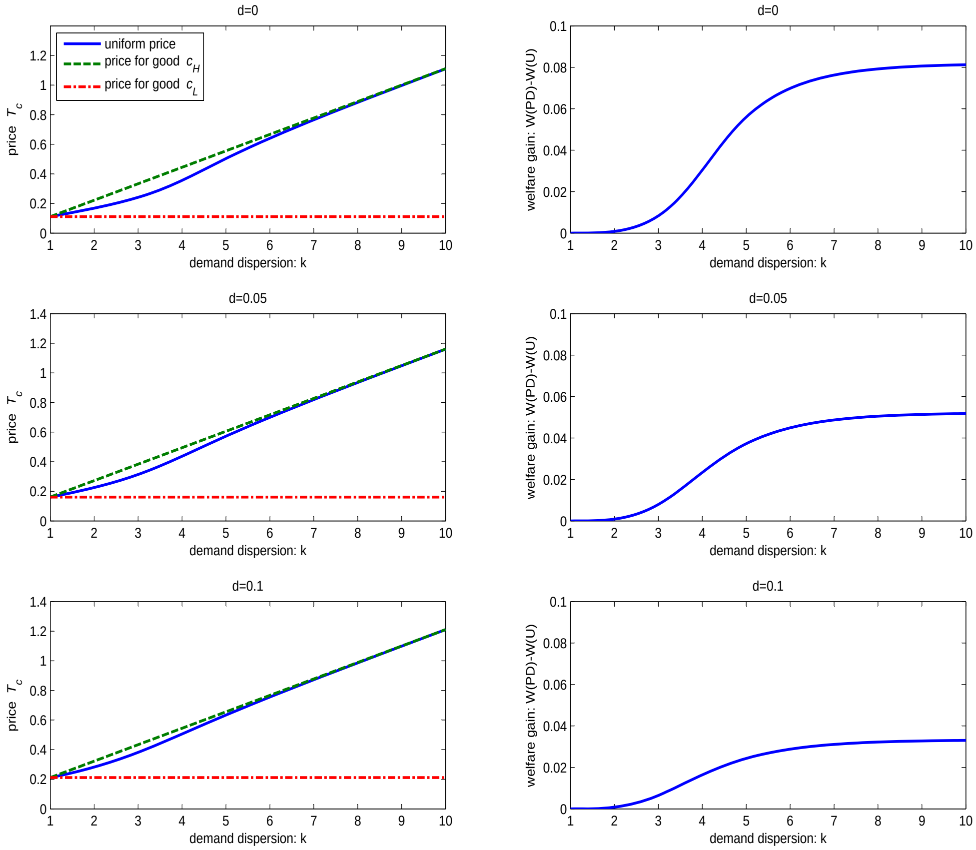
Figure 1: Monopoly prices with linear demand (two-market case)



for which welfare is higher under price discrimination than under uniform pricing is $k > 3.5$, so quantitatively we do not require unreasonably high dispersion in the demand across markets to get the welfare result.[2] In the exponential or log-convex case, Figure 3 shows that welfare is always higher under price discrimination regardless of the level of dispersion k. For both cases, Figure 3 shows that the welfare finding extends to $d > 0$.

[2]While not shown in the figure, the critical level of k for which welfare becomes higher under price discrimination declines as σ decreases below -1, so the sufficient condition $k > 3.5$ continues to hold.

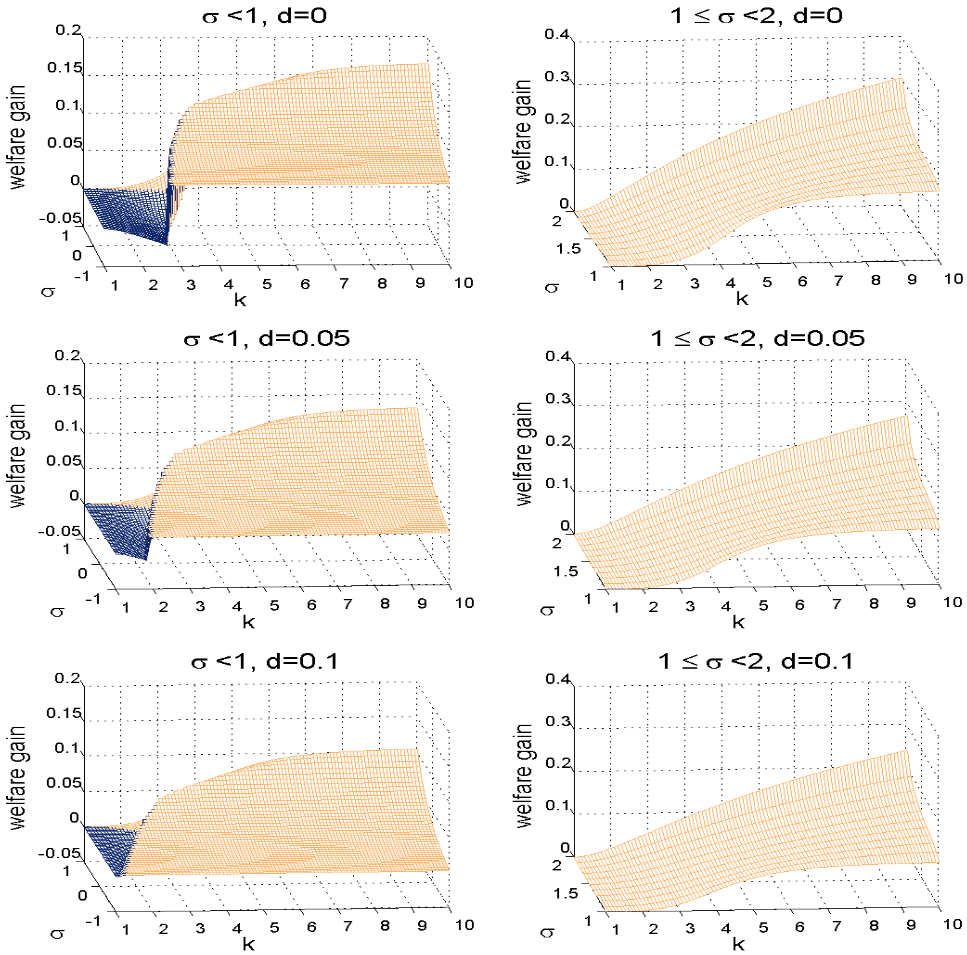
Figure 2: Monopoly prices with exponential demand (two-market case)



3 Continuum of markets

The qualitative conclusions on the welfare-gains of price discrimination (or equivalently, allowing ad valorem fees) can hold when there are many markets (or equivalently, goods) rather than just two. In this section we will assume that c is uniformly distributed between $c_L > 0$ and $c_H = kc_L$, with $k \equiv c_H/c_L$ and $k > 1$. We first derive the welfare results for the special case in which $\sigma = 0$ (so demand is linear) and $d = 0$. We then establish that welfare is always higher under price discrimination whenever demand is log-concave provided there is enough dispersion in c when $d = 0$. In particular, we show there is a cutoff level of c equal to xc_L (where $1 < x < k$) such that all markets below the cutoff will be shut down by the monopolist, provided that the dispersion in c is large enough (i.e., $k > k_0$). We show that the threshold k_0 depends only on σ, and the cutoff value x is a constant fraction of k provided $k > k_0$. Finally, we explore graphically the welfare effects of price discrimination for the full range of σ, allowing $d > 0$.

Figure 3: Welfare comparison for two markets



3.1 Linear demand

We first consider the special case with $d = 0$ and $\sigma = 0$, so demand is linear. Then the inverse demand faced by the monopolist for market c is

$$T_c\left(Q_c\right) = \frac{c\left(1 - Q_c\right)}{\lambda}.$$

Then the problem is stated in exactly the same form as the third-degree price discrimination problem analyzed by Malueg and Schwartz[3], except that we allow inverse demand to be multiplied by a constant positive parameter and we allow that the uniform distribution on c does not have to be centered at unity.[4] It turns out what matters for Malueg and Schwartz's re-

[3]Malueg, D. and M. Schwartz (1994) "Parallel Imports, Demand Dispersion, and International Price Discrimination," *Journal of International Economics*, 37: 167-195.

[4]Their specification can be obtained by setting $\lambda = 1$, $c = a$, $c_L = 1 - x$ and $c_H = 1 + x$.

sults is the ratio of the highest to lowest value of c in the support of the distribution, i.e. k. Therefore reinterpreting the relevant part of their Proposition 1 to our setting, it implies that for large enough dispersion $k > k_0$, some markets are shut down under uniform pricing; in this range, the ratio of welfare under price discrimination to welfare under uniform pricing increases monotonically with dispersion and exceeds 1 when dispersion is sufficiently large.

To calculate these points precisely, define $k_0 > 1$ which solves $1 + 2\ln k_0 = k_0$, so $k_0 \simeq 3.513$. Then the point at which dispersion is sufficiently large for welfare to increase under price discrimination arises when[5]

$$k > \frac{3k_0 - \sqrt{3k_0\,(4 - k_0)}}{k_0 - 4 + \sqrt{3k_0\,(4 - k_0)}} \simeq 4.651.$$

Thus, provided there is sufficient dispersion in c, welfare is unambiguously higher with price discrimination (or equivalently, with ad valorem fees).

The result is illustrated in Figure 4, which replicates Figure 1 for this continuum case.

3.2 Demand is log-concave

We can generalize Malueg and Schwartz's result on the positive welfare effects of price discrimination when demand across markets is sufficiently dispersed to the case in which demand is log-concave and from the class of generalized Pareto demands. The demand in each market c is given by (1) and inverse demand is

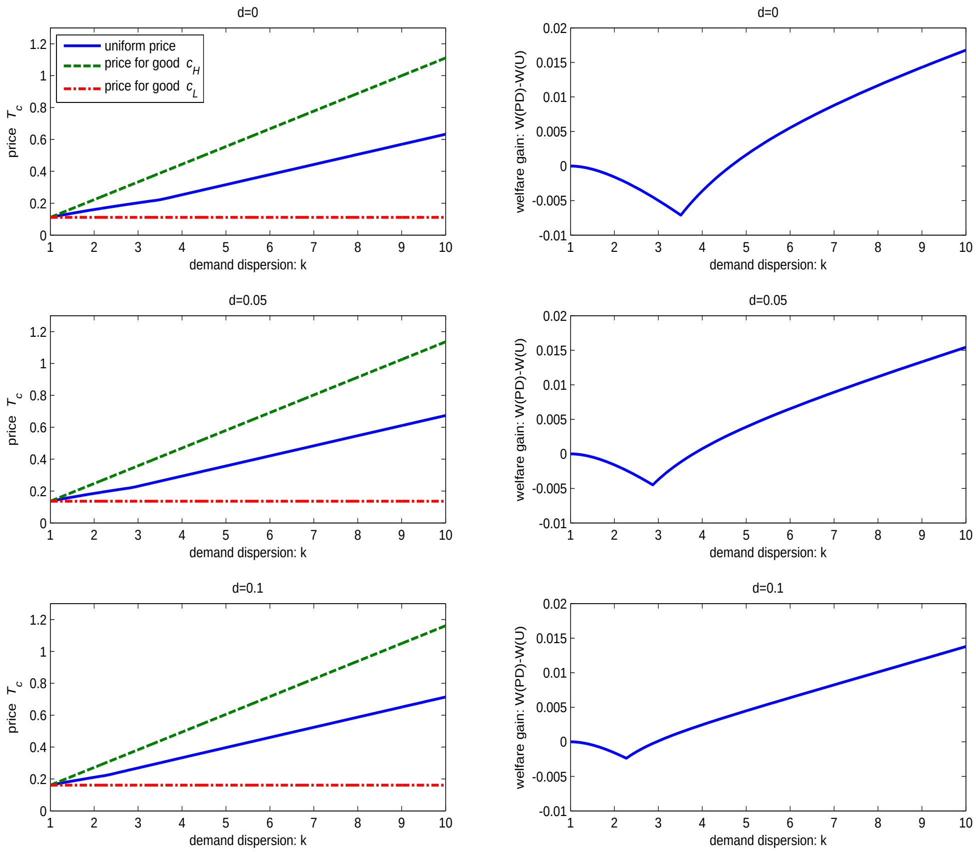
$$T_c\,(Q_c) = \frac{c\,(1 - Q_c^{1-\sigma})}{\lambda(1 - \sigma)},$$

where $\sigma < 1$ given demand is log-concave. With this specification, we obtain the following result on the welfare effects of price discrimination.

Proposition A. ***(Welfare effects with a continuum of markets for log-concave demands)***: *Assume demand is given by (1) and the monopolist has zero marginal costs (i.e.,*

[5] There is a typo in Malueg and Schwartz's stated formula for this threshold (in their footnote 17) which does not generate the approximate numerical value they state in the footnote. However, their stated numerical value corresponds to ours, which we derived directly with our specification. I.e. if their threshold is denoted x_e and ours is denoted k_e, then it can be checked that $k_e = (1 + x_e)\,/\,(1 - x_e)$.

Figure 4: Monopoly prices with linear demand (continuum case)



$d = 0$). If there are a continuum of markets, uniformly distributed between c_L and c_H, then banning price discrimination across the markets lowers welfare if demand is log-concave ($\sigma < 1$) provided $k \equiv c_H/c_L$ is sufficiently large.

Proof. We break the proof up into three steps.

(i) Price discrimination is allowed:

If price discrimination is allowed, the monopolist solves the following problem for each market c:

$$\underset{T_c}{Max}\ \Pi_c = T_c \left(1 - \frac{\lambda\,(1-\sigma)\,T_c}{c}\right)^{\frac{1}{1-\sigma}}.$$

The first-order condition yields the optimal price

$$T_c^* = \frac{c}{(2-\sigma)\lambda}.$$

The corresponding demand in market c is

$$Q_c\left(T_c^*\right)=\left(\frac{1}{2-\sigma}\right)^{\frac{1}{1-\sigma}},$$

and the monopolist's profit is

$$\Pi_c=\frac{c}{\lambda}\left(\frac{1}{2-\sigma}\right)^{\frac{2-\sigma}{1-\sigma}}.$$

The resulting welfare from market c is

$$\begin{aligned} W_c &= \int_0^{Q_c} T_c\left(Q\right)dQ=\int_0^{Q_c}\frac{c\left(1-Q^{1-\sigma}\right)}{\lambda(1-\sigma)}dQ \\ &= \frac{c}{\lambda(1-\sigma)}\left[\left(\frac{1}{2-\sigma}\right)^{\frac{1}{1-\sigma}}-\left(\frac{1}{2-\sigma}\right)^{2+\frac{1}{1-\sigma}}\right]. \end{aligned}$$

Therefore, the monopolist's profit from all markets is

$$\Pi^{PD}=\left(\frac{1}{c_H-c_L}\right)\int_{c_L}^{c_H}\Pi_c dc=\left(\frac{1}{2-\sigma}\right)^{\frac{2-\sigma}{1-\sigma}}\frac{\left(c_H+c_L\right)}{2\lambda},$$

and the overall social welfare is

$$W^{PD}=\left(\frac{1}{c_H-c_L}\right)\int_{c_L}^{c_H}W_c dc=\frac{\left(c_H+c_L\right)}{2\lambda\left(1-\sigma\right)}\left[\left(\frac{1}{2-\sigma}\right)^{\frac{1}{1-\sigma}}-\left(\frac{1}{2-\sigma}\right)^{2+\frac{1}{1-\sigma}}\right].$$

(ii) Price discrimination is not allowed:

If price discrimination is not allowed, the monopolist solves for the following problem:

$$\underset{x,T}{Max}\ \Pi^U=\left(\frac{T}{c_H-c_L}\right)\int_{xc_L}^{c_H}\left(1-\frac{\lambda\left(1-\sigma\right)T}{c}\right)^{\frac{1}{1-\sigma}}dc$$

$$s.t.\quad x\geq 1.$$

The Lagrangian is

$$L=\left(\frac{T}{c_H-c_L}\right)\int_{xc_L}^{c_H}\left(1-\frac{\lambda\left(1-\sigma\right)T}{c}\right)^{\frac{1}{1-\sigma}}dc+\gamma(x-1),$$

where γ is the Lagrangian multiplier.

The first-order condition for x when $x \geq 1$ is not binding is

$$\frac{\partial L}{\partial x} = 0 \Longrightarrow xc_L = (1-\sigma)\lambda T.$$

The first-order condition for T is

$$\frac{\partial L}{\partial T} = 0 \Longrightarrow \int_{(1-\sigma)\lambda T}^{c_H} \left(1 - \frac{(1-\sigma)\,\lambda T}{c}\right)^{\frac{1}{1-\sigma}} \left(\frac{c-(2-\sigma)\lambda T}{c-(1-\sigma)\,\lambda T}\right) dc = 0. \tag{8}$$

Define $c/(\lambda T) = t$. We can rewrite (8) as follows:

$$\lambda T \int_{1-\sigma}^{\frac{c_H}{\lambda T}} \left(1 - \frac{(1-\sigma)}{t}\right)^{\frac{1}{1-\sigma}} \left(\frac{t-(2-\sigma)}{t-(1-\sigma)}\right) dt = 0.$$

Let the optimal fee be denoted $\widehat{T}$. Accordingly, the optimal solution requires c_H and $\widehat{T}$ always being proportional, i.e. $\widehat{T} = c_H/(z\lambda)$, where z is a constant satisfying

$$\int_{1-\sigma}^{z} \left(1 - \frac{(1-\sigma)}{t}\right)^{\frac{1}{1-\sigma}} \left(\frac{t-(2-\sigma)}{t-(1-\sigma)}\right) dt = 0.$$

Therefore, the larger the c_H, the larger the $\widehat{T}$ and x. Define the threshold $k_0 = \frac{z}{(1-\sigma)}$ for a given σ. When $k = \frac{c_H}{c_L} > k_0$, some low-$c$ markets are shut down because

$$xc_L = (1-\sigma)\lambda\widehat{T} > c_L. \tag{9}$$

Given $\widehat{T} = c_H/(z\lambda)$, (9) implies that the cutoff value x is a constant fraction of k, i.e.

$$x = \frac{(1-\sigma)}{z}k, \tag{10}$$

which implies that x is uniquely determined by σ but not λ (i.e., λ is a scale parameter which does not affect x).

In the following discussion, we assume $k > k_0$, so some low-c markets are shut down. The corresponding welfare from market c is

$$W_c^U = \int_0^{Q_c(\widehat{T})} \frac{c\left(1-Q^{1-\sigma}\right)}{\lambda(1-\sigma)} dQ = \frac{c}{\lambda(1-\sigma)} \left[Q_c\left(\widehat{T}\right) - \frac{Q_c\left(\widehat{T}\right)^{2-\sigma}}{2-\sigma} \right],$$

and the total welfare is

$$\begin{aligned} W^U &= \frac{1}{c_H - c_L} \int_{(1-\sigma)\lambda\widehat{T}}^{c_H} \frac{c}{\lambda(1-\sigma)} \left[Q_c\left(\widehat{T}\right) - \frac{Q_c\left(\widehat{T}\right)^{2-\sigma}}{2-\sigma} \right] dc \\ &= \frac{1}{c_H - c_L} \int_{(1-\sigma)\lambda\widehat{T}}^{c_H} \frac{c}{\lambda(1-\sigma)} \left[\left(\frac{c-(1-\sigma)\lambda\widehat{T}}{c} \right)^{\frac{1}{1-\sigma}} - \frac{\left(\frac{c-(1-\sigma)\lambda\widehat{T}}{c}\right)^{\frac{2-\sigma}{1-\sigma}}}{2-\sigma} \right] dc \\ &= \frac{1}{c_H - c_L} \int_{\frac{(1-\sigma)c_H}{z}}^{c_H} \frac{c}{\lambda(1-\sigma)} \left[\left(1 - \frac{(1-\sigma)c_H}{cz} \right)^{\frac{1}{1-\sigma}} - \frac{\left(1-\frac{(1-\sigma)c_H}{cz}\right)^{\frac{2-\sigma}{1-\sigma}}}{2-\sigma} \right] dc. \end{aligned} \quad (11)$$

Define $c/c_H = s$. We can rewrite (11) as

$$W^U = \frac{Rc_H^2}{c_H - c_L},$$

where R is a constant satisfying

$$R = \int_{\frac{1-\sigma}{z}}^{1} \frac{s}{\lambda(1-\sigma)} \left[\left(1 - \frac{1-\sigma}{sz} \right)^{\frac{1}{1-\sigma}} - \frac{\left(1-\frac{1-\sigma}{sz}\right)^{\frac{2-\sigma}{1-\sigma}}}{2-\sigma} \right] ds.$$

(iii) Welfare Comparison:

As shown above, the welfare under price discrimination is

$$W^{PD} = ac_H + ac_L,$$

where

$$a = \frac{1}{2\lambda(1-\sigma)} \left[\left(\frac{1}{2-\sigma}\right)^{\frac{1}{1-\sigma}} - \left(\frac{1}{2-\sigma}\right)^{2+\frac{1}{1-\sigma}} \right]. \quad (12)$$

In contrast, the welfare under uniform price is

$$W^U = \frac{Rc_H^2}{c_H - c_L},$$

where

$$R = \int_{\frac{1-\sigma}{z}}^{1} \frac{s}{\lambda(1-\sigma)} \left[\left(1 - \frac{1-\sigma}{sz}\right)^{\frac{1}{1-\sigma}} - \frac{\left(1 - \frac{1-\sigma}{sz}\right)^{\frac{2-\sigma}{1-\sigma}}}{2-\sigma} \right] ds, \tag{13}$$

and z is a constant satisfying

$$\int_{1-\sigma}^{z} \left(1 - \frac{(1-\sigma)}{t}\right)^{\frac{1}{1-\sigma}} \left(\frac{t-(2-\sigma)}{t-(1-\sigma)}\right) dt = 0. \tag{14}$$

Normalize $c_L = 1$, so $c_H = k$ and the welfare difference is

$$W^{PD} - W^U = ak + a - \frac{Rk^2}{k-1}.$$

Given that $a > R$ for $\sigma < 1$, we have

$$W^{PD} - W^U > 0 \Leftrightarrow k > \sqrt{\frac{a}{a-R}}.$$

Hence, welfare is always higher under price discrimination when there is enough demand dispersion across markets; i.e. $k > \sqrt{\frac{a}{a-R}}$.

Note from above, when there is a continuum of uniformly distributed markets and demand is log-concave, we find the monopolist that is not allowed to price discriminate will set the price such that markets below the cutoff level xc_L are shut down, provided that the dispersion of demand across markets is large enough (i.e., $k > k_0$). As (10) suggests, the cutoff value x is a constant fraction of the dispersion k and is unique for each given σ, i.e.

$$x = \frac{(1-\sigma)}{z} k.$$

Accordingly, the fraction of markets shut down is

$$\frac{\frac{k(1-\sigma)}{z} - 1}{k-1},$$

which increases in k given $(1-\sigma)/z$ is a fraction less than one.

Again, take the linear demand $\sigma = 0$ as an example. Equation (12) can be written as

$$a = \frac{1}{2\lambda}\left[\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^3\right] = \frac{3}{16\lambda}. \tag{15}$$

Equation (13) can be rewritten as

$$R = \frac{1}{2\lambda}\int_{\frac{1}{z}}^{1}\left[s - \frac{1}{sz^2}\right] ds = \frac{1}{2\lambda}\left[\frac{1}{2} - \frac{1}{2z^2} + \frac{1}{z^2}\ln\left(\frac{1}{z}\right)\right]. \tag{16}$$

Note that z is a constant satisfying (14):

$$\int_1^z \left(1 - \frac{2}{t}\right) dt = 0,$$

which implies

$$z - 1 - 2\ln z = 0,$$

so $z \simeq 3.513$, which corresponds to k_0 in the analysis of Section 2. For any $k > z$, there is a cutoff level k/z such that any markets $c < k/z$ will be shut down.

Given $z \simeq 3.513$, we can also compare (15) and (16),

$$W^{PD} - W^U > 0 \Leftrightarrow k > \sqrt{\frac{a}{a-R}} = \sqrt{\frac{3}{4/3.513 - 1}} \simeq 4.651,$$

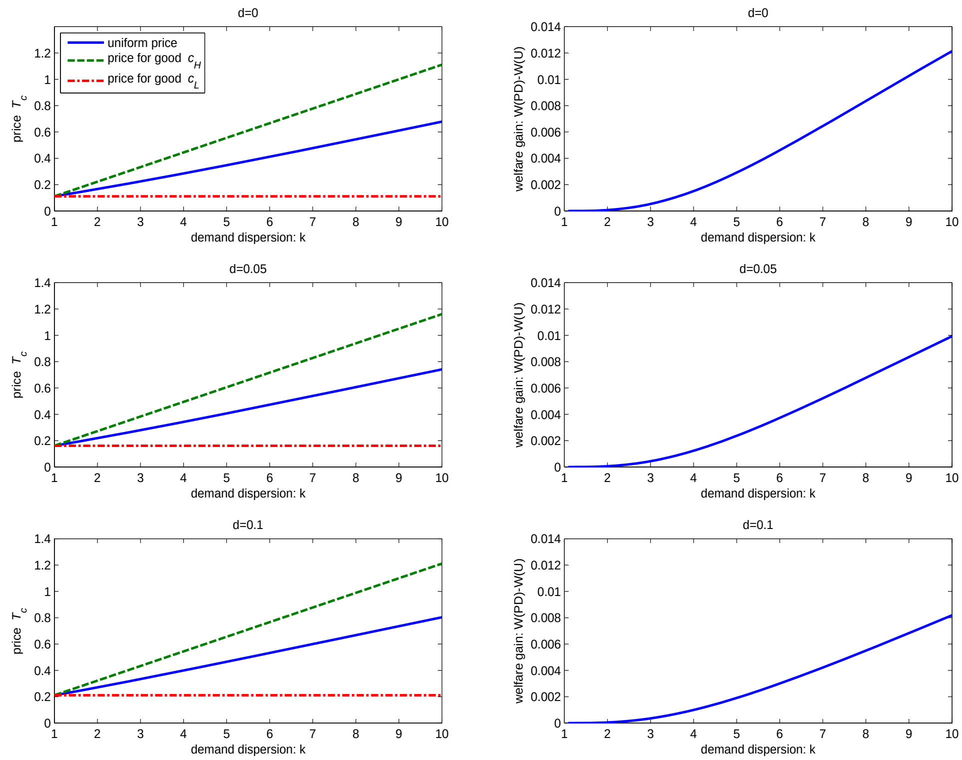
as we found for the linear demand case.

In conclusion, we have shown for the continuum case, that when the inverse demand implied by (1) is log-concave, welfare is higher under price discrimination provided there is sufficient dispersion of demand across markets. □

3.3 Exponential and log-convex demand

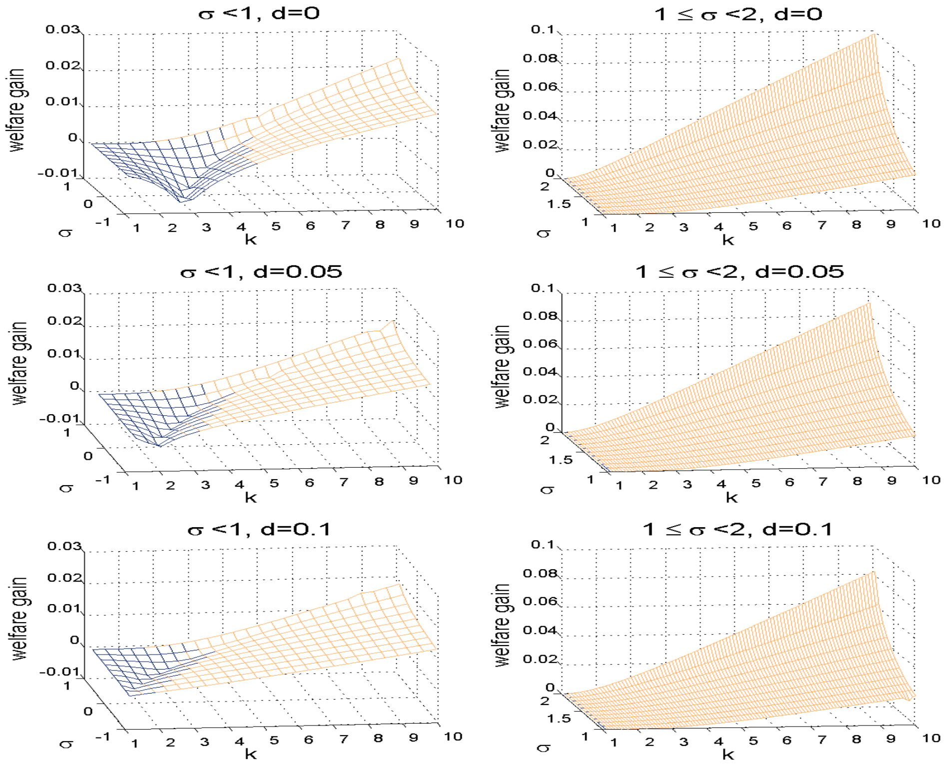
For cases with exponential and log-convex demand, we present the results graphically. The case corresponding to Figure 2, with exponential demand, is given in Figure 5.

Figure 5: Monopoly prices with exponential demand (continuum case)



More generally, Figure 6 shows that provided k is large enough when demand is log-concave, and for all k when demand is exponential or log-convex, welfare is higher under price discrimination (and so when a platform can use ad valorem fees). In the log-concave case with $d = 0$, the critical level of k for which welfare is higher under price discrimination than under uniform pricing is $k > 5$, and the critical value of k declines as d increases, so the dispersion in demand across market does not have to be very high for price discrimination to generate higher welfare than uniform pricing. In the exponential or log-convex case, Figure 6 suggests welfare is always higher under price discrimination regardless of the level of k and d.

Figure 6: Welfare comparison for a continuum of goods



4 Consumer surplus and profit effects

In this section we decompose the change in welfare from allowing ad valorem fees into changes in platform profit and consumer surplus for the calibrated models of Section 4.2 and 4.3 from the main text. Specifically, if we denote W is welfare, Π is profit and CS is consumer surplus, then we have $W(D) = \Pi(D) + CS(D)$ and $W(ND) = \Pi(ND) + CS(ND)$, where we use D to indicate the case where ad valorem fees are allowed (so price discrimination is possible) and ND to indicate the no-discrimination case that arises when ad valorem fees are banned. Then we decompose the welfare gain $\frac{W(D)-W(ND)}{W(ND)}$ into the profit gain $\frac{\Pi(D)-\Pi(ND)}{W(ND)}$ and the consumer surplus gain $\frac{CS(D)-CS(ND)}{W(ND)}$. The results are presented in Figures 7 and 8.

Figure 7: Visa signature debit cards: Decomposition of welfare gain from ad valorem fees

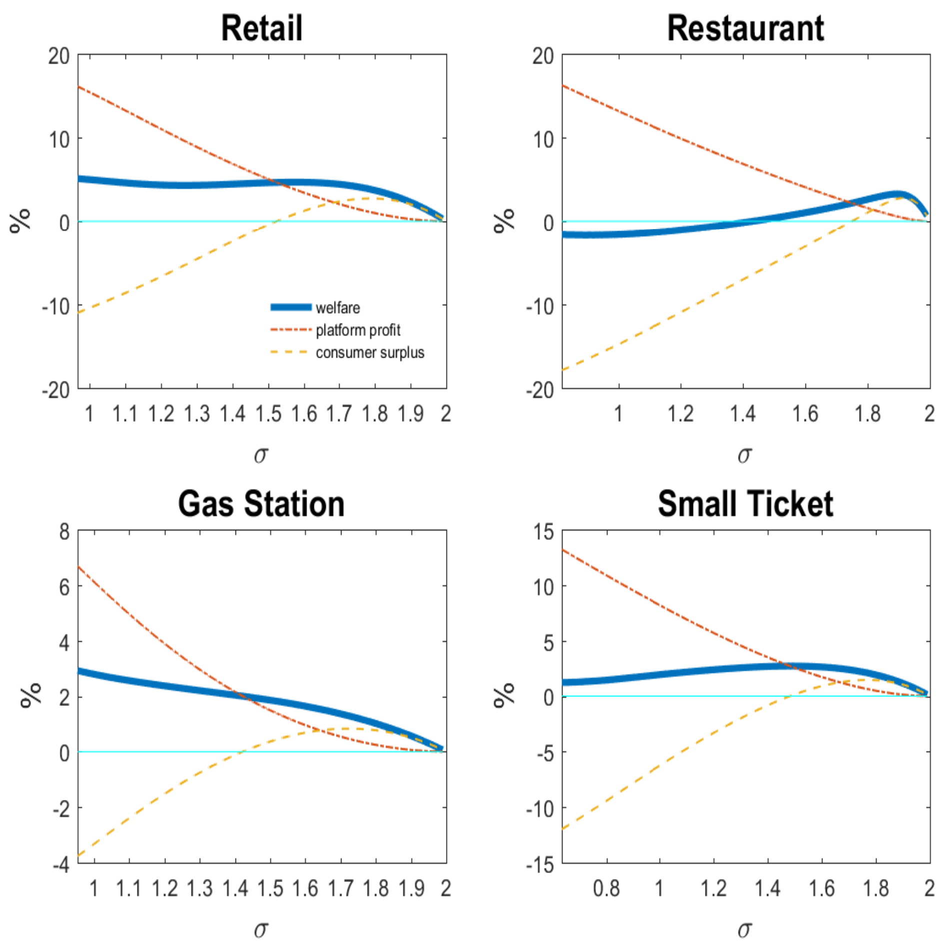


Figure 8: Amazon DVDs: Decomposition of welfare gain from ad valorem fees