Supplementary Material: What Inventory Behavior Tells Us About How Business Cycles Have Changed

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Contents

1 The Planner's Problem

The allocations in the model can be found as the solution to the following planner's problem:

$$\max\ E_0 \sum_{t=0}^{\infty} \left[\prod_{0}^{t-1} \zeta_s \right] \beta^t \left[\kappa \ln C_t + (1-\kappa) \ln \left(1 - \Upsilon_t L_t \right) \right],\ 1 > \kappa > 0,$$

subject to (Lagrangians are in parenthesis)

$$C_t = \prod_{j=1}^{N} \Lambda C_{j,t}^{\eta_j},\ \eta_j \geq 0,\ \sum_{j=1}^{N} \eta_j = 1,\ \Lambda > 0,$$

$$C_{j,t} + \sum_{i=1}^{N} I_{ji,t} + \sum_{i=1}^{N} M_{ji,t} = Y_{j,t},\ j = \{1, .., N\},\ \left(\lambda_{j,t} \beta^t \prod_{0}^{t-1} \zeta_s \right), \tag{1}$$

$$K_{j,t+1} = X_{j,t} + (1-\delta) K_{j,t}, j = \{1, .., N\},\ \left(\mu_{j,t} \beta^t \prod_{0}^{t-1} \zeta_s \right),\ \delta \in [0,1],$$

where

$$X_{j,t} = \Xi_j \prod_{i=1}^{N} I_{ij,t}^{\theta_{ij}},\ \theta_{ij} \geq 0, \sum \theta_{ij} = 1, \Xi_j > 0,\ j = \{1, .., N\}, \tag{2}$$

$$Y_{j,t} = \left(B_j \sum_{s=0}^{S} \omega_j(s)^{\frac{1}{\varrho}} Z_{j,t-s|t}^{\frac{\varrho-1}{\varrho}} \right)^{\frac{\varrho}{\varrho-1}}, \ \varrho > 0, \omega_j(s) > 0, \ B_j > 0, \ \ j = \{1,..,N\},$$

where

$$\begin{aligned} Z_{j,t|t+s} &= K_{j,t|t+s}^{\alpha_j} \Pi_{i=1}^{N} M_{ij,t|t+s}^{\gamma_{ij}} \left(A_{j,t} L_{j,t|t+s} \right)^{\xi_j}, \\ \alpha_j &\geq 0, \ \gamma_{ij} \geq 0, \ \xi_i \geq 0, \ \alpha_j + \sum_{i=1}^{N} \gamma_{ij} + \xi_j = 1, \ j = \{1,..,N\}, \ s = \{0,...,S\}, \end{aligned} \tag{3}$$

where $\left\{ Z_{j,-s|-s+v} \right\}_{\{s,v\}=\{1,s\}}^{\{S,S\}}$ are predetermined.

Additional constraints are:

$$\begin{aligned} \sum_{j=1}^{N} \sum_{s=0}^{S} L_{j,t|t+s} &= L_t, \ \left(\nu_t \beta^t \prod_{0}^{t-1} \zeta_s \right), \\ \sum_{s=0}^{S} K_{j,t|t+s} &= K_{j,t}, \left(\chi_{j,t} \beta^t \prod_{0}^{t-1} \zeta_s \right), \\ A_{j,t} &= u_t \Gamma_t a_{j,t}, \ \ j = \{1,..,N\}, \end{aligned} \tag{4}$$

where

$$\frac{\Gamma_t}{\Gamma_{t-1}} = g_t,$$

and $\{u_t, g_t, \zeta_t, \Upsilon_t, A_{jt}\}$ are random variables with unconditional mean $\{1, g, 1, 1, 1\}$ and values known at t or after

The variable definitions are as follows:

ζ_t	:	Preference shock affecting discount rates between dates t and $t+1$
Υ_t	:	Preference shock affecting labor supply at date t
C_t	:	Aggregate Consumption at date t
L_t	:	Aggregate Hours Worked at date t
$C_{j,t}$	:	Consumption in sector j at date t
$I_{ji,t}$	:	Good j used for investment in capital of sector i at date t
$M_{ji,t\|t+s}$	:	Good j used at t as inputs in sector i for production of good available at $t+s$
$K_{j,t\|t+s}$	:	Fixed capital stock in sector j used at date t for production of good available at $t+s$
$L_{j,t\|t+s}$	:	Hours worked in sector j at date t for production of good available at $t+s$
$X_{j,t}$	:	Gross fixed investment in sector j performed at date t
$Y_{j,t}$	:	Goods or Services from sector j used at date t
$Z_{j,t\|t+s}$	:	Output of sector j produced at t and available for use at date $t+s$
$A_{j,t}$	:	Hicks neutral TFP shock capturing efficiency wedge in the production of sector j at date t
u_t	:	Common transitory component of Hicks neutral TFP shock at date t
Γ_t	:	Common permanent component of Hicks neutral TFP shock at date t
$a_{j,t}$	:	Sector specific component of Hicks neutral TFP shock at date t

2 First-order necessary conditions

The F.O.C.'s are:

$$\begin{aligned} L_t &: \quad (1-\kappa)\frac{\Upsilon_t}{1-\Upsilon_t L_t} = \nu_t, \\ K_{j,t+1} &: \quad \mu_{j,t} = \beta\zeta_t E_t\left[(1-\delta)\,\mu_{j,t+1} + \chi_{j,t+1}\right],\ j = \{1,..,N\}, \end{aligned} \tag{5}$$

$$C_{jt} : \kappa\frac{\eta_j}{C_{j,t}} = \lambda_{j,t},\ j = \{1,..,N\},$$

$$I_{ijt} : \lambda_{i,t} = \mu_{j,t}\frac{\theta_{ij}X_{j,t}}{I_{ij,t}},\ i,j = \{1,..,N\}.$$

Furthermore, for $i,j = \{1,..,N\}$ and $s \in \{1,..,S\}$,

$$M_{ij,t|t+s} \quad : \quad \lambda_{i,t} = \gamma_{ij} \frac{Z_{j,t|t+s}}{M_{ij,t|t+s}} \beta^s E_t \left[\left(\prod_{u=0}^{s-1} \zeta_{t+u} \right) B_j \left[\frac{Z_{j,t|t+s}}{\omega_j(s) Y_{j,t+s}} \right]^{-\frac{1}{\varrho}} \lambda_{j,t+s} \right], \tag{6}$$

$$L_{j,t|t+s} \quad : \quad \nu_t = \xi_j \frac{Z_{j,t|t+s}}{L_{j,t|t+s}} \beta^s E_t \left[\left(\prod_{u=0}^{s-1} \zeta_{t+u} \right) B_j \left[\frac{Z_{j,t|t+s}}{\omega_j(s) Y_{j,t+s}} \right]^{-\frac{1}{\varrho}} \lambda_{j,t+s} \right], \tag{7}$$

$$K_{j,t|t+s} \quad : \quad \chi_{j,t} = \alpha_j \frac{Z_{j,t|t+s}}{K_{j,t|t+s}} \beta^s E_t \left[\left(\prod_{u=0}^{s-1} \zeta_{t+u} \right) B_j \left[\frac{Z_{j,t|t+s}}{\omega_j(s) Y_{j,t+s}} \right]^{-\frac{1}{\varrho}} \lambda_{j,t+s} \right], \tag{8}$$

$$\begin{aligned}
M_{ij,t|t} \quad &: \quad \lambda_{i,t} = \gamma_{ij} \frac{Z_{j,t|t}}{M_{ij,t|t}} B_j \left[\frac{Z_{j,t|t}}{\omega_j(s) Y_{j,t}} \right]^{-\frac{1}{\varrho}} \lambda_{j,t} \ , \ i,j = \{1,..,N\}\,, \\
L_{j,t} \quad &: \quad \nu_t = \xi_j \frac{Z_{j,t|t}}{L_{j,t|t}} B_j \left[\frac{Z_{j,t|t}}{\omega_j(s) Y_{j,t}} \right]^{-\frac{1}{\varrho}} \lambda_{j,t} \ , \ j = \{1,..,N\}\,, \\
K_{j,t} \quad &: \quad \chi_{j,t} = \alpha_j \frac{Z_{j,t|t}}{K_{j,t|t}} B_j \left[\frac{Z_{j,t|t}}{\omega_j(s) Y_{j,t}} \right]^{-\frac{1}{\varrho}} \lambda_{j,t} \ , \ j = \{1,..,N\}\,.
\end{aligned}$$

3 Aggregation of Composite Goods into General Sectoral Output $Z_{j,t}$

We assume that, for the production of any given $Z_{j,t|t+v}$, the factor elasticities α_j, γ_{ij} and ξ_j do not depend on v. This allows us to linearly aggregate the composite goods associated with each production stage into a general output, $Z_{j,t}$, that we can then use to considerably simplify notation and better highlight the mechanisms behind the model. In particular, we now show that under the assumption that factor elasticities are independent of the stages of production, the aggregate of output at different stages may be described as arising directly from a Cobb-Douglas production technology such that,

$$Z_{j,t} = \sum_{s=0}^{S} Z_{j,t|t+s}, \tag{9}$$

where

$$Z_{j,t} = K_{j,t}^{\alpha_j} \Pi_{i=1}^{S} M_{ij,t}^{\gamma_{ij}} \left(A_{j,t} L_{j,t}\right)^{\xi_j}, \tag{10}$$

with

$$L_{j,t} = \sum_{s=0}^{S} L_{j,t|t+s},\ K_{j,t} = \sum_{s=0}^{S} K_{j,t|t+s},\ M_{ij,t} = \sum_{s=0}^{S} M_{ij,t|t+s}. \tag{11}$$

We demonstrate the result by showing that the first order conditions of the original problem are equivalent to those of a problem featuring the constraints (9) through (11) above.

To see this, note that, we can rearrange the first order conditions (6) through (8) and substitute them in the production function for $Z_{j,t|t+s}$'s (3) to write for all $s \in \{0, ..., S\}$

$$Z_{j,t|t+s} = \left(\frac{\alpha_j}{\chi_{j,t}}\right)^{\alpha_j} \left(A_t \frac{\xi_j}{\nu_t}\right)^{\xi_j} \Pi_{i=1}^{N} \left(\frac{\gamma_{ij}}{\lambda_{i,t}}\right)^{\gamma_{ij}} \beta^s E_t \left[\left(\prod_{u=0}^{s-1} \zeta_{t+u}\right) B_j \left[\frac{Z_{j,t|t+s}}{\omega_j(s) Y_{j,t+s}}\right]^{-\frac{1}{\varrho}} \lambda_{j,t+s}\right] Z_{j,t|t+s}.$$

Since $Z_{j,t|t+s}$ appears on both sides of the equations, it cancels out. With some rearrangement,

$$\begin{aligned} &\beta^s E_t \left[\left(\prod_{u=0}^{s-1} \zeta_{t+u}\right) B_j \left[\frac{Z_{j,t|t+s}}{\omega_j(s) Y_{j,t+s}}\right]^{-\frac{1}{\varrho}} \lambda_{j,t+s}\right] \\ = &\left(\frac{\alpha_j}{\chi_{j,t}}\right)^{-\alpha_j} \left(A_t \frac{\xi_j}{\nu_t}\right)^{-\xi_j} \Pi_{i=1}^{N} \left(\frac{\gamma_{ij}}{\lambda_{i,t}}\right) Z_{j,t|t+s}. \end{aligned} \tag{12}$$

Define $\phi_{j,t}$ as:

$$\phi_{j,t} \equiv \frac{\left(\frac{\alpha_j}{\chi_{j,t}}\right)^{-\alpha_j} \left(\frac{\xi_j}{\nu_t}\right)^{-\xi_j} \Pi_{i=1}^{N} \left(\frac{\gamma_{ij}}{\lambda_{i,t}}\right)^{-\gamma_{ij}}}{\lambda_{j,t}}. \tag{13}$$

This can be interpreted as the ratio between the unit cost index for production in sector j (in the numerator) and the marginal value for a household of consuming one more unit of that same good (in the denominator). Therefore, for any $s \in \{0, ..., S\}$, we can rewrite equation (12) as:

$$\beta^s E_t \left[\left(\prod_{u=0}^{s-1} \zeta_{t+u}\right) B_j \left[\frac{Z_{j,t|t+s}}{\omega_j(s) Y_{j,t+s}}\right]^{-\frac{1}{\varrho}} \lambda_{j,t+s}\right] = \phi_{j,t} \lambda_{j,t} Z_{j,t|t+s}. \tag{14}$$

With some slight rearrangement we can use equation (14) to rewrite the first order conditions for $L_{j,t|t+s}$, $K_{j,t|t+s}$ and $M_{ij,t|t+s}$ in equations (6) through (8) more succinctly as:

$$\begin{aligned}
M_{ij,t|t+s} &: \lambda_{i,t} = \gamma_{ij}\frac{Z_{j,t|t+s}}{M_{ij,t|t+s}}\phi_{j,t}\lambda_{j,t},\ i,j = \{1,..,N\},\ s \in \{0,..,S\},\\
L_{j,t} &: \nu_t = \xi_j\frac{Z_{j,t|t+s}}{L_{j,t|t+s}}\phi_{j,t}\lambda_{j,t}\ ,\ j = \{1,..,N\},\ s \in \{0,..,S\},\\
K_{j,t} &: \chi_{j,t} = \alpha_j\frac{Z_{j,t|t+s}}{K_{j,t|t+s}}\phi_{j,t}\lambda_{j,t}\ ,\ j = \{1,..,N\},\ s \in \{0,..,S\}.
\end{aligned}$$

With some rearrangement, we can aggregate the conditions to get for $i,j = \{1,..,N\}$ and $s \in \{0,..,S\}$

$$\begin{aligned}
M_{ij,t|t+s} &: \lambda_{i,t}\sum_{s=0}^{S} M_{ij,t|t+s} = \gamma_{ij}\left(\sum_{s=0}^{S} Z_{j,t|t+s}\right)\phi_{j,t}\lambda_{j,t},\\
L_{j,t} &: \nu_t\left(\sum_{s=0}^{S} L_{j,t|t+s}\right) = \xi_j\left(\sum_{s=0}^{S} Z_{j,t|t+s}\right)\phi_{j,t}\lambda_{j,t},\\
K_{j,t} &: \chi_{j,t}\left(\sum_{s=0}^{S} K_{j,t|t+s}\right) = \alpha_j\left(\sum_{s=0}^{S} Z_{j,t|t+s}\right)\phi_{j,t}\lambda_{j,t},
\end{aligned}$$

which, given the definitions (11) above are the same as:

$$\begin{aligned}
M_{ij,t|t+s} &: \lambda_{i,t}M_{ij,t} = \gamma_{ij}Z_{j,t}\phi_{j,t}\lambda_{j,t},\ i,j = \{1,..,N\},\ s \in \{0,..,S\},\\
L_{j,t} &: \nu_t L_{j,t} = \xi_j Z_{j,t}\phi_{j,t}\lambda_{j,t}\ ,\ j = \{1,..,N\},\ s \in \{0,..,S\},\\
K_{j,t} &: \chi_{j,t}K_{j,t} = \alpha_j Z_{j,t}\phi_{j,t}\lambda_{j,t}\ ,\ j = \{1,..,N\},\ s \in \{0,..,S\}.
\end{aligned}$$

To complete the proof, write the alternative planner's problem:

$$\max\ E_t\sum_{t=0}^{\infty}\left[\prod_{0}^{t-1}\zeta_s\right]\beta^t\left[\kappa\ln C_t + (1-\kappa)\ln\left(1-\Upsilon_t L_t\right)\right],\ 1 > \kappa > 0$$

subject to (Lagrangians are in parenthesis)

$$\begin{aligned}
C_t &= \prod_{j=1}^{N}\Lambda C_{j,t}^{\eta_j},\ \eta_j \geq 0,\ \sum_{j=1}^{N}\eta_j = 1,\ \Lambda > 0,\\
C_{j,t} + \sum_{i=1}^{N} I_{ji,t} + \sum_{i=1}^{N} M_{ji,t} &= Y_{j,t},\ j = \{1,..,N\},\ \left(\lambda_{j,t}\beta^t\prod_{0}^{t-1}\zeta_s\right),
\end{aligned}$$

$$K_{j,t+1} = X_{j,t} + (1-\delta)K_{j,t}, j = \{1,..,N\}\,,\ \left(\mu_{j,t}\beta^t \prod_0^{t-1} \zeta_s\right), \, \delta \in [0,1]\,,$$

where, for $j \in \{1, ..., N\}$,

$$X_{j,t} = \Xi_j \prod_{i=1}^{N} I_{ij,t}^{\theta_{ij}},\ \theta_{ij} \geq 0, \sum \theta_{ij} = 1, \Xi_j > 0,$$

$$Y_{j,t} = \left(B_j \sum_{s=0}^{S} \omega_j(s)^{\frac{1}{\varrho}} Z_{j,t-s|t}^{\frac{\varrho-1}{\varrho}}\right)^{\frac{\varrho}{\varrho-1}},\ \varrho > 0, \omega_j\,(s) > 0,\ B_j > 0,$$

where for $j = \{1, .., N\}$ and $s = \{0, ..., S\}$

$$Z_{j,t} = K_{j,t}^{\alpha_j} \Pi_{i=1}^{N} M_{ij,t}^{\gamma_{ij}} \left(A_{j,t}L_{j,t}\right)^{\xi_i},\ \alpha_j \geq 0,\ \gamma_{ij} \geq 0,\ \xi_i \geq 0,\ \alpha_j + \sum_{i=1}^{N} \gamma_{ij} + \xi_i = 1,$$

and

$$\begin{aligned}
\sum_{s=0}^{S} Z_{j,t|t+s} &= Z_{j,t},\ \left(\phi_{j,t}\lambda_{j,t}\right), \\
\sum_{j=1}^{N} L_{j,t} &= L_t,\ \left(\nu_t \beta^t \prod_0^{t-1} \zeta_s\right),
\end{aligned}$$

$$A_{j,t} = u_t \Gamma_t a_{j,t},\ j = \{1, .., N\}\,,$$

where

$$\frac{\Gamma_t}{\Gamma_{t-1}} = g_t,$$

and $\{u_t, g_t, \zeta_t, \Upsilon_t, A_{jt}\}$ are random variables with unconditional mean $\{1, g, 1, 1, 1\}$ and values known at t or after.

The first order conditions for that problem are:

$$\begin{aligned}
L_t &: \ (1-\kappa)\frac{\Upsilon_t}{1-\Upsilon_t L_t} = \nu_t, \qquad (15)\\
K_{j,t+1} &: \ \mu_{j,t} = \beta\zeta_t E_t\left[(1-\delta)\,\mu_{j,t+1} + \chi_{j,t+1}\right],\ j = \{1, .., N\}\,,
\end{aligned}$$

$$C_{jt} : \kappa \frac{\eta_j}{C_{j,t}} = \lambda_{j,t},\ j = \{1, .., N\}\,, \qquad (16)$$

$$I_{ijt} : \lambda_{i,t} = \mu_{j,t} \frac{\theta_{ij} X_{j,t}}{I_{ij,t}},\ i,j = \{1,..,N\},$$

$$\begin{aligned}
M_{ij,t} &: \lambda_{i,t} = \gamma_{ij} \frac{Z_{j,t}}{M_{ij,t}} \phi_{j,t} \lambda_{j,t},\ i,j = \{1,..,N\}, \\
L_{j,t} &: \nu_t = \xi_j \frac{Z_{j,t}}{L_{j,t}} \phi_{j,t} \lambda_{j,t}\ ,\ j = \{1,..,N\}, \\
K_{j,t} &: \chi_{j,t} = \alpha_j \frac{Z_{j,t}}{K_{j,t}} \phi_{j,t} \lambda_{j,t}\ ,\ j = \{1,..,N\},
\end{aligned}$$

$$\begin{aligned}
\phi_{j,t} \lambda_{j,t} &= \beta^s E_t \left[\left(\prod_{u=0}^{s-1} \zeta_{t+u} \right) B_j \left[\frac{Z_{j,t|t+s}}{\omega_j(s) Y_{j,t+s}} \right]^{-\frac{1}{\varrho}} \lambda_{j,t+s} \right],\ j \in \{1,.,N\},\ s \in \{1,.,S\}, \\
\phi_{j,t} &= B_j \left[\frac{Z_{j,t|t}}{\omega_j(s) Y_{j,t}} \right]^{-\frac{1}{\varrho}},\ j \in \{1,...,N\}.
\end{aligned}$$

4 Decentralization

We now show how to decentralize the model. The decentralization serves two purposes:

i) it establishes an equivalence between the lagrangians in the planner's problem and prices. This is important since it allows to construct "real" aggregates analogous to the ones in the NIPAs that can be compared to the data.

ii) this decentralization establishes an equivalence between the preference shocks and tax wedges similar to the ones considered by Chari, Kehoe, and McGrattan (2007a), and Christiano and Davis (2006). In particular, we show that the discount rate shock ζ_t is observationally equivalent to a wedge reflecting a tax on capital income, which Christiano and Davis (2006) show best captures the financial frictions in Carlstrom and Fuerst (1997), as well as Bernanke, Gertler, and Gilchrist (1999), rather than a tax on investment emphasized by Chari, Kehoe, and McGrattan (2007a). Chari, Kehoe, and McGrattan (2007b) show that, in the context of their model, capital and investment wedges have similar implications, so that we refer to the two wedges interchangeably.

The decentralized model features a household and N firms (one in each sector). We now describe the problems that each solves.

4.1 The Representative Household

The household solves:

$$\max_{\left\{C_t,\{C_{j,t}\}_{j=1}^N,L_t,V_{j,t}\right\}_{t=0}^{\infty}} E_0\sum_{t=0}^{\infty}\beta^t\left[\kappa\ln C_t+(1-\kappa)\ln(1-L_t)\right],\ 1>\kappa>0,$$

subject to (lagrangians in parenthesis)

$$\begin{aligned}C_t &= \prod_{j=1}^N \Lambda C_{j,t}^{\eta_j},\ \eta_j\geq 0,\ \sum_{j=1}^N\eta_j=1,\ \Lambda>0,\\ \sum_{j=1}^N p_{j,t}C_{j,t}+\sum_{j=1}^N q_{j,t}V_{j,t+1} &= \sum_{j=1}^N(1-\tau_{K,t})(d_{j,t}+q_{j,t})V_{j,t}+(1-\tau_{L,t})w_tL_t,\ \left(\beta^t\pi_t^C\right)\end{aligned}$$

where $V_{j,t}^H$ is the quantity of claims to sector j profits owned by the household, $q_{j,t}$ is the price of those claims, $d_{j,t}$ are the dividends paid by those claims, w_t is the wage rate, $\tau_{L,t}$ is a tax on labor income (the labor wedge) and $\tau_{K,t}$ is a tax on capital income (the investment wedge). The first order conditions for the household are:

$$C_{jt}:\kappa\frac{\eta_j}{C_{j,t}}=p_{j,t}\pi_t^C,\ j=\{1,..,N\},$$

$$\begin{aligned}L_t &: (1-\kappa)\frac{1}{1-L_t}=(1-\tau_{L,t})w_t\pi_t^C, \qquad (17)\\ V_{j,t+1} &: \pi_t^C q_{j,t}=\beta E_t\left[(1-\tau_{K,t+1})\pi_{t+1}^C(q_{j,t+1}+d_{j,t+1})\right],\ j=\{1,..,N\}.\end{aligned}$$

Note that, iterating forward and applying the no-bubble/transversality condition $\lim_{T\to\infty}$ $\beta^T\pi_T^C q_{j,T}=0$, we have that

$$q_{j,t}=E_t\sum_{s=1}^{\infty}\beta^s\left(\prod_{v=1}^{s}(1-\tau_{K,t+v})\right)\frac{\pi_{t+s}^C}{\pi_t^C}d_{t+s}.$$

This last expression is the object that firms ultimately seek to maximize.

4.2 Firms

Firms are owned by households and solve their problem subject to the household's marginal value of consumption in each period. They seek to maximize their market price, that is, to ensure that $q_{j,t}$ is as large as possible while hiring labor and renting capital from households. The problem of a firm in sector $j\in\{1,...,N\}$ is

$$\max_{d_{t+s},} q_{j,t} = E_t\sum_{s=1}^{\infty}\beta^s\left(\prod_{v=1}^{s}(1-\tau_{K,t+v})\right)\frac{\pi_{t+s}^C}{\pi_t^C}d_{t+s},$$
$$\text{given }\left\{Z_{j,t-s|t-s+v}\right\}_{\{s,v\}=\{1,s\}}^{\{S,S\}},$$

where

$$d_{j,t} = p_{j,t}Y_{j,t} - \left(w_t L_{j,t} + \sum_{i=1}^{N} p_{ij,t} \left(M_{ij,t} + I_{ij,t} \right) \right),$$

subject to the constraints (lagrangians in parenthesis)

$$Y_{j,t} = \left(B_j \sum_{s=0}^{S} \omega_j(s)^{\frac{1}{\varrho}} Z_{j,t-s|t}^{\frac{\varrho-1}{\varrho}} \right)^{\frac{\varrho}{\varrho-1}}, \ \varrho > 0, \omega_j(s) > 0, B_j > 0, \ j = \{1,..,N\},$$

$$\sum_{s=0}^{S} Z_{j,t|t+s} = Z_{j,t} \quad \left(\beta^t \left(\prod_{v=1}^{s} (1 - \tau_{K,t+v}) \right) \pi_{j,t}^{Z} \right),$$

$$\begin{aligned} Z_{j,t} &= \left(K_{j,t}^{F}\right)^{\alpha_j} \Pi_{i=1}^{N} M_{ij,t}^{\gamma_{ij}} \left(A_{j,t} L_{j,t}\right)^{\xi_i}, \\ \alpha_j &\geq 0, \ \gamma_{ij} \geq 0, \ \xi_i \geq 0, \ \alpha_j + \sum_{i=1}^{N} \gamma_{ij} + \xi_i = 1, \ \left(\beta^t \left(\prod_{v=1}^{s} (1 - \tau_{K,t+v}) \right) \pi_{j,t}^{Z} \right) \end{aligned}$$

and

$$\begin{aligned} K_{j,t+1} &= (1-\delta) K_{j,t} + I_{j,t} \quad \left(\beta^t \left(\prod_{v=1}^{s} (1 - \tau_{K,t+v}) \right) \pi_{j,t}^{K} \right), \\ I_{j,t} &= \Xi_j \prod_{i=1}^{N} I_{ij,t}^{\theta_{ij}} \quad \left(\beta^t \left(\prod_{v=1}^{s} (1 - \tau_{K,t+v}) \right) \pi_{j,t}^{Z} \right). \end{aligned}$$

The set of first order conditions are:

$$\begin{aligned} M_{ij,t} &: \pi_t^C p_{i,t} = \gamma_{ij} \frac{Z_{j,t}}{M_{ij,t}} \pi_{j,t}^{Z}, \ i,j = \{1,..,N\}, \ s \in \{0,..,S\}, \\ L_{j,t} &: \pi_t^C w_t = \xi_j \frac{Z_{j,t}}{L_{j,t}} \pi_{j,t}^{Z}, \ j = \{1,..,N\}, \ s \in \{0,..,S\}, \\ K_{j,t} &: \pi_{j,t}^{K} = \beta E_t \left[(1 - \tau_{K,t+1}) \left(\pi_{j,t+1}^{K} + \alpha_j \frac{Z_{j,t+1}}{K_{j,t+1}} \pi_{j,t+1}^{Z} \right) \right], \end{aligned}$$

$$\begin{aligned} \pi_{j,t}^{Z} &= \beta^s \left(\prod_{v=1}^{s} (1 - \tau_{K,t+v}) \right) E_t \left[B_j \left[\frac{Z_{j,t|t+s}}{\omega_j(s) Y_{j,t+s}} \right]^{-\frac{1}{\varrho}} p_{j,t+s} \pi_{t+s}^{C} \right], j \in \{1,.,N\}, s \in \{1,.,S\}, \\ \pi_{j,t}^{Z} &= B_j \left[\frac{Z_{j,t|t}}{\omega_j(s) Y_{j,t}} \right]^{-\frac{1}{\varrho}} p_{j,t} \pi_t^{C}, j \in \{1,..,N\}. \end{aligned}$$

4.3 Equilibrium

Equilibrium is given by a set of choices by the households, prices and lagrangian terms, that also satisfy

$$\sum_{j=1}^{N} L_{j,t} = L_t,$$

$$C_{j,t} + \sum_{i=1}^{N} I_{ji,t} + \sum_{i=1}^{N} M_{ji,t} = Y_{j,t},\ j = \{1,..,N\}.$$

4.4 Equivalence with the Decentralized Economy

To see the equivalence with the decentralized economy, note that, with the exception of equation (17) describing labor supply, the system of equations describing the first order conditions and equilibrium conditions of the decentralized problem are identical to those describing the allocations in the centralized problem with aggregation in $Z_{j,t}$, as described in Section 3, given the following relabeling of variables:

$$\begin{aligned}
\pi_t^C p_{j,t} &= \lambda_{j,t}, \\
\alpha_j \frac{Z_{j,t+1}}{K_{j,t+1}} \pi_{j,t+1}^Z &= \chi_{j,t}, \\
\pi_t^C w_t &= \nu_t, \\
\pi_{j,t}^K &= \mu_{j,t}, \\
\pi_{j,t}^Z &= \phi_{j,t}\lambda_{j,t}, \\
1 - \tau_{K,t} &= \zeta_t.
\end{aligned}$$

There is no relabeling that will make equation (17) identical to the labor demand condition. However, we solve the model up to a first order approximation, in which case such an equivalence does exist. To see this, log-linearize equation (17) around a steady state with $\tau_{L,t} \cong 0$ to get[1]

$$\frac{L}{1-L}\widehat{L}_t = -\widehat{\tau}_{L,t} + \widehat{w_t \pi_t^C}.$$

Given the relabeling, this expression reduces to

[1]Similarly, the condition $\tau_t^X = -\widehat{\zeta}_t$ requires τ_t^X to be close to zero in steady state.

$$\frac{L}{1-L}\widehat{L}_t = -\widehat{\tau}_{L,t} + \widehat{\nu}_t.$$

Recall that, in steady-state, $\Upsilon = 1$ (the unconditional mean). Hence, if we log-linearize the first order condition for aggregate employment in the centralized problem (15), we have that

$$\frac{L}{1-L}\widehat{L}_t + \frac{1}{1-L}\widehat{\Upsilon}_t = \widehat{\nu}_t.$$

It follows that the two equations are equivalent so long as $\widehat{\tau}_{L,t} = \frac{1}{1-L}\widehat{\Upsilon}_t$.

5 The Stationary Model

The model as written is non-stationary, since labor productivity in each sector, $A_{j,t}$, depends on the non-stationary term $\Gamma_{j,t}$. It is possible to describe the economy with a stationary system using the following variable redefinitions. For any given variable V_t let $\tilde{V}_t$ denote the detrended value of the variable, with $\tilde{V}_t = \frac{V_t}{\Gamma_t}$ if $V_t \in \left\{C_t, C_{j,t}, I_{j,t}, M_{ij,t}K_{j,t}, X_{j,t}, Z_{j,t}, Z_{j,t|t+s}, Y_{j,t}, A_{j,t}\right\}$ and $\tilde{V}_t = \Gamma_t V_t$ if $V_t \in \{\lambda_{j,t}, \chi_{j,t}, \mu_{j,t}\}$.

The economy is thus described by the system of equations given by the detrended resource constraints and the detrended first order conditions:

$$\tilde{C}_t = \prod_{j=1}^{N} \Lambda \tilde{C}_{j,t}^{\eta_j},\ \eta_j \geq 0,\ \sum_{j=1}^{N} \eta_j = 1,\ \Lambda > 0, \tag{18}$$

$$\tilde{C}_{j,t} + \sum_{i=1}^{N} \tilde{I}_{ji,t} + \sum_{i=1}^{N} \tilde{M}_{ji,t} = \tilde{Y}_{j,t},\ j = \{1,..,N\},$$

$$g_{t+1}\tilde{K}_{j,t+1} = \tilde{X}_{j,t} + (1-\delta)\tilde{K}_{j,t}, \quad j = \{1,..,N\}, \delta \in [0,1], \tag{19}$$

where, for $j = \{1, ..., N\}$,

$$\tilde{X}_{j,t} = \Xi_j \prod_{i=1}^{N} \tilde{I}_{ij,t}^{\theta_{ij}},\ \theta_{ij} \geq 0, \sum \theta_{ij} = 1, \Xi_j > 0, \tag{20}$$

$$\tilde{Y}_{j,t} = \left(B_j \sum_{s=0}^{S} \omega_j(s)^{\frac{1}{\varrho}} \left(\frac{\tilde{Z}_{j,t-s|t}}{\prod_{v=0}^{s-1} g_{t-v}} \right)^{\frac{\varrho-1}{\varrho}} \right)^{\frac{\varrho}{\varrho-1}},\ \varrho > 0, \omega_j(s) > 0,\ B_j > 0, \tag{21}$$

where, for $j = \{1, ..., N\}$ and $s = \{0, ..., S\}$,

$$\tilde{Z}_{j,t} = \tilde{K}_{j,t}^{\alpha_j} \Pi_{i=1}^{N} \tilde{M}_{ij,t}^{\gamma_{ij}} \left(\tilde{A}_{j,t} L_{j,t|t+s} \right)^{\xi_i}, \ \alpha_j \geq 0, \ \gamma_{ij} \geq 0, \ \xi_i \geq 0, \ \alpha_j + \sum_{i=1}^{N} \gamma_{ij} + \xi_i = 1, \tag{22}$$

and

$$\tilde{Z}_{j,t} = \sum \tilde{Z}_{j,t|t+s}, \tag{23}$$

$$\sum_{j=1}^{N} \tilde{L}_{j,t} = L_t, \tag{24}$$

$$\tilde{A}_{j,t} = u_t a_{j,t}, \ j = \{1, .., N\}.$$

The detrended first order conditions are:

$$\begin{aligned} (1-\kappa) \frac{\Upsilon_t}{1 - \Upsilon_t L_t} &= \nu_t, & (25) \\ \tilde{\mu}_{j,t} &= \beta \frac{\zeta_t}{g_t} E_t \left[(1-\delta) \tilde{\mu}_{j,t+1} + \tilde{\chi}_{j,t+1} \right], \ j = \{1, .., N\}, & (26) \end{aligned}$$

$$\kappa \frac{\eta_j}{\tilde{C}_{j,t}} = \tilde{\lambda}_{j,t}, \ j = \{1, .., N\}, \tag{27}$$

$$\tilde{\lambda}_{i,t} = \tilde{\mu}_{j,t} \frac{\theta_{ij} \tilde{X}_{j,t}}{\tilde{I}_{ij,t}}, \ i, j = \{1, .., N\}, \tag{28}$$

$$\begin{aligned} \tilde{\lambda}_{i,t} &= \gamma_{ij} \frac{\tilde{Z}_{j,t}}{\tilde{M}_{ij,t}} \phi_{j,t} \tilde{\lambda}_{j,t}, \ i, j = \{1, .., N\}, \ s = \{0, .., S\}, & (29) \\ \nu_t &= \xi_j \frac{\tilde{Z}_{j,t}}{L_{j,t}} \phi_{j,t} \tilde{\lambda}_{j,t}, \ j = \{1, .., N\}, \ s = \{0, .., S\}, & (30) \\ \chi_{j,t} &= \alpha_j \frac{\tilde{Z}_{j,t}}{K_{j,t}} \phi_{j,t} \tilde{\lambda}_{j,t}, \ j = \{1, .., N\}, \ s = \{0, .., S\}, & (31) \end{aligned}$$

and for $j = \{1, ..., N\}$ and $s = \{1, ..., S\}$

$$\phi_{j,t} \tilde{\lambda}_{j,t} = \beta^s E_t \left[\left(\prod_{u=0}^{s-1} \zeta_{t+u} \right) B_j \left[\frac{\tilde{Z}_{j,t|t+s}}{\omega_j(s) \tilde{Y}_{j,t+s}} \right]^{-\frac{1}{\varrho}} \left(\frac{1}{\prod_{v=1}^{s} g_{t+v}} \right)^{1-\frac{1}{\varrho}} \tilde{\lambda}_{j,t+s} \right]. \tag{32}$$

For $j = \{1, ..., N\}$ and $s = 0$,

$$\phi_{j,t} = B_j \left[\frac{\tilde{Z}_{j,t|t}}{\omega_j(s)\tilde{Y}_{j,t}} \right]^{-\frac{1}{\varrho}}, \tag{33}$$

and $\{u_t, g_t, \zeta_t, \Upsilon_t, A_{jt}\}$ are random variables with unconditional mean $\{1, g, 1, 1, 1\}$ and values known at t or after.

From this point on, we work only with the detrended model and do away with the tildes.

6 Steady State

We now show how to calculate the steady state of the model analytically. As usual, we denote the steady state values of the various variables by dropping the t's. For the case of $Z_{j,t|t+s}$, this implies that we denote their steady state values by $Z_{j|+s}$. The steady state comprises a set of $3 + 8N + 2N^2 + 5N(S+1)$ variables represented in the same number of equations. To calculate the steady state, it is convenient to add to the system the following set of $3N + 1$ normalizing restrictions:

$$\begin{aligned} C_j &= \eta_j C,\ j = \{1, ..., N\}, & (34)\\ I_{1j} &= \theta_{1j} X_j,\ j = \{1, ..., N\}, & (35)\\ Z_{j|+0} &= \omega_j(0) Y_j,\ j = \{1, ..., N\}, & (36)\\ \frac{1-\kappa}{\kappa}\frac{C}{1-L} &= 1. & (37) \end{aligned}$$

We set the $3N + 1$ multiplicative parameters $\{B_j\}_{j=1}^N$, $\{A_j\}_{j=1}^N$,$\{\Xi_j\}_{j=1}^N$ and Λ so that these restrictions are satisfied. These parameters reflect the choice of unit for C and, for all $j \in \{1, ..., N\}$, for C_j (which is the same as Y_j and I_{ji}), X_j and $Z_{j|+0}$. Also, as we will show, since A_j does not depend on s - the choice of units for $Z_{j|+0}$ pins down the units for the other $Z_{j|+s}$. We proceed in seven steps, described below:

Step 1: Recognizing the implications of the normalizations for the lagrangians

The first step is to recognize that the normalization above has implications for the steady state values of the lagrangians. Imposing the restrictions $C_1 = \eta_1 C$ and $C_2 = \eta_2 C$ the F.O.C. for individual consumption goods (27) reduces to:

$$\lambda_j = \kappa \frac{\eta_j}{C_j} = \kappa \frac{\eta_j}{\eta_j C} = \frac{\kappa}{C}. \tag{38}$$

Note that the right hand side is the same for all j. Hence, in steady state, λ_j is the same for all j.

We now show that, for all j, $\lambda_j = \mu_j = \nu$. To establish that $\lambda_j = \mu_j$, we start from the steady-state version of equation (28)[2]:

$$\lambda_i = \mu_j \frac{\theta_{ij} X_j}{I_{ij}},\ i, j = \{1, .., N\}. \tag{39}$$

Combining equation (39) with the normalization in equation (35), we have that

$$\lambda_1 = \mu_j.$$

Since $\lambda_j = \lambda_1$ for all j, this establishes that $\lambda_j = \mu_j$.

Finally, to establish that $\mu_j = \nu$, note first that the steady-state version of the aggregate labor supply described in equation (25) is

$$(1-\kappa)\frac{1}{1-L} = \nu.$$

Combine this with equation (38) to get

$$\frac{1-\kappa}{\kappa}\frac{C}{1-L} = \frac{\nu}{\lambda_j}.$$

Given the normalization in equation (37), this establishes that, for all j, $\lambda_j = \nu$

Lastly, substituting the normalization $Z_{j|+0} = \omega_j(0) Y_j$ in the steady-state version of the F.O.C. for $Z_{j,t|t}$, (32) yields $\phi_j = B_j$.

Step 2: Finding steady state factor inputs as functions of Y_j's:

Given the F.O.C.'s for the detrended system (29) through (31), factor inputs can be expressed as log-linear functions of Y_j's, Z_j's and lagrangians:

$$\begin{aligned} 1 &= \gamma_{ij}\frac{Z_j}{M_{ij}} B_j,\ i, j = \{1, .., N\}, s = \{0, .., S\}, && (40) \\ 1 &= \xi_j \frac{Z_j}{L_j} B_j\ ,\ j = \{1, .., N\}, s = \{0, .., S\}, && (41) \\ g\beta^{-1} - (1-\delta) &= \alpha_j \frac{Z_j}{K_j} B_j\ ,\ j = \{1, .., N\}, s = \{0, .., S\}, && (42) \end{aligned}$$

where we use the fact that equation (26) implies that in steady state, $\chi_j = \left(g\beta^{-1} - (1-\delta)\right)\mu_j$ and that for all j, $\lambda_j = \mu_j = \nu$ to simplify out the lagrangians. These equations yields determine the steady-state values for factor inputs as functions of Z_j's. Substituting $\phi_j = B_j$ back in the F.O.C.'s for $Z_{j,t|t+s}$ (32) yields, with some rearrangement,

$$Z_{j|+s} = \omega_j(s)\,\beta^{\varrho s} g^{(1-\varrho)s} Y_j. \tag{43}$$

[2] Recall that, for notational simplicity, we do not use the tildes.

This result, in turn, can be used to substitute out $Z_{j|+s}$ from the steady-state version of equation (23) to yields Z_j as a function of Y_j:

$$\frac{Z_j}{Y_j} = \sum_{s=0}^{S} \omega_j(s)\, \beta^{\varrho s} g^{(1-\varrho)s}. \tag{44}$$

Step 3: Solving for the vectors of normalizing constants B_j, A_j, Ξ_j and Λ.

The normalizations imply restrictions on the constants B_j, A_j, Ξ_j and Λ:

- To recover A_j, substitute out the factor inputs, determined as functions of Z_j in equations (40) through (42), in the production function for Z_j given by the steady-state version of equation (22) Given constant returns to scale, Z_j will cancel out and it is then possible to solve for A_j.

- To recover B_j, follow the same procedure, substituting out the choices of $Z_{j|+s}$ as a function of Y_j given by equation (43) in the production function for Y_j given by the steady state version of equation (21), canceling out Y_j, and solving out for B_j.

-To recover Λ, substitute $C_j = \eta_j C$ in the steady-state version of the definition for C (equation 18) and solve out for Λ.

- To recover Ξ_j, note that, since $\lambda_1 = \mu_j$ for all $j \in \{1, ..., N\}$, then equation (39) implies that $I_{ij} = \theta_{ij} X_j$ for all $\{i, j\} \in \{1, ..., N\}$. Substitute this in the steady-state version of the production function for investment goods described in equation (20). Since X_j is a constant returns to scale function of I_{ij}'s, X_j cancels out and Ξ_j can be solved for as a function of parameters.

Step 4: Using goods market clearing conditions to solve for $\frac{Y_j}{Y}$

The goods market clearing conditions provide a system of linear equations in consumption C_j, investment I_{ji}, materials, M_{ji}, and sectoral output Y_j. We have solved for materials and capital as a linear function of sectoral output Y_j. It is also straightforward to recover I_{ji} from K_j given the steady-state version of the law of motion for capital (equation 19). Moreover, given the normalization $C_j = \eta_j C$, we can write the goods market clearing conditions as a system of N equations on $N+1$ unknowns, $\{Y_j\}_{j=1}^{N}$ and C. Thus, the unknowns are indeterminate. We can, however, solve for $\frac{Y_j}{Y}$, by dividing through by $Y = \sum Y_j$ and adding the equation $\sum \frac{Y_j}{Y} = 1$. This system of $N+1$ equations in $N+1$ unknowns can then be solved for $\frac{Y_j}{Y}$ (the fraction of output in sector j in total output) and $\frac{C}{Y}$.

Step 5: Using the labor supply equation to solve for L **as a function of** Y

Substitute the steady-state versions of the labor demand equations (30) with all the substitutions allowed for by Steps 1-5 above into the labor market clearing condition (equation 24). These substitutions yield:

$$L = \sum_{j=1}^{N} \xi_j B_j Z_j.$$

From equation (44), we can find $\frac{L}{Y}$ as a function of $\left\{\frac{Y_j}{Y}\right\}_{j=1}^{N}$ calculated in Step 4 above.

$$\frac{L}{Y} = \sum_{j=1}^{N} B_j \left(\sum_{s=0}^{S} \omega_j(s) \beta^{\varrho s} g^{(1-\varrho)s} \right) \xi_j \frac{Y_j}{Y}.$$

Step 6: Using L, the expression for C and the normalization for wages to solve for Y:

From the optimality conditions for the representative household, in steady state it is the case that

$$\frac{1-\kappa}{\kappa} C = 1 - L.$$

Given $\frac{C}{Y}$ calculated in step 4 and $\frac{L}{Y}$ calculated in step 5, it is straightforward to solve this equation for Y.

Step 7: Using Y, find aggregate consumption, labor, sectoral output, and factor inputs using previously calculated values for $\frac{C}{Y}$, $\frac{L}{Y}$, $\frac{Y_j}{Y}$ and $\frac{Z_j}{Y_j}$.

7 Log-Linearization:

We now log-linearize the model around the non-stochastic steady state. We take variables with carets to denote log deviations from steady state.

7.1 Resource constraints

7.1.1 Production Functions

For all $j \in \{1, ..., N\}$

$$\begin{aligned}
\widehat{Y}_{j,t} &= \sum_{s=0}^{S} B_j \omega_i(s) \left(\frac{\beta}{g}\right)^{(\varrho-1)s} \left(\widehat{Z}_{j,t-s|t} - \sum_{u=0}^{s-1} g_{t-u} \right), \\
\widehat{Z}_{j,t} &= \alpha_j \widehat{K}_{j,t} + \xi_j \widehat{L}_{j,t} + \sum_{i=1}^{N} \gamma_{ij} \widehat{M}_{ij,t} + \xi_j \widehat{A}_{j,t}.
\end{aligned}$$

The values for $\widehat{Z}_{j,t|t+s}$ for $j \in \{1, ..., N\}$ and $s \in \{0, ..., S\}$ satisfy the linearized resource constraint

$$\widehat{Z}_{j,t} = \frac{\omega_j(s)\,\beta^{\varrho s} g^{(1-\varrho)s}}{\sum_{s=0}^{S} \omega_j(s)\,\beta^{\varrho s} g^{(1-\varrho)s}} \widehat{Z}_{j,t|t+s}.$$

7.1.2 Goods Market Clearing Conditions:

The goods market clearing condition is, for all $j \in \{1, ..., N\}$

$$\frac{C_j}{Y_j}\widehat{C}_{i,t} + \sum_i \frac{I_{ji}}{Y_j}\widehat{I}_{j,t} + \sum_i \frac{M_{ji}}{Y_j}\widehat{M}_{ji,t} = \widehat{Y}_{j,t}.$$

7.1.3 Resource Constraints for Capital:

The capital accumulation equation is:

$$\widehat{g}_t + \widehat{K}_{j,t} = \left[1 - \frac{1-\delta}{g}\right] \sum_i \theta_{ij} \widehat{I}_{ij,t} + \frac{1-\delta}{g}\widehat{K}_{j,t-1}.$$

7.2 First Order Conditions:

7.2.1 Consumption allocation:

For all $j \in \{1, ..., N\}$

$$-\widehat{C}_{j,t} = \widehat{\lambda}_{j,t}. \tag{45}$$

7.2.2 Labor Supply:

$$\frac{L}{1-L}\widehat{L}_t + \frac{1}{1-L}\widehat{\Upsilon}_t = \widehat{\nu}_t, \tag{46}$$

with

$$\widehat{L}_t = \sum_{j=1}^{N} \frac{L_j}{L}\widehat{L}_{j,t}.$$

7.2.3 Investment in Fixed Capital:

For all $j \in \{1, ..., N\}$

$$\widehat{\mu}_{j,t} = E_t \left[\frac{\beta}{g} (1-\delta) \widehat{\mu}_{j,t+1} - \widehat{g}_{t+1} + \left(1 - \frac{\beta}{g} (1-\delta) \right) \widehat{\chi}_{j,t+1} \right] + \widehat{\zeta}_t. \tag{47}$$

7.2.4 Composition of investment:

For all $j \in \{1, ..., N\}$ and $i \in \{1, ..., N\}$

$$\widehat{\lambda}_{j,t} = \widehat{\mu}_{i,t} + \sum_j \theta_{ji} \widehat{I}_{ji,t} - \widehat{I}_{ji,t}. \tag{48}$$

7.2.5 Input demands:

For $j \in \{1, ..., N\}$,

Materials:

$$\widehat{\lambda}_{i,t} = \widehat{\phi}_{j,t} + \widehat{\lambda}_{j,t} + \widehat{Z}_{j,t} - \widehat{M}_{ij,t}.$$

Labor:

$$\widehat{\nu}_t = \widehat{\phi}_{j,t} + \widehat{\lambda}_{j,t} + \widehat{Z}_{j,t} - \widehat{L}_{j,t}. \tag{49}$$

Capital:

$$\widehat{\chi}_{jt} = \widehat{\phi}_{j,t} + \widehat{\lambda}_{j,t} + \widehat{Z}_{j,t} - \widehat{K}_{j,t}. \tag{50}$$

Composite inputs:

for $s = \{1, ..., S\}$

$$\widehat{\phi}_{j,t} + \widehat{\lambda}_{j,t} = E_t \sum_{u=0}^{s-1} \widehat{\zeta}_{t+u} + \frac{1}{\varrho} \left(E_t \widehat{Y}_{j,t+s} - \widehat{Z}_{j,t|t+s} \right) - \left(1 - \frac{1}{\varrho} \right) E_t \sum_{u=0}^{s-1} \widehat{g}_{t+u} + E_t \widehat{\lambda}_{j,t+s}, \tag{51}$$

and

$$\widehat{\phi}_{j,t} = \frac{1}{\varrho} \left(\widehat{Y}_{j,t} - \widehat{Z}_{j,t|t} \right).$$

7.3 Exogenous shock processes:

$$
\begin{aligned}
\widehat{u}_t &= \rho_u \widehat{u}_{t-1} + \varepsilon_t^u, \\
\widehat{a}_{j,t} &= \rho_{P_j} \widehat{a}_{j,t-1} + \varepsilon_t^{P_j}, \\
\widehat{\zeta}_t &= \rho_\zeta \widehat{\zeta}_{t-1} + \varepsilon_t^\zeta, \\
\widehat{\Upsilon}_t &= \rho_\Upsilon \widehat{\Upsilon}_{t-1} + \varepsilon_t^\Upsilon, \\
g_t &= \rho_g g_{t-1} + \varepsilon_t^g.
\end{aligned}
$$

8 Wedges and Aggregate Variables with Multiple Stages of Production

This section provides calculations that highlight the relationships between the wedges defined in section 4 above and macroeconomic aggregates. In particular, it compares the wedges implied by the multi-sector/multi-stage model with those in the canonical one-sector model, $N = 1$, $S = 0$ model. Unlike the simple one sector growth case with no stages of production, the determination of all wedges in our benchmark case requires fully estimating the model, since these wedges all depend on forecasts or otherwise unobserved components. Nevertheless, it is possible, with some manipulation, to show how the wedges in our model relate to aggregate variables and how this relationship is different from the $N = 1$, $S = 0$ benchmark. The results in this Section provide underlying detail for the differences between the decompositions for the time paths of U.S. Recessions under different assumptions, presented in Section 5.3 of the main text.

8.1 The Labor Wedge

The labor wedge is defined as the deviation of the marginal product of labor from the marginal rate of substitution between consumption and leisure. This object is of interest since it measures the role of other forces affecting labor markets, given a correct specification of preferences and technology and a lack of exogenous shocks or shifts in the primitives. The existence of this wedge is typically motivated by distortions, such as labor taxes. As an example, given the preferences adopted in the paper, the labor wedge in the prototypical one-sector growth model can be expressed as:

$$\widetilde{\tau}_{L,t} = \underbrace{\left(\widehat{Z}_t - \widehat{L}_t\right)}_{\text{Labor Productivity}} - \left(\widehat{C}_t + \frac{L}{1-L}\widehat{L}_t\right), \tag{52}$$

where the caret represents percentage deviations from steady state, and variables without a "t" subscript are evaluated at the steady state.[3]

As defined in section 3 of this appendix and equation (13), $\phi_{j,t}$ denotes the ratio between the marginal cost of the inputs involved in the production of good j and its marginal value to a household consuming it. This ratio varies over time since the marginal rate of transformation of output into current sales depends on past production decisions. The appropriate labor wedge in our framework with multiple sectors and multiple stages of production, derived below, is:

$$\tau_{L,t} = \sum_{j=1}^{N} \underbrace{\eta_j\left(\widehat{Z}_{j,t} - \widehat{L}_{j,t}\right)}_{\text{Sectoral Labor Productivity}} - \left(\widehat{C}_t + \frac{L}{1-L}\widehat{L}_t\right) + \sum_{j=1}^{N}\eta_j\widehat{\phi}_{j,t}. \tag{53}$$

There are two key differences between equations (52) and (53). First, the relevant notion of a labor wedge involves the marginal contribution of labor to the generation of current and future sales, rather than just the marginal product of labor. This fact is captured by the term $\widehat{\phi}_{j,t}$ and relies on the assumption that in the absence of distribution and sales, current output cannot be directly used for consumption or investment. Second, in a multi-sector economy, the marginal product of labor in a given sector affects aggregate labor supply decisions more the larger the share of that sector is in aggregate consumption. Hence, calculating the labor wedge as a function of aggregate consumption requires averaging labor productivity in different sectors by using their consumption shares, η_j, as weights. In the one-sector case with no production lags ($N = 1$ and $S = 0$), $\widehat{\phi}_{j,t} = 0$ and the labor wedge in (53) reduces to the conventional wedge (52).

We note that in our framework, the labor wedge is approximated by $\tau_t^L = \widehat{\Upsilon}_t/(1-L)$. Time-variation in the preference parameter Υ_t gives our model the necessary flexibility to generate a time-varying labor wedge. Alternatively, a labor tax in the decentralized version of the model would play the same role.

Given the calculations for the decentralized model discussed in section 4, we have that the labor wedge $\tau_{L,t} = \frac{1}{1-L}\hat{\Upsilon}_t$. Using this fact, and combining equations (46) and (49), yields:

$$\frac{L}{1-L}\widehat{L}_t + \tau_{L,t} = \widehat{\phi}_{j,t} + \widehat{\lambda}_{j,t} + \widehat{Z}_{j,t} - \widehat{L}_{j,t}.$$

[3] The formal derivation of this equation assumes a steady state value for $\Upsilon_t = 1$.

The first order condition for consumption (45) implies that $\widehat{\lambda}_{j,t} = -\widehat{C}_{j,t}$, so that with a slight rearrangement:

$$\tau_{L,t} = \widehat{Z}_{j,t} - \widehat{L}_{j,t} - \left(\widehat{C}_{j,t} + \frac{L}{1-L}\widehat{L}_t\right) + \hat{\phi}_{j,t}.$$

Taking a weighted average of both sides across j,'s with the weights given by $\eta_{j,t}$ yields the expression in (53),

$$\tau_{L,t} = \sum \eta_{j,t}\left(\widehat{Z}_{j,t} - \widehat{L}_{j,t}\right) - \left(\widehat{C}_t + \frac{L}{1-L}\widehat{L}_t\right) + \sum_{j=1}^{N} \eta_j \hat{\phi}_{j,t},$$

where we are using the fact that $\sum \eta_j \widehat{C}_{j,t} = \widehat{C}_t$. With $N = 1$, the labor wedge collapses to

$$\tau_{L,t} = \left(\widehat{Z}_t - \widehat{L}_t\right) - \left(\widehat{C}_t + \frac{L}{1-L}\widehat{L}_t\right) + \hat{\phi}_t.$$

Furthermore, with $S = 0$, $\widehat{Y}_t = \widehat{Z}_{t|t}$, so that from equation (51), $\widehat{\phi}_{j,t} = 0\ \forall j$. Hence, with $N = 1$ and $S = 0$, the labor wedge is as in (52)

$$\tau_{L,t} = \left(\widehat{Z}_t - \widehat{L}_t\right) - \left(\widehat{C}_t + \frac{L}{1-L}\widehat{L}_t\right).$$

Note that, in the case with $S > 0$, the labor wedge depends on $\hat{\phi}_{j,t}$, which is itself a function of expected values of future output, allocations of stages of production and lagrangian terms. Thus unlike the $S = 0$, $N = 1$ case where the labor wedge is only a function of current period aggregates, in the multi-stage model, calculation of the labor wedge requires imputing values for $\widehat{Z}_{j,t}$ which, given readily available aggregate data, can only be obtained by fully estimating the model.

8.2 The Investment Wedge

The investment wedge measures the deviation between households' intertemporal rate of substitution and the physical return to investment. This wedge is typically associated with distortions to credit markets that arise from informational or limited commitment problems, or more simply taxes on investment. Given a one-sector model with logarithmic preferences and no production lags, the investment wedge is

$$\widetilde{\tau}_{K,t} = E_t \underbrace{\left[\left(1-\widetilde{\beta}\right)\left(\widehat{V}_{t+1} - \widehat{K}_{t+1}\right)\right]}_{\text{Marginal Return to Investment}} - E_t\left(\Delta\widehat{C}_{t+1}\right), \tag{54}$$

where $\Delta\widehat{C}_{t+1} = \widehat{C}_{t+1} - \widehat{C}_t$ and $\widetilde{\beta} = \beta(1-\delta)$.

The investment wedge takes a more complicated form in the multi-sector, multi-stage production setup,

$$\tau_{K,t} = \underbrace{E_t\left(1-\widetilde{\beta}\right)\sum_{j=1}^{N}\eta_j\left(\widehat{Z}_{j,t+1}-\widehat{K}_{j,t+1}+\widehat{\phi}_{j,t+1}\right)-\sum_{j=1}^{N}\sum_{i=1}^{N}\left(\eta_j\theta_{ij}-\eta_j\eta_i\right)\left(\widehat{\lambda}_{i,t}-\widetilde{\beta}\widehat{\lambda}_{i,t+1}\right)}_{\text{Marginal Return to Investment}}$$
$$-E_t\left(\Delta\widehat{C}_{t+1}\right), \qquad (55)$$

where, as mentioned earlier, $\widehat{\lambda}_{j,t}$ is the Lagrange multiplier associated with the economy's aggregate resource constraint.

There are three main differences between the conventional wedge (54) and the one in equation (55). The first two differences are analogous to those for the labor wedge. They are related to the distinction between output and sales and the calculation of the marginal product of capital in a multi-sector model. These differences are captured by the terms $\widehat{\phi}_{j,t}$ and the consumption weights η_j. In addition, different sectors contribute differently to investment in other sectors given the technology described by equation (2). The marginal return on investment therefore incorporates changes in the cost of producing investment goods relative to consumption in the current and following period. This is reflected by the θ_{ij}'s in the last term of the marginal return to investment. As shown in section 4 of this appendix, it is also the case in our model that $\tau_{K,t}=-\widehat{\zeta}_t$, so that shifts in the households' discount factor can be thought of as a stand-in for a tax on investment. Observe that the investment wedge in the generalized model cannot be readily identified from aggregate consumption, investment, and output data because of the variable, $\widehat{\phi}_{j,t+1}$, in the marginal return to investment.

Equations (54) and (55) give the investment wedge in terms of the Euler equation governing fixed investment. However, to the degree that the wedge captures distortions in credit markets, it will also appear in the Euler equation for inventory investment. It is in this sense that inventory data are informative with respect to the investment wedge. Given a process for $\tau_{K,t}$, optimal inventory investment in sector j requires that

$$\widehat{\phi}_{j,t}+\frac{1}{\varrho}\left(\frac{\Delta N_{j,t}}{Y_{j,t}}\right)=E_t\left(\sum_{s=1}^{S}\psi_j\left(s\right)\left[\Delta\widehat{\lambda}_{j,t+s}+\frac{1}{\varrho}\Delta\widehat{Y}_{j,t+s}-\left(\sum_{u=0}^{s-1}\tau_{K,t+u}\right)\right]\right), \qquad (56)$$

where $\psi_j\left(s\right)$ is the steady state ratio of inputs dedicated to production s periods hence to current production, $\psi_j\left(s\right)=Z_{t|t+s}/Z_t=Z_s/Z$. The ratio $\Delta N_{j,t}/Y_{j,t}$ is that of inventory investment to sales in sector j.

The left-hand side shows the marginal cost of increasing inventories. For given sales, inventory investment requires raising output, with associated marginal cost given by $\widehat{\phi}_{j,t}$. The marginal inventory investment necessary to generate future sales increases as total investment rises, a fact captured by the term $\frac{1}{\varrho}\left(\frac{\Delta N_{j,t}}{Y_{j,t}}\right)$. The right-hand side of (56) summarizes the marginal benefits of inventory investment. These benefits are weighted by the share of currently accumulated inventories that is dedicated to production in each period, $\psi_j(s)$. Inventory investment helps increase sales in future periods relative to the current period, an effect that on the margin is valued relative to current sales at $\Delta\widehat{\lambda}_{j,t+s}$. Moreover, the marginal contribution of current inventory investment to future sales is larger when future sales are expected to be large, an effect reflected in the term $\frac{1}{\varrho}\Delta\widehat{Y}_{j,t+s}$. Finally, accumulated expected future investment wedges, $\sum_{u=0}^{s-1}\tau_{t+u}^{X}$, lower the benefits of increased inventory investment. We now derive the expressions in (55) and (56) explicitly in more detail.

Given the calculations for the decentralized model discussed above, we have that, up to a first order approximation, the capital wedge $\tau_{K,t} = -\widehat{\zeta}_t$. Using this fact, we can back out the investment wedge from equation (47). We can substitute out the lagrangian terms $\widehat{\mu}_{j,t}$ and $\widehat{\mu}_{j,t+1}$ using the first order condition for investment (48). To see that, take the weighted average of both sides of that equation over j using θ_{ji} as weights:

$$\sum_{j=1}^{N}\theta_{ji}\left(\widehat{\mu}_{i,t}+\sum_{j}\theta_{ji}\widehat{I}_{ji,t}-\widehat{I}_{ji,t}\right)=\sum_{j=1}^{N}\theta_{ji}\widehat{\lambda}_{j,t},$$

so that

$$\widehat{\mu}_{j,t}=\sum_{i=1}^{N}\theta_{ij}\widehat{\lambda}_{i,t}.$$

Furthermore, we can substitute out $\widehat{\chi}_{j,t+1}$ using the F.O.C. for capital demand (50) and applying $\tau_{K,t} = -\widehat{\zeta}_t$. This yields:

$$\tau_{K,t}=E_t\left[\left(1-\tilde{\beta}\right)\left(\widehat{\lambda}_{j,t+1}+\widehat{Z}_{j,t+1}-\widehat{K}_{j,t+1}+\widehat{\phi}_{j,t+1}\right)-\left(\sum_{i=1}^{N}\theta_{ij}\left(\widehat{\lambda}_{i,t}-\tilde{\beta}\widehat{\lambda}_{i,t+1}\right)+\widehat{g}_{t+1}\right)\right],$$

with $\tilde{\beta}\equiv\frac{\beta}{g}(1-\delta)$. The definition of $\tilde{\beta}$ in the text assumes for expositional simplicity that $g=1$. Rearrange and add and subtract $\widehat{\lambda}_{j,t}$ to get:

$$\tau_{K,t}=E_t\left[\begin{array}{c}\left(1-\tilde{\beta}\right)\left(\widehat{Z}_{j,t+1}-\widehat{K}_{j,t+1}+\widehat{\phi}_{j,t+1}\right)-\left(\sum_{i=1}^{N}\theta_{ij}\left(\widehat{\lambda}_{i,t}-\tilde{\beta}\widehat{\lambda}_{i,t+1}\right)+\widehat{g}_{t+1}\right)+\\ \widehat{\lambda}_{j,t+1}-\widehat{\lambda}_{j,t}+\widehat{\lambda}_{j,t}-\tilde{\beta}\widehat{\lambda}_{j,t+1}\end{array}\right].$$

Alternatively,

$$\tau_{K,t} = E_t \left[\begin{array}{c} \left(1-\tilde{\beta}\right)\left(\widehat{Z}_{j,t+1} - \widehat{K}_{j,t+1} + \widehat{\phi}_{j,t+1}\right) - \\ \left(\sum_{i=1}^{N} \theta_{ij}\widehat{\lambda}_{i,t} - \widehat{\lambda}_{j,t} - \tilde{\beta}\left(\sum_{i=1}^{N} \theta_{ij}\widehat{\lambda}_{i,t+1} - \widehat{\lambda}_{j,t+1}\right) + \widehat{\lambda}_{j,t+1} - \widehat{\lambda}_{j,t} + \widehat{g}_{t+1}\right) \end{array} \right].$$

Use the fact that, $\widehat{\lambda}_{j,t} = -\widehat{C}_{j,t}$ to get

$$\tau_{K,t} = E_t \left[\left(1-\tilde{\beta}\right)\left(\widehat{Z}_{j,t+1} - \widehat{K}_{j,t+1} + \widehat{\phi}_{j,t+1}\right) - \left(\begin{array}{c} \sum_{i=1}^{N} \theta_{ij}\widehat{\lambda}_{i,t} - \widehat{\lambda}_{j,t} - \tilde{\beta}\left(\sum_{i=1}^{N} \theta_{ij}\widehat{\lambda}_{i,t+1} - \widehat{\lambda}_{j,t+1}\right) - \\ \left(\widehat{C}_{j,t+1} - \widehat{C}_{j,t} + \widehat{g}_{t+1}\right) \end{array} \right) \right].$$

Average both sides of the equation using η_j as weights and use the fact that $\sum \eta_j \widehat{C}_{j,t} = \widehat{C}_t$ to get

$$\tau_{K,t} = E_t \left[\begin{array}{c} \left(1-\tilde{\beta}\right)\sum \eta_j \left(\widehat{Z}_{j,t+1} - \widehat{K}_{j,t+1} + \widehat{\phi}_{j,t+1}\right) - \\ \left(\sum_{j=1}^{N}\sum_{i=1}^{N} \theta_{ij}\eta_j \left(\widehat{\lambda}_{i,t} - \tilde{\beta}\widehat{\lambda}_{i,t+1}\right) - \sum_{j=1}^{N} \eta_j \left(\widehat{\lambda}_{j,t} - \tilde{\beta}\widehat{\lambda}_{j,t+1}\right) - \Delta\widehat{C}_{t+1}\right) \end{array} \right],$$

where

$$\Delta\widehat{C}_{t+1} \equiv \widehat{C}_{t+1} - \widehat{C}_t + \widehat{g}_{t+1}$$

is the growth in aggregate consumption, including the stochastic trend. This definition differs slightly from that in (55) in that, for expositional simplicity, we abstract from the trend in that equation.

Lastly, use the fact that $\sum_{i=1}^{N}\left(\eta_i \widehat{\lambda}_{j,t+1}\right) = \left(\sum_{i=1}^{N} \eta_i\right)\widehat{\lambda}_{j,t+1} = \widehat{\lambda}_{j,t+1}$ to write, as in (55),

$$\tau_{K,t} = E_t \left[\begin{array}{c} \left(1-\tilde{\beta}\right)\sum \eta_j \left(\widehat{Z}_{j,t+1} - \widehat{K}_{j,t+1} + \widehat{\phi}_{j,t+1}\right) - \\ \left(\begin{array}{c} \sum_{j=1}^{N}\sum_{i=1}^{N} \left(\eta_j\theta_{ij} - \eta_j\eta_i\right)\left(\widehat{\lambda}_{i,t} - \tilde{\beta}\widehat{\lambda}_{i,t+1}\right) - \\ \Delta\widehat{C}_{t+1} \end{array} \right) \end{array} \right].$$

We turn now to the relationship between $\widehat{\zeta}_t$ and inventory investment. Consider the log-linearized version of the first order condition for the individual stages of production in equation (51). Substituting out $\widehat{\zeta}_{t+u} = -\tau_{K,t+u}$, it follows that, for $s = \{1, ..., S\}$

$$\widehat{\phi}_{j,t} + \widehat{\lambda}_{j,t} = -E_t \sum_{u=0}^{s-1} \tau_{K,t+u} + \frac{1}{\varrho}\left(E_t \widehat{Y}_{j,t+s} - \widehat{Z}_{j,t|t+s}\right) - \left(1 - \frac{1}{\varrho}\right) E_t \sum_{u=0}^{s-1} \widehat{g}_{t+u} + E_t \widehat{\lambda}_{j,t+s}.$$

and

$$\widehat{\phi}_{j,t}=\frac{1}{\varrho}\left(\widehat{Y}_{j,t}-\widehat{Z}_{j,t|t}\right).$$

Let $\psi_j(s)\equiv\frac{\omega_j(s)\beta\varrho^s g^{(1-\varrho)s}}{\sum_{s=0}^{S}\omega_j(s)\beta\varrho^s g^{(1-\varrho)s}}$, the steady state ratio between $Z_{t|t+s}$ and Z_t. Averaging up both sides over $s\in\{0,...,S\}$'s using $\psi_j(s)$ as weights yields

$$\widehat{\phi}_{j,t}+\widehat{\lambda}_{j,t}=\frac{1}{\varrho}\left(\sum\psi_j(s)\,E_t\widehat{Y}_{j,t+s}-\widehat{Z}_{j,t}\right)+E_t\left(\sum\psi_j(s)\left(\widehat{\lambda}_{j,t+s}-\sum_{u=0}^{s-1}\tau_{K,t+u}-\left(1-\frac{1}{\varrho}\right)\sum_{u=0}^{s-1}\widehat{g}_{t+u}\right)\right).$$

Add and subtract $\frac{1}{\varrho}\widehat{Y}_{j,t}$ to get

$$\widehat{\phi}_{j,t}+\widehat{\lambda}_{j,t}=\frac{1}{\varrho}\left(\widehat{Y}_{j,t}-\widehat{Z}_{j,t}\right)+E_t\left(\sum\psi_j(s)\left(\widehat{\lambda}_{j,t+s}-\sum_{u=0}^{s-1}\tau_{K,t+u}+\frac{1}{\varrho}\widehat{Y}_{j,t+s}-\frac{1}{\varrho}\widehat{Y}_{j,t}-\left(1-\frac{1}{\varrho}\right)\sum_{u=0}^{s-1}\widehat{g}_{t+u}\right)\right).$$

Rearranging,

$$\widehat{\phi}_{j,t}+\frac{1}{\varrho}\left(\widehat{Z}_{j,t}-\widehat{Y}_{j,t}\right)=E_t\left(\sum\psi_j(s)\left(\Delta\widehat{\lambda}_{j,t+s}+\frac{1}{\varrho}\Delta\widehat{Y}_{j,t+s}-\sum_{u=0}^{s-1}\tau_{K,t+u}\right)\right),$$

where $\Delta\widehat{\lambda}_{j,t+s}\equiv\widehat{\lambda}_{j,t+s}-\widehat{\lambda}_{j,t}-\sum_{u=0}^{s-1}\widehat{g}_{t+u}$ and $\Delta\widehat{Y}_{j,t+s}\equiv\widehat{Y}_{j,t+s}-\widehat{Y}_{j,t}+\sum_{u=0}^{s-1}\widehat{g}_{t+u}$ are the growth rates in $\widehat{\lambda}_{j,t+s}$ and $\widehat{Y}_{j,t+s}$ including the effect of the stochastic trend.

Finally, we have that

$$\begin{aligned}\widehat{Z}_{j,t}&\cong\ln(Z_{j,t})-\ln(Z_j),\\ \widehat{Y}_{j,t}&\cong\ln(Y_{j,t})-\ln(Y_j).\end{aligned}$$

In steady state, $Z_j\cong Y_j$ (with some small adjustment for the non-stochastic trend), so that

$$\begin{aligned}\widehat{Z}_{j,t}-\widehat{Y}_{j,t}&=\ln(Z_{j,t})-\ln(Y_{j,t}),\\ &=\ln\left(\frac{Z_{j,t}}{Y_{j,t}}\right),\\ &\cong\frac{Z_{j,t}-Y_{j,t}}{Y_{j,t}}.\end{aligned}$$

Given the national accounting definition for the change in inventories, $\Delta N_{j,t}$ is given by

$$\Delta N_{j,t} = Z_{j,t} - Y_{j,t},$$

and

$$\widehat{Z}_{j,t} - \widehat{Y}_{j,t} \cong \frac{\Delta N_{j,t}}{Y_{j,t}}.$$

It follows that

$$\widehat{\phi}_{j,t} + \frac{1}{\varrho}\frac{\Delta N_{j,t}}{Y_{j,t}} = E_t\left(\sum \psi_j\left(s\right)\left(\Delta\widehat{\lambda}_{j,t+s} + \frac{1}{\varrho}\Delta\widehat{Y}_{j,t+s} - \sum_{u=0}^{s-1}\tau_{K,t+u}\right)\right)$$

as in equation (56).

8.3 Efficiency Wedges

Efficiency wedges capture all factors that influence the efficiency with which inputs, that is, capital, labor and materials, are transformed into output. As such, they reflect changes in production possibilities associated with technological progress, as well as changes in taxes and regulations which distort the composition of intermediate inputs or the allocation of resources across firms. In a multi-sector model, efficiency wedges are defined separately for each sector,

$$\tau^A_{j,t} = \widehat{Z}_{j,t} - \alpha_j\widehat{K}_{j,t} - \xi_j\widehat{L}_{j,t} - \sum_i \gamma_{ij}\widehat{M}_{ij,t}. \tag{57}$$

In our environment, given the specification of production shocks (4), these wedges are a function of productivity components

$$\tau^A_{j,t} = \xi_j\left(\widehat{u}_t + \widehat{A}_t + \widehat{a}_{j,t}\right), \tag{58}$$

where $\widehat{u}_t$ is a transitory aggregate component, $\widehat{A}_t$ is a permanent component, and $\widehat{a}_{j,t}$ is a sector-specific component. The RBC literature has traditionally placed great emphasis on efficiency wedges as drivers of business cycles, albeit typically in the form of a single, transitory aggregate Hicks-neutral component. In specifying a more flexible form of efficiency wedges, combined with a production structure that generalizes technological trade offs, we position the model to explain shifts in comovement and relative volatilities over the post-war period with productivity shocks alone.

9 Details of Calibration and Estimation

The model economy used in the estimation/calibration exercises has two sectors ($N = 2$) and four stages of production ($S = 3$). We calibrate sector 1 to capture the production of durable goods and sector 2 the production of non-durable goods. All the data used in the calibration refers to 1997. Sector 1 is defined to include durable goods, which are the goods produced in the Construction Sector together with all the goods that the BEA classifies as being durables (including Wood Products, Nonmetallic Mineral Products, Primary Metals, Fabricated Metal Products, Machinery, Computer and Electronics, Electrical Equipment, Motor Vehicles, Other Transportation, Furniture and Related and Miscellaneous Manufacturing). Sector 2 comprises non-durable goods, which includes all private non-agricultural output not included in Sector 1. Non-durable output also includes the services from durable goods, calculated using the formula in the Technical Appendix to Chari, Kehoe, and McGrattan (2007a). This is the sum of the current cost depreciation (presented in 8.4 of the Fixed Assets Accounts) and a service flow equal to 4 percent of the total stock of durable consumer goods.

9.1 Calibrated Parameters

To calibrate the preference parameters, we set $\beta = 0.99$ and $\kappa = 0.4$. We also choose $g = 1.005$ and $\delta = 0.025$. All share parameters are calibrated based on available data. In order to calibrate the share parameters, we proceed in the following steps:

1) We add the services from durables to both output and profits of the non-durable sector.

2) We consolidate the input-output matrix of the U.S. economy into a 2x2 matrix using the classification described above. We can then calculate α_1, α_2, ξ_1 and ξ_2, γ_{11}, γ_{12}, γ_{21} and γ_{22} by dividing profits, wages and purchases of materials from the consolidated input-output matrix by total output in each sector net of the value paid in taxes.

4) We consolidate the capital-flows matrix of the U.S. economy into a 2x2 matrix using the classification described above. Note that there are non-durable inputs that contribute to investment, including Wholesale Trade, Retail Trade, Air Transportation, Rail Transportation, Water Transportation, Truck Transportation, Software, Telecommunications, Offices of Real Estate Agents, Engineering Services, Custom Computer Programming Services and Computer systems Design Services. We add the purchase of consumer durables as part of the durable part of investment in the capital of the non-durable sector. We then use the resulting 2x2 capital flows matrix to calibrate θ_{11}, θ_{12}, θ_{21}, θ_{22}.

5) To calibrate the $\omega_i(s)$'s:

i) we assume that $\omega_i(s) = \phi_i^s$.

ii) we calculate the ratio between inventories and sales of finished goods for durables and non-durables in all quarters for which data is available (use chained 2005 dollars).

iii) We take the average overtime of the ratios. This is 1.49 for durables and 0.37 for non-durables.

iv) We choose ϕ_i^s so that the steady-state inventory/sales ratios in the model for the two sectors match those obtained from the data.

6) Households only consume non-durables, so that we have that $\eta_1 = 0$ and $\eta_2 = 1$.

Table 1 below shows the calibrated parameters

Table 1: Calibrated Parameters

	N=2, S=3	N=1, S=0
g	1.004	1.004
β	0.993	0.993
κ	0.400	0.400
δ	0.025	0.025
α_1	0.121	0.428
α_2	0.296	-
$\theta_{1,1}$	0.746	1.000
$\theta_{1,2}$	0.896	-
$\theta_{2,1}$	0.254	-
$\theta_{2,2}$	0.104	-
η_1	0.000	1.000
η_2	1.000	-
ϕ_1	0.605	-
ϕ_2	0.148	-

9.2 Details of the Estimation

The parameters to be estimated include the elasticity of substitution, ϱ, which governs the curvature of the production function for Y_t, as well as the parameters governing the processes for the shocks to the different wedges. We assume the following processes:

$$
\begin{aligned}
u_t &= \rho_u u_{t-1} + \sigma_u \varepsilon_t^u, \\
a_{2,t} &= \rho_{a_2} a_{2,t-1} + \sigma_{a_2} \varepsilon_t^{a_2}, \\
g_t &= (1-\rho_g)\, g + \rho_g g_{t-1} + \sigma_g \varepsilon_t^g, \\
\Upsilon_t &= \rho_\Upsilon \Upsilon_{t-1} + \sigma_\Upsilon \varepsilon_t^\Upsilon, \\
\zeta_t &= \rho_\zeta \zeta_{t-1} + \sigma_\zeta \varepsilon_t^\zeta.
\end{aligned}
$$

where we allow ε_t, $\varepsilon_t^{a_2}$, $\varepsilon_t^g, \varepsilon_t^\Upsilon$ and ε_t^ζ to be correlated. We also allow the covariance matrix to change between the pre 1984 period and the post 1984 period.

When estimating the covariance matrix of these various shocks we need to restrict it to be positive definite. Let Σ be the covariance matrix of the shocks. We can write it as:

$$
\Sigma = TDT',
$$

where T is upper triangular with ones as diagonal elements and D is diagonal. We put inverse-gamma priors on the diagonal elements of D with mean 0.2 and standard-deviation 1 and normal priors on the off-diagonal elements of T, with mean 0 and standard-deviation 20. There are $\frac{5\times4}{2} = 10$ off-diagonal elements in T and 5 elements in D to be estimated, so that there are 15 parameters governing the covariance matrix. Since we estimate two covariance matrices, one for pre 1984 data and the other for post 1984 data, this gives us a total of 30 parameters to be estimated.

One issue with the estimation is that the scale of the shocks may be different. Since the scale of the priors on the parameters governing the covariance matrix are not different, this assumption could skew the results by making the priors effectively much tighter for certain shocks than for others. As a preliminary step in the estimation, we do a trial run with $\sigma_u = \sigma_{a_2} = \sigma_g = \sigma_\Upsilon = \sigma_\zeta = 1$. Based on the estimated variances for the different shocks, we calibrate those parameters to ensure that all variances are of the same scale. Specifically, we set $\sigma_u = \sigma_{a_2} = 1$, $\sigma_g = \sigma_\Upsilon = 10$ and $\sigma_\zeta = 100$.

Table 2 shows the posterior mode estimate for the covariance matrices pre and post 1984.

9.3 Impulse Responses with Error Bands

Figure 1 in the appendix presents the posterior means for the impulse response functions reported in Section 5.1, together with a 90% probability error bands. Unlike the impulse response functions reported in Section 5.1, which underscore the implied relative volatility of different aggregates, these are responses to one standard deviation innovations to the

Table 2: **Covariance Matrices for the Innovations**

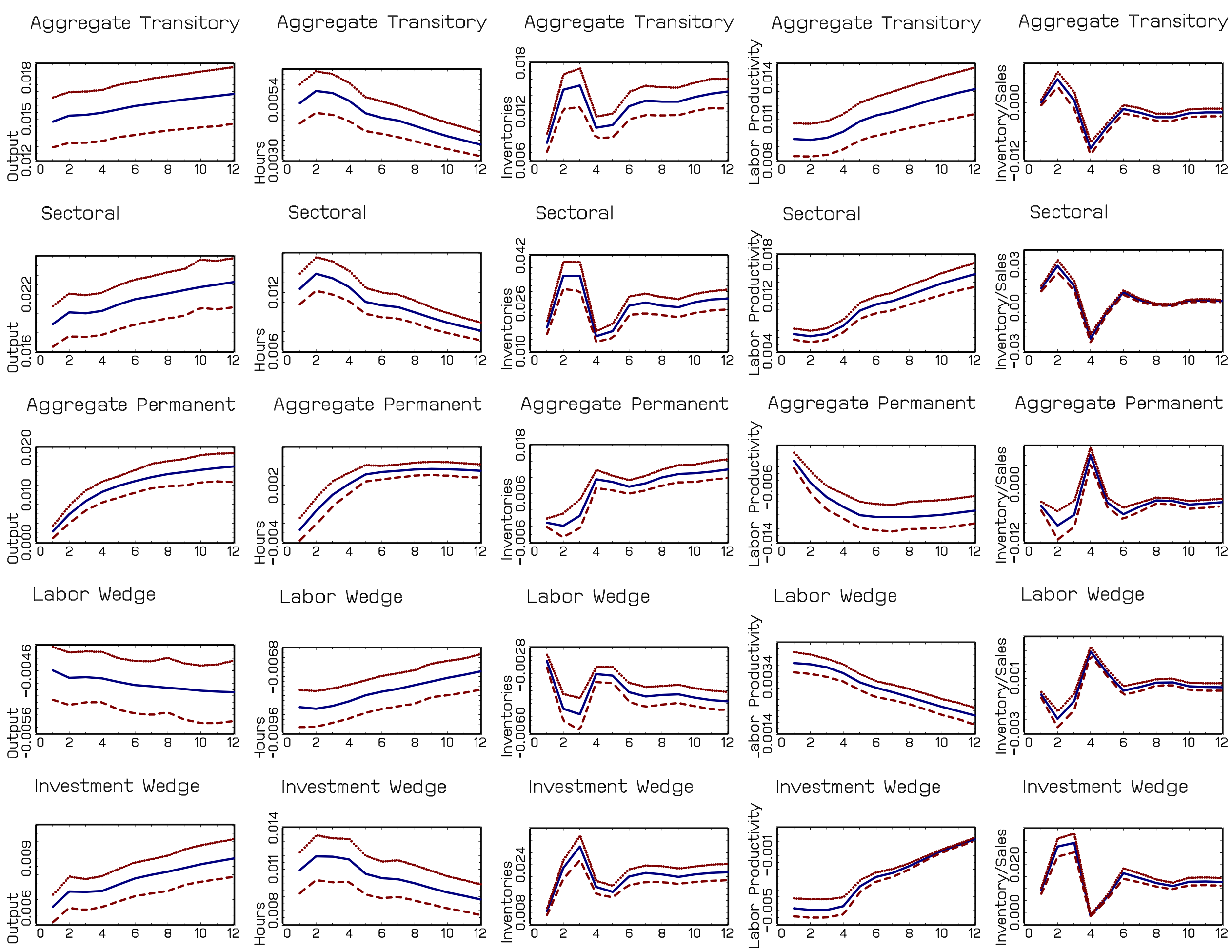
a. Pre 1984					
	u	$a_{2,t}$	g_t	Υ_t	ζ_t
u	0.05%	-0.17%	-0.12%	0.14%	0.10%
$a_{2,t}$	-0.17%	0.81%	0.52%	-0.44%	-0.50%
g_t	-0.12%	0.52%	0.86%	-0.67%	-0.20%
Υ_t	0.14%	-0.44%	-0.67%	0.65%	0.22%
ζ_t	0.10%	-0.50%	-0.20%	0.22%	0.37%
b. Post 1984					
	u	$a_{2,t}$	g_t	Υ_t	ζ_t
u	0.04%	-0.10%	-0.04%	0.05%	0.07%
$a_{2,t}$	-0.10%	0.31%	0.16%	-0.10%	-0.20%
g_t	-0.04%	0.16%	0.22%	-0.15%	-0.08%
Υ_t	0.05%	-0.10%	-0.15%	0.28%	0.05%
ζ_t	0.07%	-0.20%	-0.08%	0.05%	0.15%

different components of preferences and technology. The error bands confirm the robustness of the shape of the different impulse response functions to estimation uncertainty.

9.4 Decomposition of the 2001 recession

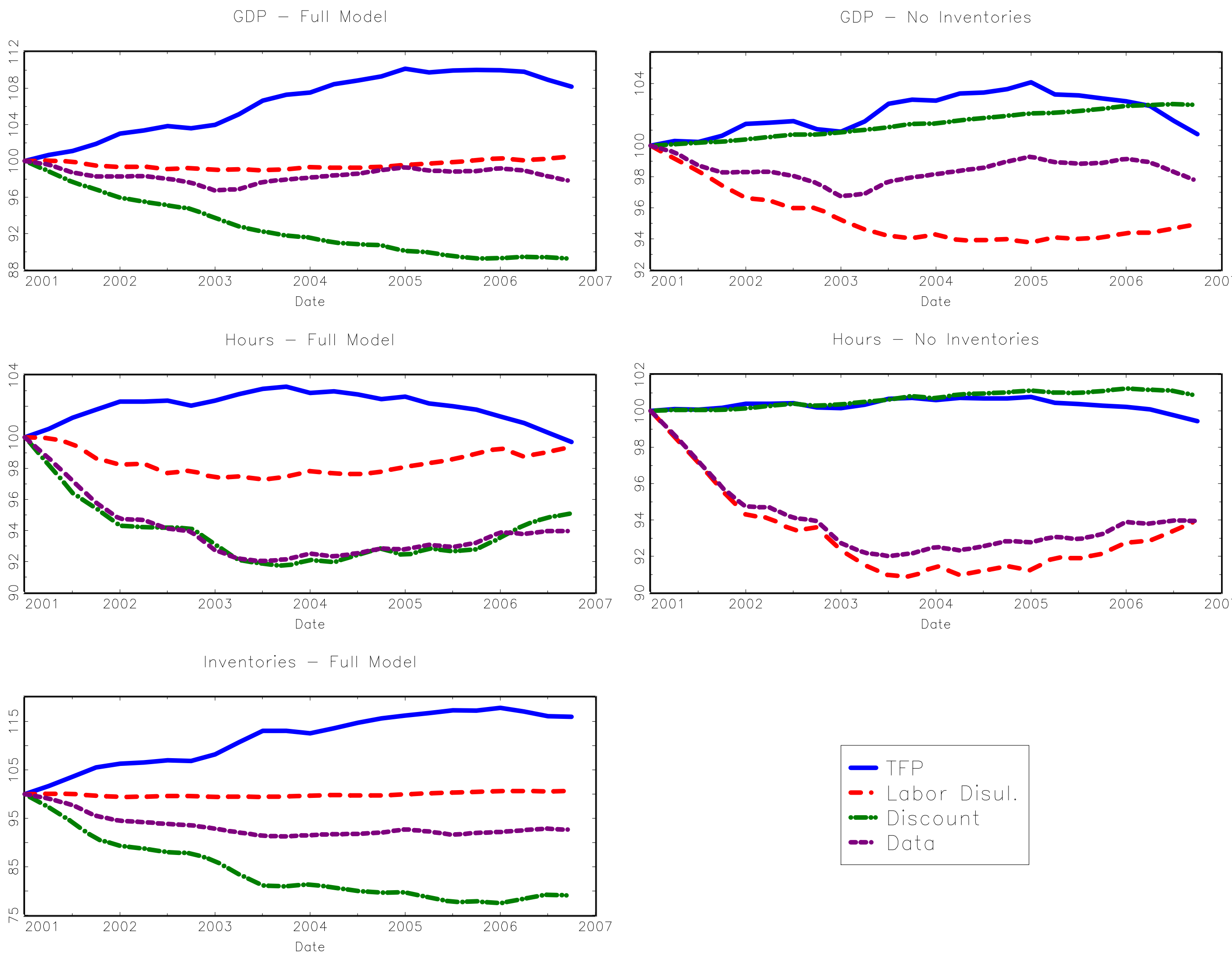
Figure 2 of the appendix presents a decomposition of the time path of the 2001 recession, similar to the decompositions presented in Figure 2 of the main text. The 2001 recession shares with the 1991 recession a diminished role for technology shocks. In effect, technology alone would seem to drive the economy towards an expansion rather than a recession. The fall in GDP, hours and inventories relative to trend were thus driven by a combination of an increased disutility of labor and reduced discount factors. For the 2001 recession, the main differences between the decomposition in the full model from the decomposition that one would obtain from a benchmark model without sectors or stages of production are that in the latter case, the model assigns a distinctly more prominent role for productivity variations and virtually no role for changes in the discount factor. Thus, as in the 1991 recession, the benchmark model without inventories would tend to underestimate the contribution of changes in the discount factor for the recession outcome.

Figure 1: **Impulse Response Functions**



Note: Based on posterior mode estimate of model parameters.

Figure 2: **Historical Decomposition of the 2001 Recession - Benchmark and One Sector Models**



Note: Based on posterior mode estimate of model parameters.

10 Calculation of Population Moments

The population moments implied by the model described in Table 6 were calculated from the state-space representation of the model. The log-linearized stationary model laid out in Section 7 has the following state-space representation:

$$\begin{aligned} x_{t+1} &= T_1 x_t + T_0 \Sigma^{1/2} w_{t+1}, \\ y_t &= Z x_t \end{aligned}$$

where x_{t+1} is an $M \times 1$ column vector of state variables, y_t is a $R \times 1$ vector of observed variables, T_1 is an $M \times M$ matrix, w_{t+1} is a vector of five innovations (corresponding to the five shocks), Σ is the covariance matrix for the innovations, T_0 is $M \times 5$ matrix and Z is an $R \times M$ matrix. In order to obtain the state-space representation of the model, we solve the linear rational expectations equilibrium using Chris Sims's GENSYS and expand the resulting system to include lags of variables used to calculate differences and time-aggregation. The latter is necessary in order to compare the model generated data, which is quarterly, with the moments reported by Ramey and West (1999), which are based on yearly data. This procedure provides us with T_1 and T_0, but in order to obtain the full state-space representation of the model we also need the covariance matrix of the shocks, Σ. In our estimation procedure, we estimate two such covariance matrices, one for the period before 1984 and the other for the period after. When calculating population moments, we rely on an arithmetic average of those two matrices.

Given the state space representation, the spectral density of the stationary distribution of x at frequency ω is given by (see Ljungqvist and Sargent (2004), for example):

$$S_y(\omega) = Z \left(I - T_1 e^{-i\omega}\right)^{-1} T_0 \Sigma T_0' \left(I - T_1' e^{i\omega}\right)^{-1} Z', \forall \omega \in [-\pi, \pi]$$

We then calculate the variance of any variable between frequencies ω_0 and ω_1 by numerically integrating $S_y(\omega)$. In table 6, we denote the frequencies by the number of quarters per cycle, i.e., by $\frac{\pi}{\omega}$. The overall variance of any observed variable y can be obtained by integrating S_y between 0 and π.

11 Subsample estimation

In the end of Section 6.3 in the main text, we present results from an estimate of the model in subsamples. In order to obtain the estimates, we assumed that the shock processes were as estimated in the full sample procedure. In terms of the calibration, the only change was the choice of $\omega_i(s)$'s. We redid step 5 in the calibration procedure described in Section 9.1 above using as targets for the inventories/sales of finished goods ratio of 1.57 for durables and 0.37 for non-durables in the pre-1984 sample and 1.32 for durables and 0.35 for non-durables in the post-1984 sample. As described above, we assume that $\omega_i(s) = \phi_i^s$ for some scalar ϕ_i. The corresponding values for ϕ_i were $\phi_D = 0.605$ and $\phi_{ND} = 0.1475$ pre-1984 and $\phi_D = 0.542$ and $\phi_{ND} = 0.145$ post-1984.

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