SUPPLEMENTARY APPENDIX*

Indeterminacy and Imperfect Information

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I Extended Discussion: Simple Example Economy

I.1 Nonlinear Restrictions on the Model Solution

The solution to our modelling framework with imperfect information under rational expectations involves the interplay of two nonlinear restrictions on top of the standard methods for solving linear rational expectations models. Specifically, the Kalman gain is an endogenous coefficient and has to be computed from the nonlinear Riccati equation, while the projection condition for mutual consistency of expectation formation of the two agents imposes a second moment restriction. Any solution has to obey both sets of restrictions simultaneously. In order to give a sense how these elements interact in determining a solution, we plot the two roots of the Riccati equation for the projection forecast error matrix Σ, the Kalman gain κ_r, and the projection condition for the specification with an endogenous signal in Figure A.1.

We choose a standard parameterization for illustration purposes and set $\gamma_b = 0$ for simplicity. The depicted restrictions are plotted as functions of the loading on the real rate innovation γ_ε over the range $[-10, 10]$, which in turn imply values for the loading on the measurement error γ_ν as given by (27) in the main text. The hyperbola of the projection condition reflects the quadratic on the measurement error loading γ_ν. Existence of a solution requires that the positive root of the Riccati equation and the projection condition both hold. In this specific example, however, they never intersect or even touch. Therefore, under this parametrization, no equilibrium exists since the Kalman filter does not exist in the sense that the projection equations for the variables outside of the central bank's information set are explosive and mutually inconsistent.

Varying parameter values we find that the projection condition is consistent with a positive Σ when we reduce the serial correlation of the real rate ρ, increase the policy coefficient ϕ, but also for a large measurement error variance σ_ν^2. Moreover, there is one special equilibrium, as discussed in section 2.4.1 of the main text, where the projection condition and both roots of the Riccati equation all intersect at $\Sigma = 0$. In this case, the equilibrium is one where belief shocks do not affect aggregate outcomes and the equilibrium is seemingly determinate although it is one of many. It has the special characteristic that the Kalman gain is such that it leads to an explosive root in the system which pins down the endogenous forecast error. Figure A.1 also shows that the equilibrium would be indeterminate, as in the case with an exogenous information set, since the Kalman gain is small and negative.

The previous exercise illustrates the challenges inherent to computing a solution in our imperfect information environment in that we have to solve a set of nonlinear restrictions on top of a linear rational expectations system, whereby the solution to the latter depends on the solution to the former. Appendix B of the main text describes the general solution algorithm to this complicated fixed point problem. As discussed in the main text, an interesting implication of this environment is that the set of multiple equilibria is restricted in the sense that the loadings on the innovations in the decomposition of the endogenous forecast error solution are constrained to lie within certain bounds (see Propositions 1 and 2 in the main text for analytical results).

Figure A.2 reports the set of multiple equilibria as indexed by the loadings on the fundamental shock to the real rate ε, the measurement error innovation ν, and the belief shock b. We assume the same parameterization as before. Results for other specifications are available on request. The figure depicts ranges where an equilibrium exists for permutations of the three innovations applied to the endogenous forecast error on inflation. Each dot in the graphs represents a specific equilibrium indexed by the values of the γ loadings on the innovations. Each equilibrium is found by running the solution algorithm 2,000 times for random draws for starting values. The dot plot then traces out existence regions for equilibria under a given parameterization for the structural parameters. The algorithm does not find any other equilibria outside of the depicted regions.[1]

What stands out from the figure is that the range of equilibria is fairly tightly restricted. For instance, loadings on the belief shock lie within a range of ± 2 around zero. We also find that the set of equilibria in this case is symmetric around the zero line which confirms the findings from the impulse response functions depicted in section 4 of the main text (see Figure 1). All equilibria impose a negative value on the measurement error loading (see the top and the bottom panel) so that a positive innovation implies that observed inflation is perceived to be higher than it actually is. This leads to a policy-driven decline in inflation in the model. At the same time, the response to the real rate innovation can lie on either side of zero (see the top and middle panel of the figure).

I.2 Example Economy with Wicksell Rule

We also consider an alternative specification of our simple example model that differs from the baseline case in terms of policy rule. Specifically, we assume that the reaction function contains a time-varying intercept, namely the real interest rate. This policy rule may be labeled a Wicksellian rule as it tracks the behavior of the natural real rate. The equation system is therefore given by:

$$
\begin{aligned}
i_t &= r_t + E_t \pi_{t+1}, & \text{(A.1)}\\
i_t &= r_t + \phi \pi_t, & \text{(A.2)}\\
r_t &= \rho r_{t-1} + \varepsilon_t, & \text{(A.3)}
\end{aligned}
$$

where as before we assume that $\varepsilon_t \sim iid\ N(0, \sigma_\varepsilon^2)$, $|\rho| < 1$, and $|\phi| > 1$. Substituting the policy rule into the Fisher equation results in an autonomous expectational difference equation in inflation only, the solution to which is $\pi_t = 0$, and consequently $i_t = r_t$. In contrast to our baseline case, the inflation rate is always held at zero in a unique equilibrium under full information.

We can also derive a solution for the conditioned-down system where we evaluate expectations conditional on a nested information set. The solution to this specification is isomorphic to the full information case; specifically, we have: $\pi_{t|t} = 0$, $i_t = r_{t|t}$. As discussed in the main text, these are the optimal projections of the policymaker which have to be consistent with any equilibrium path in the imperfect information economy. This gives rise to the projection condition which restricts the set of multiple equilibria under indeterminacy. Moreover, it is also the case in this alternative specification that the central bank does not project indeterminate outcomes.

We now turn to the imperfect information economy. We assume that the central bank uses a certainty-equivalent policy rule that responds to projected variables:

$$
i_t = r_{t|t} + \phi \pi_{t|t.} \tag{A.4}
$$

Furthermore, we specialize this case to an *iid* real rate process with $r_t = \varepsilon_t$, which admits a convenient analytical solution that allows further insights. As in the main text, we consider both an exogenous and an endogenous signal, whereby we frame the presentation in terms of real rate projections. Specifically, we can write:

$$
r_{t|t} = \kappa_r Z_t, \text{ with } Z_t = r_t + \nu_t \text{ or } Z_t = \pi_t + \nu_t, \tag{A.5}
$$

where Z_t is the information set, $\nu_t \sim iid\ N(0, \sigma_\nu^2)$ is the measurement error, and the Kalman updating equation already imposes $r_{t|t-1} = 0$ since all innovations are *iid*. We can then rewrite the equation for inflation dynamics as follows:

$$
\pi_t = -\left(r_{t-1} - r_{t-1|t-1}\right) + \gamma_\varepsilon \varepsilon_t + \gamma_\nu \nu_t + \gamma_b b_t. \tag{A.6}
$$

[1] Although we cannot show it analytically, the larger white areas within the hyperbolas are likely not regions of non-existence but are due to numerical inaccuracies, as they do tend to shift with repeated simulations.

In the case of the information set with an exogenous signal, $Z_t = r_t + \nu_t$, we can compute the Kalman gain as before:

$$\kappa_r = \frac{cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)}{var(\widetilde{Z}_t)} = \frac{cov\left(r_t - r_{t|t-1}, Z_t - Z_{t|t-1}\right)}{var(Z_t - Z_{t|t-1})} = \frac{cov\left(r_t, r_t + \nu_t\right)}{var(r_t + \nu_t)} = \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\nu^2}. \tag{A.7}$$

It immediately follows that $0 \leq \kappa_r \leq 1$, with $\kappa_r \to 1$ as $\sigma_\nu^2 \to 0$ and $\kappa_r \to 0$ as $\sigma_\nu^2 \to \infty$. The projection error variance is then:

$$\Sigma = var(r_t - r_{t|t}) = \frac{\sigma_\varepsilon^2 \sigma_\nu^2}{\sigma_\varepsilon^2 + \sigma_\nu^2} = \kappa_r \sigma_\nu^2. \tag{A.8}$$

We note that as $\sigma_\nu^2 \to 0$, $\Sigma \to \infty$. The projection condition follows immediately as:

$$cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right) = cov\left(\pi_t, r_t + \nu_t\right) = \gamma_\nu \sigma_\nu^2 + \gamma_\varepsilon \sigma_\varepsilon^2 = 0, \tag{A.9}$$

which imposes the following linear restriction on the equilibrium path:

$$\gamma_\nu = -\gamma_\varepsilon \frac{\sigma_\varepsilon^2}{\sigma_\nu^2}. \tag{A.10}$$

Finally, the solution for inflation is given by:

$$\pi_t = \gamma_\varepsilon \varepsilon_t + \gamma_\nu \nu_t + \gamma_b b_t - \left(1 - \kappa_r\right) \varepsilon_{t-1} + \kappa_r \nu_{t-1}, \tag{A.11}$$

where the Kalman gain is given by the above expression, the loading on the belief shock is unrestricted, and there is one linear restriction on the loadings for the real rate innovation and the measurement error. Obviously, the solution is indeterminate as in the baseline case.

Turning to the case of an endogenous signal, $Z_t = \pi_t + \nu_t$, we compute the Kalman gain as:

$$\kappa_r = \frac{cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)}{var(\widetilde{Z}_t)} = \frac{\gamma_\varepsilon \sigma_\varepsilon^2}{\Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \left(1 + \gamma_\nu\right)^2 \sigma_\nu^2 + \gamma_b^2 \sigma_b^2}. \tag{A.12}$$

The projection error variance is calculated from the Riccati equation:

$$\Sigma = -\frac{\left(\gamma_\varepsilon \sigma_\varepsilon^2\right)^2}{\Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \left(1 + \gamma_\nu\right)^2 \sigma_\nu^2 + \gamma_b^2 \sigma_b^2} + \sigma_\varepsilon^2. \tag{A.13}$$

As before, the projection condition is:

$$cov\left(\pi_t, \pi_t + \nu_t\right) = \Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_\nu \left(1 + \gamma_\nu\right) \sigma_\nu^2 + \gamma_b^2 \sigma_b^2 = 0, \tag{A.14}$$

which imposes a more complicated nonlinear restriction on the shock loadings. Finally, the dynamics of the model with an endogenous signal are given by:

$$r_{t|t} = \kappa_r r_{t-1|t-1} - \kappa_r \varepsilon_{t-1} + \gamma_\varepsilon \kappa_r \varepsilon_t + \left(1 + \gamma_\nu\right) \kappa_r \nu_t + \gamma_b b_t, \tag{A.15}$$

$$\pi_t = r_{t-1|t-1} - \varepsilon_{t-1} + \gamma_\varepsilon \varepsilon_t + \gamma_\nu \nu_t + \gamma_b b_t, \tag{A.16}$$

with the restrictions and definitions as given above. We also note that in this case $|\kappa_r| < 1$ so that the possibility of a special equilibrium discussed in the main text does not arise.

I.3 A Comparison with Svensson and Woodford (2004)

In this section of the Supplementary Appendix we describe a minimum-state-variable (MSV) approach to solving the imperfect information model as in Svensson and Woodford (2004). A key difference between their work and ours is that they endeavour to characterize optimal policy in a New Keynesian model while we study a general environment for a given feedback rule. However, for a given set of first-order conditions to the optimal policy problem under imperfect information their approach falls into the class of expectational linear difference equations studied here as well. Section 3 of the main text provides a more general discussion. Moreover, Svensson and Woodford (2004) are not alone in pursing a MSV approach in such models: other examples are Aoki (2003, 2008) or Nimark (2008).

We now apply this approach to our simple example economy. The notation follows Svensson and Woodford (2004) for ease of comparison. It begins with a guess that the equilibrium process for inflation has the following form:

$$\pi_t = g\ r_t^* + \bar{g}\ r_{t|t} = g\ r_t + (\bar{g} - g)\ r_{t|t}, \text{ where } \bar{g} \equiv \frac{1}{\phi - \rho}. \tag{A.17}$$

For any choice of g, this guess automatically satisfies the projection condition $\pi_{t|t} = \bar{g} r_{t|t}$. What remains to be seen is which values for g, if any, are consistent with the rest of the dynamic system, specifically the innovations version of the Fisher equation. We note that the proposed solution in the vein of Svensson and Woodford's (2004) solution approach excludes belief shocks.

We now proceed by deriving the dynamics for r_t^* and $r_{t|t}$ implied by (A.17) for a given value of g. The initial guess for inflation in (A.17) depends on the projected real rate and thereby on the history of measurements Z^t, which in turn depends on the history of inflation:

$$Z_t = g\ r_t + (\bar{g} - g)\ r_{t|t} + \nu_t. \tag{A.18}$$

This complicates setting up the Kalman filter, but we note that the term in $r_{t|t}$ does not add any new information to Z_t. In fact, Z_t provides an implicit definition of an information set spanned by:

$$W_t = g\ r_t + \nu_t \tag{A.19}$$

in the sense that $E(x_t|Z^t) = E(x_t|W^t)$ for any variable x_t. While projections of variables onto W^t and Z^t are equivalent, the associated Kalman gains differ, however, by a factor of proportionality.[2]

Now consider the Kalman gain involved in projecting the real rate onto W^t, $\tilde{r}_{t|t} = \kappa \tilde{W}_t$. Define $R^2 \equiv g\ K$, so that $0 \leq R^2 \leq 1$. We can then write:

$$\begin{aligned} \tilde{\pi}_t &= g\ \tilde{r}_t + (\bar{g} - g)\,\kappa\,\tilde{W}_t = \\ &= \left[g\ (1 - R^2) + \bar{g}\ R^2\right] \tilde{r}_t + (\bar{g} - g)\,\kappa\,\nu_t = \\ &= \left[g\ (1 - R^2) + \bar{g}\ R^2\right] \rho\, r_{t-1}^* + \underbrace{\left[g\ (1 - R^2) + \bar{g}\ R^2\right] \varepsilon_t + (\bar{g} - g)\,\kappa\,\nu_t}_{=\eta_t}, \end{aligned} \tag{A.20}$$

where the last line uses $\tilde{r}_t = \rho\, r_{t-1}^* + \varepsilon_t$, and η_t is the endogenous forecast error as defined in the paper and set equal to the shock components of (A.20) as indicated above.

[2] Let K_r continue to denote the Kalman gain of r_t onto Z_t and we have:

$$Z_t = W_t + (\bar{g} - g)\ K_r\ Z_t = W_t / \left[1 - (\bar{g} - g)\ K_r)\right].$$

We need to find a value for g that sets the loading on r^*_{t-1} in (A.20) equal to negative one:

$$\left[g\ (1-R^2)+\bar{g}\ R^2\right]\rho=-1,\ g\leq 0, \tag{A.21}$$

where the inequality follows from $\bar{g}$, R^2 and ρ being positive numbers. As an additional condition, the solution approach of Svensson and Woodford (2004) requires the roots of the characteristic equation that describe the joint dynamics of π_t, $r_{t|t}$ and r_t to satisfy the usual counting rule for values inside and outside the unit circle. In this model, there is one backward-looking variable, r_t, and two forward-looking variables, π_t and $r_{t|t}$, which requires finding one stable and two unstable eigenvalues for equilibrium determinacy. However, as we show in the main part of the paper, in the present example the Kalman filter will always stabilize the dynamics of $r_t - r_{t|t}$ causing the system to have two stable and only one unstable root.

It follows that the set of MSV candidate solutions, described by (A.17) for any given value of g, does not span the set of all candidate solutions, which include those described by any combination of weights γ for the linear combination of shocks that make up the endogenous forecast error η_t. Moreover, the set of candidate solutions in Svensson and Woodford (2004) does not even span the restricted set of candidates for η_t where $\gamma_b = 0$. To see this, note that the MSV candidate is parametrized by a single unknown coefficient g, which places a restriction on the weights γ_ε and γ_ν implied by the associated specification of η_t as seen in (A.20).

II The Variance Bound in the General Case

In section 2 of the main text, the analysis of the simple example model highlights the existence of an upper bound on the variance of endogenous variables, specifically inflation in this example, that holds across all of the equilibria under consideration. We now show how these arguments can be extended to the general case when the backward-looking variables are exogenous. Analogous to the example from section 2, we consider the case of an endogenous signal, where the measurement vector conveys a noisy signal about the vector of forward-looking variables:

$$\boldsymbol{Z}_t = \boldsymbol{Y}_t + \boldsymbol{\nu}_t \qquad\qquad \boldsymbol{\nu}_t \sim N(\boldsymbol{0}, \boldsymbol{\Omega}_{\nu\nu}) \tag{A.22}$$

where $\boldsymbol{\nu}_t$ is a vector of *iid* measurement errors. In terms of our general framework, these measurement errors are usually tracked as part of the vector of backward-looking variables, $\boldsymbol{X}_t$. To facilitate the derivation of the variance bound, it is, however, more convenient to track the measurement errors $\boldsymbol{\nu}_t$ separately from the vector of backward-looking variables $\boldsymbol{X}_t$.

To establish the variance bound, we assume that the backward-looking variables are purely exogenous; that means $\boldsymbol{X}_t = \boldsymbol{A}_{xx}\boldsymbol{X}_{t-1} + \boldsymbol{B}_{x\varepsilon}\boldsymbol{\varepsilon}_t$, $\boldsymbol{\varepsilon}_t \sim N(\boldsymbol{0}, \boldsymbol{I})$, $\boldsymbol{A}_{xx}$ stable, and $\mathrm{Var}\,(\boldsymbol{X}_t)$ can be computed independently from any particular equilibrium for the endogenous variables.[3]

In addition, we assume that there is no dependence of the policy instrument $\boldsymbol{i}_t$ on its own lag in the policyrule, $\boldsymbol{\Phi}_i = \boldsymbol{0}$ in equation (60) of the paper, so that the exogenous vector $\boldsymbol{X}_t$ completely describes outcomes under full information. As the measurement errors have no role in the full-information version of the model, the projection condition reduces to $\boldsymbol{Y}_{t|t} = \boldsymbol{\mathcal{G}}_{yx}\ \boldsymbol{X}_{t|t}$.

By construction, we have $\boldsymbol{Z}_{t|t} = \boldsymbol{Z}_t$ and can thus deduce that $\boldsymbol{Y}^*_t = -\boldsymbol{\nu}^*_t$.[4] Together with the

[3]Given values for $\boldsymbol{A}_{xx}$ and $\boldsymbol{B}_{x\varepsilon}$, and requiring that $\boldsymbol{A}_{xx}$ be stable, $\mathrm{Var}\,(\boldsymbol{X}_t)$ can be obtained using standard methods as the solution to a Lyapunov equation, $\mathrm{Var}\,(\boldsymbol{X}_t) = \boldsymbol{A}_{xx}\,\mathrm{Var}\,(\boldsymbol{X}_t)\boldsymbol{A}'_{xx} + \boldsymbol{B}_{x\varepsilon}\boldsymbol{B}'_{x\varepsilon}$ (Sargent and Ljungqvist 2004, Hamilton 1994).

[4]Asterisks continue to denote projection residuals $\boldsymbol{Y}^*_t \equiv \boldsymbol{Y}_t - \boldsymbol{Y}_{t|t}$ and $\boldsymbol{\nu}^*_t \equiv \boldsymbol{\nu}_t - \boldsymbol{\nu}_{t|t}$.

projection condition and the law of total variance, we then obtain:[5]

$$\boldsymbol{Y}_t = \boldsymbol{\mathcal{G}}_{yx}\ \boldsymbol{X}_{t|t} - \boldsymbol{\nu}_t^* \tag{A.23}$$

$$\Rightarrow \quad \text{Var}\left(\boldsymbol{Y}_t\right) = \boldsymbol{\mathcal{G}}_{yx}\ \text{Var}\left(\boldsymbol{X}_{t|t}\right) \boldsymbol{\mathcal{G}}_{yx}' + \text{Var}\left(\boldsymbol{\nu}_t^*\right) \tag{A.24}$$

$$\Rightarrow \quad \text{Var}\left(\boldsymbol{Y}_t\right) \leq \boldsymbol{\mathcal{G}}_{yx}\ \text{Var}\left(\boldsymbol{X}_t\right) \boldsymbol{\mathcal{G}}_{yx}' + \text{Var}\left(\boldsymbol{\nu}_t\right) \tag{A.25}$$

where the weak inequality is understood as indicating a semi-definite difference between matrices. The absence of covariance terms in (A.24) follows from the optimality of projections, which requires projection residuals, like $\boldsymbol{\nu}_t^*$ to be orthogonal to $\boldsymbol{Z}^t$ or any functions thereof (like $\boldsymbol{x}_{t|t}$). In addition, the law of total variance implies $\text{Var}\left(\boldsymbol{X}_t\right) = \text{Var}\left(\boldsymbol{X}_{t|t}\right) + \text{Var}\left(\boldsymbol{X}_t|\boldsymbol{Z}^t\right) \geq \text{Var}\left(\boldsymbol{X}_{t|t}\right)$ and $\text{Var}\left(\boldsymbol{\nu}_t\right) \geq \text{Var}\left(\boldsymbol{\nu}_t|Z^t\right) = \text{Var}\left(\boldsymbol{\nu}_t^*\right)$.

In contrast to the simple Fisher example, where $\pi_t = \bar{g}r_t - \nu_t$ was also a valid equilibrium, its analogue in the general case, given by $\boldsymbol{Y}_t^S \equiv \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{X}_t - \boldsymbol{\nu}_t$, will typically not be a possible outcome in equilibrium. Nevertheless, (A.25) provides an upper bound on the variability of equilibria for $\boldsymbol{Y}_t$ across all potential equilibria in our environment. To demonstrate why $\boldsymbol{Y}_t^S$ is generally not an equilibrium outcome, we consider the following, simplified version of the general setup known described in section 3 of the paper, together with the signal given by (A.22):

$$\boldsymbol{X}_{t+1} = \boldsymbol{A}_{xx}\boldsymbol{X}_t + \boldsymbol{B}_{x\varepsilon}\boldsymbol{\varepsilon}_{t+1} \tag{A.26}$$

$$E_t\boldsymbol{Y}_{t+1} = \boldsymbol{A}_{yx}\boldsymbol{X}_t + \boldsymbol{A}_{yy}\boldsymbol{Y}_t + \hat{\boldsymbol{A}}_{yx}\boldsymbol{X}_{t|t} + \hat{\boldsymbol{A}}_{yy}\boldsymbol{Y}_{t|t} \tag{A.27}$$

$$\boldsymbol{Z}_t = \boldsymbol{Y}_t + \boldsymbol{\nu}_t, \quad \boldsymbol{\nu}_t \sim N(\boldsymbol{0}, \boldsymbol{\Omega}_{\nu\nu}) \tag{A.22}$$

We proceed by showing that evaluation of (A.22) using the guess $\boldsymbol{Y}_t^S \equiv \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{X}_t - \boldsymbol{\nu}_t$ yields a contradiction when considering two ways of forming $E_t\boldsymbol{Y}_{t+1}$. First, we plug in the specific guess $\boldsymbol{Y}_t^S \equiv \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{X}_t - \boldsymbol{\nu}_t$ and then take expectations. In the second alternative, we substitute the guess into the right-hand side of (A.22):

$$E_t\left(\boldsymbol{Y}_{t+1}^S\right) = E_t\left(\boldsymbol{\mathcal{G}}_{yx}\boldsymbol{X}_{t+1}\right) = \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{A}_{xx}\,\boldsymbol{X}_t \tag{A.28}$$

$$\neq \quad E_t\left(\boldsymbol{Y}_{t+1}\right) = \left(\boldsymbol{A}_{yx} + \boldsymbol{A}_{yy}\boldsymbol{\mathcal{G}}_{yx}\right)\boldsymbol{X}_t + \left(\hat{\boldsymbol{A}}_{yx} + \hat{\boldsymbol{A}}_{yx}\boldsymbol{\mathcal{G}}_{yx}\right)\boldsymbol{X}_{t|t} - \boldsymbol{A}_{yy}\boldsymbol{\nu}_t\,. \tag{A.29}$$

while $E_t\boldsymbol{Y}_{t+1}$ becomes a function of $\boldsymbol{X}_t$ alone in (A.28), it depends on $\boldsymbol{X}_t$, $\boldsymbol{X}_{t|t}$ and $\boldsymbol{\nu}_t$ in (A.29). The candidate $\boldsymbol{Y}_t^S$ implies that the signal becomes a function of $\boldsymbol{X}_t$ alone, $\boldsymbol{Z}_t = \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{X}_t$, so that the projections $\boldsymbol{X}_{t|t}$ are independent from $\boldsymbol{\nu}_t$. (A.28) and (A.29) cannot be generally consistent with each other, unless $\boldsymbol{A}_yy = \boldsymbol{0}$ which was the case in the simple Fisher example.[6] However, in general, $\boldsymbol{A}_{yy}$ is not zero, as illustrated, for example, in the case of the New Keynesian model analyzed in section 4 of the paper.

III Additional Results: New Keynesian Model

This section of the appendix shows impulse response functions and second moments of outcomes in the New Keynesian model of the main paper under an alternative calibration. In this specification,

[5]The projection condition, stated in Definition 2 of the paper, requires

$$\boldsymbol{\mathcal{Y}}_{t|t} = \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{X}}_{t|t}\,, \qquad \text{and} \quad \boldsymbol{\mathcal{X}}_{t+1|t} = \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{X}}_{t|t}\,.$$

[6]In the Fisher example, (A.27) collapses to a combination of the Fisher equation and the Taylor rule, $E_t\pi_{t+1} = \phi\pi_{t|t} - r_t$, which does not feature the current value of the forward-looking variable π_t.

the noise variances on inflation and output are scaled down to one tenth of the values used in the baseline calibration described in the paper.

Figure A.3 reports impulse responses to the model shocks for the three endogenous variables. The figure is the analogue of Figure 3 in the main text but with much lower measurement error variance. The responses to GDP growth rate shocks under imperfect information in the first column are tightly clustered, even indistinguishable from the pattern under full information. This indicates that the set of multiple equilibria is small and that therefore the loadings on this specific shock fall in line in what they would be under full information. The responses to the two measurement error shocks and the two belief shocks in the remaining columns of the figure are an order of magnitude smaller than the corresponding ones in the baseline case. What is notable, however, is that the qualitative pattern of the responses is different from the baseline. This reflects the essential nonlinear nature of the model as the set of equilibria is restricted by the Riccati equation and the projection condition, both of which are functions of the belief shock and measurement error loadings.

Figure A.4 reports analogous results to Figure 5 in the main text for the case of a small measurement error. As expected, the set of autocorrelation functions for the various equilibria is concentrated around the full information case (see the upper panel of the figure); whereas the relative standard deviations are comparatively smaller. Taken together, the simulation results from this exercise indicate that the set of equilibria appears to be continuous in the size of the measurement error which separates the full information from the imperfect information economy. While the quantitative implications of introducing noise are small, as evidenced by Figure A.4, qualitatively the nature of equilibrium determination changes considerably.

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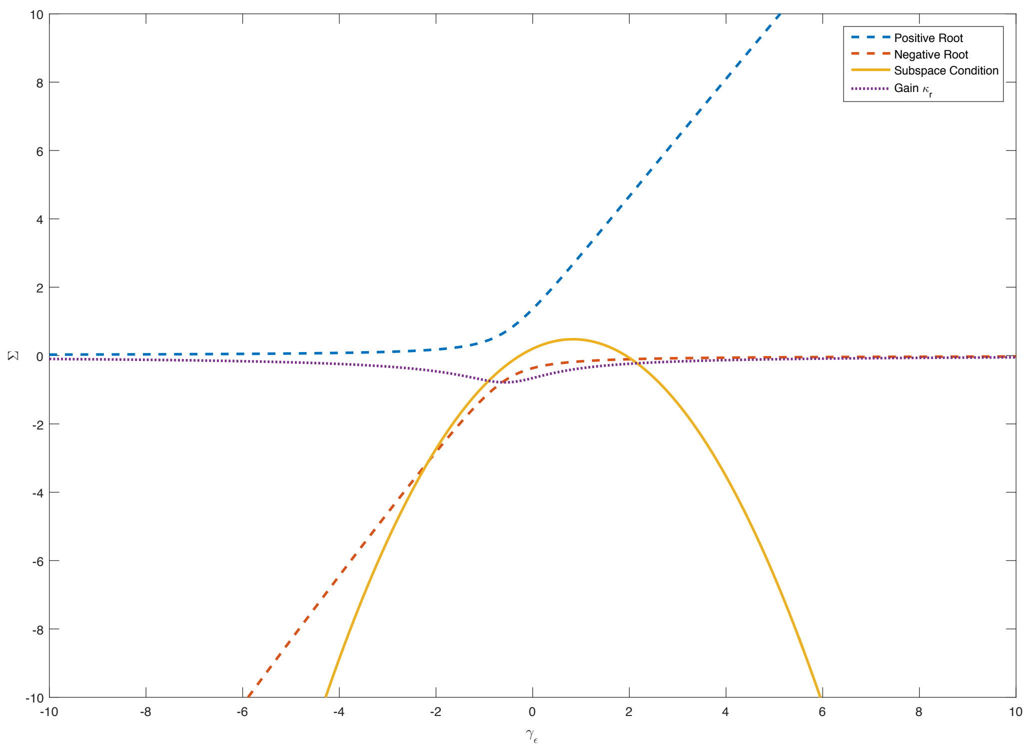
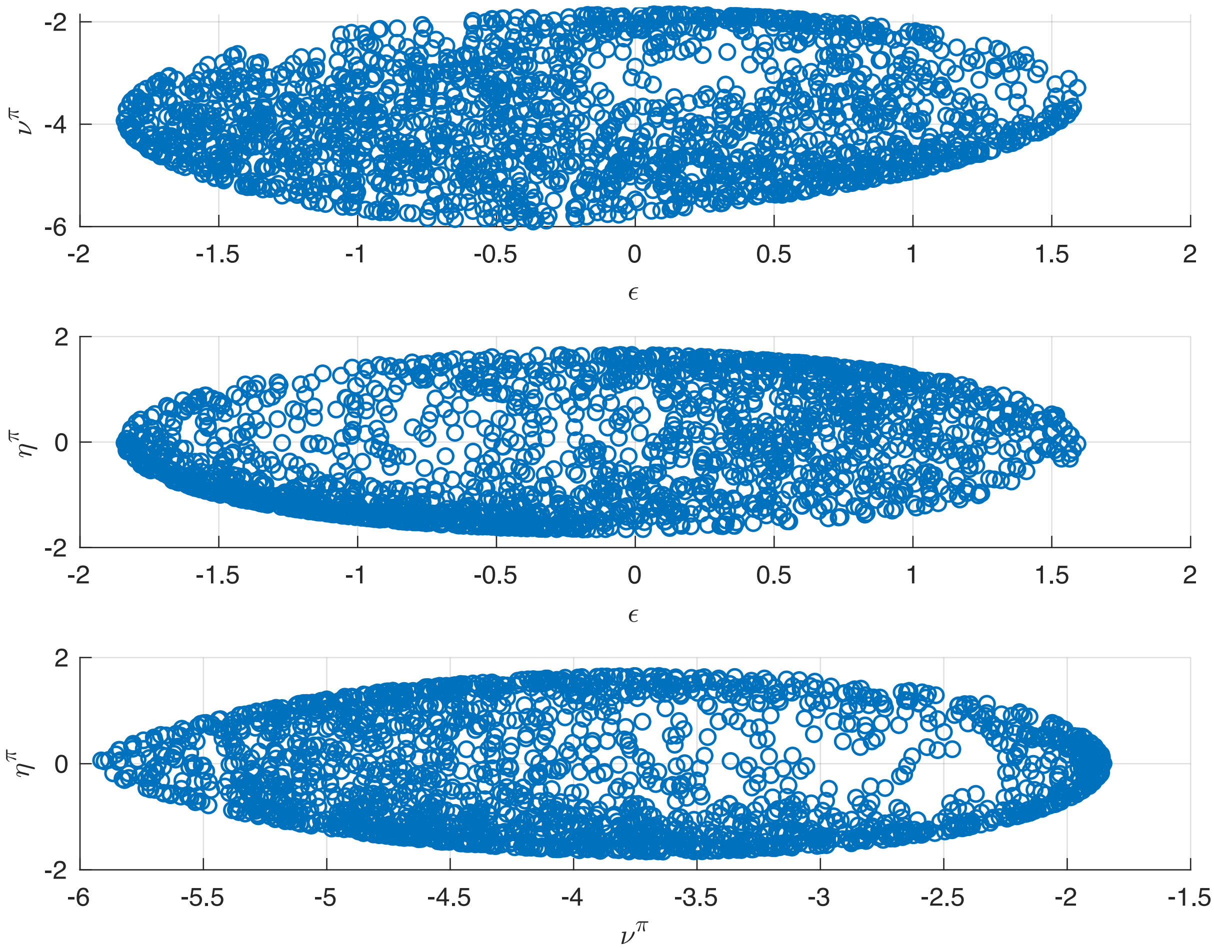
Figure A.1: Nonlinear Restrictions on the Model Solution in the Simple Example



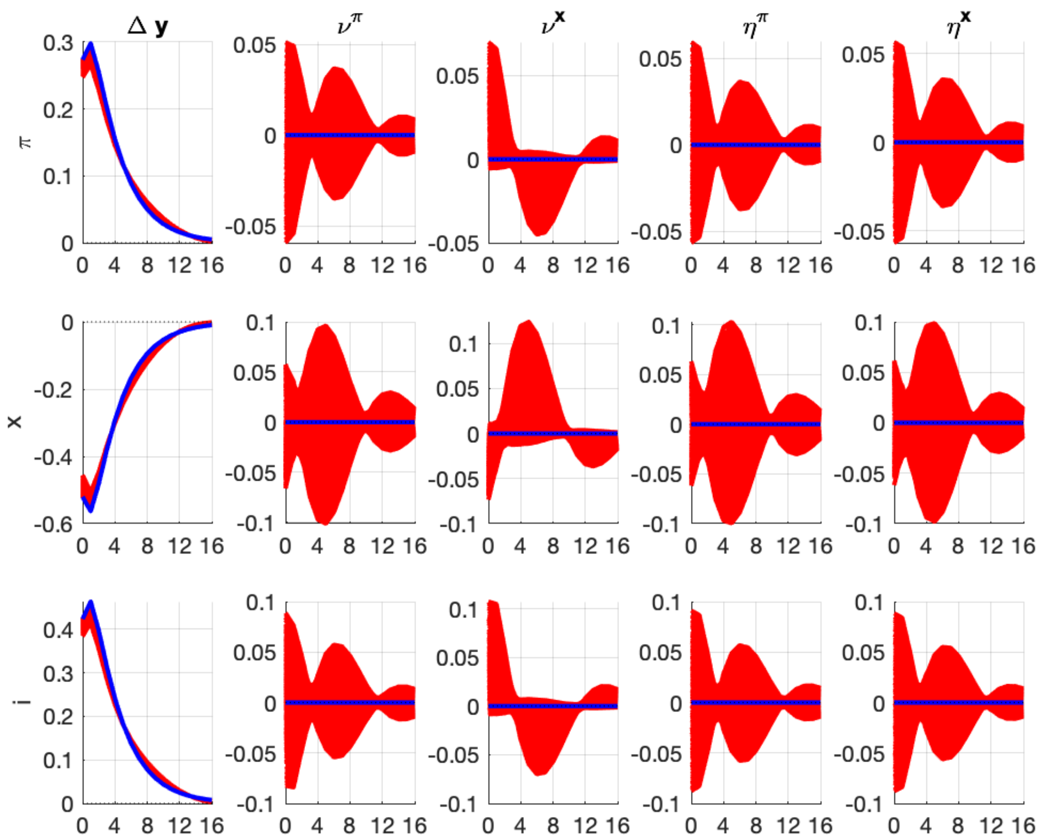
Note: The figure shows the nonlinear restrictions on the model solution for the simple example economy. It depicts the two roots of the Riccati equation, the constraint imposed by the projection condition and the implied Kalman gain.

Figure A.2: Existence Regions for Shock Loadings



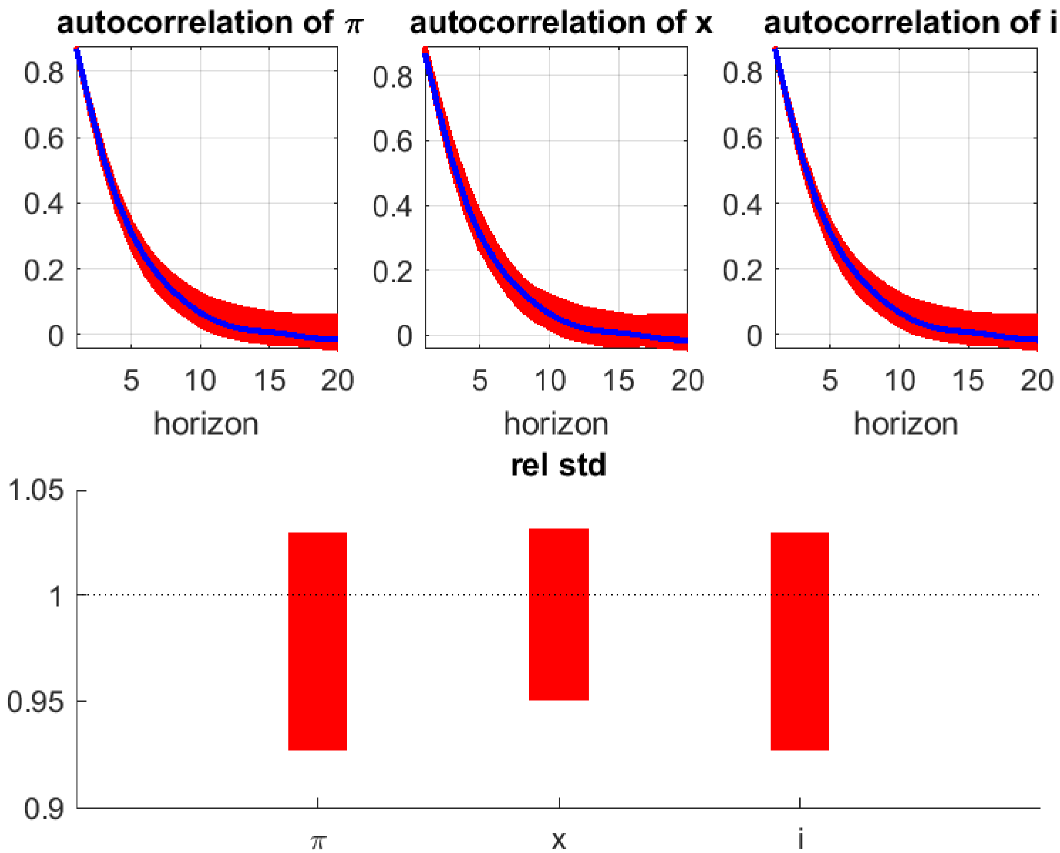
Note: The three graphs in the figure shows combinations of shocks loadings consistent with an equilibrium for a specific parameterization of the simple example economy with an endogenous signal.

Figure A.3: IRFs of Various Equilibria in New Keynesian model when Measurement Error is Small



Note: Impulse response functions (IRF) for the New Keynesian model under full information (blue) as well as various limited-information equilibria (red). Each row represents the response of a specific variable to the shocks in the model whereas each column represent the responses of the endogenous variables to a specific shock. The equilibria are obtained from an alternative calibration where the noise variances on inflation and output are scaled down to one tenth of the values used in the baseline calibration described in the paper.

Figure A.4: Second moments of limited-information equilibria in New Keynesian model when Measurement Error is Small



Note: Top panels show moments of endogenous variables for the New Keynesian model under full information (blue) as well as various limited information equilibria (red). Bottom panel reports ranges of relative standard deviations of outcomes under limited information relative to the full-information outcomes. The equilibria are obtained from an alternative calibration where the noise variances on inflation and output are scaled down to one tenth of the values used in the baseline calibration described in the paper.