Bubbly Recessions

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Abstract

We develop a tractable rational bubble model with downward nominal wage rigidity. We show that the collapse of a bubble can push the economy into a persistent recession with involuntary unemployment and depressed economic activities. The collapse can even push the economy into a liquidity trap, where a deflationary pressure exacerbates the wage rigidity and the recession. Our model highlights a novel and important welfare tradeoff between the boom and bust phases of bubbles, which warrants policy intervention.

1 Introduction

Throughout history, the collapse of asset and credit bubbles often precedes crises and protracted recessions (Jordà et al., 2015). A prominent example is the collapse of the Japanese bubble in the early 1990s and the subsequent "lost decade." Both housing prices and stock prices in Japan experienced a dramatic boom in the 1980s; the Nikkei index roughly tripled, and the housing price index nearly doubled in the second half of the decade, as seen in the bottom right panel of figure 1. The asset prices reached the peak in 1990, when the total market value of land in Japan famously exceeded four times that in the U.S. (Martin and Ventura, 2012). However, the boom turned into the bust of the early 1990s, with asset prices

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starting to fall in 1991. This coincided with the onset of a protracted period of low economic growth and high unemployment that lasted several decades until the 2000s. As seen in the top left panel of figure 1, a trend of high GDP growth abruptly ended in 1991, and the unemployment rate more than doubled from around 2% in 1991 to around 5.5% in 2002. Despite the rising unemployment rate, both nominal and real wages persisted near the peak levels of the boom, as seen in the figure's top right panel. The collapse of the asset price bubble in 1991 also coincided with abrupt changes to the nominal interest rate and inflation, as seen in the bottom left panel. The combination of falling asset prices, low nominal interest rates near the zero lower bound, disinflation, and rigid wages is a prominent feature of the onset and persistence of the Japanese lost decade.

More recently, the collapse of the U.S. housing bubble in the late 2000s precipitated the worst economic recession since the Great Depression. As seen in the bottom right panel of figure 2, the S&P/Case-Shiller U.S. National Home Price Index rose by more than 60% between 2000 and 2006. However the boom abruptly turned into the bust of 2007. As seen in the figure's top left panel, the collapse in asset prices coincided with the onset of the Great Recession, where the real GDP per capita declined by more than $2,000 between 2007 and 2009 and only recovered to the pre-recession level in 2013. The average unemployment rate doubled from the lowest of about 5% in 2007 to about 10% in 2009 and remained above the pre-recession rate until 2015. The collapse in asset prices also coincided with abrupt changes in the nominal interest rate and inflation (the bottom left panel). The nominal interest rate effectively hit the zero lower bound between 2009 and 2015, and the economy slipped into deflation between 2009 and 2010. In the mean time, the average nominal wage continued to grow at the pre-recession trend, while the average real wage actually increased between 2009 and 2010 due to deflation (the top right panel). Therefore, as in the case of Japan, a prominent feature of the Great Recession is the combination of falling asset prices, nominal interest rates near the zero lower bound, disinflation, and rigid wages.

Explaining these striking features of persistent recessions and unemployment in the wake of bursting bubbles is an open problem for the general equilibrium bubble literature. The literature has been largely silent about the interaction of bubbles with frictions in the labor market.[1] Moreover, many models predict a relatively benign economic transition after the collapse of bubbles: a standard prediction is that while bubbles give rise to economic booms, their collapse simply precedes a gradual reversion to the pre-bubble trend while the economy retains full employment (e.g., Hirano and Yanagawa, 2017).

This paper attempts to address this problem. We embed downward nominal wage rigidity

[1]For surveys of recent developments in the bubble literature, see Barlevy (2012), Miao (2014) and Brunnermeier (2016).

(à-la Schmitt-Grohé and Uribe, 2016) into a rational bubbles framework with infinite-lived agents (à-la Hirano et al., 2015 and Hirano and Yanagawa, 2017). In this framework, a rational bubble is an asset that is traded above its fundamental value. The trading in the bubble market facilitates the reallocation of resources across time, as the bubble asset can act as a savings vehicle, and across agents, as the bubble asset increases entrepreneurs net worth and hence their ability to borrow. Downward wage rigidity has been well documented (see, e.g., Kimura and Ueda, 2001 for Japan, Holden and Wulfsberg, 2009 for the OECD, Babecký et al., 2010 for European economies, and Daly et al., 2012 for the U.S.).

Given this context, we first show that under sufficient financial frictions an asset price bubble crowds in credit and investment, as in a standard expansionary bubble model. This is because the trading of the bubbly asset helps alleviate the frictions that prevent the allocation of resources from less productive entrepreneurs to more productive ones. The boom phase of the bubble is associated with increases in the capital stock, output, consumption, and most importantly, increases in wages.

Then, we show that the presence of downward wage rigidity leads to drastically different post-bubble dynamics. When the expansionary bubble collapses, entrepreneurs' net worth also collapses, leading to contractions in credit and investment. Thus, the demand for labor from firms also contracts. In a flexible labor market, wages will fall to clear the labor market. However, when wages cannot flexibly fall, there is rationing in the labor market, i.e., involuntary unemployment.

An increase in unemployment can in turn lead to an endogenous and protracted recession by eroding the intertemporal allocation of resources. This is because the drop in employment reduces the return to capital investment, which then lowers entrepreneurs' net worth. This leads to a contraction in capital investment, since entrepreneurs' ability to borrow and invest depends critically on their net worth. Therefore, the future capital stock will decline causing further downward pressure on labor demand and wages, thus reducing future capital accumulation. The vicious cycle repeats and only stops when the capital stock has fallen enough. Then the speed of capital decumulation slows, and eventually the declining rigid wage constraint falls below the wage level consistent with full employment. At that point the economy exits the unemployment spell and enters a process of gradual recovery towards the bubble-less steady state.

Our model allows for analytical characterizations of the depth and duration of the recession, facilitating policy analyses. It highlights an important tradeoff between the economic gains during the boom due to the bubble and the (potentially deep and persistent) losses from the bust. We show that if the bubble is sufficiently risky and there are sufficient labor market frictions then bubbles become welfare-reducing. Policy interventions are thus war-

ranted. We show that an expansionary monetary policy after the collapse of the bubble can help alleviate the post-bubble losses. Additionally, we show that if monetary authorities are constrained from raising inflation, then a macroprudential policy that imposes ex-ante taxation on bubble speculation can help mitigate the impacts of the bust, although the policy also weakens the boom.

We further extend the model with cash holding to allow for the possibility of a liquidity trap. When entrepreneurs can save by holding cash, the nominal interest rate is bounded from below by the zero lower bound. We then show that the collapse of a large expansionary bubble triggers a sharp drop in the real interest rate, pushing the nominal interest rate against the lower bound. The intuition is as follows. By crowding in capital investment, the bubble leads to an investment boom. Thus, after the bubble collapses, the economy enters the post-bubble phase with a capital stock above the steady state, a situation we refer to as "investment hangover" (Rognlie et al., 2014). The high capital stock implies a low marginal product of capital and a low real interest rate. If the bubble is sufficiently large, then the real net interest rate in the period of the collapse can be negative. If inflation cannot increase above a certain threshold (such as an inflation target set by a monetary authority) — an assumption that we impose — then the zero lower bound on the nominal interest rate becomes binding.

When the nominal interest rate is bounded at zero, the inflation rate must rise to be consistent with a negative real interest rate. Then, as pointed out by Krugman (1998) and Eggertsson and Krugman (2012), if future price levels are fixed, then the rise in inflation must be due to a drop in the current price level. In other words, the collapse of the bubble causes a deflationary pressure. The deflationary pressure in turn exacerbates the downward nominal wage rigidity and exacerbates involuntary unemployment. Furthermore, as capital and labor are complementary, the reduction in employment reduces the marginal product of capital and consequently the interest rate. Therefore there is a bidirectional relationship between the binding zero lower bound and the binding nominal wage rigidity.

In summary, our model shows how the collapse of bubbles can lead to an endogenous and protracted recession with involuntary unemployment, and even a liquidity trap. During the recession, aggregate economic activities are persistently below the pre-bubble trend and interest rates are depressed, consistent with the stylized features of recent bubble boom-bust episodes.

Related literature. To the best of our knowledge, our paper is the first to show that the collapse of bubbles can trigger long recessions and liquidity traps. Our paper thus makes contributions to several strands of the literature.

First, we help formalize the popular notion among policymakers that the collapse of risky bubbles can trigger inefficient recessions. A large number of papers emphasize the positive aspect of (rational) bubbles in reducing dynamic inefficiencies (e.g., Samuelson, 1958, Diamond, 1965 and Tirole, 1985) or reducing intratemporal inefficiencies in the allocation of resources (e.g., Farhi and Tirole, 2011, Miao and Wang, 2011, 2012, Martin and Ventura, 2012, Ikeda and Phan, 2015, Bengui and Phan, 2016 and Graczyk and Phan, 2016).[2] Other papers emphasize potential ex-ante inefficiencies of speculative bubble investment in diverting resources away from productive investment (e.g., Saint-Paul, 1992, Grossman and Yanagawa, 1993, King and Ferguson, 1993, and Hirano et al., 2015)[3] or generates excessive volatility (Caballero and Krishnamurthy, 2006 and Ikeda and Phan, 2016). Our paper complements this literature and highlights the ex-post inefficiency of bubbles by showing that their collapse can cause persistent involuntary unemployment. As a consequence, our paper formalizes the policy-relevant tradeoff between the gains during a bubble's boom and the losses during the bubble's bust.

Furthermore, our paper is one of the first to embed downward wage rigidity into a rational bubbles framework. To the best of our knowledge, the only other paper that does this is our earlier work, Hanson and Phan (2017).[4] There, we developed a simple overlapping generations model based on the classic frameworks of Tirole (1985). A major limitation of the overlapping generations model is that a period represents twenty or thirty years. This makes the model less appropriate for policy analyses at the business cycles frequency. In contrast, in the current paper, agents are fully forward-looking and infinitely-lived and a period can be interpreted as a quarter or a year. More importantly, while the earlier work is limited to an expositional positive analysis, in this paper we derive an explicit welfare function in closed form and conduct policy analyses.[5]

Second, a large literature investigates possible sources of shocks that trigger long recessions and liquidity traps in environments with New Keynesian frictions. Many papers have emphasized demand shocks driven by household deleveraging or tightening borrowing constraints (Krugman, 1998, Eggertsson and Krugman, 2012, Christiano et al., 2015, Schmitt-Grohé and Uribe, 2016), long-run factors such as aging demographics or safe asset

[2]Besides the rational bubble literatures, see Abreu and Brunnermeier (2003), Doblas-Madrid and Lansing (2014) and Barlevy (2014) for alternative approaches to modeling bubbles based on heterogeneous information or beliefs.

[3]Also see Miao et al. (2014), where collateralizable housing bubbles can excessively crowd in capital investment.

[4]For a complementary approach to modeling post-bubble unemployment using a search and matching model à-la Diamond-Mortensen-Pissarides, see Kocherlakota (2011) and Miao et al. (2016).

[5]For a related and emerging body of literature that analyzes the effects of monetary policies on rational bubbles, see Gali (2014, 2016), Asriyan et al. (2016), Ikeda (2016), and Dong et al. (2017).

shortages (Summers, 2013, Caballero and Farhi, 2014, Eggertsson and Mehrotra, 2014, Eggertsson et al., 2016), or over-investment of capital (Rognlie et al., 2014). By highlighting the role of rational asset bubbles, our analysis offers a complementary narrative to those in the literature. In our model, the collapse of bubbles reduces borrowers' net worth and thus leads to an *endogenous* tightening of borrowing constraints in equilibrium. Similarly, in our model, expansionary bubbles lead to an endogenous boom in capital investment, thus giving a microfoundation to the investment overhang in Rognlie et al. (2014).

Third, we conduct a normative analysis with macroprudential policies on speculative bubble investment. This analysis complements a recent literature on macroprudential policies in environments with aggregate demand externalities (e.g., Farhi and Werning, 2016 and Korinek and Simsek, 2016) or environments with financial frictions (e.g., Lorenzoni, 2008, Olivier and Korinek, 2010, He and Krishnamurthy, 2011, Bianchi, 2011, Eberly and Krishnamurthy, 2014, and Bianchi and Mendoza, forthcoming).

The plan for the paper is as follows. Section 2 describes the model. Section 3 describes the equilibrium dynamics and steady states. 4 conducts welfare and policy analyses. Section 5 introduces the zero lower bound. Section 6 concludes. Detailed derivations and proofs are in the appendix.

2 Model

Consider an economy with two types of good: a perishable consumption good and a capital good, and three types of agents: entrepreneurs, workers, and firms, each with constant unit population. Entrepreneurs and workers have the same preferences over consumption, given by

$$E_0\left(\sum_{t=0}^{\infty}\beta^t \ln c_t^j\right)$$

where c_t^j is the consumption of an individual j in period t, $\beta \in (0,1)$ is the subjective discount factor, and $E_0(\cdot)$ is the expected value conditional on information in period 0.

2.1 Entrepreneurs

Entrepreneurs are the only producers of the capital good, and face idiosyncratic productivity shocks. In each period, an entrepreneur meets a high-productivity investment project (and becomes the *H-type*) with probability $h \in (0,1)$, and a low-productivity one (and becomes the *L-type*) with probability $1-h$. The idiosyncratic productivity shock is independent across agents and time. For stationarity, we assume that the initial ($t = 0$) population measure of

each type is h and $1-h$. In each period, we denote the set of H-type entrepreneurs by $\mathcal{H}_t$ and the set of L-type entrepreneurs by $\bar{\mathcal{H}}_t$, where $\mathcal{H}_t \cup \bar{\mathcal{H}}_t = [0,1]$.

After knowing the type of her investment project at the beginning of each period, an entrepreneur j produces the capital good according to the following technology:

$$k_{t+1}^j = a_t^j i_t^j,$$

where $i_t^j \geq 0$ is the investment in units of the consumption good in period t, k_{t+1}^j is the amount of the capital good produced in the subsequent period, and $a_t^j \in \{a^H, a^L\}$ is the productivity of the project, where $a^H > a^L > 0$. For tractability, we assume capital depreciates completely after each period.[6]

Financial frictions: In a frictionless world, L-type entrepreneurs would like to lend and thus delegate investment to H-type entrepreneurs. However, following Kiyotaki et al. (1997), we assume there are frictions in the financial market so entrepreneurs can pledge at most an exogenous fraction $\theta \in [0,1]$ of the future return from investment to creditors. Thus, they face the following credit constraint:

$$R_{t+1} d_t^j \leq \theta q_{t+1} k_{t+1}^j, \tag{1}$$

where R_{t+1} is the state-contingent gross interest rate between t and $t+1$, d_t^i is the amount borrowed in period t, and q_{t+1} is the price of capital (in units of the consumption good) in period $t+1$. A lower θ represents a financial market with more frictions, while $\theta = 1$ represents a frictionless credit market. Throughout the paper we assume θ sufficiently small so that constraint (1) always binds for H-types.

Following the literature (e.g., Tirole, 1985), we introduce (pure) asset bubbles, which are durable and perfectly divisible assets in fixed unit supply that do not generate any dividend, but can be traded at positive equilibrium prices under some conditions. Such bubbles are inherently fragile as they require coordination of beliefs across agents and time. To model this fragility, we follow the literature (e.g., Weil, 1987) and assume that in each period the bubble persists with a probability $\rho \in (0,1)$ and collapses with the complementary probability $1-\rho$, where a lower ρ means a riskier bubble. Formally, let $\tilde{p}_t^b$ denote the period t price per unit of the bubble asset in units of the consumption good, and p_t^b denote the price conditional on

[6] Our result does not change qualitatively if capital depreciates more slowly.

the bubble persisting in t. Then

$$\tilde{p}_t^b = \begin{cases} p_t^b & \text{if bubble persists} \\ 0 & \text{if bubble bursts} \end{cases},$$

and

$$\Pr(\tilde{p}_{t+1}^b = 0 | \tilde{p}_t^b > 0) = 1 - \rho$$
$$\Pr(\tilde{p}_{t+1}^b = 0 | \tilde{p}_t^b = 0) = 1, \forall t \geq 0.$$

The first assumption states that if the bubble has not collapsed, then it will collapse in the next period with probability $1 - \rho$. The second states that if the bubble has collapsed, then it is expected not to re-emerge.

Macroprudential policy: We assume the government can set a (constant) tax rate τ on bubble speculation. For simplicity, we assume the government transfers the revenue $T_t = \tau \tilde{p}_t^b$ from bubble tax in lump-sums to workers.[7]

Let b_t^j denote a share of a bubble asset held by entrepreneur j. Then the entrepreneur's flow budget constraint is written as

$$c_t^j + i_t^j + (1 + \tau)\tilde{p}_t^b b_t^j = q_t k_t^j + d_t^j - R_t d_{t-1}^j + \tilde{p}_t^b b_{t-1}^j. \tag{2}$$

The left hand side of this budget constraint consists of expenditure on consumption, investment, and the purchase of bubble assets. The right hand side is the available funds at date t, which consists of the return from investment in the previous period, new borrowing minus the debt repayment, and the return from selling bubble assets. We assume agents cannot short sell the bubble asset, i.e.,

$$b_t^j \geq 0, \forall t.$$

[7]Redistribution of tax revenue to entrepreneurs would affect their net worth and hence complicate the inter-temporal equilibrium dynamics. In contrast, redistribution to workers would not affect inter-temporal dynamics, as workers are hand-to-mouth.

2.2 Workers

Workers do not have access to capital production technologies. Without loss of generality, we assume workers are hand-to-mouth,[8] i.e.,

$$c_t^w = w_t l_t + T_t, \tag{3}$$

where w_t is the wage rate, l_t is the employment level per worker, and T_t is the lump-sum transfer from the government.

2.3 Firms

In each period, there is a continuum of competitive firms that produce the consumption good using the standard production function:

$$y_t^i = (k_t^i)^\alpha (l_t^i)^{1-\alpha},\ 0 < \alpha < 1,$$

where k_t^i and l_t^i are capital and labor inputs of a representative firm i. For simplicity, we have abstracted away from exogenous TFP shocks. Real competitive factor prices are given by:

$$q_t = \alpha \left(\frac{L_t}{K_t}\right)^{1-\alpha} \tag{4}$$

$$w_t = (1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha}, \tag{5}$$

where K_t and L_t are the aggregate capital stock and employment.

Downward wage rigidity: The last and a very important element is labor market friction. Following Schmitt-Grohé and Uribe (2016), we assume that nominal wages are downwardly rigid:

$$P_{t+1} w_{t+1} \geq \gamma_n P_t w_t, \forall t \geq 0$$

[8] Alternatively, we can assume workers cannot borrow against their future labor income. Thus the optimization problem of workers is to maximize lifetime utility $E_0 \left(\sum_{t=0}^{\infty} \beta^t \ln c_t^w\right)$ subject to:

$$c_t^w + p_t^b b_t^w = w_t l_t + d_t^w - R_t d_{t-1}^w + p_t^b b_{t-1}^w + T_t$$

and $d_t^w \leq 0$ and $b_t^w \geq 0$. In equilibrium, it is straightforward to show that workers will be effectively hand to mouth, i.e., $c_t^w = w_t l_t + T_t$. Intuitively, due to financial friction, the interest rate (and the returns from bubble speculation) will be too low relative to the discount factor, and thereby it will be suboptimal for workers to save or to buy the bubbly asset.

where $\gamma_n \geq 0$ governs the degree of rigidity ($\gamma_n = 0$ implies full flexibility). Equivalently:

$$w_{t+1} \geq \frac{\gamma_n}{\Pi_{t+1}} w_t, \forall t \geq 0, \tag{6}$$

where $\Pi_{t+1} \equiv \frac{P_{t+1}}{P_t}$ is the gross inflation rate between t and $t+1$. Downward wage rigidity has been well documented (see, inter alia, Kimura and Ueda, 2001, Holden and Wulfsberg, 2009, Babecký et al., 2010, Daly et al., 2012). The presence of rigid wages implies that *the labor market does not necessarily clear*. In each period, even though each worker inelastically supplies one unit of labor, the actual employment L_t per worker in equilibrium is determined by two conditions: feasibility constraint

$$L_t \leq 1, \tag{7}$$

and complementary-slackness condition

$$(1 - L_t)(w_t - \frac{\gamma_n}{\Pi_t} w_{t-1}) = 0. \tag{8}$$

These equations state that involuntary unemployment ($L_t < 1$) must be accompanied by a binding rigidity (6). Conversely, when (6) does not bind, the economy must be in full employment ($L_t = 1$).

2.4 Monetary policy

To close the model, we need to describe how prices are determined. We specify monetary policy in the simplest possible way by following Krugman (1998) and assuming that a monetary authority sets inflation at an exogenous constant target $\bar{\Pi} \geq 1$. This could be rationalized either via cash-in-advance constraint or via nominal interest rate rule. Thus, the wage rigidity is effectively real and can be rewritten as:

$$w_{t+1} \geq \gamma w_t, \forall t \geq 0 \tag{9}$$

where

$$\gamma \equiv \frac{\gamma_n}{\bar{\Pi}}.$$

We focus on the case where $\gamma \leq 1$, so that the wage rigidity constraint does not bind in steady state.

2.5 Equilibrium

Definition. Given τ, $k_0^j = K_0$, $d_0^j = 0$, $b_0^j = 1$, p_0^b, a competitive equilibrium consists of prices $\{w_t, q_t, R_{t+1}, p_t^b\}_{t\geq 0}$ and quantities $\{\{i_t^j, k_{t+1}^j, c_t^j\}_{j\in\mathcal{H}_t\cup\bar{\mathcal{H}}_t}, c_t^w, K_{t+1}, L_t\}_{t\geq 0}$ such that:

- Entrepreneurs and firms optimize,
- Workers' consumption is given by (3),
- Credit market clears: $\int_{j\in\mathcal{H}_t} d_t^j + \int_{j\in\bar{\mathcal{H}}_t} d_t^j = 0$,
- Bubble market clears: $\int_{j\in\mathcal{H}_t} b_t^j + \int_{j\in\bar{\mathcal{H}}_t} b_t^j = 1$ if $\tilde{p}_t^b > 0$,
- Goods market clears: $\int_{j\in\mathcal{H}_t}(c_t^j + i_t^j) + \int_{j\in\bar{\mathcal{H}}_t}(c_t^j + i_t^j) + c_t^w = K_t^\alpha L_t^{1-\alpha}$,
- And labor market conditions: (9) and

$$L_t \leq 1$$
$$(1 - L_t)(w_t - \gamma w_{t-1}) = 0.$$

As usual, a steady state is an equilibrium where quantities and prices are time-invariant.

3 Equilibrium dynamics

3.1 Bubble-less equilibrium

As standard in the rational bubble literature, there are multiple equilibria. Let us first characterize the bubble-less equilibrium, where the price of the bubble asset is equal to its fundamental value of zero throughout. Detailed derivations are delegated to the appendix.

Throughout the paper, we assume the initial capital stock K_0 is small so that the capital stock (and hence wage) will grow towards the steady state, and thus with $\gamma \leq 1$, the downward wage rigidity constraint will *never* bind. Therefore, in this section we can set:

$$L_t = 1, \forall t.$$

With full employment, the price of capital from (4) becomes:

$$q_{t+1} = \alpha K_{t+1}^{\alpha-1}$$

and the price of labor (the real wage) from (5) becomes:

$$w_t = (1-\alpha)K_t^{\alpha}.$$

Furthermore, throughout the paper, we make the following parametric assumption:

$$\theta < \frac{(1-h)a^L}{a^H}. \tag{10}$$

This assumption states that there is sufficient financial friction (small θ) that the credit market cannot completely absorb the L-type's demand for savings. Hence, in the bubble-less equilibrium, the L-type is making a positive capital investment (the non-negative constraint $i_t^j \geq 0$ does not bind for the L-type):

$$i_t^j = \beta e_t^j + d_t^j, \forall j \in \bar{\mathcal{H}}_t,$$

where the net worth e_t^j is:

$$e_t^j \equiv q_t k_t^j - R_{t-1} d_{t-1}^j$$

and the equilibrium interest rate will be given by:

$$R_{t+1} = q_{t+1} a^L.$$

Turning to the H-type, their credit constraint (1) will bind, leading to the following investment equation:

$$i_t^j = \underbrace{\frac{1}{1-\frac{\theta q_{t+1} a^H}{R}}}_{\text{leverage}} \times \beta e_t^j, \forall j \in \mathcal{H}_t.$$

Combining the investment expressions above for both types yields the following law of motion for the aggregate capital stock:

$$K_{t+1} = \Omega q_t K_t,$$

where

$$\Omega \equiv \left(\frac{h(a^H - a^L)}{1-\frac{\theta a^H}{a^L}} + a^L \right) \beta.$$

Given the equilibrium dynamics above, the bubble-less steady state (denoted with the subscript nb, which stands for "no bubble") is characterized by the following interest rate

and factor prices:

$$R_{nb} = q_{nb}a^L = \frac{a^L}{\Omega} \tag{11}$$
$$q_{nb} = \alpha K_{nb}^{\alpha-1}$$
$$w_{nb} = (1-\alpha)K_{nb}^{\alpha}$$

and the following capital stock and employment:

$$K_{nb} = (\alpha\Omega)^{\frac{1}{1-\alpha}} \tag{12}$$
$$L_{nb} = 1.$$

3.2 Bubble equilibrium

We now summarize the transition dynamics and the steady state of a bubble equilibrium, with detailed derivations delegated to the appendix. As the bubble is stochastic, the dynamics will consist of that before the bubble bursts and that after the burst.

3.2.1 Before bubble bursts

We focus on equilibria where the downward wage rigidity does not bind as long as the bubble persists (i.e., $L_t = 1$ if $\tilde{p}_t^b > 0$). This supposition is verified as long as the initial capital stock K_0 and the initial bubble price p_0^b are small.

Suppose the bubble persists in t, i.e., $\tilde{p}_t^b = p_t^b > 0$. As L-type entrepreneurs face a non-negativity constraint on capital investment ($i_t^j \geq 0$), it follows that the return from lending must weakly dominates the return from capital investment:

$$R_{t+1} \geq q_{t+1}a^L,$$

where the inequality must hold with equality if $i_t^j > 0$.

Furthermore, let the bubble size (relative to aggregate savings) be defined as:

$$\phi_t \equiv \frac{p_t^b}{\beta(q_t K_t + p_t^b)}.$$

Then from the no-arbitrage condition for the L-type between bubble investment and lending,

the bubble size evolves according to:

$$\phi_{t+1} = \begin{cases} \frac{1}{\beta} \frac{\frac{1-h-(1+\tau)\phi_t}{\rho(1-h)(1+\tau)-\phi_t}(1+\tau)\phi_t}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)+\frac{(1-\rho)(1-h)}{\rho(1-h)-(1+\tau)\phi_t}(1+\tau)\phi_t} & \text{if } \phi_t \leq \phi^* \text{(small bubble)} \\ \frac{\theta}{\beta} \frac{(1+\tau)\phi_t}{\rho(1-h)-(1-\theta)(1+\tau)\phi_t} & \text{if } \phi_t > \phi^* \text{(large bubble)} \end{cases}, \tag{13}$$

and combined with market clearing conditions, the interest rate is given by:

$$R_{t+1} = \begin{cases} q_{t+1}a^L & \text{if } \phi_t \leq \phi^* \\ q_{t+1}\frac{\theta a^H(1-(1+\tau)\phi_t)}{1-h-(1+\tau)\phi_t} & \text{if } \phi_t > \phi^* \end{cases}. \tag{14}$$

where threshold ϕ^* is defined as:

$$\phi^* \equiv \frac{(1-h)a^L - \theta a^H}{(1+\tau)(a^L - \theta a^H)}.$$

Above this threshold, the bubble is "large," and below it, the bubble is "small." When $\phi_t \leq \phi^*$, the bubble is small in the sense that it cannot completely crowd out the L-type's (relatively inefficient) investment in capital. When this is the case, the interest rate is given by the indifference condition for the L-type between lending and capital investment. However, when $\phi_t > \phi^*$, the bubble is large in the sense that it completely absorbs and crowds out the L-type's investment in capital (the Lagrange multiplier on the constraint $i_t^t \geq 0$ is strictly positive for L-types). When this is the case, the bubble raises the interest rate, making the L-type strictly prefer lending to capital investment ($R_{t+1} > q_{t+1}a^L$).

Similar to the bubble-less analysis, by using the credit market clearing condition, the binding credit constraint for the H-type, and the budget constraint, we can derive the following transition dynamics for the aggregate capital stock:

$$K_{t+1} = \begin{cases} \Omega(q_tK_t + p_t^b) - a^L(1+\tau)p_t^b & \textit{if } R_t = q_{t+1}a^L \\ a^H\beta(q_tK_t + p_t^b) - a^H(1+\tau)p_t^b & \textit{if } R_t > q_{t+1}a^L \end{cases}.$$

The expressions above take into account the fact that some of the entrepreneurs' resources will be invested into the bubble asset (the terms involving $(1+\tau)p_t^b$). This is known as the "crowd-out" effect of bubbles on capital accumulation. In the mean time, the expressions also show how the return from bubble speculation raises entrepreneurs' aggregate net worth from q_tK_t to $q_tK_t + p_t^b$. This is known as the "crowd-in" effect of bubbles. Combined with the expressions for the bubble size and the interest rate, the law of motion of the aggregate

capital stock can be rewritten as:

$$K_{t+1} = \begin{cases} \frac{\left(1+\frac{(a^H-a^L)}{a^L-\theta a^H}h\right)\beta a^L - a^L\beta(1+\tau)\phi_t}{1-\beta\phi_t}\alpha K_t^\alpha & \text{if } \phi_t \leq \phi^* \\ \frac{a^H\beta[1-(1+\tau)\phi_t]}{1-\beta\phi_t}\alpha K_t^\alpha & \text{if } \phi_t > \phi^* \end{cases}. \tag{15}$$

From (13, 14, 15), we can derive the following expressions for the bubble steady state (as functions of the bubble tax τ):

$$\phi_b = \begin{cases} \phi_{sb} \equiv \frac{\rho - \frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}}{1-\frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}}\frac{1-h}{1+\tau} & \text{if } \tau \geq \bar{\tau} \text{ (small bubble)} \\ \phi_{lb} \equiv \frac{\rho(1-h)}{(1-\theta)(1+\tau)} - \frac{\theta}{\beta(1-\theta)} & \text{if } \tau < \bar{\tau} \text{ (large bubble)} \end{cases}$$

$$K_b = \begin{cases} K_{sb} \equiv \left(\frac{\left(1+\frac{a^H-a^L}{a^L-\theta a^H}\right)h}{1+\tau-\beta\rho(1-h)+\tau\frac{(1-h)(1-\rho)\beta}{\left(1+\frac{(a^H-a^L)}{a^L-\theta a^H}h\right)\beta-1}}\beta a^L\alpha\right)^{\frac{1}{1-\alpha}} & \text{if } \tau \geq \bar{\tau} \\ K_{lb} \equiv \left((1+\tau)\frac{\beta[1-\rho(1-h)]+(1+\tau-\beta)\theta}{1+\tau-\beta\rho(1-h)}a^H\alpha\right)^{\frac{1}{1-\alpha}} & \text{if } \tau < \bar{\tau} \end{cases} \tag{16}$$

$$R_b = \begin{cases} R_{sb} \equiv \left(\frac{\left(1+\frac{a^H-a^L}{a^L-\theta a^H}\right)h}{1+\tau-\beta\rho(1-h)+\tau\frac{(1-h)(1-\rho)\beta}{\left(1+\frac{(a^H-a^L)}{a^L-\theta a^H}h\right)\beta-1}}\beta a^L\alpha\right)^{\frac{1}{1-\alpha}} & \text{if } \tau \geq \bar{\tau} \\ R_{lb} \equiv \left((1+\tau)\frac{\beta[1-\rho(1-h)]+(1+\tau-\beta)\theta}{1+\tau-\beta\rho(1-h)}a^H\alpha\right)^{\frac{1}{1-\alpha}} & \text{if } \tau < \bar{\tau} \end{cases}$$

where the tax threshold is:

$$\bar{\tau} = \max\left\{0,\ \hat{\tau} \equiv \frac{\beta[\rho(1-h)-(1-\theta)]}{\theta} - 1 + \frac{\beta(1-\theta)ha^L}{\theta(a^L-\theta a^H)}\right\} \tag{17}$$

($\hat{\tau}$ is the solution to $\phi(\hat{\tau}) = \phi^*$). If the tax is above this threshold, the bubble will be small, and when it is below, the bubble will be large. Furthermore, the macroprudential policy can have a direct effect on the size of the bubble as well as the bubble steady state capital stock: an increase in τ reduces ϕ_b and K_b.

Even though we still have to solve for the equilibrium dynamics after the bubble collapses (and this will be done in the next section), from the analysis above we can already characterize the existence of the bubble steady state. The proposition below also shows how the macroprudential policy can affect the bubble existence conditions: a higher τ restricts the parameter space in which a bubble steady state exists.

Proposition 1. *A bubble steady state exists if and only if there is sufficient financial friction:*

$$\theta < \frac{\beta\rho(1-h)}{1+\tau},$$

and the bubble is not too risky (the persistent probability is sufficient):

$$\rho > \frac{a^L - \theta a^H}{\beta(a^L - \theta a^H) + \beta h(a^H - a^L)}.$$

Proof. Appendix. □

3.2.2 After the bubble bursts

For the rest of the paper, we focus on parameters such that bubbles are expansionary, i.e.,

$$K_b > K_{nb}$$

where K_b and K_{nb} are given by (16) and (12), respectively.

Suppose the bubble collapses at T (i.e., $\tilde{p}^b_{T+s} = 0,\ \forall s \geq 0$). As we consider expansionary bubbles, the post-bubble capital stock and wage will decline towards the bubble-less SS levels. However, if the downward wage rigidity constraint binds, then wage cannot flexibly fall to clear the labor market. Instead, employment is determined by the demand of firms. The rigidly high wage thus leads to involuntary unemployment. The contraction in employment has two effects on the inter-temporal equilibrium dynamics: it reduces the return from capital and it reduces entrepreneurs' net worth. Both of these effects in turn reduce entrepreneurs' accumulation of capital. The wage rigidity thus amplifies and propagates the shock of bursting bubbles.

Given the tractability of our model, we can completely characterize the post-bubble dynamics, including the depth and duration of the post-bubble unemployment episode. Let

$$s^* \equiv \min\{s \geq 0 | L_{T+s} = 1\},$$

then $T + s^*$ is the first post-bubble period when full employment is recovered. If $s^* > 0$, then we say the economy is in a *slump* (i.e., a recession with involuntary unemployment) between T and $T + s^* - 1$.

The combination of the binding wage rigidity and the labor demand curve determines employment as:

$$L_{T+s} = \left[\frac{1-\alpha}{w_{T+s}}\right]^{\frac{1}{\alpha}} K_{T+s} < 1, \forall 0 < s < s^*. \tag{18}$$

(Note that the wage rigidity does not bind right away at T, because capital at T is predetermined.) Based on this equality, the post-bubble dynamics can be characterized as follows:

Proposition 2. [Post-bubble slump] *If the bubble collapses in period T, then given the capital stock K_T in that period,*

1. *The duration of the post-bubble slump is:*

$$s^*(K_T) = \begin{cases} 0 & \text{if } \gamma = 0 \\ \max\left\{0,\ \lceil \omega(\gamma) - 2\alpha \log_\gamma K_T \rceil\right\} & \text{if } \gamma \in (0,1) \\ \infty & \text{if } \gamma = 1 \end{cases} \tag{19}$$

where the ceiling function $\lceil x \rceil$ denotes the least integer greater than or equal to x

$$\omega(\gamma) \equiv \frac{2\alpha}{1-\alpha} \log_\gamma(\alpha\Omega) - \frac{3-\alpha}{1-\alpha}.$$

2. *During the slump ($0 \leq s < s^*$), the equilibrium dynamics can be summarized by:*

$$\begin{aligned} w_{T+s} &= \gamma^s w_T \\ q_{T+s} &= \alpha \left(\frac{1-\alpha}{w_{T+s}}\right)^{\frac{1-\alpha}{\alpha}} \\ R_{T+s} &= a^L q_{T+s} \\ K_{T+s+1} &= \alpha\Omega \left(\frac{w_{T+s}}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}} K_{T+s} \\ L_{T+s} &= \gamma^{-\frac{s}{\alpha}} \frac{K_{T+s}}{K_T} < 1. \end{aligned} \tag{20}$$

After the slump ($s \geq s^*$), the dynamics can be summarized by:

$$\begin{aligned} w_{T+s} &= (1-\alpha) K_{T+s}^{\alpha} \\ q_{T+s} &= \alpha K_{t+s}^{\alpha-1} \\ R_{T+s} &= a^L q_{T+s} \\ K_{T+s+1} &= \alpha\Omega K_{T+s}^{\alpha} \\ L_{T+s} &= 1. \end{aligned}$$

Proof. Appendix. □

Interestingly, the proposition implies that *a larger boom is associated with a longer and deeper bust.* Specifically, the expression (19) for the slump length and (20) for capital stock during the slump show that a larger capital stock K_T is associated with a longer slump and more depressed capital accumulation during the slump. We will analyze this tradeoff in the policy analysis.

Figure 3 illustrates the equilibrium dynamics of aggregate variables. In the simulations, we assume that the economy starts in bubble-less steady state. Then the bubble (unanticipatedly) arises in period 20. Then the economy reaches the bubble steady state. Then the bubble bursts in period 100. The parameters are $\alpha = 0.36$, $a^H = 1.5$, $a^L = 1$, $\beta = 0.96$, $h = 0.35$, $\theta = 0.1$ and $\rho = 0.98$ and $\tau = 0$. Under these parameter values, we have a large expansionary bubble.

With $\gamma = 0$ (the blue dashed lines), the labor market is flexible and thus the post-bubble economy simply converges back to the bubble-less steady state.

However, with a $\gamma = 0.985$ (the red solid lines),[9] the downward wage rigidity constraint binds after the bubble bursts. Hence, the collapse of the bubble pushes the economy into a recession with involuntary unemployment (the recession for the baseline simulation is highlighted by the grey bar), as wage cannot flexibly fall, causing rationing in the labor market. The drop in employment not only reduces the economy's production, but also have important inter-temporal effects. On the one hand, it reduces the net worth of entrepreneurs. On the other hand, it reduces the return rate on capital. Both of these effects depress capital accumulation. This process explains the contractions of aggregate economic activities during the slump.

Remark 1. An interesting feature in figure 3 is that the collapse of a large expansionary bubble can lead to a sharp drop in the real interest rate. There are two mechanisms behind this feature. First, after the large bubble collapses, the marginal producer of the capital good switches from the H-type to the L-type, and thus instead of the identity $R_{T+1} = \frac{\theta a^H(1-(1+\tau)\phi)}{1-h-(1+\tau)\phi} q_{T+1}$ that could have prevailed if the bubble did not collapse in T, the interest would be $R_{T+1} = a^L q_{T+1}$, where $a^L < \frac{\theta a^H(1-(1+\tau)\phi)}{1-h-(1+\tau)\phi}$, as in the world without bubbles. This explains why the real interest rate drop relative to that in the bubble steady state. Second, as the bubble is expansionary, the post-bubble economy begins at an aggregate net worth that is higher than that in the bubble-less steady state. If the expansion is sufficient, then the capital stock K_{T+1} will exceed the bubble-less stock K_{nb}, leading to a marginal product of capital $q_{T+1} = \alpha K_{T+1}^{\alpha-1}$ that is smaller than $q_{nb} = \alpha K_{nb}^{\alpha-1}$ in the bubble-less steady state. The combined effects of the two mechanism can lead to an "overshooting" of the real interest rate: $R_{T+1} < R_{nb}$, as seen in the figure. This overshooting could potentially push the economy

[9] Schmitt-Grohé and Uribe (2016) also use values close to one.

into a binding zero lower bound on the nominal interest rate, as we later formalize in section 5.

4 Welfare and policy analyses

Welfare-reducing bubbles

Our model highlights an important policy tradeoff: the gain (in output, investment and consumption) from the bubble episode on the one hand, and the loss when the bubble collapses on the other. To analyze this tradeoff, we need to define a welfare objective.

For simplicity, we assume that the policymaker chooses the bubble tax to maximize the lifetime expected utility of workers in steady state.[10] As shown below, the lifetime utility of workers in the bubble steady state can be decomposed into the utility during the bubble episode, in the period the bubble collapses, during the slump, and after the slump. Thanks to the tractability of our model, each component can be solved analytically.

Proposition 3. [Welfare functions]

1. *The lifetime expected utility of workers in the bubble-less steady state is $W_{nb}(K_{nb})$, where*

$$W_{nb}(K) \equiv \Gamma_2 + \frac{\alpha}{1-\beta\alpha}\log K,$$

and

$$\Gamma_2 \equiv \frac{1}{1-\beta}\log(1-\alpha) + \frac{1}{1-\beta}\frac{\beta\alpha}{1-\beta\alpha}\log(\alpha\Omega).$$

2. *The lifetime expected utility of workers in the stochastic bubble steady state is*

$$W_b(K_b) = \frac{\log c_b^w + \beta(1-\rho)W_{burst}(K_b)}{1-\beta\rho}$$

where the bubble steady state worker's consumption (with transfer) is:

$$c_b^w = (1-\alpha)(K_b)^\alpha + T = \left[(1-\alpha) + \frac{\beta\phi_b}{1-\beta\phi_b}\alpha\tau\right](K_b)^\alpha$$

[10] This assumption is motivated simply by the fact that it is much easier to characterize in closed form the welfare of workers than that of entrepreneurs.

and:

$$W_{burst}\left(K_b\right) \equiv \underbrace{\log\left[(1-\alpha)(K_b)^{\alpha}\right]}_{\textit{contemporaneous utility}} + \underbrace{\sum_{s=1}^{s^*-1} \beta^s \log c^w_{T+s}}_{\textit{slump utility}} + \underbrace{\beta^{s^*} W_{nb}\left(\gamma^{-\left(\frac{1-\alpha}{\alpha}\right)\frac{s^*(s^*+1)}{2}}\left[\alpha\Omega\cdot(K_b)^{\alpha-1}\right]^{s^*} K_b\right)}_{\textit{post-slump continuation value}}.$$

where

$$\log c^w_{T+s} = \Gamma_1(s) - ((1-\alpha)s-\alpha)\log K_b,\ \forall 1 \le s \le s^*-1$$
$$\Gamma_1(s) \equiv \frac{\alpha-1}{\alpha}\frac{s^2+3s}{2}\log\gamma + \log(1-\alpha) + s\log\left(\alpha\Omega\right).$$

Proof. Appendix. □

From the expression for W_b above, the welfare gain of bubbles can be seen by the fact that the consumption during the expansionary bubble episode is higher than that in the bubble-less steady state ($c^w_b > (1-\alpha)K^{\alpha}_b > c^w_{nb} = (1-\alpha)K^{\alpha}_{nb}$). The loss, however, comes from the depressed post-bubble consumption due to the slump.

If there is no wage rigidity ($\gamma = 0$), as in much of the literature (e.g., Hirano et al., 2015), then there is no welfare loss due to the slump, and thus no welfare loss due to the collapse of bubbles. From this perspective, (expansionary) bubbles improve welfare: they generate an increase in K_b and an economic boom, and when the boom eventually comes to a bust, the economy will simply converge back to the bubble-less steady state.

However, with binding wage rigidity, then the welfare loss due to the slump becomes relevant. The larger γ is, the more relatively important is the slump utility term in the welfare calculation. To see how this loss necessitates policy intervention, let us assume $\tau = 0$ (no bubble tax). Then the following proposition shows that when $\gamma = 1$ (i.e., the rigidity prevents the real wage from declining), and the bubble is sufficiently risky, then bubble will strictly reduce welfare:

Proposition 4. [Welfare reducing bubble] *Suppose $\tau = 0$ (no bubble tax) and the bubble is sufficiently risky ($1-\rho$ is sufficiently high):*

$$\beta(\beta-\alpha)(1-\rho) > \alpha\left(1-\beta\right)^2.$$

Then there exists $\bar{\gamma} < 1$ such that for all $\gamma > \bar{\gamma}$, the bubble reduces welfare in steady state, i.e.,

$$W_{nb} > W_b.$$

Proof. Appendix. □

Effect of monetary policy

Now we turn to how policies could help attenuate the negative welfare effects of bubbles. Given our reduced-form approach to modeling monetary policy, an immediate implication of the model is that if the monetary authority is willing to raise the inflation target $\bar{\Pi}$, then they can alleviate the post-bubble recession. That is, a rise in $\bar{\Pi}$ would reduce the effective wage rigidity parameter $\gamma = \frac{\gamma_n}{\bar{\Pi}}$.

Figure 4 illustrates this point. It plots three simulations: one with a high γ (the dotted lines), another with a smaller γ (the red solid lines), and the flexible wage benchmark with $\gamma = 0$ (the blue dashed lines). The bubble tax is set to zero throughout. The figure shows how a smaller rigidity parameter, which can be thought as associated with a higher inflation target, leads to a shorter and less severe recession.

Of course, a weakness of the current model is that it does not feature any cost of inflation. For instance, one could embed standard New Keynesian staggered price setting into our framework to generate an endogenous cost of inflation. And thus, the monetary authority would face a tradeoff between the cost and benefit setting a higher inflation target. This would be a quantitative question that we leave for future research. For tractability, we intentionally do not include such staggered price setting in this paper. We interpret the model's prediction as pointing out an important gain from setting a higher inflation target in response to the collapse of bubbles.

Effects of macroprudential policy

In practice, there are also constraints on monetary policy against creating inflation, either because of inflation targeting regimes (such as in the U.S.) or because of fixed exchange rates (such as those in the Euro Zone). This motivates our exercise below.

We assume target inflation $\bar{\Pi}$ is fixed, but instead the authority can change τ. Figure 5 illustrates the effects of the macroprudential policy. It shows the equilibrium dynamics when the tax τ is raised from zero in the baseline (the solid red line) to a small positive value of 2 percent (the dotted line).

As seen in the figure, the policy effectively reduces the bubble size as well as the expansionary effects of the bubble on the capital stock. Consequently, the policy attenuates

the recession: the capital stock, output level, and employment level all fall by less, and the recession also gets shorter when τ is raised.

Our model thus also highlights the tradeoff of macroprudential policies: the bubble tax reduces the gains from the boom, but it also reduces the costs associated with the bust.

5 Bubbles and liquidity trap

5.1 Effect of bubble bursting on the real interest rate

Another important constraint on monetary policy is the zero lower bound on the nominal interest rate, which was relevant in the post-bubble economies of Japan and the U.S. Our model provides a natural environment where this constraint becomes relevant. This is because the collapse of a large expansionary bubble can lead to a sharp drop in the real interest rate (recall remark 1). In fact, we can show that the collapse can push the economy into a liquidity trap, a situation where some agents find it optimal to save in cash as an alternative to lending.

Formally, suppose the economy has reached the bubble steady state. The proposition below shows that if the bubble steady state K_b is sufficiently high (i.e., the boom in the capital stock is sufficiently strong), then the collapse of the steady state bubble can push the net nominal interest rate $R_{T+1}\bar{\Pi} - 1$ between periods T and $T+1$ below zero:

Proposition 5. [Effect of bubble's collapse on real interest rate] *Suppose the economy has reached the steady state with a large expansionary bubble, and then the bubble collapses. Let T denote the period that the bubble collapses. If $K_{lb} > \bar{K}$, where K_{lb} is given by (16) and $\bar{K} \equiv \left(a^L \bar{\Pi}\right)^{\frac{1}{\alpha(1-\alpha)}} \alpha^{\frac{1}{1-\alpha}} \Omega^{-1/\alpha}$, then the nominal interest rate between T and $T+1$ is negative:*

$$R_{T+1}\bar{\Pi} < 1.$$

Proof. Appendix. □

Intuitively, during the bubble episode, the bubble expands the capital stock to a level higher than that in the bubble-less steady state. Thus, when the bubble collapses and the economy reverts to the bubble-less dynamics, the post-bubble economy effectively starts at a capital stock that is "too high" (in the sense that $K_T > K_{nb}$), leading to a low marginal product of capital ($q_T < q_{nb}$). If the over-investment is sufficiently large, then the real interest rate will fall sufficiently low that, given an inflation rate fixed at the target $\bar{\Pi}$, the net nominal interest rate will be negative. One could think of this as corresponding to a situation of "investment hangover" at the end of an economic boom (Rognlie et al., 2014).

The difference between our paper and Rognlie et al. (2014) is that the over-investment is endogenous in our framework, while it is imposed exogenously in theirs.

5.2 Model with nominal asset holding

Now we make a small modification to the model in order to introduce a zero lower bound on the nominal interest rate, so that inefficiencies can arise when the real interest rate falls too low. Specifically, we assume *that entrepreneurs can also save by holding cash.* An entrepreneur j's modified budget constraint is:

$$c_t^j + i_t^j + (1+\tau)\tilde{p}_t^b b_t^j + \underbrace{\frac{M_t^j - M_{t-1}^j}{P_t}}_{\text{change in cash holding}} = q_t k_t^j + d_t^j - R_{t-1,t} d_{t-1}^j + \tilde{p}_t^b b_{t-1}^j,$$

where M_t^j denotes the entrepreneur's cash holding. We assume a simple cash in advance (CIA) constraint as in Asriyan et al. (2016): entrepreneurs must hold cash to fulfill a small quantity $\epsilon > 0$ of consumption (representing the need to hold cash for certain transactions such as shopping or paying rent):

$$\frac{M_t^j}{P_t} \geq \epsilon. \tag{CIA}$$

The CIA assumption guarantees that there is a positive demand for money at all times. As usual, we focus on the cash-less limit by assuming that $\epsilon \to 0+$. The monetary authority controls the supply of money M_t (in equilibrium, money market clearing requires that $\int_0^1 M_t^j dj = M_t$). To abstract away from possible complicating net worth effects, we assume the monetary authority transfers all seignorage to workers.

The fact that entrepreneurs can save in cash implies that in equilibrium the returns from lending must not be dominated by the returns from holding cash:

$$E_t\left[u'(c_{t+1}^j) R_{t,t+1}\right] \geq E_t\left[u'(c_{t+1}^j)\frac{P_t}{P_{t+1}}\right], \forall t \geq 0. \tag{ZLB}$$

In any period without uncertainty the inequality can be rewritten as $R_{t,t+1}\frac{P_{t+1}}{P_t} \geq 1$, i.e., the net nominal interest rate cannot be negative. Thus inequality (ZLB) corresponds to the zero lower bound in the literature (e.g., Eggertsson and Krugman, 2012, Korinek and Simsek, 2016). When the CIA constraint binds for the L-type the zero lower bound slackens, as the return from cash holding is dominated by the return from lending, and entrepreneurs only hold cash for transactional purposes. Then the supply of money determines the price level, and inflation is simply equal to the growth rate of the money supply. However, if the CIA

constraint does not bind for the L-type then (ZLB) holds with equality, as the L-type is indifferent between lending and saving in cash. Thus inflation can no longer be pinned down by the growth rate of money supply.

Definition: We say that the economy is in a *liquidity trap* in period t if the CIA constraint does not bind for the L-type in that period.[11]

Assumption on monetary policy: To show the equilibrium effects of the zero lower bound in the clearest possible way, we assume that the money supply grows at an exogenous growth rate $\bar{\Pi} \geq 1$, i.e., $M_t = \bar{M}_t \equiv (\bar{\Pi})^t M_0$, where $\bar{\Pi} \geq \gamma_n$ is chosen by the monetary authority to be sufficiently high such that the duration of the slump s^* in (19) is always zero and the CIA constraint binds in steady state.[12] Then, if there were no constraint on monetary policy due to the zero lower bound, the path of the money supply above would be an optimal policy since it would generate sufficient inflation to restore full employment. Note that our analysis would carry through under more general assumptions on the monetary policy, as long as the inflation target is bounded above by some threshold $\bar{\Pi}$. In practice, this bound could be due to constraints on monetary policy against creating high inflation.

Other assumptions: We continue to assume that the initial bubble price p_0^b is small, the bubble is expansionary, and $K_0 = K_{nb}$ (the economy starts at the bubble-less steady state capital stock), so that as long as the bubble persists, the economy grows towards the bubble steady state. Furthermore, we assume that the bubble is large and for now set the bubble tax to zero (i.e., $\tau = 0 < \bar{\tau}$).

5.3 Post-bubble equilibrium dynamics

We now analyze equilibrium dynamics with bubbles. First, note that the equilibrium dynamics are the same as before *as long as the bubble persists.* (See the appendix for detailed derivations.) This is because, by construction, neither the nominal wage rigidity nor the ZLB constraint binds as long as the economy is growing towards or is in the steady state.

However, due to the presence of money holding, the post-bubble dynamics are different. Let T be the period that the bubble collapses. Then in each period $t \geq T$; given known state variables K_t, P_{t-1} and w_{t-1}; we have the following system of equations that pin down

[11] As $a^H > a^L$, the CIA constraint will always bind for the H-type.

[12] Formally, we assume $\bar{\Pi}$ is sufficiently high such that $\ln \bar{\Pi} > \ln \gamma_n - \frac{2\alpha}{3-\alpha} \ln(\Omega \alpha K_b^{\alpha-1})$ and $\bar{\Pi} R_b \geq 1$.

equilibrium quantities K_{t+1}, L_t, Y_t and prices R_t, q_t, w_t, P_t:

$$
\begin{aligned}
Y_t &= K_t^{\alpha} L_t^{1-\alpha} \\
L_t &= \left(\frac{1-\alpha}{w_t}\right)^{\frac{1}{\alpha}} K_t \\
q_t &= \alpha \left(\frac{1-\alpha}{w_t}\right)^{\frac{1-\alpha}{\alpha}} && (21) \\
w_t &= \max\left\{(1-\alpha)K_t^{\alpha}, \frac{\gamma w_{t-1}}{P_t/P_{t-1}}\right\} && (22) \\
K_{t+1} &= \Omega(\alpha Y_t + \frac{\bar{M}_{t-1}}{P_t}) - a^L \frac{\bar{M}_t}{P_t} \\
R_t &= \begin{cases} \frac{\theta a^H (1-(1+\tau)\phi)}{1-h-(1+\tau)\phi} q_t & \text{if } t = T \\ a^L q_t & \text{if } t \geq T+1 \end{cases} && (23)
\end{aligned}
$$

and:

$$
\Pi_t = \max\left\{\frac{1}{R_{t-1,t}}, \bar{\Pi}\right\}. \tag{24}
$$

Note that the inflation is determined either by the money growth at $\bar{\Pi}$ or the zero lower bound as seen in equation (24). The real interest rate is pinned down by equation (23). In period $t = T-1$ the bubble has not collapsed yet, and the marginal investor is the H-type, leading to the first interest rate expression in (23). However, after the bubble collapses the marginal investor switches to the L-type, leading to the second interest rate expression.

We are now ready to show how: (i) the collapse of a large expansionary bubble period can push the economy into a liquidity trap, (ii) the liquidity trap can exacerbate the nominal wage rigidity and thus exacerbate involuntary unemployment, and (iii) the increase in unemployment can cause a further drop in the real interest rate, leading to a bidirectional relationship between the liquidity trap and the wage rigidity.

In the following analysis, we assume for simplicity that the economy reaches the bubble steady state before the bubble collapses. This allows our analysis to take advantage of the closed-form expressions for the bubble steady state.

For analytical tractability, we focus on the simplest case: we will construct an equilibrium where the collapse of a large bubble in T will tip the economy temporarily into a liquidity trap that lasts for only one period. The equilibrium is solved via guess and verify method. Detailed derivations are relegated to the appendix.

As in proposition 5, we can show that if the bubble capital stock, K_{lb}, is sufficiently large, then the collapse of the bubble will push the real interest rate between T and $T+1$ so low that:

$$R_{T+1} < \frac{1}{\bar{\Pi}},$$

triggering a liquidity trap. The intuition is the same as in proposition 5: a larger boom in the capital stock is associated with a larger over-accumulation of capital (relative to the bubble-less steady state), and hence a larger drop in the real interest rate when the bubble bursts. The last panel of figure 6 illustrates the effect of the bubble's collapse on the real interest rate.

When the economy is in the liquidity trap in period T, the L-type want to save in cash and the cash-in-advance constraint will not bind for them. Hence the price level P_T cannot be determined by a binding cash-in-advance constraint. Instead, it is determined by the L-type's indifference condition between lending and holding cash:

$$R_{T+1} = \frac{P_T}{P_{T+1}}.$$

Since the economy exits the liquidity trap in $T+1$, the price level P_{T+1} is determined by the binding cash-in-advance constraint:

$$P_{T+1} = \bar{P}_{T+1} \equiv \frac{\bar{M}_{T+1}}{\epsilon}.$$

Hence, the price level in the period the bubble collapses is:

$$P_T = R_{T+1}\bar{P}_{T+1}. \tag{25}$$

Equation (25) yields an interesting insight: *the collapse of the bubble will lead to a deflationary pressure.* This is because given a fixed price level P_{T+1} in the future, the drop in the real interest rate R_{T+1} must be associated with a drop in the current price level (relative to the target): $P_T < \bar{P}_T$. The intuition is as follows. When the real interest rate drops sufficiently low, inflation must rise to prevent negative nominal interest rate. For there to be inflation between T and $T+1$ with the price level fixed in $T+1$, the price level must drop today so that it can rise again in the future to generate inflation. This intuition is similar to that in Krugman (1998) and Eggertsson and Krugman (2012), except that in their case the shock to the interest rate comes from an unanticipated shock to borrowing capacity. In our model the shock comes from an anticipated collapse of a large bubble.

The *deflationary pressure then exacerbates the downward nominal wage rigidity* in the pe-

riod the bubble collapses. Recall that the wage rigidity can be written as $w_T \geq \frac{\gamma_n}{P_T/P_{T-1}} w_{T-1}$, where the right hand side is increasing in P_T. Thus, the deflationary pressure that leads to drop in P_T will raise the wage floor on the right hand side. The second panel of figure 6 illustrates the effect of the deflationary pressure in T on the equilibrium wage. There, the drop in the price level P_T causes the real wage w_T to *increase*. In turn, the increase in the real wage in T causes involuntary unemployment in period T, as seen in the third panel. In summary, by pushing the economy into the liquidity trap, the collapse of the bubble could exacerbate the nominal wage rigidity friction.

In periods $T+1$ onwards, since the economy exits the liquidity trap, the equilibrium dynamics are identical to post-bubble dynamics in section 3. Note that even if the economy exits the liquidity trap, the downward wage rigidity may still bind and hence the economy may still stay in a slump. This is illustrated in figure 6.

Remark 2. The preceding analysis highlights how the presence of the liquidity trap can exacerbate the involuntary unemployment problem. Recall from section 3 that without the liquidity trap, the economy always retains full employment in the period T when the bubble collapses. This is because without the liquidity trap, inflation is always at the target $\bar{\Pi}$. Hence, the nominal wage rigidity condition is $w_T \geq \frac{\gamma_n}{\bar{\Pi}} w_{T-1}$. Under our assumption that $\gamma_n/\bar{\Pi} \leq 1$, this constraint would never be binding (as $w_T^f = w_b$ always exceeds $\frac{\gamma_n}{\bar{\Pi}} w_{T-1} = \frac{\gamma_n}{\bar{\Pi}} w_b$).
However, with the liquidity trap, the deflationary pressure at T pushes the inflation Π_T below the target $\bar{\Pi}$, making it possible for the downward wage rigidity to bind in T. Thus, unlike the previous case, the economy can enter a recession with involuntary unemployment in period T when the bubble collapses.

Remark 3. The involuntary unemployment in period $T+1$ lowers the product of capital and hence the real interest rate relative to the full employment benchmark ($R_{T+1} = a^L \alpha K_{T+1}^{\alpha-1} L_{T+1}^{1-\alpha} \leq R_{T+1}^f \equiv a^L \alpha K_{T+1}^{\alpha-1}$). Thus, the involuntary unemployment in $T+1$ exacerbate the drops in the real interest rate R_{T+1}. Therefore, we have a *bidirectional relationship* between the two traps: the liquidity trap exacerbates the involuntary unemployment due to downward wage rigidity, and in turn the involuntary unemployment exacerbates the liquidity trap.

6 Conclusion

We have developed a tractable rational bubbles model with downward wage rigidity. We show that expansionary bubbles could boost economic activities, but their collapse can push

the economy into a persistent recession with involuntary unemployment and depressed investment, output, and consumption. The collapse could even push the economy into a liquidity trap, where the drop in the real interest rate generates a deflationary pressure that exacerbates the involuntary unemployment problem associated with the downward nominal wage rigidity. The model's predictions are consistent with stylized features of recent bubble episodes. Our model is one of the first in the literature to highlight the tradeoff between the economic gains during the boom due to the bubble and the (potentially deep and persistent) loss from the bust.

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A Appendix

A.1 Derivations

A.1.1 Bubble-less equilibrium

The equilibrium dynamics in the bubble-less environment follows once we solve for the H-type's investment function. The binding borrowing constraint of the H-type gives us H-type borrowing, which we can then plug directly into the budget constraint. With log utility, entrepreneurs consume a fraction $1-\beta$ of their net worth, defined as $e_t^j \equiv q_t k_t^j - R_t d_{t-1}^j$.

$$
\begin{aligned}
d_t^j &= \frac{\theta q_{t+1} k_{t+1}^j}{R_{t+1}} = \frac{\theta q_{t+1} a^H i_t^j}{R_{t+1}} \\
i_t^j - d_t^j &= \beta(q_t k_t^j - R_t d_{t-1}^j) = \beta e_t^j \\
i_t^j &= \frac{1}{1 - \frac{\theta q_{t+1} a^H}{R_{t+1}}} \beta e_t^j.
\end{aligned}
$$

We also note that aggregate wealth in a period is given by $\int_{j\in\mathcal{H}_t} e_t^j + \int_{j\in\bar{\mathcal{H}}_t} e_t^j = q_t K_t$. The idiosyncratic productivity shock is independent across time, which simplifies aggregation, and we can express aggregate H-type net worth as $\int_{j\in\mathcal{H}_t} e_t^j = h q_t K_t$. Since there is sufficient financial friction, L-types will invest a portion of their savings, which will be determined from the aggregate savings in the economy:

$$
\int_{j\in\mathcal{H}_t} i_t^j + \int_{j\in\bar{\mathcal{H}}_t} i_t^j = \beta q_t K_t.
$$

Furthermore, the equilibrium interest rate will be pinned down by the L-type's marginal return from investment, $R_{t+1} = q_{t+1}a^L$. Combining the aggregate savings, investment function, and interest rate, we are able to arrive at a law of motion for aggregate capital.

$$
\begin{aligned}
K_{t+1} &= a^H \int_{j\in\mathcal{H}_t} i_t^j + a^L \int_{j\in\bar{\mathcal{H}}_t} i_t^j = a^H \int_{j\in\mathcal{H}_t} i_t^j + a^L \left[\beta q_t K_t - \int_{j\in\mathcal{H}_t} i_t^j\right] \\
K_{t+1} &= a^H \frac{h}{1-\frac{\theta a^H}{a^L}} \beta q_t K_t + a^L \left[\beta q_t K_t - \frac{h}{1-\frac{\theta a^H}{a^L}} \beta q_t K_t\right] \\
K_{t+1} &= \underbrace{\left(h\frac{a^H - a^L}{1-\frac{\theta a^H}{a^L}} + a^L\right)\beta}_{\equiv\Omega} q_t K_t.
\end{aligned}
$$

A.1.2 Bubble equilibrium

Capital accumulation Similar to the bubble-less environment, H-type's borrowing constraint will bind. Additionally, H-types will not hold the bubble since their return to investment is greater. However, we must consider two cases.

Case 1: $R_{t+1} = q_{t+1}a^L$. We proceed as before by solving the H-type's investment function. Net worth now reflects bubble holdings from the past period, $e_t^j \equiv q_t k_t^j + p_t^b b_{t-1}^j - R_t d_{t-1}^j$.

$$
\begin{aligned}
i_t^j - d_t^j &= \beta(q_t k_t^j + p_t^b b_{t-1}^j - R_t d_{t-1}^j) = \beta e_t^j \\
i_t^j &= \frac{1}{1-\frac{\theta a^H}{a^L}} \beta e_t^j.
\end{aligned}
$$

The aggregate savings will also change to reflect the presence of the bubble and macroprudential tax:

$$
\int_{j\in\mathcal{H}_t} i_t^j + \int_{j\in\bar{\mathcal{H}}_t} i_t^j + (1+\tau)p_t^b = \beta(q_t K_t + p_t^b).
$$

As before, we combine the aggregate savings, investment function, and interest rate to determine the law of motion for capital:

$$K_{t+1} = a^H \frac{h}{1-\frac{\theta a^H}{a^L}} \beta(q_t K_t + p_t^b) + a^L \left[\beta(q_t K_t + p_t^b) - \frac{h}{1-\frac{\theta a^H}{a^L}} \beta(q_t K_t + p_t^b) - (1+\tau) p_t^b \right]$$
$$K_{t+1} = \Omega(q_t K_t + p_t^b) - a^L (1+\tau) p_t^b.$$

Case 2: $R_{t+1} > q_{t+1} a^L$. L-types do not invest since their return to lending and bubbles is greater than their return to investment. Therefore, all non-bubble savings are shifted to the H-type to invest, and:

$$K_{t+1} = a^H \left[\beta(q_t K_t + p_t^b) - (1+\tau) p_t^b \right].$$

Using the definition of bubble size, $\phi_t \equiv \frac{p_t^b}{\beta(q_t K_t + p_t^b)}$, we can re-write the above capital flows as below:

$$K_{t+1} = \begin{cases} \frac{\left(1+\frac{(a^H - a^L)}{a^L - \theta a^H} h\right) \beta a^L - a^L \beta (1+\tau) \phi_t}{1-\beta\phi_t} \alpha K_t^\alpha & \textit{if } \phi_t \leq \phi^* \\ \frac{a^H \beta [1-(1+\tau)\phi_t]}{1-\beta\phi_t} \alpha K_t^\alpha & \textit{if } \phi_t > \phi^* \end{cases}.$$

In the notation above, we define small bubbles as $\phi_t \leq \phi^*$. Small bubbles arise when the L-type is still investing, therefore the small bubble condition is equivalent to $R_{t+1} = q_{t+1} a^L$. On the other hand, large bubbles, $\phi_t > \phi^*$, arise when L-types no longer invest, and thus is equivalent to $R_{t+1} > q_{t+1} a^L$. We show the derivation for ϕ^* in the interest rate derivation below.

Interest rate Using the definition of bubble size, the H-type's investment function, and aggregate savings, we can solve for R_{t+1} when $R_{t+1} > q_{t+1} a^L$. Recall that in this case, L-types do not invest, so:

$$\underbrace{h \frac{\beta(q_t K_t + p_t^b)}{1 - \frac{\theta q_{t+1} a_t^H}{R_{t+1}}}}_{i_t^H} + (1+\tau) p_t^b = \beta(q_t K_t + p_t^b),$$

Solving for interest rate, we get the following expression.

$$R_{t+1} = \begin{cases} q_{t+1} a^L & \text{if } \phi_t \leq \phi^* \\ q_{t+1} \frac{\theta a^H (1-(1+\tau)\phi_t)}{1-h-(1+\tau)\phi_t} & \text{if } \phi_t > \phi^* \end{cases}.$$

Here, ϕ^*, is defined as the threshold bubble size that equates the two different values of interest rate:

$$a^L = \frac{\theta a^H(1-(1+\tau)\phi^*)}{1-h-(1+\tau)\phi^*}$$
$$\phi^* \equiv \frac{(1-h)a^L - \theta a^H}{(1+\tau)(a^L - \theta a^H)}.$$

Bubble growth In the stochastic bubble environment, the expected returns from holding the bubble must equal the expected returns from lending. In the notation below, terms with superscript ρ represent values in the state that bubble persists, and terms with superscript $1-\rho$ represent values in the state that the bubble bursts.

$$E_t[u'(c_{t+1}^{i,\rho})\frac{p_{t+1}^b}{(1+\tau)p_t^b}] = E_t[u'(c_{t+1}^{i,\rho})R_{t+1}]$$
$$\Rightarrow \rho\frac{1}{c_{t+1}^{i,\rho}}\frac{p_{t+1}^b}{(1+\tau)p_t^b} = R_{t+1}\rho\frac{1}{c_{t+1}^{i,\rho}} + R_{t+1}^{1-\rho}(1-\rho)\frac{1}{c_{t+1}^{i,1-\rho}}$$
$$\Rightarrow \rho\frac{p_{t+1}^b}{(1+\tau)p_t^b} = R_{t+1} + (1-\rho)\frac{p_{t+1}^b b_t^j}{\beta e_t^j - (1+\tau)p_t^b b_t^j}.$$

We guess that L-types hold a portion η of their savings in bubble, that is $\eta\beta e_t^j = (1+\tau)p_t^b b_t^j$, and then solve for η to get L-type bubble demand:

$$(1+\tau)p_t^b b_t^j = \underbrace{\frac{\rho\frac{p_{t+1}^b}{(1+\tau)p_t^b} - R_{t+1}}{\frac{p_{t+1}^b}{(1+\tau)p_t^b} - R_{t+1}}}_{\eta}\beta e_t^j.$$

Plugging the expression for L-type bubble demand into the Euler equation above, we get the following no-arbitrage condition:

$$\frac{p_{t+1}^b}{p_t^b} = \frac{R_{t+1}(1-h-(1+\tau)\phi_t)}{\rho(1-h)-(1+\tau)\phi_t}(1+\tau).$$

Next, we define the evolution of wealth, using the transition dynamics for aggregate

capital.

$$q_{t+1}K_{t+1}+p^b_{t+1}=\begin{cases} q_{t+1}\left[\left(\frac{h(a^H-a^L)}{1-\frac{\theta a^H}{a^L}}+a^L\right)\beta(q_tK_t+p^b_t)-a^L(1+\tau)p^b_t\right]+\frac{p^b_{t+1}}{p^b_t}\phi_t\beta(q_tK_t+p^b_t) & \text{if } \phi_t\leq \phi \\ q_{t+1}\left[a^H\beta(q_tK_t+p^b_t)-a^H(1+\tau)p^b_t\right]+\frac{p^b_{t+1}}{p^b_t}\phi_t\beta(q_tK_t+p^b_t) & \text{if } \phi_t> \phi \end{cases}$$

$$\Rightarrow \frac{q_{t+1}K_{t+1}+p^b_{t+1}}{q_tK_t+p^b_t}=\begin{cases} \beta q_{t+1}\left[\left(\frac{h(a^H-a^L)}{1-\frac{\theta a^H}{a^L}}+a^L\right)-a^L(1+\tau)\phi_t\right]+\frac{p^b_{t+1}}{p^b_t}\phi_t & \text{if } \phi_t\leq\phi^* \\ \beta q_{t+1}\left[a^H-a^H(1+\tau)\phi_t\right]+\beta\frac{p^b_{t+1}}{p^b_t}\phi_t & \text{if } \phi_t>\phi^* \end{cases}.$$

Using the flow of bubble price, evolution of wealth, and interest rate, we characterize the evolution of bubble below:

$$\begin{aligned}\phi_{t+1}&=\frac{\frac{p^b_{t+1}}{p^b_t}}{\frac{q_{t+1}K_{t+1}+p^b_{t+1}}{q_tK_t+p^b_t}}\phi_t \\ &=\begin{cases}\frac{1}{\beta}\frac{\frac{1-h-(1+\tau)\phi_t}{\rho(1-h)(1+\tau)-\phi_t}(1+\tau)\phi_t}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)+\frac{(1-\rho)(1-h)}{\rho(1-h)-(1+\tau)\phi_t}(1+\tau)\phi_t} & \text{if } \phi_t\leq\phi^* \\ \frac{\theta}{\beta}\frac{(1+\tau)\phi_t}{\rho(1-h)-(1-\theta)(1+\tau)\phi_t} & \text{if } \phi_t>\phi^*\end{cases}.\end{aligned}$$

Steady state bubble size First, we use the above evolution of bubble, to solve for steady state bubble size for each case of small and large bubble, as a function of macroprudential tax:

$$\phi_{sb}\equiv\frac{\rho-\frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}}{1-\frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}}\frac{1-h}{1+\tau} \qquad \text{(small bubble)}$$

$$\phi_{lb}\equiv\frac{\rho(1-h)}{(1-\theta)(1+\tau)}-\frac{\theta}{\beta(1-\theta)} \qquad \text{(large bubble).}$$

Note that in steady state, bubble size is decreasing in tax, in both cases. Therefore, we have $\phi_{sb}(\tau)\leq\phi_{sb}(\hat{\tau})$ for all $\tau\geq\hat{\tau}$, and $\phi_{lb}(\tau)>\phi_{lb}(\hat{\tau})$ for all $\tau<\hat{\tau}$. If at $\hat{\tau}$, steady state bubble size is the threshold bubble size, we can re-write the bubble threshold cutoff as a cutoff in τ. Thus, we solve for $\hat{\tau}$ by equating the steady state bubble sizes to the threshold bubble size:

$$\phi_{sb}(\hat{\tau})=\phi_{lb}(\hat{\tau})=\phi^*.$$

Lastly, we define $\bar{\tau} = \max[0, \hat{\tau}]$, and then re-write steady state bubble size as follows:

$$\phi_b = \begin{cases} \phi_{sb} & \text{if } \tau \geq \bar{\tau} \\ \phi_{lb} & \text{if } \tau < \bar{\tau} \end{cases}.$$

The remainder of the steady state values follow directly from previously derived equilibrium evolution equations and the above steady state bubble size.

A.1.3 Bubble equilibrium dynamics with zero lower bound (section 5.3)

Capital accumulation In the zero lower bound environment, we focus on the bubble equilibrium dynamics for large bubbles. With large bubbles, L-types do not invest, and all resources excluding the bubble and money holdings are invested by H-types. The following shows H-type investment, as derived from the new aggregate savings equation and money growth:

$$\int_{j \in \mathcal{H}_t} i_t^j + \frac{M_t}{P_t} + (1+\tau)p_t^b = \beta(q_t K_t + p_t^b + \frac{M_{t-1}}{P_t})$$

$$\int_{j \in \mathcal{H}_t} i_t^j = \beta(q_t K_t + p_t^b) - \frac{M_t}{P_t}\left(1 - \frac{\beta}{\Pi^*}\right) - (1+\tau)p_t^b.$$

Furthermore, noting that the CIA constraint binds during the bubble periods, and defining $\epsilon^* \equiv \left(1 - \frac{\beta}{\Pi^*}\right)\epsilon$, capital accumulation follows:

$$K_{t+1} \quad = a^H[\beta(q_t K_t + p_t^b) - \epsilon^*] - a^H(1+\tau)p_t^b.$$

The cashless limit, where $\epsilon \to 0$, implies $\epsilon^* \to 0$, and that capital accumulation remains the same as the large bubble environment:

$$K_{t+1} \quad = a^H[\beta(q_t K_t + p_t^b)] - a^H(1+\tau)p_t^b.$$

Interest rate Noting that H-types do not hold bubbles, and plugging the binding borrowing and CIA constraints into the budget constraint, we derive the following H-type investment

function.

$$i_t^j + \frac{M_t^j}{P_t} - d_t^j = \beta\left(q_t k_t^j - R_{t-1,t} d_{t-1}^j + p_t^b b_{t-1}^j + \frac{M_{t-1}^j}{P_t}\right)$$

$$i_t^j\left(1 - \frac{\theta q_{t+1} a_t^H}{R_{t,t+1}}\right) = \beta\left(q_t k_t^j - R_{t-1,t} d_{t-1}^j + p_t^b b_{t-1}^j\right) - \left(\frac{M_t^j}{P_t} - \beta\frac{M_{t-1}^j}{P_t}\right)$$

$$\int_{j\in\mathcal{H}_t} i_t^j = \frac{\beta(q_t K_t + p_t^b) - \epsilon^*}{1 - \frac{\theta q_{t+1} a_t^H}{R_{t,t+1}}}$$

Then, plugging in the above investment function into the aggregate savings equation, we derive the interest rate. We also define the bubble size with money holdings as $\phi_t = \frac{p_t^b}{\beta(q_t K_t + p_t^b) - \epsilon^*}$.

$$h\frac{\beta(q_t K_t + p_t^b) - \epsilon^*}{1 - \frac{\theta q_{t+1} a_t^H}{R_{t,t+1}}} + (1+\tau)p_t^b = \beta(q_t K_t + p_t^b) - \epsilon^*$$

$$h\frac{1}{1 - \frac{\theta q_{t+1} a_t^H}{R_{t,t+1}}} + (1+\tau)\phi_t = 1$$

$$R_{t,t+1} = q_{t+1}\frac{\theta a^H(1 - (1+\tau)\phi_t)}{1 - h - (1+\tau)\phi_t}.$$

Bubble growth Similar to the standard bubble environment, the no-arbitrage condition for L-types between lending and bubble holdings pins down the bubble price growth rate. In the notation below, terms with superscript ρ represent values in the state that bubble persists, and terms with superscript $1-\rho$ represent values in the state that the bubble bursts. Now, we must take into account the money holdings of the L-type, and under the assumption that CIA constraints bind in the pre-burst period, $\frac{M_t}{P_t} = \epsilon$.

$$E_t[u'(c_{t+1}^{i,\rho})\frac{p_{t+1}^b}{(1+\tau)p_t^b}] = E_t[u'(c_{t+1}^{i,\rho})R_{t+1}]$$

$$\Rightarrow \rho\frac{p_{t+1}^b}{(1+\tau)p_t^b} = \rho R_{t,t+1}^{\rho} + (1-\rho)R_{t,t+1}^{1-\rho}\frac{p_{t+1}^b b_t^j - R_{t,t+1}^{\rho} d_t^j + \frac{M_t^j}{P_{t+1}}}{\frac{M_t^j}{P_{t+1}} - R_{t,t+1}^{1-\rho} d_t^j}$$

$$= \rho R_{t,t+1}^{\rho} + (1-\rho)R_{t,t+1}^{1-\rho}\frac{p_{t+1}^b b_t^j - R_{t,t+1}^{\rho} d_t^j + \frac{\epsilon}{\Pi}}{\frac{\epsilon}{\Pi} - R_{t,t+1}^{1-\rho} d_t^j}$$

However, at the cashless limit, as $\epsilon \to 0$, we return to the following equation, which is equivalent to our standard bubble environment.

$$\rho \frac{p^b_{t+1}}{(1+\tau)p^b_t} = R_{t,t+1} + (1-\rho)\frac{p^b_{t+1}b^j_t}{\beta e^j_t - (1+\tau)p^b_t b^j_t},$$

Solving for the bubble price growth in terms of the interest rate and parameters,

$$\frac{p^b_{t+1}}{(1+\tau)p^b_t} = \frac{(1-h-(1+\tau)\phi_t)}{\rho(1-h)-(1+\tau)\phi_t} R_{t,t+1}.$$

Bubble growth follows from the above bubble price growth, evolution of wealth, and interest rate, which, in the cashless limit, are all equivalent to the standard large bubble environment.

$$\frac{\phi_{t+1}}{(1+\tau)\phi_t} = \frac{\theta}{\beta}\frac{1}{\rho(1-h)-(1-\theta)(1+\tau)\phi_t}$$

A.1.4 Bubble equilibrium dynamics with partial depreciation

The following re-calculates the bubble dynamics in the presence of partial depreciation, δ. First, the budget constraint and borrowing constraint must account for the depreciation:

$$\begin{aligned} R_{t+1}d^j_t &\leq \theta(q_{t+1}+1-\delta)k^j_{t+1} \\ c^j_t + i^j_t + (1+\tau)\tilde{p}^b_t b^j_t &= (q_t+1-\delta)k^j_t + d^j_t - R_t d^j_{t-1} + \tilde{p}^b_t b^j_{t-1}. \end{aligned}$$

Similar to before, small bubbles arise when the interest rate is equal to the marginal return to investment for the L-type. With partial depreciation, this results in the case $R_{t+1} = (q_{t+1}+1-\delta)a^L$. The resulting capital flow is calculated similarly as the case with full depreciation. First, we solve the H-type's investment function. Now, net worth also reflects partial depreciation, $e^j_t \equiv (q_t+1-\delta)k^j_t + p^b_t b^j_{t-1} - R_t d^j_{t-1}$.

$$\begin{aligned} i^j_t - d^j_t &= \beta((q_t+1-\delta)k^j_t + p^b_t b^j_{t-1} - R_t d^j_{t-1}) = \beta e^j_t \\ i^j_t &= \frac{1}{1-\frac{\theta a^H}{a^L}}\beta e^j_t. \end{aligned}$$

The aggregate savings will also change to reflect the presence of the bubble and macropru-

dential tax:

$$\int_{j\in\mathcal{H}_t} i_t^j + \int_{j\in\bar{\mathcal{H}}_t} i_t^j + (1+\tau)p_t^b = \beta((q_t+1-\delta)K_t + p_t^b).$$

As before, we combine the aggregate savings, investment function, and interest rate to determine the law of motion for capital:

$$K_{t+1} = a^H \frac{h}{1-\frac{\theta a^H}{a^L}}\beta((q_t+1-\delta)K_t+p_t^b) + a^L\left[\beta((q_t+1-\delta)K_t+p_t^b) - \frac{h}{1-\frac{\theta a^H}{a^L}}\beta((q_t+1-\delta)K_t+p_t^b) - \right.$$
$$K_{t+1} = \Omega((q_t+1-\delta)K_t+p_t^b) - a^L(1+\tau)p_t^b.$$

In the case of large bubbles, L-types do not invest since their return to lending and bubbles is greater than their return to investment: $R_{t+1} > (q_{t+1}+1-\delta)a^L$. Therefore, all non-bubble savings are shifted to the H-type to invest, and:

$$K_{t+1} = a^H\left[\beta((q_t+1-\delta)K_t+p_t^b) - (1+\tau)p_t^b\right].$$

Adjusting the definition of bubble size to account for partial depreciation, $\phi_t \equiv \frac{p_t^b}{\beta((q_t+1-\delta)K_t+p_t^b)}$, we can re-write the above capital flows as below:

$$K_{t+1} = \begin{cases} \frac{\left(1+\frac{(a^H-a^L)}{a^L-\theta a^H}h\right)\beta a^L - a^L\beta(1+\tau)\phi_t}{1-\beta\phi_t}\left(\alpha K_t^\alpha + (1-\delta)K_t\right) & \textit{if } \phi_t \le \phi^* \\ \frac{a^H\beta[1-(1+\tau)\phi_t]}{1-\beta\phi_t}\left(\alpha K_t^\alpha + (1-\delta)K_t\right) & \textit{if } \phi_t > \phi^* \end{cases}.$$

Interest rate Using the definition of bubble size, the H-type's investment function, and aggregate savings, we can solve for R_{t+1} when $R_{t+1} > q_{t+1}a^L$. Recall that in this case, L-types do not invest, so:

$$\underbrace{h\frac{\beta((q_t+1-\delta)K_t+p_t^b)}{1-\frac{\theta(q_{t+1}+1-\delta)a_t^H}{R_{t+1}}}}_{i_t^H} + (1+\tau)p_t^b = \beta((q_t+1-\delta)K_t+p_t^b),$$

Solving for interest rate, we get the following expression.

$$R_{t+1} = \begin{cases} (q_{t+1}+1-\delta)a^L & \text{if } \phi_t \le \phi^* \\ (q_{t+1}+1-\delta)\frac{\theta a^H(1-(1+\tau)\phi_t)}{1-h-(1+\tau)\phi_t} & \text{if } \phi_t > \phi^* \end{cases}.$$

Here, ϕ^*, is defined as the threshold bubble size that equates the two different values of interest rate:

$$a^L = \frac{\theta a^H(1-(1+\tau)\phi^*)}{1-h-(1+\tau)\phi^*}$$
$$\phi^* \equiv \frac{(1-h)a^L - \theta a^H}{(1+\tau)(a^L - \theta a^H)}.$$

Bubble growth In the stochastic bubble environment, the expected returns from holding the bubble must equal the expected returns from lending. In the notation below, terms with superscript ρ represent values in the state that bubble persists, and terms with superscript $1-\rho$ represent values in the state that the bubble bursts.

$$E_t[u'(c_{t+1}^{i,\rho})\frac{p_{t+1}^b}{(1+\tau)p_t^b}] = E_t[u'(c_{t+1}^{i,\rho})R_{t+1}]$$
$$\Rightarrow \rho\frac{1}{c_{t+1}^{i,\rho}}\frac{p_{t+1}^b}{(1+\tau)p_t^b} = R_{t+1}\rho\frac{1}{c_{t+1}^{i,\rho}} + R_{t+1}^{1-\rho}(1-\rho)\frac{1}{c_{t+1}^{i,1-\rho}}$$
$$\Rightarrow \rho\frac{p_{t+1}^b}{(1+\tau)p_t^b} = R_{t+1} + (1-\rho)\frac{p_{t+1}^b b_t^j}{\beta e_t^j - (1+\tau)p_t^b b_t^j}.$$

We guess that L-types hold a portion η of their savings in bubble, that is $\eta\beta e_t^j = (1+\tau)p_t^b b_t^j$, and then solve for η to get L-type bubble demand:

$$(1+\tau)p_t^b b_t^j = \underbrace{\frac{\rho\frac{p_{t+1}^b}{(1+\tau)p_t^b} - R_{t+1}}{\frac{p_{t+1}^b}{(1+\tau)p_t^b} - R_{t+1}}}_{\eta}\beta e_t^j.$$

Plugging the expression for L-type bubble demand into Euler equation above, we get the following no-arbitrage condition:

$$\frac{p_{t+1}^b}{p_t^b} = \frac{R_{t+1}(1-h-(1+\tau)\phi_t)}{\rho(1-h)-(1+\tau)\phi_t}(1+\tau).$$

Next, we define the evolution of wealth, using the transition dynamics for aggregate

capital.

$$(q_{t+1}+1-\delta)K_{t+1}+p^b_{t+1} = \begin{cases} (q_{t+1}+1-\delta)\left[\left(\frac{h(a^H-a^L)}{1-\frac{\theta a^H}{a^L}}+a^L\right)\beta((q_t+1-\delta)K_t+p^b_t)-a^L(1+\tau)p^b_t\right]+ \\ (q_{t+1}+1-\delta)\Big[a^H\beta((q_t+1-\delta)K_t+p^b_t)-a^H(1+\tau)p^b_t\Big]+\frac{p^b_{t+1}}{p^b_t}\phi_t\beta((q_t+ \end{cases}$$

$$\Rightarrow \frac{(q_{t+1}+1-\delta)K_{t+1}+p^b_{t+1}}{(q_t+1-\delta)K_t+p^b_t} = \begin{cases} \beta(q_{t+1}+1-\delta)\left[\left(\frac{h(a^H-a^L)}{1-\frac{\theta a^H}{a^L}}+a^L\right)-a^L(1+\tau)\phi_t\right]+\beta\frac{p^b_{t+1}}{p^b_t}\phi_t & \text{if } \phi_t \le \phi^* \\ \beta(q_{t+1}+1-\delta)\Big[a^H-a^H(1+\tau)\phi_t\Big]+\beta\frac{p^b_{t+1}}{p^b_t}\phi_t & \text{if } \phi_t > \phi^* \end{cases}.$$

Using the flow of bubble price, evolution of wealth, and interest rate, we characterize the evolution of bubble below:

$$\begin{aligned} \phi_{t+1} &= \frac{\frac{p^b_{t+1}}{p^b_t}}{\frac{(q_{t+1}+1-\delta)K_{t+1}+p^b_{t+1}}{(q_t+1-\delta)K_t+p^b_t}}\phi_t \\ &= \begin{cases} \frac{1}{\beta}\frac{\frac{1-h-(1+\tau)\phi_t}{\rho(1-h)(1+\tau)-\phi_t}(1+\tau)\phi_t}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)+\frac{(1-\rho)(1-h)}{\rho(1-h)-(1+\tau)\phi_t}(1+\tau)\phi_t} & \text{if } \phi_t \le \phi^* \\ \frac{\theta}{\beta}\frac{(1+\tau)\phi_t}{\rho(1-h)-(1-\theta)(1+\tau)\phi_t} & \text{if } \phi_t > \phi^* \end{cases}. \end{aligned}$$

Non-bubble dynamics Without the bubble, but in the presence of partial depreciation, the dynamics of capital and the interested are adjusted as following:

$$\begin{aligned} K_{t+1} &= \Omega(q_t+1-\delta)K_t = \Omega(\alpha K_t^\alpha+(1-\delta)K_t) \\ R_{t+1} &= (q_{t+1}+1-\delta)a^L \end{aligned}$$

The resulting steady state is thus:

$$\begin{aligned} K_{nb} &= \left(\frac{\Omega\alpha}{1-\Omega(1-\delta)}\right)^{\frac{1}{1-\alpha}} \\ R_{nb} &= \frac{a^L}{\Omega} \end{aligned}$$

A.1.5 Post-bubble equilibrium dynamics of section 5.3

Equilibrium Guess We guess an equilibrium where the collapse of a large bubble in T will tip the economy temporarily into a liquidity trap that lasts for only one period. Let $T+s^*$ be the first period in which the economy regains full employment. Then in equilibrium the following must hold simultaneously:

1. Downward wage rigidity binds from periods T to $T+s^*-1$:

$$w^f_{T+s} < \frac{\gamma_n}{\Pi_{T+s}} w_{T+s-1}$$
$$w_{T+s} = \frac{\gamma_n}{\Pi_{T+s}} w_{T+s-1}, \quad \forall\ s \in [0, s^*)$$

2. There is a liquidity trap in the burst period, i.e. the zero lower bound (ZLB) binds:

$$R_{T+1} < \frac{1}{\overline{\Pi}}, \tag{26}$$

3. The economy is not in a liquidity trap before the burst, and exits the liquidity trap in $T+1$:

$$E_t\left[u'(c^j_{t+1})R_{t+1}\right] \geq E_t\left[u'(c^j_{t+1})\frac{P_t}{P_{t+1}}\right], \qquad \forall\ t \leq T-1$$
$$R_{t+1} \geq \frac{1}{\overline{\Pi}} \qquad \forall\ t \geq T+1$$

We can characterize the equilibrium quantities and prices based on these properties. First, we find R_{T+1}. Recall that after the bubble collapses, the marginal capital investor will be the L-type, and so:

$$R_{T+1} = a^L q_{T+1} = a^L \alpha \left(\frac{L_{T+1}}{K_{T+1}}\right)^{1-\alpha}.$$

Note that wage rigidity may still bind in $T+1$. To find K_{T+1}, recall the bubble-less dynamics:

$$K_{T+1} = \Omega \alpha Y_T$$

where output at T is:

$$Y_T = K_b^\alpha L_T^{1-\alpha}$$

and, as in (18), we know:

$$L_T = \left[\frac{w_T}{1-\alpha}\right]^{-\frac{1}{\alpha}} K_T = \left[\frac{w_T}{1-\alpha}\right]^{-\frac{1}{\alpha}} K_b$$

so:

$$Y_T = \left[\frac{w_T}{1-\alpha}\right]^{\frac{\alpha-1}{\alpha}} K_b.$$

Combining the equations above gives an expression for R_{T+1} and P_T in w_T:

$$\begin{aligned}
\frac{P_T}{\bar{P}_{T+1}} &= R_{T+1} \\
&= a^L \alpha \left(\frac{L_{T+1}}{K_{T+1}}\right)^{1-\alpha} \\
&= a^L \alpha \left(\frac{\gamma_n}{\Pi_{T+1}} \frac{w_T}{1-\alpha}\right)^{-\frac{1-\alpha}{\alpha}} \\
\frac{P_T}{\bar{P}_{T+1}} = \frac{1}{\Pi_{T+1}} &= a^L \alpha \left(\gamma_n \frac{w_T}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}} \left[\frac{1}{\Pi_{T+1}}\right]^{\frac{\alpha-1}{\alpha}} \\
\Rightarrow \frac{P_T}{\bar{P}_{T+1}} &= R_{T+1} = \left(a^L \alpha \left(\gamma_n \frac{w_T}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}}\right)^{\alpha} \qquad (27)
\end{aligned}$$

Next, we find w_T. Under the guess that the wage rigidity constraint is binding in T, we have:

$$\begin{aligned}
w_T &= \frac{\gamma_n}{\Pi_{T-1,T}} w_{T-1} \\
w_T &= \frac{\gamma_n}{P_T/\bar{P}_{T-1}} w_{lb} \\
&= \frac{\gamma_n}{P_T/(\bar{P}_{T+1}/\bar{\Pi}^2)} w_{lb} \\
w_T &= \frac{\gamma_n/\bar{\Pi}^2}{(a^L\alpha)^{\alpha}\left(\gamma_n \frac{w_T}{1-\alpha}\right)^{\alpha-1}} w_{lb} \\
&= \frac{1}{1-\alpha}\left(\frac{a^L\alpha}{1-\alpha}\right)^{-\alpha} \frac{\gamma_n^{2-\alpha}/\bar{\Pi}^2}{w_T^{\alpha-1}} w_{lb} \\
w_T^{\alpha} &= \frac{1}{1-\alpha}\left(\frac{a^L\alpha}{1-\alpha}\right)^{-\alpha} \frac{\gamma_n^{2-\alpha}}{\bar{\Pi}^2} w_{lb} \\
w_T &= \zeta(w_{lb}, \bar{\Pi})
\end{aligned}$$

where $\zeta(w_{lb}, \bar{\Pi}) \equiv \frac{1-\alpha}{a^L\alpha\gamma_n}\left(\frac{1}{1-\alpha}\frac{\gamma_n^2}{\bar{\Pi}^2} w_{lb}\right)^{\frac{1}{\alpha}}$. is a function of w_{lb}. Combined with (27), we get

expressions for R_{T+1} and P_T in exogenous parameters:

$$\frac{P_T}{\bar{P}_{T+1}} = R_{T+1} = \left(a^L \alpha \left(\gamma_n \frac{\zeta(w_{lb}, \bar{\Pi})}{1-\alpha} \right)^{\frac{\alpha-1}{\alpha}} \right)^{\alpha}. \tag{28}$$

During the slump ($1 \leq s < s^*$), $L_{T+s} < 1$, and wages are solved from binding downward wage rigidity:

$$\begin{aligned} w_{T+s} &= \left(\frac{\gamma_n}{\bar{\Pi}}\right)^s \frac{P_T}{\bar{P}_T} w_T \\ &= \left(\frac{\gamma_n}{\bar{\Pi}}\right)^s \frac{P_T}{\bar{P}_T} \zeta(w_{lb}, \bar{\Pi}) \end{aligned}$$

From (18), we know:

$$\begin{aligned} \frac{L_{T+s}}{K_{T+s}} &= \left[\frac{w_{T+s}}{1-\alpha}\right]^{-\frac{1}{\alpha}} \\ &= \left[\frac{1}{1-\alpha}\left(\frac{\gamma_n}{\bar{\Pi}}\right)^s \frac{P_T}{\bar{P}_T}\zeta(w_{lb}, \bar{\Pi})\right]^{-\frac{1}{\alpha}} \end{aligned}$$

Thus, interest rate during the slump is as follows:

$$\begin{aligned} R_{T+s} &= a^L q_{T+s} = a^L \alpha K_{T+s}^{\alpha-1} L_{T+s}^{1-\alpha} \\ &= a^L \alpha \left(\frac{L_{T+s}}{K_{T+s}}\right)^{1-\alpha} \\ &= a^L \alpha \left(\frac{\gamma_n}{\bar{\Pi}}\right)^{-s\frac{1-\alpha}{\alpha}} \left[\frac{1}{1-\alpha}\frac{P_T}{\bar{P}_T}\zeta(w_{lb}, \bar{\Pi})\right]^{-\frac{1-\alpha}{\alpha}} \end{aligned} \tag{29}$$

The dynamics of the post-slump economy is as in section 3.2.2.

Verifying the equilibrium Now, we find a set of parametric conditions under which the above guess is correct. First, for the wage rigidity constraint to bind in T, it must be that:

$$w_T^f < \frac{\gamma_n}{P_T/\bar{P}_{T-1}} w_{T-1},$$

where recall that $w_{T-1} = w_{lb}$ and $w_T^f = (1-\alpha)K_T^\alpha = (1-\alpha)K_b^\alpha = w_{lb}$. So the inequality above is equivalent to

$$P_T < \gamma_n \bar{P}_{T-1}.$$

Recall from (25) that $\bar{\Pi}^2 \frac{\bar{P}_{T-1}}{P_T} = \frac{\bar{P}_{T+1}}{P_T} = \Pi_{T,T+1} = \frac{1}{R_{T+1}}$, so $R_{T+1} = \frac{1}{\bar{\Pi}^2} \frac{P_T}{\bar{P}_{T-1}}$. So for the wage rigidity constraint to bind at T we need:

$$R_{T+1} < \frac{\gamma_n}{\bar{\Pi}^2}. \tag{30}$$

Recall (28), so the inequality above is equivalent to:

$$\left(a^L \alpha \left(\gamma_n \frac{\zeta(w_{lb}, \bar{\Pi})}{1-\alpha} \right)^{\frac{\alpha-1}{\alpha}} \right)^{\alpha} < \frac{\gamma_n}{\bar{\Pi}^2}. \tag{31}$$

Second, we need to verify that the ZLB is binding in the burst period, i.e.,

$$R_{T+1} < \frac{1}{\bar{\Pi}}, \tag{32}$$

Intuitively, inequality (32) implies that the monetary authority can no longer set $\Pi_{T,T+1} = \bar{\Pi}$, as it would violate the ZLB, and thereby the ZLB is binding at T. However, this inequality is automatically satisfied because of (30), under our assumption:

$$\bar{\Pi} \geq \gamma_n.$$

Third, we need to verify that the CIA binds in $t \geq T+1$ (i.e., $R_{t+1}\bar{\Pi} \geq 1$ for such t). As $\bar{\Pi} \geq \gamma_n$, the interest rate R_{T+s} in (29) is weakly increasing in s. Therefore, $R_{T+2} \leq R_{T+s}\ \forall s \in (2, s^*)$. After the slump, $(s \geq s^*)$, the economy returns to full employment. Furthermore, capital is increasing; therefore, interest rate is decreasing. As the economy transitions back to the bubble-less steady state, we have that $R_{nb} \leq R_{T+s}\ \forall s \geq s^*$. Lastly, under our hypothesis, inflation is at a constant $\bar{\Pi}$ for all periods $t > T+1$. Thus, if $R_{T+2}\bar{\Pi} \geq 1$ and $R_{nb}\bar{\Pi} \geq 1$, then $R_{T+s}\bar{\Pi} \geq 1\ \forall s > 1$, that is the economy exits the liquidity trap after one period, as desired.

If the slump lasts less than two periods, $s^* < 2$, the economy reaches full employment in $T+2$. Then, following the above logic, $R_{nb} \leq R_{T+s}\ \forall s \geq 2$, and a condition on R_{nb} is sufficient to exit the liquidity trap after one period.Then, under the following sufficient condition, the economy is not in a liquidity trap after $t = T+1$:

$$\begin{aligned} \min\{R_{T+2},\ R_{nb}\} &\geq \frac{1}{\bar{\Pi}}, && \text{if } s^* \geq 2 \\ R_{nb} &\geq \frac{1}{\bar{\Pi}}, && \text{if } s^* < 2 \end{aligned}$$

or equivalently:

$$\min\{\left(a^L\alpha\right)^{\alpha^2}\gamma_n^{-(1-\alpha^2)}\left(\frac{\zeta(w_{lb},\bar{\Pi})}{1-\alpha}\right)^{-\alpha(1-\alpha)},\ \frac{a^L}{\Omega}\}\geq\frac{1}{\bar{\Pi}} \qquad \text{if } s^*\geq 2$$
$$\frac{a^L}{\Omega}\geq\frac{1}{\bar{\Pi}} \qquad \text{if } s^*<2 \tag{33}$$

Fourth, we also need to verify that the CIA binds in $T-1$. Since before the bubble bursts, there is uncertainty, the corresponding condition is $E_{T-1}[u'(c_T)P_{T-1}/P_T]\leq E_{T-1}[u'(c_T)R_{T-1,T}]$. Wage rigidity does bind in T, therefore we have unemployment, $L_T<1$. In the notation below, ρ superscripts denote values in the state that the bubble persists, and $1-\rho$ denotes values in the state that the bubble bursts.

$$E_{T-1}[u'(c_T)P_{T-1}/P_T]\leq E_{T-1}[u'(c_T)R_{T-1,T}]$$
$$\frac{\rho u'(c_T^\rho)}{\bar{\Pi}}+(1-\rho)u'(c_T^{1-\rho})\frac{\bar{P}_{T-1}}{P_T^{1-\rho}}\leq\rho u'(c_T^\rho)R_{lb}+(1-\rho)u'(c_T^{1-\rho})R_{T-1,T}^{1-\rho}$$
$$\rho\frac{1}{(1+\tau)}\leq\rho R_{lb}+\frac{\theta a^H(1-(1+\tau)\phi)}{1-h-(1+\tau)\phi}\alpha\left(\frac{w_T}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}}\rho\frac{P_T}{\bar{P}_T}\bar{\Pi}\left(\frac{1}{(1+\tau)}-\frac{1}{\bar{\Pi}}\right)$$
$$\text{where } w_T=\zeta(w_{lb},\bar{\Pi})$$

The above condition can be re-written in terms of bubble steady state values:

$$\left(\frac{\zeta(w_{lb},\bar{\Pi})}{1-\alpha}\right)^{\frac{\alpha^2-1}{\alpha}}\geq\frac{1}{\bar{\Pi}^2}\frac{1-h-(1+\tau)\phi}{\theta a^H(1-(1+\tau)\phi)}\frac{\gamma_n}{(a^L\alpha\gamma_n)^\alpha}\frac{(1+\tau)\bar{\Pi}}{\alpha\left(\bar{\Pi}-(1+\tau)\right)}\left(\frac{1}{(1+\tau)}-R_{lb}\right) \tag{34}$$

Finally, we numerically verify that the parameter space that satisfies all of the above conditions is not empty.

A.2 Proofs

A.2.1 Proof of proposition 1

Proof. Begin with the size of a large bubble ($\tau<\bar{\tau}$) in steady state:

$$\phi_{lb}(\tau)=\frac{\beta\rho(1-h)}{\beta(1-\theta)(1+\tau)}-\frac{\theta}{\beta(1-\theta)}.$$

Given that a large bubble exists, its size on the saddle path must be equal to the steady state size $\forall t$. Thus a necessary and sufficient condition for large bubble existence is:

$$\begin{aligned}
\phi_{lb}(\tau) &= \frac{\beta\rho(1-h)}{\beta(1-\theta)(1+\tau)} - \frac{\theta}{\beta(1-\theta)} > 0 \\
\frac{\theta}{\beta(1-\theta)} &< \frac{\beta\rho(1-h)}{\beta(1-\theta)(1+\tau)} \\
\theta &< \frac{\beta\rho(1-h)}{1+\tau}.
\end{aligned}$$

Now consider the size of a small bubble ($\tau \geq \bar{\tau}$) in steady state:

$$\phi_{sb}(\tau) = \frac{\rho - \frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}}{1 - \frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}} \frac{1-h}{1+\tau}.$$

Once again, given that a small bubble exists, its size on the saddle path must be equal to the steady state size in all t. Thus a necessary and sufficient condition for small bubble existence is:

$$\begin{aligned}
\phi_{sb}(\tau) &= \frac{\rho - \frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}}{1 - \frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}} \frac{1-h}{1+\tau} > 0 \\
0 &< \rho - \frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)} \\
1 &< \left(\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta\right)\rho \\
\rho &> \frac{a^L-\theta a^H}{\beta(a^L-\theta a^H)+\beta h(a^H-a^L)}.
\end{aligned}$$

□

A.2.2 Proof of proposition 2

Proof. Let T denote the period in which the bubble bursts. Define:

$$s^* \equiv \min\left\{s \geq 0 \mid w^f_{T+s}\left(K_{T+s}\right) \geq \gamma^s w_T\right\}.$$

After T: The law of motion for capital after T is identical to that in the bubble-less

environment, except that L_{T+s} may not be one:

$$A_{T+s} = q_{T+s}K_{T+s} = \alpha K_{T+s}^{\alpha} L_{T+s}^{1-\alpha}$$
$$K_{T+s+1} = ha^H \frac{\beta A_{T+s}}{1-\frac{\theta\alpha^H}{\alpha^L}} + a^L \left(\beta A_{T+s} - \frac{\beta p A_{T+s}}{1-\frac{\theta\alpha^H}{\alpha^L}}\right) = \Omega A_{T+s}, \forall s \geq s^*.$$

From the firm's first order conditions, we have $\left(\frac{K_{T+s}}{L_{T+s}}\right)^{\alpha} = \frac{w_{T+s}}{1-\alpha}$ so that the dynamics above can be rewritten in terms of wage:

$$q_{T+s} = \alpha \left(\frac{K_{T+s}}{L_{T+s}}\right)^{\alpha-1} = \alpha \left(\frac{w_{T+s}}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}}$$
$$A_{T+s} = \alpha K_{T+s}^{\alpha} L_{T+s}^{1-\alpha} = \alpha \left(\frac{w_{T+s}}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}} K_{T+s}$$
$$K_{T+s+1} = \alpha\Omega \left(\frac{w_{T+s}}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}} K_{T+s}$$
$$L_{T+s} = \left(\frac{w_{T+s}}{1-\alpha}\right)^{-\frac{1}{\alpha}} K_{T+s}.$$

Between T and $T+s^*-1$: By definition of $s^*(\tau)$, wage rigidity binds, i.e.

$$w_{T+s} = \gamma^s w_T.$$

Thus:

$$K_{T+s+1} = \alpha\Omega \left(\frac{w_T}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}} \gamma^{\frac{\alpha-1}{\alpha}s} K_{T+s}$$
$$L_{T+s} = \left(\frac{\gamma^s w_T}{1-\alpha}\right)^{-\frac{1}{\alpha}} K_{T+s} = \gamma^{-\frac{s}{\alpha}} \frac{K_{T+s}}{K_T}.$$

Proceeding by backward iteration:

$$\begin{aligned} K_{T+s+1} &= \left[\alpha\Omega \left(\frac{w_T}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}}\right]^{s+1} \left(\prod_{i=0}^{s+1} \gamma^{\frac{\alpha-1}{\alpha}i}\right) K_T \\ &= \left[\alpha\Omega \left(\frac{w_T}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}}\right]^{s+1} \gamma^{\frac{\alpha-1}{\alpha}\frac{(s+1)(s+2)}{2}} K_T \\ &= \left[\alpha\Omega K_T^{\alpha-1}\right]^{s+1} \gamma^{\frac{\alpha-1}{\alpha}\frac{(s+1)(s+2)}{2}} K_T. \end{aligned}$$

Finally, turning to the definition of slump length:

$$
\begin{aligned}
s^* &\equiv \min\left\{s \geq 0 \mid w^f_{T+s} \geq \gamma^s w_T\right\} \\
&= \min\left\{s \geq 0 \mid (1-\alpha)K^\alpha_{T+s} \geq \gamma^s w_T\right\} \\
&= \min\left\{s \geq 0 \mid K^\alpha_{T+s} \geq \gamma^s K^\alpha_T\right\} \\
&= \min\left\{s \geq 0 \mid \left[\alpha\Omega K_T^{\alpha-1}\right]^s \gamma^{\frac{\alpha-1}{\alpha}\frac{s(s+1)}{2}} K_T \geq \gamma^{\frac{s}{\alpha}} K_T\right\} \\
&= \min\left\{s \geq 0 \mid s\log_\gamma(\alpha\Omega) + (\alpha-1)s\log_\gamma K_T + \frac{\alpha-1}{\alpha}\frac{s(s+1)}{2} \leq \frac{s}{\alpha}\right\} \\
&= \min\left\{s \geq 0 \mid (\alpha-1)\frac{(s+1)}{2} \leq 1 - \alpha\log_\gamma(\alpha\Omega) - \alpha(\alpha-1)\log_\gamma K_T\right\} \\
&= \min\left\{s \geq 0 \mid s \geq \frac{2\alpha}{1-\alpha}\log_\gamma(\alpha\Omega) - \frac{3-\alpha}{1-\alpha} - 2\alpha\log_\gamma K_T\right\}.
\end{aligned}
$$

Define:

$$
\omega(\gamma) \equiv \frac{2\alpha}{1-\alpha}\log_\gamma(\alpha\Omega) - \frac{3-\alpha}{1-\alpha}.
$$

Then we have:

$$
s^* = \begin{cases} 0 & \gamma = 0 \\ \max\left\{0, \left\lceil \omega(\gamma) - 2\alpha\log_\gamma K_T \right\rceil\right\} & 0 < \gamma < 1 \\ \infty & \gamma = 1 \end{cases}.
$$

Once the slump has ended ($s > s^*$) there are no other external shocks to the economy. Thus the dynamics are identical to the bubble-less environment:

$$
\begin{aligned}
w_{T+s} &= w^f_{T+s} = (1-\alpha)K^\alpha_{T+s} \\
L_{T+s} &= 1 \\
q_{T+s} &= \alpha K^{\alpha-1}_{T+s} \\
K_{T+s+1} &= \Omega A_{T+s} = \alpha\Omega K^\alpha_{T+s}.
\end{aligned}
$$

□

A.2.3 Proof of proposition 3

Proof. Begin by considering the value function for a worker in the post-slump period ($s \geq s^*$). It satisfies:

$$
W_{nb}\left(K_{T+s}\right) = \log c_{T+s^*} + \beta W_{nb}\left(K_{T+s+1}\right)
$$

where:

$$c_{T+s} = w_{T+s} = (1-\alpha)K_{T+s}^{\alpha}$$
$$K_{T+s+1} = \Omega A_{T+s} = \alpha\Omega K_{T+s}^{\alpha}.$$

We guess and verify that the welfare function takes the following functional form: $W_{nb}(K) = f + g\log K$. Given this guess, we have:

$$f + g\log K_{T+s} = \log\left[(1-\alpha)K_{T+s}^{\alpha}\right] + \beta\left(f + g\log\left[\alpha\Omega K_{T+s}^{\alpha}\right]\right).$$

Solving for the coefficients yields:

$$\begin{aligned} g &= \alpha + \alpha\beta g = \frac{\alpha}{1-\alpha\beta} \\ f &= \log[1-\alpha] + \beta f + \beta g\log[\alpha\Omega] \\ &= \frac{1}{1-\beta}\left(\log[1-\alpha] + \frac{\alpha\beta}{1-\alpha\beta}\log[\alpha\Omega]\right). \end{aligned}$$

Thus we have verified our guess and achieve the following solution to W_{nb}:

$$W_{nb}(K) = \frac{1}{1-\beta}\left(\log[1-\alpha] + \frac{\alpha\beta}{1-\alpha\beta}\log[\alpha\Omega]\right) + \frac{\alpha}{1-\alpha\beta}\log K.$$

Recall that dynamics in the post-slump period are equivalent to those in the bubble-less environment. As such, the welfare functions will be of equivalent form.

Now consider the welfare of a worker in the stochastic bubble steady state. It satisfies:

$$W_b(K_t) = \log c_t^w + \beta\left[\rho W_b(K_t) + (1-\rho)W_{burst}(K_{t+1})\right]$$

where:

$$\begin{aligned} c_t^w &= (1-\alpha)(K_t)^{\alpha} + T = \left[(1-\alpha) + \frac{\beta\phi_b}{1-\beta\phi_b}\alpha\tau\right](K_t)^{\alpha} \\ K_{t+1} &= K_t = K_b. \end{aligned}$$

Algebra yields:

$$\begin{aligned} W_b(K_b) &= \frac{\log c_b^w + \beta(1-\rho)W_{burst}(K_b)}{1-\beta\rho} \\ c_b^w &= \left[(1-\alpha) + \frac{\beta\phi_b}{1-\beta\phi_b}\alpha\tau\right](K_b)^{\alpha}. \end{aligned}$$

The welfare during the slump period is by definition dependent on how long the slump lasts. The ex-ante welfare for the worker in the period the bubble bursts satisfies:

$$W_{burst}\left(K_T\right) = \log c_T^w + \sum_{s=1}^{s^*-1} \beta^s \log c_{T+s}^w + \beta^{s^*} W_{nb}\left(K_{T+s^*}\right)$$

where:

$$\begin{aligned}
c_{T+s} &= w_{T+s} L_{T+s} = w_{T+s}\left(\frac{1-\alpha}{w_{T+s}}\right)^{\frac{1}{\alpha}} K_{T+s} = \gamma^{\frac{\alpha-1}{\alpha}s}(1-\alpha)K_T^{\alpha-1}K_{T+s} \\
K_{T+s} &= \gamma^{\left(\frac{\alpha-1}{\alpha}\right)\frac{s(s+1)}{2}}\left[\alpha\Omega K_T(\tau)^{\alpha-1}\right]^s K_T(\tau) \\
s^* &= \begin{cases} \max\left\{0, \omega(\gamma) - 2\alpha\log_\gamma K_T\right\} & 0 \le \gamma < 1 \\ \infty & \gamma = 1 \end{cases}.
\end{aligned}$$

A proof for the form of the constants can be found in proposition 2. Substituting in constants and grouping terms yields the form in the main text. □

A.2.4 Proof of proposition 4

Proof. It is sufficient to show that $W_{nb} > W_b$ as $\gamma \to 1$. Recall that as $\gamma \to 1$, the slump length approaches infinity. Then the limiting welfare of a worker in the stochastic bubbly steady state is given by the following:

$$\begin{aligned}
W_b(K_b) &= \frac{\log c_b^w + \beta(1-\rho)W_{burst}(K_b)}{1-\beta\rho} \\
c_b^w &= \left[(1-\alpha) + \frac{\beta\phi_b}{1-\beta\phi_b}\alpha\tau\right](K_b)^\alpha.
\end{aligned}$$

However, due to an infinite slump, the worker welfare in time of burst changes to the following:

$$\begin{aligned}
W_{burst}\left(K_b\right) &\equiv \log\left[(1-\alpha)(K_b)^\alpha\right] + \sum_{s=1}^{\infty} \beta^s \log c_{T+s}^w \\
\log c_{T+s}^w &= \Gamma_1(s) - \left((1-\alpha)s - \alpha\right)\log K_b,\ \forall s \ge 1 \\
\Gamma_1(s) &\equiv \frac{\alpha-1}{\alpha}\frac{s^2+3s}{2}\log\gamma + \log(1-\alpha) + s\log\left(\alpha\Omega\right).
\end{aligned}$$

With $\tau = 0$ and $\gamma = 1$, the worker welfare reduces to the expression below:

$$\begin{aligned} W_b\left(K_b\right) =& \frac{1}{1-\beta}\log(1-\alpha) + \frac{\beta^2(1-\rho)}{(1-\beta)^2(1-\beta\rho)}\log(\alpha\Omega) \\ &+ \left(\frac{\alpha}{1-\beta\rho} + \frac{\alpha\beta(1-\rho)(\alpha-\beta)}{(1-\beta\rho)(1-\beta)^2} - \frac{\beta^2(1-\rho)}{(1-\beta\rho)(1-\beta)^2}\right)\log K_b. \end{aligned}$$

The welfare in the bubble-less environment remains the same, and is slightly re-written using the value of bubble-less steady state capital.

$$\begin{aligned} W_{nb}(K_{nb}) &= \frac{1}{1-\beta}\log(1-\alpha) + \frac{1}{1-\beta}\frac{\beta\alpha}{1-\beta\alpha}\log(\alpha\Omega) + \frac{\alpha}{1-\beta\alpha}\log \underbrace{K_{nb}}_{=(\alpha\Omega)^{\frac{1}{1-\alpha}}} \\ &= \frac{1}{1-\beta}\log(1-\alpha) + \frac{\alpha}{(1-\beta)(1-\alpha)}\log(\alpha\Omega). \end{aligned}$$

Comparing $W_b(K_b)$ to $W_{nb}(K_{nb})$, we derive the following expression. Note that since we are considering expansionary bubbles, $K_b > K_{nb}$.

$$W_{nb}(K_{nb}) - W_b(K_b) = \underbrace{\frac{1}{(1-\beta\rho)(1-\beta)^2}}_{>0}\left(\beta^2(1-\rho) + \alpha\beta(1-\beta+\rho) - \alpha\right)\underbrace{\left(\log K_b - \log K_{nb}\right)}_{>0}.$$

Then, all that is necessary for $W_{nb}(K_{nb}) > W_b(K_b)$ at $\tau = 0$ and $\gamma = 1$, is the following:

$$\beta^2(1-\rho) + \alpha\beta(1-\beta+\rho) - \alpha > 0.$$

The above condition is true if and only if the risk of the bubble bursting, $1-\rho$, satisfies:

$$\beta(\beta-\alpha)(1-\rho) > \alpha(1-\beta)^2.$$

□

A.2.5 Proof of proposition 5

Proof. Recall that $R_{T+1} = a^L\alpha K_{T+1}^{\alpha-1}L_{T+1}^{1-\alpha}$. Since $L_{T+1} \leq 1$, a sufficient condition for $R_{T+1}\bar{\Pi} < 1$ is that $a^L\alpha K_{T+1}^{\alpha-1} < \frac{1}{\bar{\Pi}}$. Furthermore, from the post-bubble equilibrium dynamics, we have $K_{T+1} = \alpha\Omega K_T^{\alpha} = \alpha\Omega K_{lb}^{\alpha}$. Therefore, $a^L\alpha K_{T+1}^{\alpha-1} < \frac{1}{\bar{\Pi}}$ if and only if $a^L\alpha\left(\alpha\Omega K_{lb}^{\alpha}\right)^{\alpha-1} < \frac{1}{\bar{\Pi}}$. Equivalently,

$$K_{lb} > \left(a^L\bar{\Pi}\right)^{\frac{1}{\alpha(1-\alpha)}}\alpha^{\frac{1}{1-\alpha}}\Omega^{-1/\alpha}.$$

$\square$

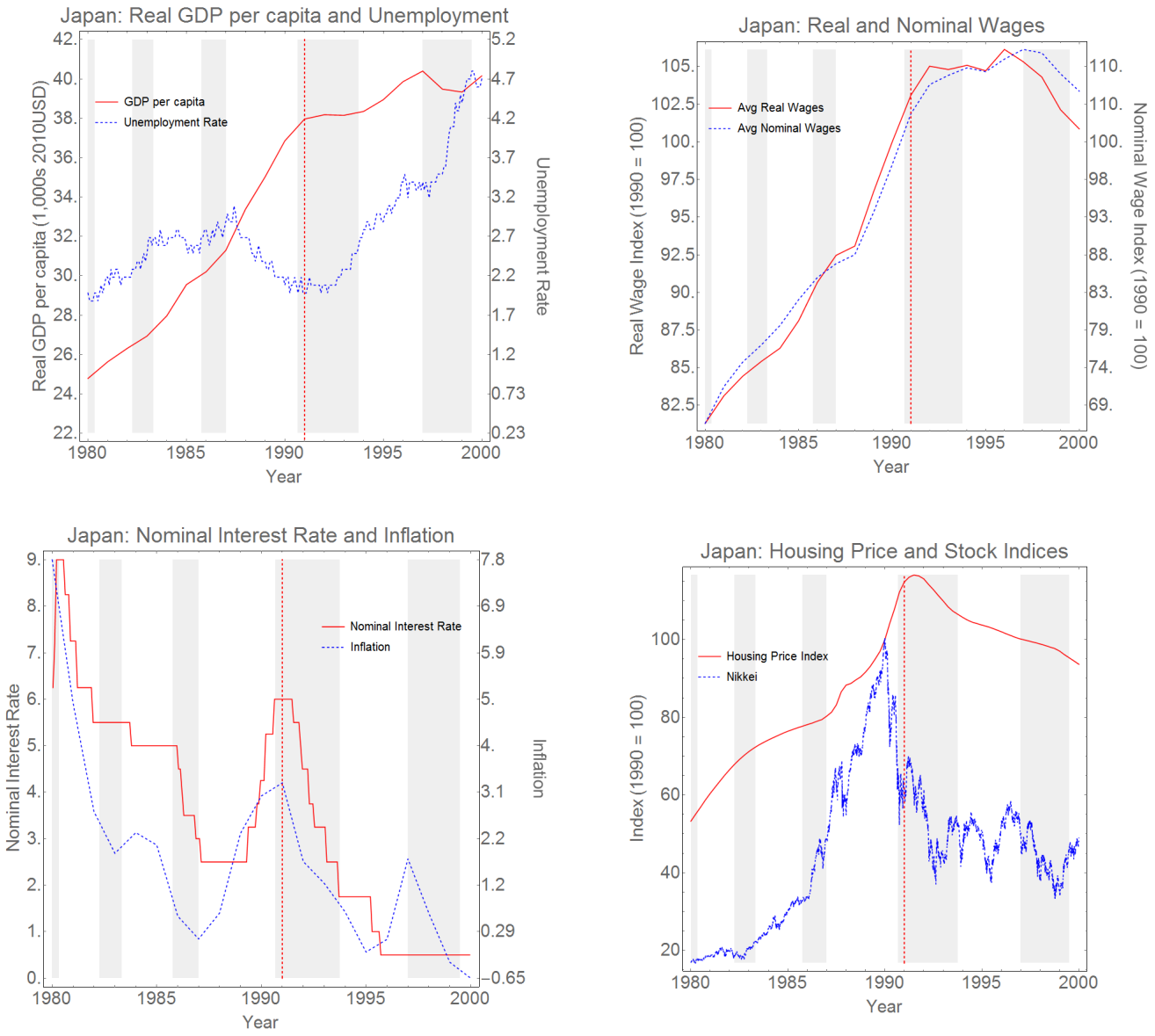

Figure 1: Japan before and after the collapse of asset prices. Dashed vertical lines indicate the approximate beginning of the collapse in asset prices (1991). Grey bars indicate recessions, according to the OECD. Real wages are calculated from nominal wages and consumer price indices. The nominal interest rate refers to the discount rate of commercial bills and interest rates on loans secured by government bonds, specially designated securities and bills corresponding to commercial bills. Sources: Statistics Bureau of Japan, OECD, IMF, FRB St. Louis, and Mack et al. (2011).

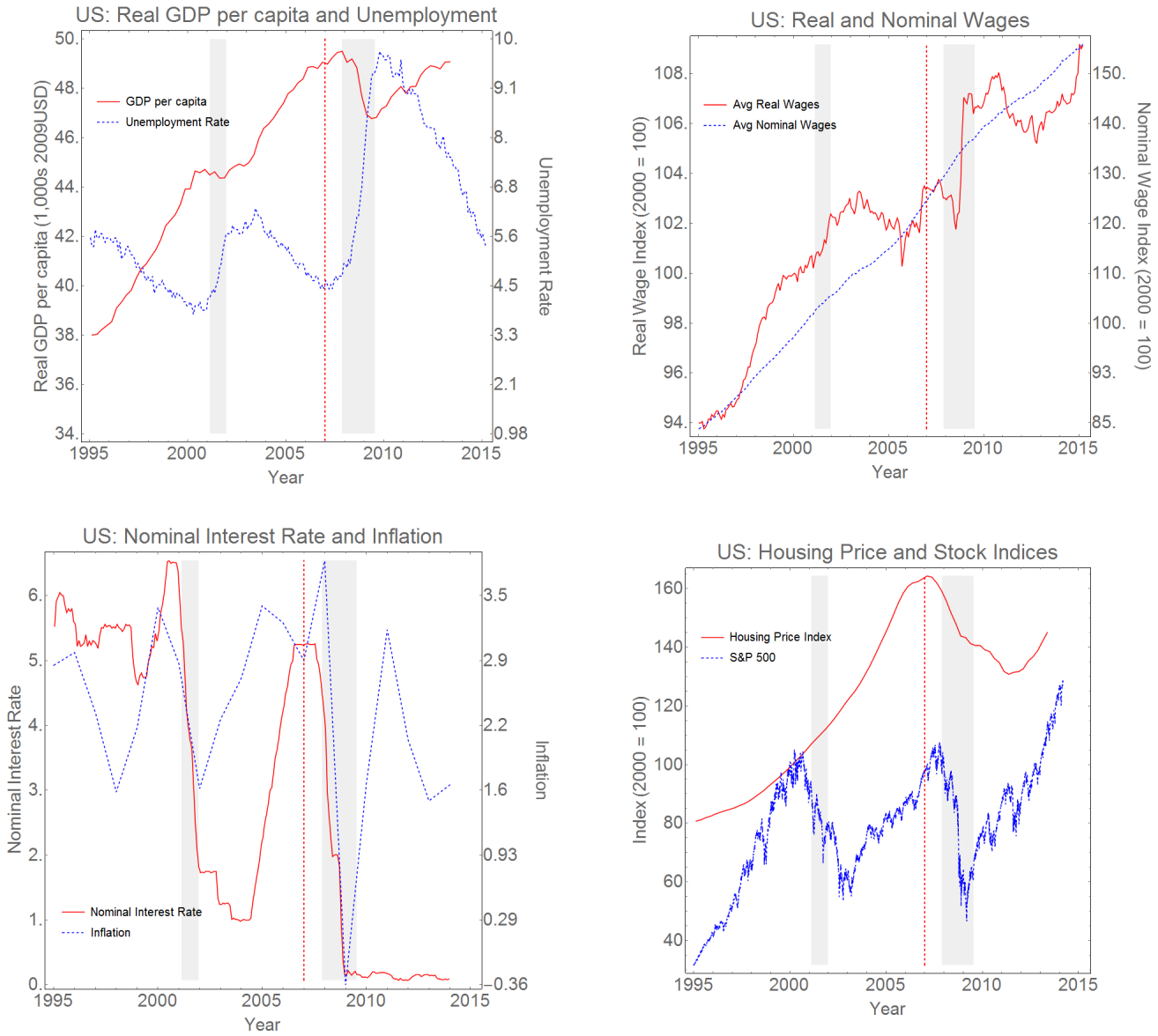

Figure 2: U.S. before and after the collapse of asset prices. Dashed vertical lines indicate the approximate beginning of the collapse in asset prices (2007). Grey bars indicate recessions, according to the NBER. Real wages are calculated from nominal wages and consumer price indices. The nominal interest rate refers to the effective federal funds rate. Sources: NBER, OECD,US Bureau of Labor Statistics, US Bureau of Economic Analysis, S&P500, US Federal Housing Finance Agency, and FRB St. Louis.

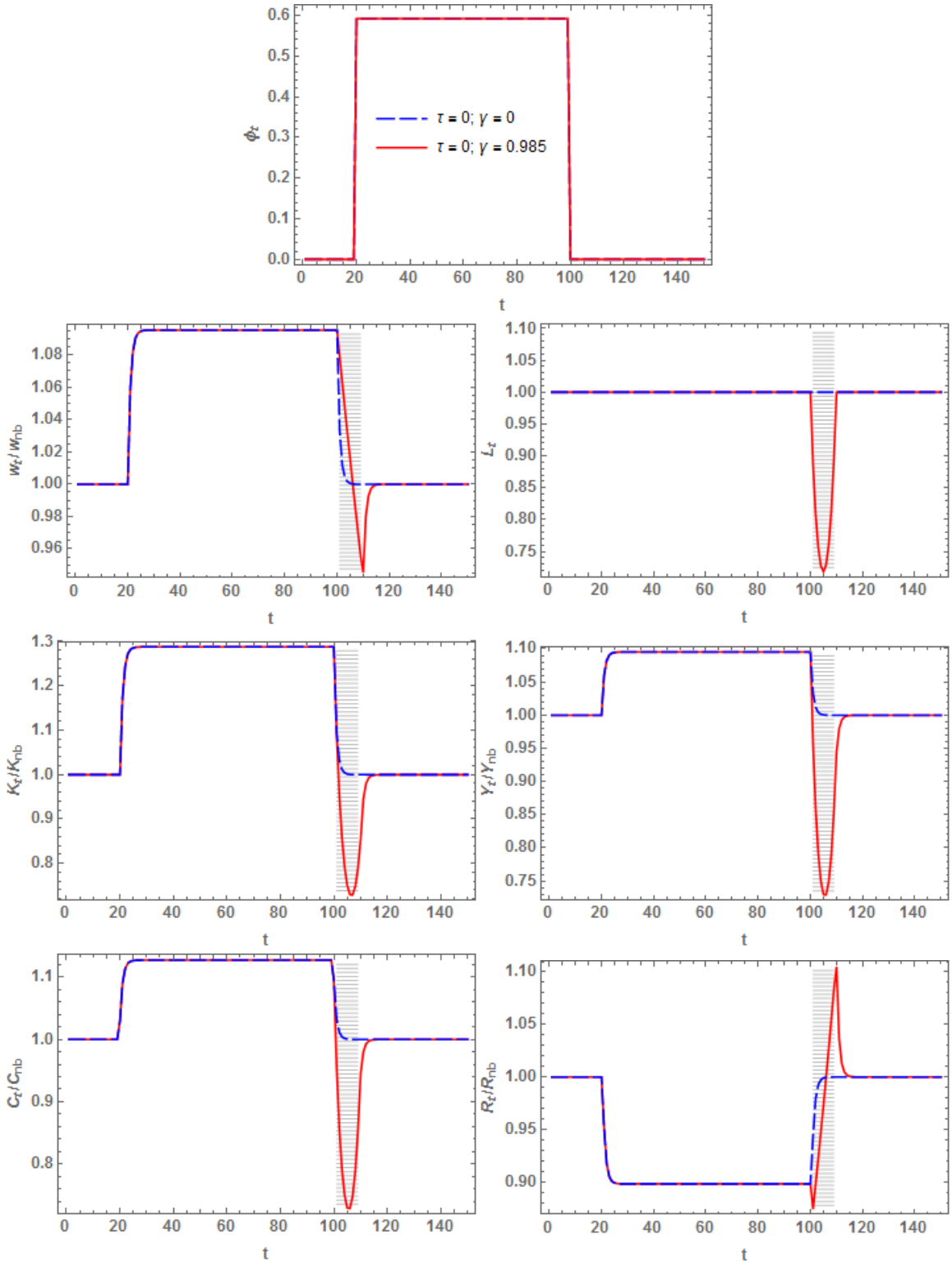

Figure 3: Equilibrium dynamics of aggregate variables through a boom-bust cycle of an expansionary bubble.

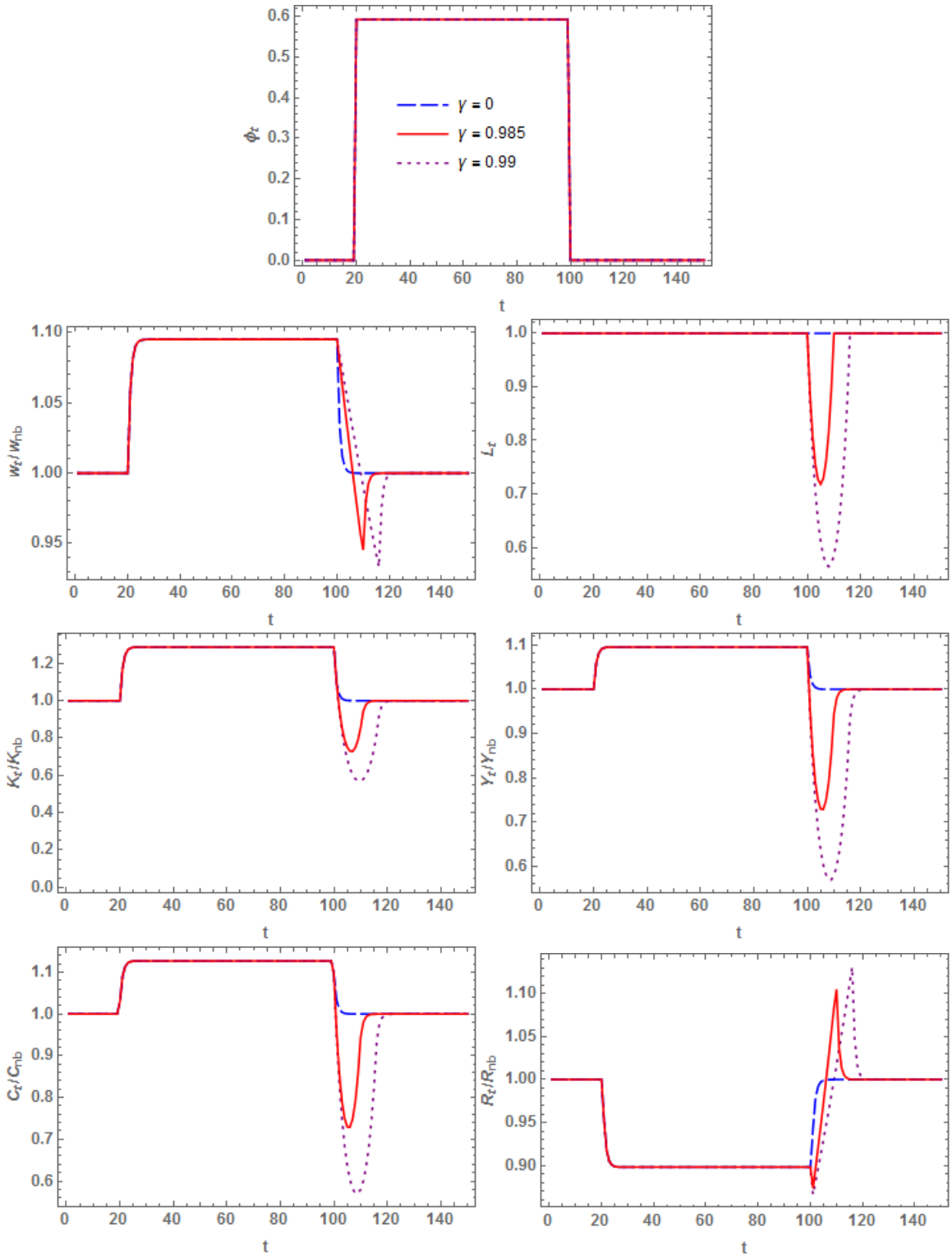

Figure 4: Equilibrium dynamics with different wage rigidity parameters.

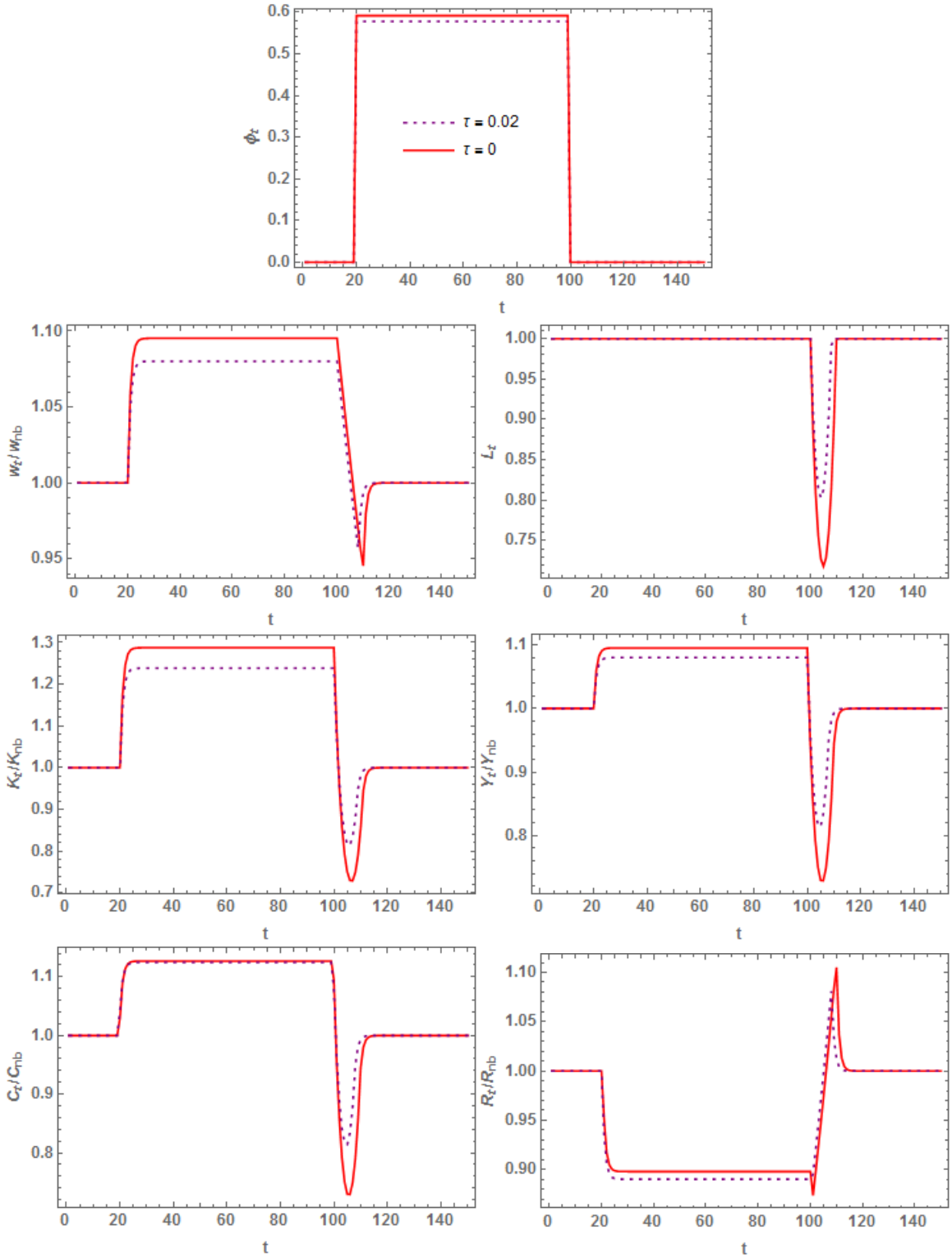

Figure 5: Effects of bubble tax.

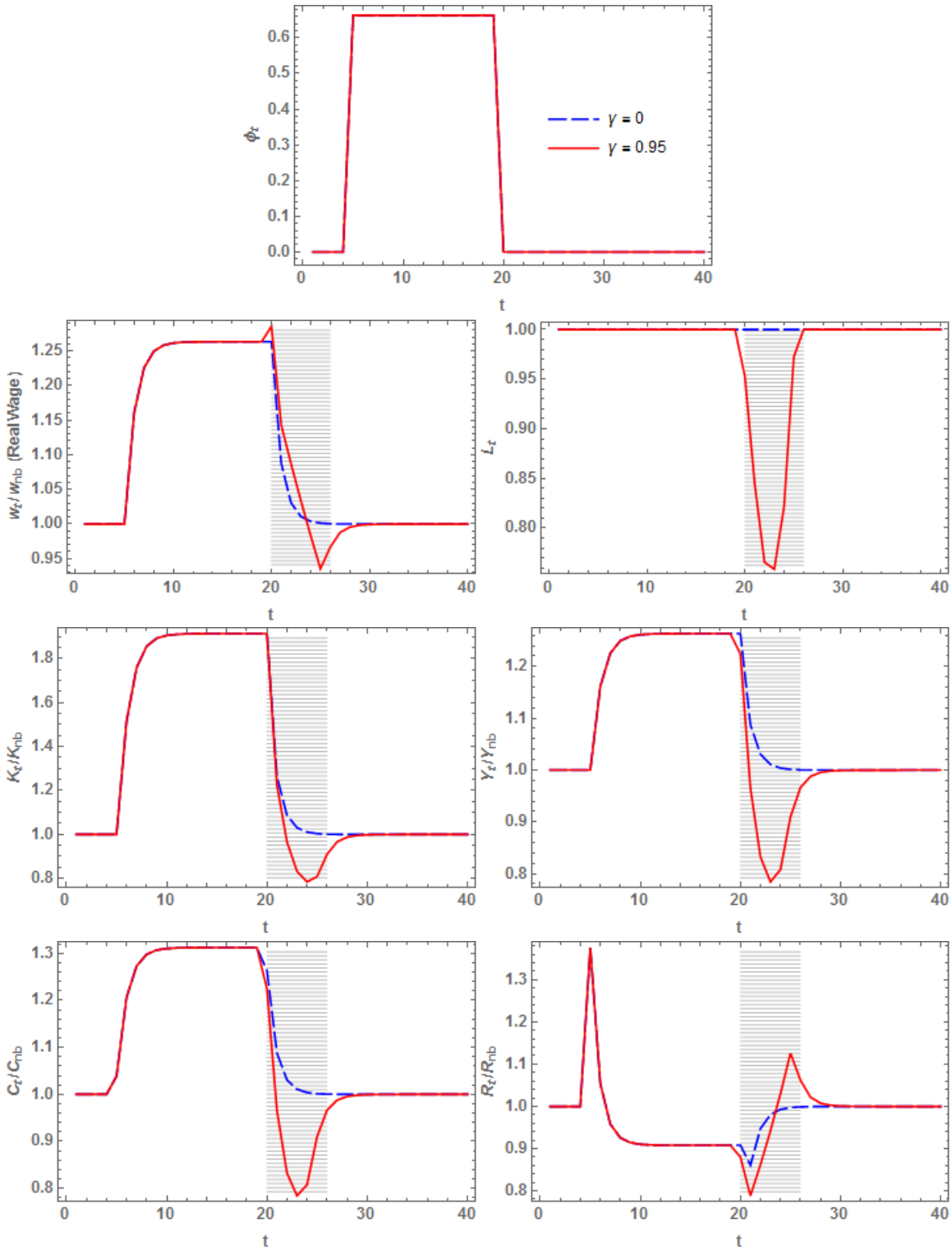

Figure 6: Equilibrium dynamics with post-bubble liquidity trap.