Appendix to Selection and Monetary Non-Neutrality in Time-Dependent Pricing Models *

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Contents

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1 The sticky-information model

We now describe a sticky-information model analogous to the sticky-price model in the main text. The comparison with this model plays a key role in the proof of Proposition 1 which, as we show, can be alternatively formulated as stating that the real effects of a monetary shock in a sticky-price model are identical to those effects in a sticky-information model, so long as the distribution of price spells in the former is identical to the distribution of price plans in the latter. One immediate implication of the proposition thus formulated is that all the results in the paper apply equally to time-dependent variants of Mankiw and Reis (2002).

We lay out a continuous-time version of Mankiw and Reis (2002) with general survival functions for price plans. The household and the market structure are identical to the one laid out in Section 2 of the paper. Firms are also identical, except that instead of choosing prices that remain in place for a period of time, firms choose *price plans*, which are insensitive to new information. In particular, under sticky information, a firm that resets its price *plan* $X_j(t, s)$ at time t solves:

$$\max_{X_j(t)} E_t \left[\int_0^\infty e^{-\rho(s-t)} \left(1 - G(s)\right) \left[X_j(t, s) Y_j(t + s) - W_j(t + s) N_j(t + s)\right] ds \right]$$

$$s.t.\ Y_j(t + s) = N_j(t + s),$$

$$Y_j(t + s) = \left(\frac{X_j(t, s)}{P(t + s)}\right)^{-\varepsilon} Y(t + s),$$

where $N_j(t + s)$ is the amount of labor demanded by the firm, and where the demand function already takes into account that goods market clearing implies $C_j(t) = Y_j(t)$.

Firms operate under perfect foresight, except for the fact that they do not anticipate the shock. Given that, the first order condition implies that:

$$X_j(t, s) = \frac{\varepsilon}{\varepsilon - 1} E_t \left[W_j(t + s)\right],$$

where $E_t[W_j(t + s)] = W_j^{old}(t + s)$ for $t < 0$ and $E_t[W_j(t + s)] = W_j^{new}(t + s)$ for $t \geq 0$.

The monetary shock is specified in the same way as in Section 2. As before, we log-linearize the model around a deterministic, zero-inflation symmetric steady state. In this log-linear environment, the optimal reset price plan for firms that change prices at time t is (lowercase variables denote log-deviations from the steady state):

$$x(t, s) = E_t \left[w_j(t + s)\right]. \tag{1}$$

A sequence of substitutions analogous to the ones laid out in Section 2 yields:

$$x(t, s) = E_t \left[\alpha m(t + s) + (1 - \alpha) p(t + s)\right],$$

where $\alpha = \frac{\sigma + \psi^{-1}}{1 + \varepsilon \psi^{-1}}$. Finally, the aggregate price level is given by:

$$p(t) = \int_{-\infty}^t \Lambda \left(1 - G(t - v)\right) x(v, s) dv.$$

In analogy to the sticky-price model, we can partition firms into those with "old" price plans and those with "new" price plans, depending on whether the price plan was set before or after the shock. Under sticky information, all firms with the same information set choose the same price

path. Hence, at any given point in time, all firms with old price plans set the same price, and all firms with new price plans also have the same price, irrespective of the path of nominal income.

It follows that, for $\alpha = 1$:

$$p(t) = \omega(t) \, m^{new}(t) + (1 - \omega(t)) \, m^{old}.$$

The real effects of a nominal shock are (using the fact that $p^{old} = m^{old}$):

$$
\begin{aligned}
\Gamma^{si} &= \int_0^\infty e^{-\rho t} \left(m^{new}(t) - p^{new}(t) - \left(m^{old} - p^{old} \right) \right) dt = \\
&= \int_0^\infty e^{-\rho t} \left(1 - \omega(t) \right) \left(m^{new}(t) - m^{old} \right).
\end{aligned}
$$

Note that Proposition 1 states that, in a sticky-price economy with strategic neutrality in price setting, $\Gamma = \int_0^\infty e^{-\rho t} \left(1 - \omega(t) \right) \left(m^{new}(t) - m^{old} \right)$. Hence, if $\alpha = 1$ and the distribution of durations of price plans in a sticky-information economy is the same as the distribution of price spells in a sticky-price economy, Proposition 1 states that the real effects of a monetary shock in both economies are the same. Importantly, since we can define selection in a sticky-information economy just as in the sticky-price economy, it follows that all the analytical results in the paper concerning the relationship between selection and real effects of nominal shocks are equally valid in a sticky-information environment.

2 Using polynomials to approximate the real effects of general shocks

Proposition 6' characterizes the real effects of general nominal shocks that can be expressed as a polynomial. If taken literally, Proposition 6' is of limited interest, since any shock characterized by polynomials of order $K = 3$ and higher will imply continuously increasing or decreasing inflation as $t \to \infty$. This would be inconsistent with the assumption of a constant inflation steady-state, as well as fairly unrealistic.

We make the case that Proposition 6' is useful as a means to calculate approximate real effects of any shock implying a continuous path for nominal income, and in which nominal income converges asymptotically to some constant growth rate path. Informally, for any given shock, an arbitrarily good approximation for the real effects of the shock can be obtained by truncating the distribution of price durations, and using Proposition 6 to calculate the real effects of a polynomial that approximates that shock between 0 and the truncation point. The approximation error can be made arbitrarily small by choosing a large enough T for the truncation point and a polynomial of high enough order.

The Proposition below makes the point formally:

Proposition A. 1. *Suppose an economy is characterized by a distribution of price spells G with $\int_0^\infty (1 - G(t)) t^2 dt < \infty$. Let the random variable τ be the realized duration of price spells. Let $E_T\left[\tau^k\right]$ be the kth moment of τ using the truncated distribution of price spells $G_T(t) = \min\left\{\frac{G(t)}{G(T)}, 1\right\}$ for some $T > 0$. Consider the real effects of a monetary shock $m^{new}(t) - m^{old}$, well defined for all $t \geq 0$ and continuous in t, and with $\int_0^\infty \left(m^{new}(t) - m^{old} - bt - \Delta \right) < \infty$ for*

3

real numbers b and Δ. Then, for any $\delta > 0$, there are $T \in \mathbb{R}^+$, $K \in \mathbb{N}$ and $\{a_k\}_{k=1}^{K} \in \mathbb{R}^K$ such that the real effects of a monetary shock satisfy:

$$\left| \lim_{\rho \to 0} \Gamma - \sum_{k=1}^{K} \frac{a_k}{k(k+1)} \frac{E_T\left[\tau^{k+1}\right]}{E_T\left[\tau\right]} \right| < \delta.$$

Proof of Proposition A.1. The Weierstrass Approximation Theorem implies that within the compact interval $[0, T]$ we can approximate the shock arbitrarily well by a polynomial of high enough order. Let $K(\varepsilon, T)$ and $\{a_k(\varepsilon, T)\}_{k=1}^{K(\varepsilon,T)}$ be values for K and a_k such that

$$\sup_{t \in [0,T]} \left| \sum_{k=1}^{K(\varepsilon,T)} a_k(\varepsilon, T) t^{k-1} - \left(m^{new}(t) - m^{old}\right) \right| \leq \varepsilon$$

Let $\omega_T(t) = \Lambda \int_0^t (1 - G_T(s)) \, ds$. Since $\omega_T(t) = 1$ for $t > T$, we have that

$$\int_0^T (1 - \omega_T(t)) \sum_{k=1}^{K(\varepsilon,T)} a_k(\varepsilon, T) t^{k-1} dt = \int_0^\infty (1 - \omega_T(t)) \sum_{k=1}^{K(\varepsilon,T)} a_k(\varepsilon, T) t^{k-1} dt$$

From propositions 1 and 6$'$, we have that

$$\int_0^\infty (1 - \omega_T(t)) \sum_{k=1}^{K(\varepsilon,T)} a_k(\varepsilon, T) t^{k-1} dt = \sum_{k=1}^{K(\varepsilon,T)} \frac{a_k(\varepsilon, T)}{k(k+1)} \frac{E_T\left[\tau^{k+1}\right]}{E_T\left[\tau\right]}$$

We can therefore prove the theorem if we are able to show that, for suitable choice of ε and T, $D \equiv \left| \int_0^T (1 - \omega_T(t)) \sum_{k=1}^{K(\varepsilon,T)} a_k(\varepsilon, T) t^{k-1} dt - \int_0^\infty (1 - \omega(t)) \left(m^{new}(t) - m^{old}\right) dt \right|$ is arbitrarily small.

First, note that we can rewrite D as

$$D = \left| \begin{array}{l} \int_0^T (1 - \omega_T(t)) \sum_{k=1}^{K(\varepsilon,T)} a_k(\varepsilon, T) t^{k-1} dt - \int_0^T (1 - \omega_T(t)) \left(m^{new}(t) - m^{old}\right) dt \\ + \int_0^T \left(\omega(t) - \omega^T(t)\right) \left(m^{new}(t) - m^{old}\right) dt - \int_T^\infty (1 - \omega(t)) \left(m^{new}(t) - m^{old}\right) \end{array} \right|$$

Using the triangular inequality

$$D \leq D_1 + D_2 + D_3$$

where

$$D_1 = \left| \int_0^T (1 - \omega_T(t)) \left[\sum_{k=1}^{K(\varepsilon,T)} a_k(\varepsilon, T) t^{k-1} - \left(m^{new}(t) - m^{old}\right) \right] dt \right|$$

$$D_2 = \left| \int_0^T (\omega(t) - \omega_T(t)) \left(m^{new}(t) - m^{old}\right) dt \right|$$

$$D_3 = \left| \int_T^\infty (1 - \omega(t)) \left(m^{new}(t) - m^{old}\right) \right|$$

We now show that, with suitable choice of ε and T we can find arbitrarily low bounds $\delta_1(\varepsilon, T)$, $\delta_2(\varepsilon, T)$ and $\delta_3(\varepsilon, T)$ for the three terms, D_1, D_2 and D_3.

4

Bounding D_1:

$$D_1 \leq \int_0^T (1 - \omega_T(t)) \left| \sum_{k=1}^{K(\varepsilon, T)} a_k(\varepsilon, T) t^{k-1} - \left(m^{new}(t) - m^{old} \right) \right| dt$$

$$\leq \int_0^T (1 - \omega_T(t)) \varepsilon dt$$

$$= \frac{1}{2} \frac{E[\tau^2]}{E[\tau]} \varepsilon.$$

where the second inequality follows from the definition of $K(\varepsilon, T)$ and $a_k(\varepsilon, T)$, and the equality from Proposition 6 in the main text. Thus, we can bound D_1 with $\delta_1(\varepsilon) = \frac{E[\tau^2]}{E[\tau]} \varepsilon$. Note that $\lim_{\varepsilon \to 0} \delta_1(\varepsilon) = 0$.

Bounding D_{12}:
In order to provide a bound for D_{12}, note first that, since $\omega(t) < 1$, we have that

$$D_{12} < \left| \left(\frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)} \right) \int_0^T \left(m^{new}(t) - m^{old} \right) dt \right|$$

Apply the triangular inequality to obtain

$$D_{12} < \left(\frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)} \right) \int_0^T \left| m^{new}(t) - m^{old} \right| dt$$

Add and subtract $bt + \Delta$ the term within the integral, and apply the triangular inequality again:

$$D_{12} \leq \frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)} \int_0^T \left| m^{new}(t) - m^{old} - bt - \Delta \right| dt$$

$$+ \frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)} \int_0^T |bt| \, dt + \frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)} \int_0^T |\Delta| \, dt$$

Define $M = \sup_{t \geq 0} \left\{ \left| m^{new}(t) - m^{old} - bt - \Delta \right| \right\}$. Recall that $\lim m^{new}(t) - m^{old} - bt = \Delta$. Therefore M is finite. We can thus bound D_{12} by

$$D_{12} \leq \frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)} \int_0^T (M + |\Delta|) \, dt + \frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)} \int_0^T |bt| \, dt$$

Solving out the integrals:

$$D_{12} \leq \delta_{12}(T)$$

where

$$\delta_{12}(T) \equiv \frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)} \left[(M + |\Delta|) T + \frac{1}{2} |b| T^2 \right]$$

By assumption, $\int_0^\infty (1 - G(t)) t^2 dt < \infty$, so that $\lim_{T \to \infty} (1 - G(T)) T^2 = 0$ and $\lim_{T \to \infty} (1 - G(T)) T = 0$ (since for $T > 0$ the latter is smaller than the former) Also $\lim_{T \to \infty} \frac{\Lambda_T}{\Lambda} = 1$. Therefore, $\lim_{T \to \infty} D_{12} = 0$.

5

Bounding D_2:

In order to provide a bound for D_2, note that we can write $\omega_T(t)$ as:

$$\omega_T(t) = \Lambda_T \int_0^{\min\{t,T\}} \left(1 - \frac{G(s)}{G(T)}\right) ds$$

with $\Lambda_T \equiv \int_0^T \left(1 - \frac{G(s)}{G(T)}\right) ds$. Adding and substracting $\frac{1}{G(T)}$ yields

$$\omega_T(t) = \Lambda_T \int_0^{\min\{t,T\}} \left(1 - \frac{1}{G(T)} + \frac{1 - G(s)}{G(T)}\right) ds$$

Since $\Lambda \int_0^t (1 - G(s)) ds = \omega(t)$, it follows that

$$\omega_T(t) = \frac{\Lambda_T}{\Lambda} \frac{\min\{\omega(t), \omega(T)\}}{G(T)} - \Lambda_T \left(\frac{1 - G(T)}{G(T)}\right) \min\{t, T\}$$

So that, for $t < T$,

$$\omega(t) - \omega_T(t) = \omega(t) \left(\frac{G(T) - \frac{\Lambda_T}{\Lambda}}{G(T)}\right) + \Lambda_T \left(\frac{1 - G(T)}{G(T)}\right) t$$

and

$$D_2 = \left| \int_0^T \left(\frac{G(T) - \frac{\Lambda_T}{\Lambda}}{G(T)} \omega(t) - \frac{G(T) - 1}{G(T)} \Lambda_T t\right) \left(m^{new}(t) - m^{old}\right) dt \right|$$

Solving the integral for the term linear in t, and applying the triangular inequality yields,

$$D_2 = \left| \begin{array}{l} \int_0^T \frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)} \omega(t) \left(m^{new}(t) - m^{old}\right) dt \\ + \frac{1 - G(T)}{G(T)} \Lambda_T \int_0^T t \left(m^{new}(t) - m^{old}\right) dt \end{array} \right|$$

$$\leq D_{21} + D_{22}$$

with

$$D_{21} \equiv \left| \frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)} \int_0^T \omega(t) \left(m^{new}(t) - m^{old}\right) dt \right|$$

$$D_{22} \equiv \left| \frac{1 - G(T)}{G(T)} \Lambda \int_0^T t \left(m^{new}(t) - m^{old}\right) dt \right|$$

Bounding D_{21}:

In order to provide a bound for D_{21}, note first that, since $\omega(t) < 1$, we have that

$$D_{21} < \left| \left(\frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)}\right) \int_0^T \left(m^{new}(t) - m^{old}\right) dt \right|$$

Apply the triangular inequality to obtain

$$D_{12} < \left(\frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)}\right) \int_0^T \left|m^{new}(t) - m^{old}\right| dt$$

Add and subtract $bt + \Delta$ the term within the integral, and apply the triangular inequality again:

$$D_{21} \leq \frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)} \int_0^T \left| m^{new}(t) - m^{old} - bt - \Delta \right| dt$$

$$+ \frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)} \int_0^T |bt| \, dt + \frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)} \int_0^T |\Delta| \, dt$$

Define $M = \sup_{t \geq 0} \left\{ \left| m^{new}(t) - m^{old} - bt - \Delta \right| \right\}$. Recall that $\lim m^{new}(t) - m^{old} - bt = \Delta$. Therefore M is finite. We can thus bound D_{21} by

$$D_{21} \leq \frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)} \int_0^T (M + |\Delta|) \, dt + \frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)} \int_0^T |bt| \, dt$$

Solving out the integrals:

$$D_{21} \leq \delta_{21}(T)$$

where

$$\delta_{21}(T) \equiv \frac{\frac{\Lambda_T}{\Lambda} - G(T)}{G(T)} \left[(M + |\Delta|) T + \frac{1}{2} |b| T^2 \right]$$

By assumption, $\int_0^\infty (1 - G(t)) t^2 dt < \infty$, so that $\lim_{T \to \infty} (1 - G(T)) T^2 = 0$ and $\lim_{T \to \infty} (1 - G(T)) T = 0$ (since for $T > 0$ the latter is smaller than the former) Also $\lim_{T \to \infty} \frac{\Lambda_T}{\Lambda} = 1$. Therefore, $\lim_{T \to \infty} \delta_{21}(T) = 0$.

Bounding D_{22}:
Add and subtract $bt + \Delta$ from $\left(m^{new}(T) - m^{old} \right)$, and apply the triangular to obtain

$$D_{22} < \delta_{22}(T)$$

where

$$\delta_{22}(T) \equiv \frac{\Lambda(1 - G(T))}{G(T)} \left| \int_0^T \left(m^{new}(t) - m^{old} - (\Delta + bt) \right) dt \right| + \frac{\Lambda(1 - G(T))}{G(T)} \int_0^T (|\Delta| + |b| \, t) \, dt$$

Since $\int_0^\infty \left(m^{new}(t) - m^{old} - (\Delta + bt) \right) dt < \infty$, it follows that:

$$\lim_{T \to \infty} \frac{\Lambda(1 - G(T))}{G(T)} \left| \int_0^T \left(m^{new}(t) - m^{old} - (\Delta + bt) \right) dt \right| = 0.$$

Furthermore, $\frac{\Lambda(1 - G(T))}{G(T)} \int_0^T (|\Delta| + |b| \, t) \, dt = \frac{\Lambda(1 - G(T))}{G(T)} \left(|\Delta| \, T + |b| \frac{T^2}{2} \right)$. Since $\lim_{T \to \infty} (1 - G(T)) T^2 = 0$ and $\lim_{T \to \infty} (1 - G(T)) T = 0$, it follows that $\lim_{T \to \infty} \delta_{22}(T) = 0$.

Bounding D_3 :
Finally, note that, applying the triangular inequality to D_3 yields

$$D_3 \leq \int_T^\infty (1 - \omega(t)) \left| m^{new}(t) - m^{old} \right|$$

7

Again, write $m^{new}(t) - m^{old} = m^{new}(t) - m^{old} - bt - \Delta + (bt + \Delta)$ and apply the triangular inequality to obtain

$$
\begin{aligned}
D_3 \quad \le \quad & \int_T^\infty (1 - \omega(t)) \left| m^{new}(t) - m^{old} - bt - \Delta \right| dt \\
& + \int_T^\infty (1 - \omega(t)) (|\Delta| + |b|\, t)\, dt
\end{aligned}
$$

Let $\varepsilon(T) = \sup \left\{ \left| m^{new}(t) - m^{old} - bt - \Delta \right| \, | t > T \right\}$. Then

$$
D_3 \le \delta_3(T)
$$

where

$$
\delta_3(T) = \int_T^\infty (1 - \omega(t)) (\varepsilon(T) + |\Delta|)\, dt + |b| \int_T^\infty (1 - \omega(t))\, t\, dt
$$

Since $\lim_{t\to\infty} m^{new}(t) - m^{old} - bt - \Delta = 0$, it follows that $\lim_{T\to\infty} \varepsilon(T) = 0$. Also, note that

$$
\begin{aligned}
\int_0^\infty (1 - \omega(t))\, dt \quad &= \quad \Lambda \int_0^\infty \int_t^\infty (1 - G(s))\, ds\, dt \\
&= \quad \Lambda \int_0^\infty (1 - G(t))\, t\, dt
\end{aligned}
$$

where the second line follows from integration by parts, and

$$
\int_0^\infty (1 - \omega(t))\, t\, dt = \Lambda \frac{1}{2} \int_0^\infty (1 - G(t))\, t^2 dt
$$

Both quantities are finite by assumption. Since $1 - \omega(t) > 0 \,\forall t$, it follows that $\lim_{T\to\infty} \int_T^\infty (1 - \omega(t))\, dt = \lim_{T\to\infty} \int_T^\infty (1 - \omega(t))\, t\, dt = 0$ and $\delta_3(T)$ declines towards 0 as T increases, with $\lim_{T\to\infty} \delta_3(T) = 0$

Wrapping up:
Let $\delta(T, \varepsilon) = \delta_1(\varepsilon) + \delta_{21}(T) + \delta_{22}(T) + \delta_3(T)$. We just showed that $D < \delta_1(\varepsilon) + \delta_{21}(T) + \delta_{22}(T) + \delta_3(T)$. Since $\lim_{\varepsilon\to 0} \delta_1(\varepsilon) = 0$ and $\lim_{T\to\infty} \delta_{12}(T) + \delta_{22}(T) + \delta_3(T) = 0$, it follows that, for any desired δ, one can find a small enough ε and a large enough T so that $\delta(T, \varepsilon) = \delta$.

$\square$

3 Implementation of the numerical simulation

The numerical analysis is based on a log-linearized, discrete-time version of the model solved using Dynare. The heart of the model is a pricing rule, dependent on α. Let x_t be the reset price chosen by firms who get to choose their prices at t in log-deviation form. The discrete time analogue of equation (9) is

$$
x_t = \Lambda \sum_{s=0}^\infty \beta^s (1 - G_s) \left[\alpha m_{t+s} + (1 - \alpha) p_{t+s} \right] ds, \tag{2}
$$

8

The price level at t is:

$$p_t = \Lambda \sum_{s=0}^{\infty} (1 - G_s) x_{t-s}$$

and output is

$$y_t = m_t - p_t$$

The law of motion for m_t is:

$$\Delta m_t = 0.5 \Delta m_{t-1} + \varepsilon_t$$

Let y_{t+s}^{IRF} be the impulse response function of a unit shock to ε_t that hits the economy at t. We calculate the cumulative real effect as

$$\Gamma = \sum_{s=0}^{\infty} \beta^s y_{t+s}^{IRF}$$

This system of four equations in four variables describes how output reacts to shocks. Note that m_t is not stationary, so, as written this model is not amenable to be solved using conventional methods. We can make the model stationary by rewriting it in terms of $\tilde{p}_t = p_t - m_t$ and $\tilde{x}_t = x_t - m_t$. Then we have the equivalent model:

$$\tilde{x}_t = \Lambda \sum_{s=0}^{\infty} \beta^s (1 - G_s) \left[\sum_{v=0}^{s} \Delta m_{t+v} + (1 - \alpha) \tilde{p}_{t+s} \right] ds, \tag{3}$$

$$\tilde{p}_t = \Lambda \sum_{s=0}^{\infty} (1 - G_s) \left(\tilde{x}_{t-s} - \sum_{v=0}^{s} \Delta m_{t-s+v} \right) \tag{4}$$

$$y_t = -\tilde{p}_t \tag{5}$$

$$\Delta m_t = 0.5 \Delta m_{t-1} + \varepsilon_t \tag{6}$$

This model as written is still not solvable in Dynare because it involves infinite sums, even if convergent. We solve it for cases where there is some J such that $G_s = 1$ for all $s > J$. In those cases the state space becomes finite.

For the truncated exponentials case, we have that, given the maximum duration T and the decay parameter θ, the average duration of price-spells is:

$$E[\tau] = \left(1 - (1 - \theta)^{T-1} \right) \frac{1}{\theta} + (1 - \theta)^{T-1}$$

For any T, we look for θ such that $E[\tau] = 2$. We consider $T = \{2, ..., 20\}$. Table (1) gives the corresponding θ's, together with the mean, variance and skewness of price durations for some of these cases. Note that the variance is increasing in T.

Table 1: Parameters and Moments for Truncated Exponential Models

T	θ	mean	variance	skewness
2	0.000	2.000	0.000	-
4	0.456	2.000	1.234	0.703
6	0.491	2.000	1.718	1.430
8	0.498	2.000	1.903	1.804
10	0.500	2.000	1.969	1.988
12	0.500	2.000	1.990	2.070
14	0.500	2.000	1.997	2.103
16	0.500	2.000	1.999	2.115

4   Introducing alternative sources of economic fluctuations

We now introduce markup and productivity shocks to the model described in Section 2 of the paper. These will be subject to analysis in Section 6. We also show how to derive the firm's optimal pricing choice as a function of those shocks.

We introduce productivity shocks by assuming that the production function for good j is $Y_j(t) = Z(t) N_j(t)$, where $Z(t)$ denotes the aggregate productivity shock. For simplicity, we adopt the steady-state normalization $Z(t) = 1$. We introduce markup shocks by allowing the θ coefficients in household preferences to be time-varying.

With these changes, the only change in the household's problem is that the composite consumption good is now given by

$$C(t) \equiv \left[\int_0^1 C_j(t)^{\frac{\varepsilon(t)-1}{\varepsilon(t)}} \, dj \right]^{\frac{\varepsilon(t)}{\varepsilon(t)-1}},$$

with the elasticity of substitution between varieties $\varepsilon(t) > 1$ time-varying. Denoting by $P_j(t)$ the price charged by firm j at time t, the corresponding consumption price index is:

$$P(t) = \left[\int_0^1 P_j(t)^{1-\varepsilon(t)} \, dj \right]^{\frac{1}{1-\varepsilon(t)}}.$$

The first-order conditions for the representative consumer's remain as before, except for the choice of varieties, which is now

$$C_j(t) = C(t) \left(\frac{P_j(t)}{P(t)} \right)^{-\varepsilon(t)}, \quad j \in [0,1]. \tag{7}$$

A firm that resets its price at time t chooses the price $X_j(t)$ to solve:

$$\max_{X_j(t)} E_t \left[\int_t^\infty e^{-\rho s} (1 - G(s)) \left[X_j(t) Y_j(t+s) - W_j(t+s) N_j(t+s) \right] ds \right]$$

$$s.t. \ Y_j(t+s) = Z(t+s) N_j(t+s), \tag{8}$$

$$Y_j(t+s) = \left(\frac{X_j(t)}{P(t+s)} \right)^{-\varepsilon(t)} Y(t+s),$$

The first-order condition now yields:

$$1 = E_t \left[\frac{\int_0^\infty e^{-\rho s} \left(1 - G\left(s\right)\right) P\left(t+s\right)^{\varepsilon(t+s)} Y\left(t+s\right) \frac{\varepsilon(t+s)}{\varepsilon(t+s)-1} \frac{W_j(t+s)}{Z(t+s)} X_j\left(t\right)^{-\varepsilon(t+s)-1} ds}{\int_0^\infty e^{-\rho s} \left(1 - G\left(s\right)\right) P\left(t+s\right)^{\varepsilon(t+s)} Y\left(t+s\right) X_j\left(t\right)^{-\varepsilon(t+s)} ds} \right].$$

Note that, unlike in the text, we cannot factor $X_j\left(t\right)$ out of the integral, since it is raised to a time-varying exponent.

We log-linearize the model around a zero-inflation steady state. In this log-linear environment, firms that change prices at time t set (lowercase variables denote log-deviations from the steady state):

$$x\left(t\right) = x_j\left(t\right) = E_t \left[\frac{\int_0^\infty e^{-\rho s} \left(1 - G\left(s\right)\right) \left(w_j\left(t+s\right) - z\left(t+s\right) + \zeta\left(t+s\right)\right) ds}{\int_0^\infty e^{-\rho s} \left(1 - G\left(s\right)\right) ds} \right], \tag{9}$$

where $\zeta\left(t+s\right)$ is the log deviation from steady-state of the desired markup $\frac{\varepsilon(t+s)}{\varepsilon(t+s)-1}$.

As in the main text,

$$w_j\left(t+s\right) = p\left(t+s\right) + \sigma c\left(t+s\right) + \psi^{-1} l_j\left(t+s\right).$$

Using the production function to substitute out $l_j\left(t+s\right) = y_j\left(t+s\right) - z\left(t+s\right)$, applying the equilibrium condition $c\left(t+s\right) = y\left(t+s\right)$ and the optimality condition for the choice of varieties $y_j\left(t+s\right) = y\left(t+s\right) - \varepsilon\left(p_j\left(t+s\right) - p\left(t+s\right)\right)$ yields the following expression for nominal wages:

$$w_j\left(t+s\right) = p\left(t+s\right) + \left(\sigma + \psi^{-1}\right) y\left(t+s\right) - \psi^{-1} z\left(t+s\right) - \varepsilon\psi^{-1}\left(x\left(t\right) - p\left(t+s\right)\right),$$

where the main difference arises because now $y_j\left(t+s\right) = l_j\left(t+s\right) - z\left(t+s\right)$.

Note that $w_j\left(t+s\right)$ is the same for all j, so that, consistent with the symmetry assumption above, $x_j\left(t\right)$ is also the same for all j. We can also use $m\left(t+s\right) = p\left(t+s\right) + y\left(t+s\right)$ to substitute out $y\left(t+s\right)$, rearrange slightly, and obtain:

$$w_j\left(t+s\right) = \left(1 + \varepsilon\psi^{-1} - \sigma - \psi^{-1}\right) p\left(t+s\right) + \left(\sigma + \psi^{-1}\right) m\left(t+s\right) - \psi^{-1} z\left(t+s\right) - \varepsilon\psi^{-1} x\left(t\right).$$

Substituting the expression above in the first-order condition for the firm's problem (equation 9) and rearranging yields:

$$x\left(t\right) = E_t \left[\frac{\int_0^\infty e^{-\rho s} \left(1 - G\left(s\right)\right) \left[\alpha m\left(t+s\right) + \left(1 - \alpha\right) p\left(t+s\right) - \frac{1+\psi^{-1}}{1+\varepsilon\psi^{-1}} z\left(t+s\right) + \frac{1}{1+\varepsilon\psi^{-1}} \zeta\left(t+s\right)\right] ds}{\int_0^\infty e^{-\rho s} \left(1 - G\left(s\right)\right) ds} \right],$$

$$\tag{10}$$

where $\alpha = \frac{\sigma+\psi^{-1}}{1+\varepsilon\psi^{-1}}$.

5   Second-order approximations for efficiency criteria

In Section 6 we discuss the implications of selection for private and social efficiency. We provide here a derivation of the second order approximation to the relevant efficiency criteria.

5.1 Private efficiency

Real Profits for firm j in any given period t are:

$$\frac{\Pi_j(t)}{P(t)} = \frac{P_j(t)}{P(t)} Y_j(t) - \frac{W_j(t)}{P(t)} L_j(t)$$

Substituting in the demand function $Y_j(t) = \left(\frac{P_j(t)}{P(t)}\right)^{-\varepsilon(t)} Y(t)$ and the production function $Y_j(t) = Z(t) L_j(t)$ we obtain

$$\frac{\Pi_j(t)}{P(t)} = \left(\frac{P_j(t)}{P(t)} - \frac{W_j(t)}{P(t)} \frac{1}{Z(t)}\right) \left(\frac{P_j(t)}{P(t)}\right)^{-\varepsilon(t)} Y(t)$$

Let $\tilde{\Pi}_j(t)$ denote the profits of a hypothetical identical firm in the same economy that does not face any nominal frictions and $\tilde{P}_j(t)$ the price chosen by that firm. Then

$$\tilde{\Pi}_j(t) = \max_{\tilde{P}_j(t)} \left(\frac{\tilde{P}_j(t)}{P(t)} - \frac{W_j(t)}{P(t)} \frac{1}{Z(t)}\right) \left(\frac{\tilde{P}_j(t)}{P(t)}\right)^{-\varepsilon(t)} Y(t)$$

Solving the maximization problem, it follows that:

$$\frac{\tilde{P}_j(t)}{P(t)} = \frac{\varepsilon(t)}{\varepsilon(t) - 1} \frac{W_j(t)}{P(t)} \frac{1}{Z(t)}$$

$$\tilde{\Pi}_j(t) = \frac{1}{\varepsilon(t)} \left(\frac{\tilde{P}_j(t)}{P(t)}\right)^{1-\varepsilon(t)} Y(t)$$

Private inefficiency occurs when firms lose profit relative to the hypothetical frictionless firm. We want to obtain an approximation for the firm's profit loss relative to the hypothetical frictionless firm. This is:

$$\hat{\Pi}_j(t) = \tilde{\Pi}_j(t) - \Pi_j(t)$$

Which we can factor as:

$$\hat{\Pi}_j(t) = \left(1 - \frac{\Pi_j(t)}{\tilde{\Pi}_j(t)}\right) \tilde{\Pi}_j(t)$$

The profit-ratio $\frac{\Pi_j(t)}{\tilde{\Pi}_j(t)}$ can be written as:

$$
\begin{aligned}
\frac{\Pi_j\left(t\right)}{\tilde{\Pi}_j\left(t\right)} &= \frac{\left(\frac{P_j(t)}{P(t)} - \frac{W_j(t)}{P(t)}\frac{1}{Z(t)}\right)\left(\frac{P_j(t)}{P(t)}\right)^{-\varepsilon(t)}Y\left(t\right)}{\left(\frac{\tilde{P}_j(t)}{P(t)} - \frac{W_j(t)}{P(t)}\frac{1}{Z(t)}\right)\left(\frac{\tilde{P}_j(t)}{P(t)}\right)^{-\varepsilon(t)}Y\left(t\right)} \\[2mm]
&= \frac{\left(\frac{P_j(t)}{P(t)} - \frac{\varepsilon(t)-1}{\varepsilon(t)}\frac{\tilde{P}_j(t)}{P(t)}\right)\left(\frac{P_j(t)}{P(t)}\right)^{-\varepsilon(t)}Y\left(t\right)}{\left(\frac{\tilde{P}_j(t)}{P(t)} - \frac{\varepsilon(t)-1}{\varepsilon(t)}\frac{\tilde{P}_j(t)}{P(t)}\right)\left(\frac{\tilde{P}_j(t)}{P(t)}\right)^{-\varepsilon(t)}Y\left(t\right)} \\[2mm]
&= \varepsilon\left(t\right)\frac{\left(\frac{P_j(t)}{P(t)} - \frac{\varepsilon(t)-1}{\varepsilon(t)}\frac{\tilde{P}_j(t)}{P(t)}\right)\left(\frac{P_j(t)}{P(t)}\right)^{-\varepsilon(t)}Y\left(t\right)}{\left(\frac{\tilde{P}_j(t)}{P(t)}\right)^{1-\varepsilon(t)}Y\left(t\right)} \\[2mm]
&= \varepsilon\left(t\right)\left(\frac{P_j\left(t\right)}{\tilde{P}_j\left(t\right)}\right)^{1-\varepsilon(t)} - \left(\varepsilon\left(t\right)-1\right)\left(\frac{P_j\left(t\right)}{\tilde{P}_j\left(t\right)}\right)^{-\varepsilon(t)}
\end{aligned}
$$

Thus,

$$
\hat{\Pi}_j\left(t\right) = \left(1 - \varepsilon\left(t\right)\left(\frac{P_j\left(t\right)}{\tilde{P}_j\left(t\right)}\right)^{1-\varepsilon(t)} + \left(\varepsilon\left(t\right)-1\right)\left(\frac{P_j\left(t\right)}{\tilde{P}_j\left(t\right)}\right)^{-\varepsilon(t)}\right)\frac{1}{\varepsilon\left(t\right)}\left(\frac{\tilde{P}_j\left(t\right)}{P\left(t\right)}\right)^{1-\varepsilon(t)}Y\left(t\right).
$$

Rewriting the loss function in terms of log-deviations of $P_j\left(t\right)$, $\tilde{P}_j\left(t\right)$ and $Y\left(t\right)$ (denoted by lower-case letters) from the deterministic steady-state yields:

$$
\hat{\Pi}_j\left(t\right) = \left(1 - \varepsilon\left(t\right)e^{(1-\varepsilon(t))(p_j(t)-\tilde{p}_j(t))} + \left(\varepsilon\left(t\right)-1\right)e^{-\varepsilon(t)(p_j(t)-\tilde{p}_j(t))}\right)\frac{1}{\varepsilon\left(t\right)}e^{(1-\varepsilon(t))(\tilde{p}_j(t)-p(t))}e^{y(t)}Y.
$$

First, we show that, up to a second order approximation around the steady-state, profit losses do not depend directly on the elasticity $\varepsilon\left(t\right)$. To see this, write

$$
\hat{\Pi}_j\left(t\right) = G\left(\varepsilon\left(t\right),p_j\left(t\right)-\tilde{p}_j\left(t\right)\right)H\left(\varepsilon\left(t\right),\tilde{p}_j\left(t\right)-p\left(t\right),y\left(t\right)\right),
$$

with

$$
\begin{aligned}
G\left(\varepsilon\left(t\right),p_j\left(t\right)-\tilde{p}_j\left(t\right)\right) &\equiv \left(1 - \varepsilon\left(t\right)e^{(1-\varepsilon(t))(p_j(t)-\tilde{p}_j(t))} + \left(\varepsilon\left(t\right)-1\right)e^{-\varepsilon(t)(p_j(t)-\tilde{p}_j(t))}\right) \\[2mm]
H\left(\varepsilon\left(t\right),\tilde{p}_j\left(t\right)-p\left(t\right),y_t\right) &\equiv \frac{1}{\varepsilon\left(t\right)}e^{(1-\varepsilon(t))(\tilde{p}_j(t)-p(t))}e^{y(t)}Y.
\end{aligned}
$$

Then a second order approximation of$\hat{\Pi}_j\left(t\right)$ around the non-stochastic steady state (which features $p_j - p_j^* = 0$) is:

$$
\begin{aligned}
\hat{\Pi}_j(t) \cong\ & [G_1(\varepsilon,0)\,H(\varepsilon,0,0) + G(\varepsilon,0)\,H_1(\varepsilon,0,0)]\,\varepsilon(t) \\
& + G_2(\varepsilon,0)\,H(\varepsilon,0,0)\,(p_j(t) - \tilde{p}_j(t)) \\
& + G(\varepsilon,0)\,H_2(\varepsilon,0,0)\,(\tilde{p}_j(t) - p(t)) \\
& + G(\varepsilon,0)\,H_3(\varepsilon,0,0)\,y(t) \\
& + \left[\frac{1}{2}G_{11}(\varepsilon,0)\,H(\varepsilon,0,0) + G_1(\varepsilon,0)\,H_1(\varepsilon,0,0) + \frac{1}{2}G(\varepsilon,0)\,H_{11}(\varepsilon,0,0)\right]\varepsilon(t)^2 \\
& + \frac{1}{2}G_{22}(\varepsilon,0)\,H(\varepsilon,0,0)\,(p_j(t) - \tilde{p}_j(t))^2 \\
& + \frac{1}{2}G(\varepsilon,0)\,H_{22}(\varepsilon,0,0)\,(\tilde{p}_j(t) - p(t))^2 \\
& + \frac{1}{2}G(\varepsilon,0)\,H_{33}(\varepsilon,0,0)\,y(t)^2 \\
& + [G_{21}(\varepsilon,0)\,H(\varepsilon,0,0) + G_2(\varepsilon,0)\,H_1(\varepsilon,0,0)]\,\varepsilon(t)\,(p_j(t) - \tilde{p}_j(t)) \\
& + [G(\varepsilon,0)\,H_{12}(\varepsilon,0,0) + G_1(\varepsilon,0)\,H_2(\varepsilon,0,0)]\,\varepsilon(t)\,(\tilde{p}_j(t) - p(t)) \\
& + G_2(\varepsilon,0)\,H_2(\varepsilon,0,0)\,(p_j(t) - \tilde{p}_j(t))\,(\tilde{p}_j(t) - p(t)) \\
& + G_2(\varepsilon,0)\,H_3(\varepsilon,0,0)\,(p_j(t) - \tilde{p}_j(t))\,y(t) \\
& + G(\varepsilon,0)\,H_{23}(\varepsilon,0,0)\,(\tilde{p}_j(t) - p(t))\,y(t) \\
& + (G_1(\varepsilon,0)\,H_3(\varepsilon,0,0) + G(\varepsilon,0)\,H_{13}(\varepsilon,0,0))\,y(t)\,\varepsilon(t)
\end{aligned}
$$

Now, note that

$$
\begin{aligned}
G_1\left(\varepsilon, p_j - p_j^*\right) =\ & -e^{(1-\varepsilon)\left(p_j - p_j^*\right)} + \varepsilon\left(p_j - p_j^*\right)e^{(1-\varepsilon)\left(p_j - p_j^*\right)} \\
& + e^{-\varepsilon\left(p_j - p_j^*\right)} - (\varepsilon - 1)\left(p_j - p_j^*\right)e^{-\varepsilon\left(p_j - p_j^*\right)}, \\
G_2\left(\varepsilon, p_j - p_j^*\right) =\ & -\varepsilon(1-\varepsilon)e^{(1-\varepsilon)\left(p_j - p_j^*\right)} - \varepsilon(\varepsilon - 1)e^{-\varepsilon\left(p_j - p_j^*\right)}, \\
G_{11}\left(\varepsilon, p_j - p_j^*\right) =\ & \left(p_j - p_j^*\right)e^{(1-\varepsilon)\left(p_j - p_j^*\right)} + \left(p_j - p_j^*\right)e^{(1-\varepsilon)\left(p_j - p_j^*\right)} \\
& - \varepsilon\left(p_j - p_j^*\right)^2 e^{(1-\varepsilon)\left(p_j - p_j^*\right)} \\
& - 2\left(p_j - p_j^*\right)e^{-\varepsilon\left(p_j - p_j^*\right)} + (\varepsilon - 1)e^{-\varepsilon\left(p_j - p_j^*\right)}\left(p_j - p_j^*\right)^2, \\
G_{22}\left(\varepsilon, p_j - p_j^*\right) =\ & -\varepsilon(1-\varepsilon)^2 e^{(1-\varepsilon)\left(p_j - p_j^*\right)} + \varepsilon^2(\varepsilon - 1)e^{-\varepsilon\left(p_j - p_j^*\right)}, \\
G_{21}\left(\varepsilon, p_j - p_j^*\right) =\ & -(1 - 2\varepsilon)e^{(1-\varepsilon)\left(p_j - p_j^*\right)} - (2\varepsilon - 1)e^{-\varepsilon\left(p_j - p_j^*\right)} \\
& + \varepsilon(1-\varepsilon)\left(p_j - p_j^*\right)e^{(1-\varepsilon)\left(p_j - p_j^*\right)} \\
& + \varepsilon(\varepsilon - 1)\left(p_j - p_j^*\right)e^{-\varepsilon\left(p_j - p_j^*\right)}.
\end{aligned}
$$

so that,

$$
\begin{aligned}
G\left(\varepsilon,0\right) &= 0, \\
G_1\left(\varepsilon,0\right) &= 0, \\
G_2\left(\varepsilon,0\right) &= 0, \\
G_{11}\left(\varepsilon,0\right) &= 0, \\
G_{21}\left(\varepsilon,0\right) &= 0, \\
G_{22}\left(\varepsilon,0\right) &= \varepsilon\left(\varepsilon-1\right).
\end{aligned}
$$

Also,

$$
\begin{aligned}
H_3\left(\varepsilon, p_j^* - p, y\right) &= \frac{1}{\varepsilon}e^{(1-\varepsilon)\left(p_j^*-p\right)}e^y Y, \\
H_{33}\left(\varepsilon, p_j^* - p, y\right) &= \frac{1}{\varepsilon}e^{(1-\varepsilon)\left(p_j^*-p\right)}e^y Y, \\
H_{13}\left(\varepsilon, p_j^* - p, y\right) &= -\frac{1}{\varepsilon^2}e^{(1-\varepsilon)\left(p_j^*-p\right)}e^y Y - \frac{1}{\varepsilon}\left(p_j^* - p\right)e^{(1-\varepsilon)\left(p_j^*-p\right)}e^y Y,
\end{aligned}
$$

so that

$$
\begin{aligned}
H_3\left(\varepsilon,0,0\right) &= \frac{1}{\varepsilon}Y, \\
H_{33}\left(\varepsilon,0,0\right) &= \frac{1}{\varepsilon}Y, \\
H_{13}\left(\varepsilon,0,0\right) &= -\frac{1}{\varepsilon^2}Y,
\end{aligned}
$$

and the second order approximation for profits is:

$$
\hat{\pi}_j\left(t\right) = \frac{1}{2}\left(\varepsilon-1\right)Y\left(p_j\left(t\right)-\tilde{p}_j\left(t\right)\right)^2 + \frac{1}{\varepsilon}Yy\left(t\right) + \frac{1}{2}\frac{1}{\varepsilon}Yy\left(t\right)^2 + \frac{1}{\varepsilon^2}Yy\left(t\right)\varepsilon\left(t\right).
$$

Note that the last three terms do not depend on selection, since they are out of the control of the firm so we can write $\hat{\pi}_j\left(t\right)$ as

$$
\hat{\pi}_j\left(t\right) = \frac{1}{2}\left(\varepsilon-1\right)Y\left(p_j\left(t\right)-\tilde{p}_j\left(t\right)\right)^2 + t.i.s.,
$$

where $t.i.s.$ stands for terms independent of selection. Finally, note that we can write $\tilde{p}_j\left(t\right)$ as:

$$
\tilde{p}_j\left(t\right) = p\left(t\right) + \zeta\left(t\right) + w_j\left(t\right) - p\left(t\right) - z\left(t\right),
$$

where $\zeta\left(t\right) = \ln\left(\frac{\varepsilon(t)}{\varepsilon(t)-1}\right) - \ln\left(\frac{\varepsilon}{\varepsilon-1}\right)$. Now, from the log-linearized version of the household's labor supply condition,

$$
w_j\left(t\right) - p\left(t\right) = \sigma c\left(t\right) + \psi^{-1}l_j\left(t\right).
$$

Imposing the equilibrium condition $c\left(t\right) = y\left(t\right)$, the production function $z\left(t\right) + l_j\left(t\right) = y_j\left(t\right)$ and the household's optimality condition for the choice of varieties $y_j\left(t\right) = y\left(t\right) - \varepsilon\left(p_j\left(t\right) - p\left(t\right)\right)$ yields,

$$w_j(t) - p(t) = \left(\sigma + \psi^{-1}\right) y(t) - \psi^{-1} \left(\varepsilon \left(p_j(t) - p(t)\right) + z(t)\right),$$

so that

$$\tilde{p}_j(t) = p(t) + \zeta(t) + \left(\sigma + \psi^{-1}\right) y(t) - \psi^{-1}\varepsilon \left(p_j(t) - p(t)\right) - \left(1 + \psi^{-1}\right) z(t),$$

Imposing the definition of nominal income $m(t) = p(t) + y(t)$,

$$\tilde{p}_j(t) = \left(1 + \psi^{-1}\varepsilon - \left(\sigma + \psi^{-1}\right)\right) p(t) + \left(\sigma + \psi^{-1}\right) m(t) + \zeta(t) - \left(1 + \psi^{-1}\right) z(t) - \psi^{-1}\varepsilon p_j(t),$$

With some rearranging, we can write:

$$\left(1 + \psi^{-1}\varepsilon\right) p_j(t) - \tilde{p}_j(t) = \left(1 + \psi^{-1}\varepsilon - \left(\sigma + \psi^{-1}\right)\right) p(t) + \left(\sigma + \psi^{-1}\right) m(t) + \zeta(t) - \left(1 + \psi^{-1}\right) z(t)$$

This implies that:

$$\hat{\pi}(t) = \frac{1}{2} (\varepsilon - 1) \sqrt{1 + \varepsilon\psi^{-1}} Y \left(p_j(t) - p_j^*(t)\right)^2,$$

where:

$$p_j^*(t) \equiv (1 - \alpha) p(t) + \alpha m(t) + \frac{1}{1 + \psi^{-1}\varepsilon}\zeta(t) - \frac{1 + \psi^{-1}}{1 + \psi^{-1}\varepsilon} z(t).$$

5.2 Social efficiency

We now derive the quadratic approximation to the welfare function in the presence of monetary shocks as well as productivity and markup shocks.

The utility function of the representative household in a given period t is:

$$U\left(C(t), \{L_j(t)\}_{j \in [0,1]}\right) = \frac{C(t)^{1-\sigma} - 1}{1 - \sigma} - \frac{\lambda}{1 + \psi^{-1}} \int_0^1 L_j(t)^{1 + \psi^{-1}} \, dj$$

In equilibrium, $C(t) = Y(t)$, $L_j(t) = \frac{Y_j(t)}{Z(t)}$, $Y_j = \left(\frac{P_j(t)}{P(t)}\right)^{-\varepsilon(t)} Y(t)$. Substituting those in yields:

$$U(Y(t), Z(t), \Delta(t)) = \frac{Y(t)^{1-\sigma} - 1}{1 - \sigma} - \frac{\lambda \left(\frac{Y(t)}{Z(t)}\right)^{1+\psi^{-1}}}{1 + \psi^{-1}} \Delta(t),$$

where $\Delta(t) \equiv \int_0^1 \left(\frac{P_j(t)}{P(t)}\right)^{-\varepsilon(t)\left(1+\psi^{-1}\right)} dj$.

We start by characterizing $\Delta(t)$. First, note that $P_j(t) = X(t-s)$ if firm j last changed its price in time $t - s$. At any time t, a fraction $\Lambda(1 - G(t-s))$ will have last changed their price at $t - s$. Hence,

$$\Delta(t) \equiv \int_0^\infty (1 - G(t-s)) \left(\frac{X(t-s)}{P(t)}\right)^{-\varepsilon(t)\left(1+\psi^{-1}\right)}$$

Taking a second-order approximation of $\Delta(t)$ around the steady-state yields:

$$\Delta(t) \cong -\varepsilon\left(1+\psi^{-1}\right)\int_0^\infty \Lambda\left(1-G(t-s)\right)\left(x(t-s)-p(t)\right)ds$$

$$+\frac{1}{2}\varepsilon^2\left(1+\psi^{-1}\right)^2\int_0^\infty \Lambda\left(1-G(t-s)\right)\left(x(t-s)-p(t)\right)^2 ds.$$

where terms in the log of $\varepsilon(t)$ drop out since $X=P$ in steady-state.

Also, from equation (6) in the text, $p(t)=\int_{-\infty}^t \Lambda\left(1-G(t-s)\right)x(s)\,ds$. Since $\int_{-\infty}^t \Lambda\left(1-G(t-s)\right)ds = 1$, it follows that $\varepsilon\left(1+\psi^{-1}\right)\int_{-\infty}^t \Lambda\left(1-G(t-s)\right)\left(x(s)-p(t)\right)ds = 0$ and

$$\hat{\Delta}(t) = \frac{1}{2}\varepsilon^2\left(1+\psi^{-1}\right)^2\int_0^\infty \Lambda\left(1-G(t-s)\right)\left(x(t-s)-p(t)\right)^2 ds.$$

The second-order approximation of the utility function is therefore (using the fact that in steady-state $Z(t)=1$ and $\Delta=1$ in steady-state),

$$U\left(Y(t),Z(t),\Delta(t)\right)\cong Y^{1-\sigma}\left[\begin{array}{l} y(t)-\lambda Y^{\psi^{-1}+\sigma}\left(y(t)-z(t)\right)-\frac{\lambda Y^{\sigma+\psi^{-1}}}{1+\psi^{-1}}\hat{\Delta}(t) \\ +(1-\sigma)\frac{y(t)^2}{2}+\lambda Y^{1+\psi^{-1}}\left(1+\psi^{-1}\right)\frac{(y(t)-z(t))^2}{2} \end{array}\right]$$

There are no quadratic or cross-terms in $\hat{\Delta}(t)$, since $\hat{\Delta}(t)$ only includes a second order term to begin with, implying that those terms are third order or higher.

In steady-state, firms set all prices to be the same and to price to be equal to a constant markup over marginal cost

$$P_j = \frac{\varepsilon}{\varepsilon-1}W_j\ \forall j$$

Also, steady-state household labor supply implies that

$$\lambda L_j^{\psi^{-1}}C^\sigma = \frac{W_j}{P}$$

We focus on a symmetric steady-state equilibrium, so that $Y=Y_j=L_j$ and $P=P_j$. Therefore, using those facts, together with the equilibrium condition $C=Y$, yields

$$\lambda Y^{\sigma+\psi^{-1}} = \frac{\varepsilon-1}{\varepsilon}$$

Substituting back into the quadratic approximation to the utility function yields:

$$U\left(Y(t),Z(t),\Delta(t)\right)\cong Y^{1-\sigma}\left[\begin{array}{l} y(t)-\frac{\varepsilon-1}{\varepsilon}\left(y(t)-z(t)\right)-\frac{\varepsilon-1}{\varepsilon}\frac{1}{1+\psi^{-1}}\hat{\Delta}(t) \\ +(1-\sigma)\frac{y(t)^2}{2}+\frac{\varepsilon-1}{\varepsilon}\left(1+\psi^{-1}\right)\frac{(y(t)-z(t))^2}{2} \end{array}\right]$$

Let $Y^*(t)$ be the efficient level of output at any point in time, that is, the one that would hold in the absence of price rigidities or monopolistic distortions. It is the solution to:

$$\max \frac{Y^*(t)^{1-\sigma}-1}{1-\sigma}-\frac{\lambda}{1+\psi^{-1}}\int_0^1 L_j^*(t)^{1+\psi^{-1}}dj$$

$$s.t.\ :\ Y_j^*(t)=Z(t)L_j^*(t)\ \forall j\in[0,1]$$

$$Y^*(t)=\left[\int Y_j^*(t)^{\frac{\varepsilon(t)-1}{\varepsilon(t)}}dj\right]^{\frac{\varepsilon(t)}{\varepsilon(t)-1}}\ \forall j\in[0,1]$$

17

The first order conditions are (letting $\mu_j^*(t)$ be the lagrange multiplier for the first set of constraints and $\nu_j^*(t)$ for the second):

$$L_j^*(t) \quad : \quad \lambda L_j^*(t)^{\psi^{-1}} = Z(t)\,\mu_j^*(t)\,, \,\forall j \in [0,1]$$

$$Y_j^*(t) \quad : \quad \mu_j^*(t) = \left(\frac{Y_j^*(t)}{Y^*(t)}\right)^{-\frac{1}{\varepsilon(t)}} \nu_j^*(t)\,, \,\forall j \in [0,1]$$

$$Y^*(t) \quad : \quad \nu_j^*(t) = Y^*(t)^{-\sigma}\,, \,\forall j \in [0,1]$$

Eliminating the lagrangians yields the set of optimality conditions:

$$\lambda L_j^*(t)^{\psi^{-1}} = Z(t)\left(\frac{Y_j^*(t)}{Y^*(t)}\right)^{-\frac{1}{\varepsilon(t)}} Y^*(t)^{-\sigma}\,, \,\forall j \in [0,1]$$

Substituting in the production function:

$$\lambda \left(\frac{Y_j^*(t)}{Z(t)}\right)^{\psi^{-1}} = Z(t)\left(\frac{Y_j^*(t)}{Y^*(t)}\right)^{-\frac{1}{\varepsilon(t)}} Y^*(t)^{-\sigma}\,, \,\forall j \in [0,1]$$

It follows that $Y_j^*(t)$ is the same for all j, so that $Y_j^*(t) = Y^*(t)$. Hence,

$$\lambda \left(\frac{Y^*(t)}{Z(t)}\right)^{\psi^{-1}} = Z(t)\,Y^*(t)^{-\sigma}$$

Log-linearizing and rearranging

$$y^*(t) = \frac{1+\psi^{-1}}{\sigma+\psi^{-1}} z(t)$$

Now we use the expression for frictionless output to substitute out $z(t)$ from the approximate utility function:

$$U(Y(t),Z(t),\Delta(t)) = Y^{1-\sigma}\left[\begin{array}{c} y(t) - \frac{\varepsilon-1}{\varepsilon}(y(t)-\beta y^*(t)) - \frac{\varepsilon-1}{\varepsilon}\frac{1}{1+\psi^{-1}}\hat{\Delta}(t) \\ + (1-\sigma)\frac{y(t)^2}{2} - \frac{\varepsilon-1}{\varepsilon}(1+\psi^{-1})\frac{(y(t)-\tilde{b}y^*(t))^2}{2} \end{array}\right],$$

where $\beta \equiv \frac{\sigma+\psi^{-1}}{1+\psi^{-1}}$. Expanding the quadratic term $(y(t)-\beta y^*(t))^2$, collecting terms and completing the squares, we can rewrite the expression as:

$$U(Y(t),Z(t),\Delta(t)) = Y^{1-\sigma}\left[\begin{array}{c} y(t) - \frac{\varepsilon-1}{\varepsilon}(y(t)-\beta y^*(t)) - \frac{\varepsilon-1}{\varepsilon}\frac{1}{1+\psi^{-1}}\hat{\Delta}(t) \\ -\frac{1}{2}\left(\sigma+\psi^{-1}-\frac{1}{\varepsilon}(1+\psi^{-1})\right)(y(t)-by^*(t))^2 \\ +t.i.s. \end{array}\right]$$

where terms dependent on $y^*(t)$ only are collected under t.i.s., standing for "terms independent of selection".

Finally, we can use the fact that $\hat{\Delta}(t) = \frac{1}{2}\varepsilon^2\left(1+\psi^{-1}\right)^2\int_0^\infty (p_i(t)-p(t))^2$ to write the approximate utility function as in the text:

$$U(Y(t),Z(t),\Delta(t)) = Y^{1-\sigma}\left[\begin{array}{c} \frac{1}{\varepsilon}y(t) - \frac{1}{2}(\varepsilon-1)\varepsilon\left(1+\psi^{-1}\right)\int_0^\infty (p_i(t)-p(t))^2 \\ -\frac{1}{2}\left(\sigma+\psi^{-1}-\frac{1}{\varepsilon}(1+\psi^{-1})\right)(y(t)-by^*(t))^2 \end{array}\right]$$
$$+t.i.s.,$$

In Section 6.2 we only consider the impact of monetary shocks, so that we assume $y^*(t) = 0$ for the purpose of the analysis.

6   Private and social efficiency under different shock dynamics

The analysis in the paper suggests that there is no clear relationship between selection and inefficiencies caused by monetary shocks. Here we present some numerical examples that show that this is indeed the case. In what follows, we compare losses in an environment with maximal selection (Taylor pricing) and an environment with no selection (Calvo pricing).

6.1   Private efficiency

We start with an analysis of mean-reverting shocks to nominal income. One might expect Calvo pricing to be associated with smaller profit losses since, with mean reversion, prices set right after the shock might well become more misaligned than prices set before the shock. The Lemma below provides algebraic expressions for the profit losses associated with such shocks under both Taylor and Calvo pricing:

Lemma 1. *Suppose $m^{new}(t) - m^{old} = \Delta m \times e^{-\alpha t}$ for $t > 0$. Let $\eta \equiv \frac{\alpha}{\Lambda}$. It follows that*
  1) Under Taylor pricing, $lim_{\rho \to 0} \int_0^\infty e^{-\rho t} \hat{\pi}(t)\, dt = -\chi \frac{\eta^2 + 2e^{-\eta} - 1 - e^{-2\eta}}{2\eta^3 \Lambda} \Delta m^2$
  2) Under Calvo pricing, $lim_{\rho \to 0} \int_0^\infty e^{-\rho t} \hat{\pi}(t)\, dt = -\chi \frac{2+\eta}{2(1+\eta)^2 \Lambda} \Delta m^2$

    The proof of the lemma follows from direct application of optimal price-setting (equation 5 in the main paper), and of the second order approximation for average profits (equation 28). The details of the calculations can be seen in the Mathematica workbook, submitted together with the paper. Note that Δm^2 and Λ scale both quantities equally, so that the ratio of the two only depends on η. Figure 1 shows the ratio of expected discounted profit losses of Calvo to Taylor for different values of η. For the whole range of values for η, profit losses are smaller for firms operating under Taylor pricing. Thus, for the particular case of mean reverting shocks considered here, maximal selection tends to induce smaller losses than no selection.

    The numerical example conforms with the intuition that, by allowing firms to better track their preferred prices, higher selection increases private efficiency. However, this is not necessarily the case. For example, for a shock characterized by $\Delta m^2 m(t) = e^{-\delta t} \cos(\alpha t)$, we find that efficiency losses under Calvo pricing are smaller than under Taylor pricing for certain values of α (see figure ### for $\delta = .01$ and different values of α).[1]

6.2   Social efficiency

We now return to the case of mean-reverting shocks, and compare the two components of households' expected utility under Calvo and Taylor pricing:

Lemma 2. *Lemma: Suppose $m(t) = me^{-\Lambda \eta t}$ for $t > 0$. It follows that,*
  1) Under Taylor pricing, $\lim_{\rho \to 0} \int_0^\infty e^{-\rho t} y(t)^2\, dt = \frac{1}{\Lambda} \frac{2e^{-3\eta} - 2e^{-2\eta}(3 - \eta + \eta^2) + e^{-\eta}(6 - 4\eta + 4\eta^2) - 2 + 2\eta - 2\eta^2 + \eta^4}{2\eta^5} m^2$
and $\lim_{\rho \to 0} \int_0^\infty \int_0^1 (p_j(t) - p(t))^2\, dj\, dt = \frac{1}{\Lambda} \frac{(1 - e^{-\eta})^2 (2 - 2\eta + \eta^2 - 2e^{-\eta})}{2\eta^5} m^2$;
  2) Under Calvo pricing, $\lim_{\rho \to 0} \int_0^\infty e^{-\rho t} y(t)^2\, dt = \frac{1}{\Lambda} \frac{1 + 3\eta + \eta^2}{2(1+\eta)^3} m^2$ and $\lim_{\rho \to 0} \int_0^\infty \int_0^1 (p_j(t) - p(t))^2\, dj\, dt = \frac{1 + 3\eta + \eta^2}{2(1+\eta)^3} m^2$.

---

[1]This analysis follows a suggestion by the referee that we look into non-monotonic shocks (e.g., with negative autocorrelation).
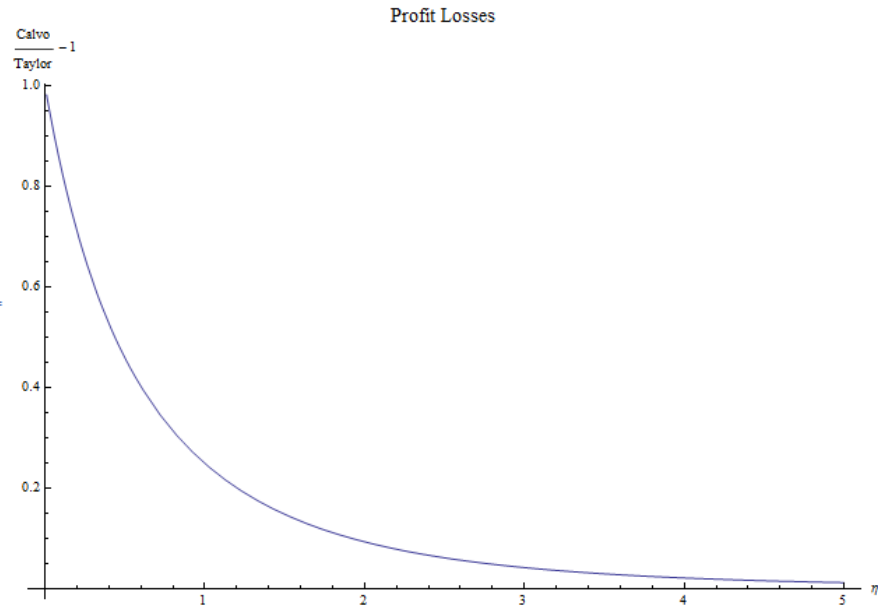
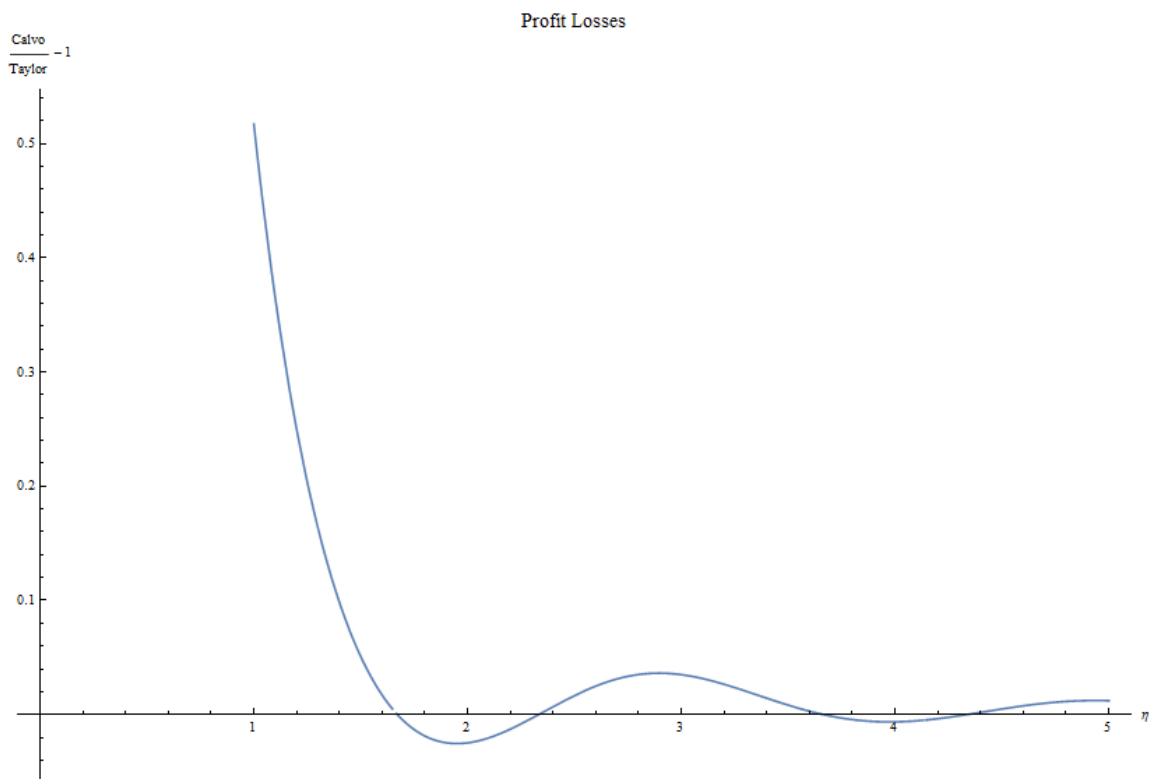Figure 1: Cumulative profit losses from a shock with mean reversion parameter η.

Figure 2: Cumulative profit losses from a shock with oscillating shocks of different frequencies η.

Again, Λ is only a scaling parameter. Figure 2 shows the values for $\int_0^\infty y(t)^2 \, dt$ (lower panel) and $\int_0^\infty \int_0^1 (p_j(t) - p(t))^2 \, dj \, dt$ (upper panel) for both Calvo and Taylor pricing. In the range showed in the figure, the consumption volatility component of welfare loss is always lower for Taylor than for Calvo. However, this is not true for the price dispersion component: Calvo pricing implies larger price dispersion for lower values of η but lower price dispersion for shocks that mean revert faster. Thus, if $a_\Delta \equiv \frac{1}{2}(\varepsilon - 1)\varepsilon(1 + \psi^{-1})$ is much larger than $a_{yy} \equiv \frac{1}{2}(\sigma + \psi^{-1} - \frac{1}{\varepsilon}(1 + \psi^{-1}))$, and the mean reversion parameter η is high enough, an economy with no selection might be socially preferable to one with high selection.

7 Proofs of Lemmas and Propositions in the text

Lemma A. 1. *Let $G(t)$, $\omega(t)$, $\mu(t)$ and $\Xi(t)$ be as defined in Section 3. Then*

$$1 - \omega(t) = e^{-\Lambda t - \Lambda \int_0^t \mu(v)dv} = e^{-\Lambda t - \Lambda \Xi(t)}.$$

Proof of Lemma A.1. The second equation follows from the definition of cumulative selection. We prove that the first equation holds.

Let t_1 be the smallest t such that $\omega(t) = 1$. Since $\omega(t)$ is monotonically increasing, $\omega(t) = 1$ for all $t \geq t_1$ and $\omega(t) < 1$ otherwise.

We consider two cases: $t \in [0, t_1)$ and $t \in [t_1, \infty)$. We use a guess and verify procedure.

1) $t \in [0, t_1)$:

First, note that for $t = 0$, $1 - \omega(t) = 1$ and $e^{-\Lambda t - \int_0^t \mu(v)dv} = 1$. Thus, we confirm that the guess works for $t = 0$. We verify the guess also holds for $t \in [0, t_1)$ if we can show that the derivative of both sides of the equation with respect to t are identical over that range.

The derivative of the left-hand-side is:

$$\frac{\partial(1 - \omega(t))}{\partial t} = -\frac{\partial \omega(t)}{\partial t} = -\Lambda(1 - G(t)),$$

and the derivative of the right-hand-side is:

$$\frac{\partial e^{-\Lambda t - \Lambda \int_0^t \mu(v)dv}}{\partial t} = -\Lambda(1 + \mu(t)) e^{-\Lambda t - \Lambda \int_0^t \mu(v)dv}$$

Given our guess, $e^{-\Lambda t - \Lambda \int_0^t \mu(v)dv} = 1 - \omega(t)$, so that

$$-\Lambda(1 + \mu(t)) e^{-\Lambda t - \Lambda \int_0^t \mu(v)dv} = -\Lambda(1 + \mu(t))(1 - \omega(t)).$$

By definition, $1 + \mu(t) = \frac{1 - G(t)}{1 - \omega(t)}$ so that

$$\frac{\partial e^{-\Lambda t - \Lambda \int_0^t \mu(v)dv}}{\partial t} = -\Lambda(1 - G(t)).$$

2) $t \geq t_1$:

We know that $1 - \omega(t) = 0$ for $t \geq t_1$. We need to show that this also the case for $e^{-\Lambda t - \Lambda \int_0^t \mu(v)dv}$. First, note that, given the result in (1) above, we know that, taking the left-limit as $t \to t_1^-$, we find that $\lim_{t \to t_1^-} e^{-\Lambda t - \Lambda \int_0^t \mu(v)dv} = \lim_{t \to t_1^-} (1 - \omega(t)) = 0$.

21

Since $1 - \omega(t) = 0$ for $t \geq t_1$, it remains to show that it is also the case that $e^{-\Lambda t - \Lambda \int_0^t \mu(v)dv} = 0$ for all $t \geq t_1$. From the proposed solution, for any t and t^{prime} such that $t > t'$, $1 - \omega(t) = (1 - \omega(t'))e^{-\Lambda(t-t') - \Lambda \int_{t'}^t \mu(v)dv}$. Taking the left-limit again and applying the definition of μ, so that $\mu(t) = 0$ for $t > t_1$, it follows that, for $t > t_1$:

$$
\begin{aligned}
1 - \omega(t) &= \lim_{t' \to t_1^-} (1 - \omega(t'))e^{-\Lambda(t-t') - \Lambda \int_{t'}^t \mu(v)dv} \\
&= \lim_{t' \to t_1^-} (1 - \omega(t'))e^{-\Lambda(t-t')} = 0.
\end{aligned}
$$

Given that $e^{-\Lambda t - \Lambda \int_0^t \mu(v)dv} = 0$ for all $t > t_1$, it follows that $\lim_{t' \to t_1^+} e^{-\Lambda t - \Lambda \int_0^t \mu(v)dv} = 0$. Thus, both the left and right limits coincide and $e^{-\Lambda t_1 - \Lambda \int_0^{t_1} \mu(v)dv} = 0$.

$\square$

Proof of Lemma 1. We check that f is given by

$$G(t) = 1 - (1 + \mu(t))e^{-\Lambda t - \Lambda \int_0^t \mu(s)ds} \; \forall t.$$

Integrating equation (8) in the main text we find that

$$1 - \omega(t) = 1 - \Lambda \int_0^t (1 - G(s)) \, ds.$$

From the sequence of equalities (9) in the main text, it follows that

$$1 - \Lambda \int_0^t (1 - G(s)) \, ds = 1 - \omega(t) = e^{-\Lambda t - \Lambda \int_0^t \mu(s)ds}.$$

Thus, we can write

$$G(t) = 1 - (1 + \mu(t))\left[1 - \Lambda \int_0^t (1 - G(s)) \, ds\right].$$

Use the definition of $\mu(t)$ to substitute it out, to get, for t such that $\omega(t) < 1$

$$
\begin{aligned}
G(t) &= 1 - \left(1 + \frac{1 - G(t)}{1 - \Lambda \int_0^t (1 - G(s)) \, ds} - 1\right)\left[1 - \Lambda \int_0^t (1 - G(s)) \, ds\right] \\
&= G(t).
\end{aligned}
$$

For t such that $\omega(t) = 1$,

$$G(t) = 1 - (1 - 1)\left[1 - \Lambda \int_0^t (1 - G(s)) \, ds\right] = 1 \tag{11}$$

Since $\omega(t) = 1 - \Lambda \int_0^t (1 - G(s)) \, ds$, it follows that if $\omega(t) = 1$, $\int_0^t (1 - G(s)) \, ds = \Lambda^{-1}$. But, by definition, $\Lambda^{-1} = \int_0^\infty (1 - G(s)) \, ds$. Hence, if $\omega(t) = 1$, $G(t) = 1$, as implied by equation (11).

$\square$

Proof of Lemma 2. 1) and 2) follow from inspection of the sequence of equations (13) in the main text. $\qquad\square$

Proof of Proposition 1. Let $\Gamma^{sp}\left(m^{old}, m^{new}\left(t\right); G\right)$ and $\Gamma^{si}\left(m^{old}, m^{new}\left(t\right); G\right)$ be the cumulative real effects of an unexpected change in the path of nominal aggregate demand from m to $m^{new}\left(t\right)$ in, respectively, a sticky price as defined in the text and in a sticky information economy as defined in Appendix 1, with G the c.d.f that summarizes the arrival of adjustment opportunities in each of these economies. We show that, if $\alpha = 1$, $\Gamma^{sp}\left(m_0\left(t\right), m_1\left(t\right); G\right) = \Gamma^{si}\left(m_0\left(t\right), m_1\left(t\right); G\right) = \int_0^\infty \left(1 - \omega\left(t\right)\right)\left(m^{new}\left(t\right) - m^{old}\right) dt$, where the second equality follows from the discussion in Appendix 1. We denote the "new" and "old" paths of the various endogenous variables in the sticky information and sticky price economies by $y^{si,new}\left(t\right)$, $y^{si,old}$, $y^{sp,new}\left(t\right)$, $y^{sp,old}$, etc.

Define $\Delta_\Gamma \equiv \Gamma^{si} - \Gamma^{sp}$. Then:

$$
\begin{aligned}
\Delta_\Gamma &= \int_0^\infty e^{-\rho t}\left(y^{si,new}\left(t\right) - y^{si,old}\right) dt - \int_0^\infty e^{-\rho t}\left(y^{sp,new}\left(t\right) - y^{sp,old}\right) dt = \\
&= \int_0^\infty e^{-\rho t}\left(m^{new}\left(t\right) - p^{si,new}\left(t\right) - \left(m^{old} - p_0^{si,old}\right)\right) dt \\
&\quad - \int_0^\infty e^{-\rho t}\left(m^{new}\left(t\right) - p^{sp,new}\left(t\right) - \left(m^{old} - p^{sp,old}\right)\right) dt = \\
&= \int_0^\infty e^{-\rho t}\left(\left[p^{sp,new}\left(t\right) - p^{sp,old}\right] - \left[p^{si,new}\left(t\right) - p^{si,old}\right]\right) dt.
\end{aligned}
$$

Since m^{old} is a constant, it follows that $p^{si,old} = p^{sp,old} = m^{old}$ and

$$
\Delta_\Gamma = \int_0^\infty e^{-\rho t}\left(p^{sp,new}\left(t\right) - p^{si,new}\left(t\right)\right).
$$

Recall that

$$
\begin{aligned}
p^{sp,new}\left(t\right) &= \int_0^t \Lambda\left(1 - G\left(t - v\right)\right) x^{sp,new}\left(v\right) dv + \int_{-\infty}^0 \Lambda\left(1 - G\left(t - v\right)\right) x^{sp,old}\left(v\right) dv \\
&= \int_0^t \Lambda\left(1 - G\left(t - v\right)\right) x^{sp,new}\left(v\right) dv + p_0^{sp,old} = \\
&= \int_0^t \Lambda\left(1 - G\left(t - v\right)\right) x^{sp,new}\left(v\right) dv + m_0,
\end{aligned}
$$

and, analogously,

$$
p^{si,new}\left(t\right) = \int_0^t \Lambda\left(1 - G\left(t - v\right)\right) x^{si,new}\left(v, t\right) dv + m_0,
$$

so that:

$$
\Delta_\Gamma = \int_0^\infty e^{-\rho t}\left(\left[\int_0^t \Lambda\left(1 - G\left(t - v\right)\right)\left[x^{sp,new}\left(v\right) - x^{si,new}\left(v, t\right)\right] dv\right]\right) dt.
$$

We prove that $\int_0^\infty e^{-\rho t} \int_0^t \Lambda\left(1 - G\left(t - v\right)\right)\left[x^{sp,new}\left(v\right) - x_1^{si,new}\left(v, t\right)\right] dv dt = 0$. First write the integral as:

$$
\int_0^\infty \int_0^\infty \mathbb{1}\left(v \leq t\right) e^{-\rho t} \Lambda\left(1 - G\left(t - v\right)\right)\left[x^{sp,new}\left(v\right) - x^{si,new}\left(v, t\right)\right] dv dt,
$$

23

where $\mathbb{1}(.)$ is the indicator function. Now do the substitution: $v = z$ and $t = z + w$:

$$\int_0^\infty \int_0^\infty e^{-\rho(z+w)} \mathbb{1}\left(z \leq z + w\right) \Lambda \left(1 - G\left(w\right)\right) \left[x^{sp,new}\left(z\right) - x^{si,new}\left(z, z + w\right)\right] dw dz$$

Note that the indicator function is always equal to 1 for all z and $w \geq 0$, so we can write:

$$\int_0^\infty \int_0^\infty e^{-\rho(z+w)} \Lambda \left(1 - G\left(w\right)\right) \left[x^{sp,new}\left(z\right) - x^{si,new}\left(z, z + w\right)\right] dw dz$$

Now, we show that the inner integral is equal to zero. We can take all multiplicative terms that only depend on z out of the inner integral and rearrange it to get

$$\int_0^\infty e^{-\rho(z+w)} \Lambda \left(1 - G\left(w\right)\right) \left[x^{sp,new}\left(z\right) - x^{si,new}\left(z, z + w\right)\right] dw =$$

$$= \Lambda e^{-\rho z} \int_0^\infty e^{-\rho w} \left(1 - G\left(w\right)\right) dw \times x^{sp,new}\left(z\right) - \Lambda e^{-\rho z} \int_0^\infty e^{-\rho w} \left(1 - G\left(w\right)\right) x^{si,new}\left(z, z + w\right) dw =$$

$$= \Lambda e^{-\rho z} \int_0^\infty e^{-\rho w} \left(1 - G\left(w\right)\right) dw \times \left[x^{sp,new}\left(z\right) - \frac{\int_0^\infty e^{-\rho w} \left(1 - G\left(w\right)\right) x^{si,new}\left(z, z + w\right) dw}{\int_0^\infty e^{-\rho(z+w)} \Lambda \left(1 - G\left(w\right)\right) dw}\right].$$

Optimal price-setting implies:

$$x^{sp,new}\left(z\right) = \frac{\int_0^\infty e^{-\rho w} \left(1 - G\left(w\right)\right) m^{new}\left(z + w\right) dw}{\int_0^\infty e^{-\rho w} \left(1 - G\left(w\right)\right) dw},$$

$$x^{si,new}\left(z, z + w\right) = m^{new}\left(z + w\right),$$

so that

$$x^{sp,new}\left(z\right) = \frac{\int_0^\infty e^{-\rho w} \left(1 - G\left(w\right)\right) x^{si,new}\left(z, z + w\right) dw}{\int_0^\infty e^{-\rho w} \left(1 - G\left(w\right)\right) dw}.$$

It follows that

$$\int_0^\infty \int_0^t \Lambda \left(1 - G\left(t - v\right)\right) \left[x^{sp,new}\left(v\right) - x^{si,new}\left(v, t\right)\right] dv = 0.$$

$\square$

Proof of Proposition 2. 1) and 2) follow from inspection of equation (14) in the main text and from the fact that 1) implies 2).
$\square$

Proof of Lemma 3. Consider any alternative pricing rule G with the average frequency of price changes Λ. If $G(t)=1$, $\mu(t) = 0$ and the result follows trivially. Otherwise, we denote the corresponding selection at t as $\mu(t)$:

$$\mu\left(t\right) = \frac{1 - G\left(t\right)}{1 - \omega\left(t\right)} - 1. \tag{12}$$

The proof of the result follows from a comparison of numerators and denominators in equation (14) in the main text and equation (12) in this appendix for a given Λ. First, it is immediate that $1 - G\left(t\right) \leq 1$. Second, we can show that $1 - \omega\left(t\right) \geq 1 - \Lambda t$:

$$1 - \omega\left(t\right) = 1 - \Lambda \int_0^t \left(1 - G\left(s\right)\right) ds = 1 - \Lambda t + \int_0^t G\left(s\right) ds \geq 1 - \Lambda t.$$

Thus, wherever $\mu^{Taylor}(t)$ is positive, $\mu^{Taylor}(t) \geq \mu(t)$ for any G.

Cumulative selection under Taylor pricing is:

$$\Xi^{Taylor}(t) = \begin{cases} -\frac{\ln(1-\Lambda t)}{\Lambda} - t & \text{if } t < \Lambda^{-1}, \\ \infty & \text{otherwise.} \end{cases}$$

Since under Taylor pricing selection is the highest possible for $t < \Lambda^{-1}$, and for $t \geq \Lambda^{-1}$ cumulative selection is infinite, it follows that cumulative selection is the highest possible everywhere. $\qquad\square$

Proof of Lemma 4. We verify that the statement is true through a sequence of substitutions. Substituting out $\Psi_t(s)$ in the right-hand-side, we get that $\mu(t) = \int_t^{t_1} \frac{h(s)}{\Lambda} \frac{1-G(s)}{\int_t^{\infty}(1-G(v))dv} ds - 1$. From equation (8) in the main text it is easy to verify that $\Lambda \int_t^{t_1}(1-G(v))dv = 1-\omega(t)$. Since it does not depend on s we can take it out of the integral to get $\mu(t) = \frac{1}{1-\omega(t)} \int_t^{t_1} h(s)(1-G(s))ds - 1$. Finally, note that $h(s)(1-G(s)) = \frac{\frac{\partial G(s)}{\partial s}}{1-G(s)}(1-G(s)) = \frac{\partial G(s)}{\partial s}$, so that $\mu(t) = \frac{1}{1-\omega(t)} \int_t^{t_1} \frac{\partial G(s)}{\partial s} ds - 1 = \frac{1}{1-\omega(t)} \int_t^{\infty} \frac{\partial G(s)}{\partial s} ds - 1 = \frac{1-G(t)}{1-\omega(t)} - 1$, where the second equality follows from the fact that $G(t) = 1$ for $t > t_1$. This is exactly how $\mu(t)$ is defined in **Definition 1** in the main text for $\omega(t) < 1$. $\qquad\square$

Proof of Lemma 5. First, note that $\mu(0) = 0$ always, since $G(0) = \omega(0) = 0$. Second, we can write $\mu(t) = E_{\Psi_0}\left[\frac{h(s)}{\Lambda}|s \geq t\right] - 1$, where the conditional expectation is taken with respect to the probability measure Ψ_0. If the hazard function is (weakly) increasing, it follows that $\mu(t) = E_{\Psi_0}\left[\frac{h(s)}{\Lambda}|h(s) \geq h(t)\right] - 1 > 0$. Also, since, by assumption, $h(t)$ increases in t, so does $\mu(t)$. Since $\mu(0) = 0$, $\mu(t) > 0 \ \forall t > 0$. $\qquad\square$

Proof of Proposition 3. 1) We prove this in two steps:

i) *There is a unique $t^{**} > 0$ such that $G_A(t^{**}) = G_B(t^{**}) < 1$, $1 - G_A(t) < 1 - G_B(t)$ if $t < t^*$ and $1 - G_A(t) > 1 - G_B(t)$ if $t > t^*$.*

First we show that a t^{**} with $G_A(t^{**}) = G_B(t^{**}) < 1$ exists. Suppose not, then $G_A(t) > G_B(t) \ \forall t$ or vice versa (otherwise, since G is differentiable, it is continuous and t^{**} must exist by the intermediate point theorem). But this contradicts the assumption that $\Lambda_A = \Lambda_B$, since $\Lambda_A = \int_0^{\infty}(1-G_A(t))dt$ and $\Lambda_B = \int_0^{\infty}(1-G_B(t))$.

Second, we show that $t^{**} > t^*$. Note that:

$$1 - G_A(t) = e^{-\int_0^t h_A(s)ds},$$
$$1 - G_B(t) = e^{-\int_0^t h_B(s)ds}.$$

Since $h_A(t) > h_B(t) \ \forall t < t^*$, it follows that

$$1 - G_A(t) < 1 - G_B(t) \ \forall t < t^*.$$

It follows that $t^{**} > t^*$.

Let t^{**} be the first crossing point. Since $e^{-\int_0^{t^{**}} h_A(s)ds} = e^{-\int_0^{t^{**}} h_B(s)ds}$, we can write

$$
\begin{aligned}
1 - G_A(t) - (1 - G_B(t)) &= e^{-\int_0^t h_A(s)ds} - e^{-\int_0^t h_B(s)ds} \\
&= e^{-\int_0^{t^{**}} h_A(s)ds}\left(e^{-\int_{t^{**}}^t h_A(s)ds} - e^{-\int_{t^{**}}^t h_B(s)ds}\right) \text{ if } t > t^{**}.
\end{aligned}
$$

Now, recall that $t^{**} > t^*$, so that if $t > t^{**}$, then $h_A(t) < h_B(t)$. Thus, the expression in parenthesis is strictly positive. Thus there is no crossing point to the right of t^{**}. There is also no crossing point to the left of t^{**}, since in that case we could repeat the exercise above to show that t^{**} cannot exist. Thus, t^{**} is unique.

ii) *If there is t^{**} so that $1 - G_A(t) < 1 - G_B(t)$ for $t < t^{**}$ and $1 - G_A(t) > 1 - G_B(t)$ for $t > t^{**}$, then $\Xi_A(t) > \Xi_B(t)$ $\forall t$.*

For $t < t^{**}$ it follows trivially that $\int_0^t G_B(s)\,ds < \int_0^t G_A(s)\,ds$ $\forall t$ with the inequality strict for t above a certain range. For $t > t^{**}$,

$$
\frac{\partial \int_0^t [G_B(s) - G_A(s)]\,ds}{\partial t} = G_B(t) - G_A(t) > 0.
$$

This means that we can bound $\int_0^t [G_B(s) - G_A(s)]\,ds$ above as follows:

$$
\int_0^t [G_B(s) - G_A(s)]\,ds < \int_0^\infty [G_B(s) - G_A(s)]\,ds \ \forall t \geq t^{**}
$$

Using integration by parts plus the condition that the expected values are the same implies that the bound is zero:

$$
\int_0^\infty [G_B(s) - G_A(s)]\,ds = -\left(\Lambda_B^{-1} - \Lambda_A^{-1}\right) = 0
$$

Thus $\int_0^t G_B(s)ds < \int_0^t G_A(s)ds$ $\forall t$. We can verify from equation (8) that $\omega(t) = \Lambda \int_0^t (1 - G(s))\,ds$. Hence, it follows that $1 - \omega_B(s) > 1 - \omega_A(s)$ $\forall t$. Since, from equation (9) in the main text, $1 - \omega(t) = e^{-\Lambda t - \Lambda \Xi(t)}$, it follows that $\Xi_A(t) > \Xi_B(t)$ $\forall t$.

2) Suppose the two functions do not cross. The either $h_A(t) > h_B(t)$ for all t or vice versa. In the first case, we have that $G_A(t) = 1 - e^{-\int_0^t h_A(v)dv} > 1 - e^{-\int_0^t h_B(v)dv} = G_B(t)$ for all t (and vice versa in the opposite case). But both of these violate the condition that $\int_0^\infty (1 - G_A(t))\,dt = \int_0^\infty (1 - G_B(t))\,dt$ which is necessary for $\Lambda_A^{-1} = \Lambda_B^{-1}$.

Let t^* be a crossing point. Then, for any $t < t^*$, $h_A(t) = h_A(t^*) - \int_t^{t^*} \frac{\partial h_A(s)}{\partial s}ds$ and $h_B(t) = h_B(t^*) - \int_t^{t^*} \frac{\partial h_B(s)}{\partial s}ds$. Since $\frac{\partial h_A(s)}{\partial s} < \frac{\partial h_B(s)}{\partial s}$ and $h_A(t^*) = h_B(t^*)$ it follows that $h_A(t) > h_B(t)$. Likewise, for any $t > t^*$, $h_A(t) = h_A(t^*) + \int_t^{t^*} \frac{\partial h_A(s)}{\partial s}ds$ and $h_B(t) = h_B(t^*) + \int_t^{t^*} \frac{\partial h_B(s)}{\partial s}ds$ so that $h_B(t) > h_A(t)$. Thus there is a single crossing point and part 1) applies. $\qquad \square$

Proof of Proposition 2'. The proposition relies on the fact that we can build a one-sector economy characterized by $\tilde{G}(t) = E\left[\frac{\Lambda_k}{E[\Lambda_k]}G_k(t)\right]$ and $\tilde{\Lambda} = E[\Lambda_k]$ in which monetary shocks have the same real effects as in the heterogeneous economy.[2] Given $\alpha = 1$, firms in each sector do not

---

[2] Note that this is not the counterfactual one-sector economy discussed subsequently.

interact with one another, so that each sector behaves as if it were a separate economy. From **Proposition 1** in the main text the real impact of the shock in an economy characterized by G_k is $\Gamma_k = \int_0^\infty e^{-\rho t} \left(1 - \omega_k(t)\right) \left(m^{new}(t) - m^{old}\right) dt$, where $\omega_k(t) = \Lambda_k \int_0^t \left(1 - G_k(t)\right) dt$. The real impact of the shock in the multisector economy is just the cross-sectoral average of the real effects in each sector:

$$\Gamma^{het} = E\left[\Gamma_k\right] = \int_0^\infty e^{-\rho t} \left(1 - E\left[\omega_k(t)\right]\right) \left(m^{new}(t) - m^{old}\right) dt.$$

We now show that the real effects of any given monetary shock in the heterogeneous economy described above are identical to the effects in a one sector economy with $\tilde{G}(t) = \frac{E[\Lambda_k G_k(t)]}{E[\Lambda_k]}$. The real impact of the shock in that economy is $\tilde{\Gamma} = \int_0^\infty e^{-\rho t} \left(1 - \tilde{\omega}(t)\right) \left(m^{new}(t) - m^{old}\right) dt$, where $\tilde{\omega}(t) = E\left[\Lambda_k\right] \int_0^t \left(1 - \frac{E[\Lambda_k G_k(t)]}{E[\Lambda_k]}\right) dt$. Since

$$E\left[\omega_k(t)\right] = E\left[\Lambda_k \int_0^t \left(1 - G_k(s)\right) ds\right] = E\left[\Lambda_k\right] \int_0^t \left(1 - \frac{E\left[\Lambda_k G_k(s)\right]}{E\left[\Lambda_k\right]} ds\right) ds = \tilde{\omega}(t),$$

it follows that $\tilde{\Gamma} = \Gamma^{het}$.

Finally, we can show that the average frequency of price changes in this one-sector economy is the same as in the multisector economy. It is $\tilde{\Lambda} = \left[\int_0^\infty \frac{E[\Lambda_k(1 - G_k(t))]}{E[\Lambda_k]} dt\right]^{-1} = \left[\frac{E\left[\Lambda_k \int_0^\infty (1 - G_k(t)) dt\right]}{E[\Lambda_k]}\right]^{-1} =$

$\left[\frac{E\left[\Lambda_k \Lambda_k^{-1}\right]}{E[\Lambda_k]}\right]^{-1} = E\left[\Lambda_k\right]$.

Note that $\tilde{G}(t) = G^{het}(t)$ defined in the text, just as $\tilde{\omega}(t) = \omega^{het}(t)$. Thus, $\mu^{het}(t)$ and $\Xi^{het}(t)$ correspond to selection and cumulative selection for the one-sector economy with the same real effect and the same average frequency of price changes as the multisector economy. The last step of the proof then follows by applying Proposition 2. $\qquad\square$

Proof of Proposition 4. First, we note that for the counterfactual one-sector economy,

$$1 - \omega^{count}(t) = e^{-E[\Lambda_k]t - \Xi^{count}(t)},$$

where $\omega^{count}(t) \equiv \int_0^t \left(1 - G^{count}(s)\right) ds$ and for the heterogeneous economy,

$$1 - \omega^{het}(t) = e^{-E[\Lambda_k]t - \Xi^{het}(t)},$$

so that $\Xi^{count}(t) \geq \Xi^{het}(t)$ for all t so long as $1 - \omega^{count}(t) \leq 1 - \omega^{het}(t)$ for all t.

Use **Definition 4** and integrate both sides of equation (8) in the main text to write $1 - \omega^{count}(t)$:

$$
\begin{aligned}
1 - \omega^{count}(t) &= E\left[\Lambda_k\right] \int_t^\infty \left(1 - \bar{G}\left(E\left[\Lambda_k\right] s\right)\right) ds, \\
&= E\left[\Lambda_k\right] \int_{E[\Lambda_k]t}^\infty \left(1 - \bar{G}(v)\right) E\left[\Lambda_k\right]^{-1} dv, \\
&= \int_{E[\Lambda_k]t}^\infty \left(1 - \bar{G}(v)\right) dv, \\
&= 1 - \bar{\omega}\left(E\left[\Lambda_k\right] t\right).
\end{aligned}
$$

We can show that any $\omega(t) = \Lambda \int_0^t (1 - G(s)) ds$ is concave, i.e.:

$$\omega \left(\lambda x + (1 - \lambda) x' \right) > \lambda \omega (x) + (1 - \lambda) \omega (x').$$

This is trivial to see if $G(t)$ is differentiable, since in that case $\frac{\partial^2 \omega(t)}{(\partial t)^2} = -\Lambda \frac{\partial G(t)}{\partial t} < 0$. More generally, given its definition, ω is concave if and only if

$$\int_0^{\lambda x + (1-\lambda)x'} [1 - G(s)] ds > \lambda \int_0^x [1 - G(s)] ds + (1 - \lambda) \int_0^{x'} [1 - G(s)] ds.$$

W.l.o.g., let $x' > x$. Then, we can write

$$\int_0^{\lambda x + (1-\lambda)x'} [1 - G(s)] ds > \lambda \int_0^x [1 - G(s)] ds + (1 - \lambda) \int_0^{x'} [1 - G(s)] ds$$

$$\Longleftrightarrow \lambda \left[\begin{array}{c} \int_0^{\lambda x + (1-\lambda)x'} [1 - G(s)] ds \\ - \int_0^x [1 - G(s)] ds \end{array} \right] > (1 - \lambda) \left[\begin{array}{c} \int_0^{x'} [1 - G(s)] ds \\ - \int_0^{\lambda x + (1-\lambda)x'} [1 - G(s)] ds \end{array} \right]$$

$$\Longleftrightarrow \lambda \int_x^{\lambda x + (1-\lambda)x'} [1 - G(s)] ds > (1 - \lambda) \int_{\lambda x + (1-\lambda)x'}^{x'} [1 - G(s)] ds$$

$$\Longleftrightarrow (1 - \lambda) (x' - x) \lambda \frac{\int_x^{\lambda x + (1-\lambda)x'} [1 - G(s)] ds}{(1 - \lambda)(x' - x)} > \lambda (1 - \lambda) (x' - x) \frac{\int_{\lambda x + (1-\lambda)x'}^{x'} [1 - G(s)] ds}{\lambda (x' - x)}$$

$$\Longleftrightarrow \frac{\int_x^{\lambda x + (1-\lambda)x'} [1 - G(s)] ds}{(1 - \lambda)(x' - x)} > \frac{\int_{\lambda x + (1-\lambda)x'}^{x'} [1 - G(s)] ds}{\lambda (x' - x)}.$$

We can verify that the inequality holds since G is increasing, implying that

$$\frac{\int_x^{\lambda x + (1-\lambda)x'} [1 - G(s)] ds}{(1 - \lambda)(x' - x)} > \frac{\int_x^{\lambda x + (1-\lambda)x'} [1 - G(\lambda x + (1 - \lambda) x')] ds}{(1 - \lambda)(x' - x)} = 1 - G \left(\lambda x + (1 - \lambda) x' \right),$$

and

$$\frac{\int_{\lambda x + (1-\lambda)x'}^{x'} [1 - G(s)] ds}{\lambda (x' - x)} < \frac{\int_{\lambda x + (1-\lambda)x'}^{x'} [1 - G(\lambda x + (1 - \lambda) x')] ds}{\lambda (x' - x)} = 1 - G \left(\lambda x + (1 - \lambda) x' \right)$$

In particular, given that any ω is concave, so is $\bar{\omega}(t) \equiv 1 - \int_0^t \left(1 - \bar{G}(s) \right) ds$ with $\bar{G}(s)$ as defined in the statement of the proposition is also concave. Therefore, from Jensen's inequality, it follows that $E[\bar{\omega}(\Lambda_k t)] < \bar{\omega}(E[\Lambda_k t])$, and

$$1 - \omega^{het}(t) = 1 - E[\bar{\omega}(\Lambda_k t)] > 1 - \bar{\omega}(E[\Lambda_k] t) = 1 - \omega^{count}(t).$$

The last step of the proof follows from $1 - \omega(t) = e^{-\Lambda - \Lambda \Xi(t)}$. Since, by construction, $\Lambda^{het} = \Lambda^{count} = E[\Lambda_k]$, and since $1 - \omega^{het}(t) > 1 - \omega^{count}(t)$, it follows that $\Xi^{het}(t) < \Xi^{count}(t)$.

$\square$

28

Proof of Proposition 5. Suppose $G_A(t)$ is a mean preserving spread of $G_B(t)$. Then $\int_0^t G_A(s)\,ds > \int_0^t G_B(s)\,ds$ $\forall t$. It follows that $1 - \omega_A(t) = 1 - \Lambda \int_0^t G_A(s)\,ds < 1 - \Lambda \int_0^t G_B(s)\,ds = 1 - \omega_B(t)$. It follows that $\Xi_A(t) = -\frac{\ln(1-\omega_A(t))}{\Lambda} - t > -\frac{\ln(1-\omega_B(t))}{\Lambda} - t = \Xi_B(t)$. Thus $\Xi_A(t) > \Xi_B(t)$. □

Proposition 6 is a special case of Proposition 6', proved below:

Proof of Proposition 6'. Consider first a case with bounded support, i.e., there is z such that $\omega(t) = 1$ $\forall t \geq z$:

$$
\begin{aligned}
\lim_{\rho \to 0} \Gamma &= \int_0^\infty \sum_{k=1}^K (1 - \omega(t)) a_k t^{k-1} dt = \\
&= \int_0^z \sum_{k=1}^K (1 - \omega(t)) a_k t^{k-1} dt = \\
&= \sum_{k=1}^K a_k \left[\frac{z^k}{k}(1 - \omega(z)) - 0 \times (1 - \omega(0)) - \int_0^z \frac{t^k}{k} \frac{\partial(1-\omega(t))}{\partial t} dt \right] = \\
&= 0 + \sum_{k=1}^K a_k \int_0^z \frac{t^k}{k} \frac{\partial \omega(t)}{\partial t} dt = \\
&= \Lambda \int_0^z \sum_{k=1}^K a_k \frac{t^k}{k} (1 - G(t)) dt.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\lim_{\rho \to 0} \Gamma &= \Lambda \int_0^z \sum_{k=1}^K a_k \frac{t^k}{k} (1 - G(t)) t\,dt = \\
&= \Lambda \left[\sum_{k=1}^K a_k \frac{z^{k+1}}{k(k+1)} (1 - G(z)) - 0 \times (1 - G(0)) - \int_0^z \frac{t^{k+1}}{k(k+1)} d(1 - G(t)) \right] = \\
&= \Lambda \int_0^z \sum_{k=1}^K a_k \frac{t^{k+1}}{k(k+1)} dG(t) = \\
&= \Lambda \sum_{k=1}^K \frac{a_k}{k(k+1)} E\left[\tau^{k+1}\right] = \\
&= \sum_{k=1}^K \frac{a_k}{k(k+1)} \frac{E\left[\tau^{k+1}\right]}{E\left[\tau\right]},
\end{aligned}
$$

where the last line follows from $\Lambda^{-1} = E[\tau]$.

The case with unbounded support can be obtained by constructing a sequence of distribution functions $G_z(t)$ defined as:

$$
G_z(t) = \frac{G(t)}{G(z)} \mathbb{1}(t \leq z) + \mathbb{1}(t \geq z),
$$

29

with associated $\Lambda_z = \left[\int_0^\infty \left(1 - G_z \left(t \right) \right) dt \right]^{-1}$. We take the limit

$$\lim_{z \to \infty} \lim_{\rho \to 0} \Gamma = \lim_{z \to \infty} \Lambda_z \int_0^\infty \sum_{k=1}^{K} a_k \frac{t^k}{k} \left(1 - G_z \left(t \right) \right) dt.$$

We then use Lebesgue's dominated convergence theorem (see, for example, Kolmogorov and Fomin 1970) to show that this limit is equal to $\Lambda \int_0^\infty a_k \frac{t^k}{k} \left(1 - G \left(t \right) \right)$. Let $\{ z_1, z_2, ..., z_n, ... \}$ be an infinite sequence such that $z_{k+1} > z_k \ \forall k$ and $z_1 > 0$. Then, for all t, $\lim_{n \to \infty} \Lambda_{z_n} \left(1 - G_{z_n} \left(t \right) \right) t^k = \Lambda \left(1 - G \left(t \right) \right) t^k$. Furthermore there is a $\bar{\Lambda} < \infty$ such that, $\Lambda_{z_n} \left(1 - G_{z_n} \left(t \right) \right) t^k < \bar{\Lambda} \left(1 - G \left(t \right) \right) t^k$. That $1 - G_{z_n} \left(t \right) < 1 - G \left(t \right)$ follows trivially from the definition of G_{z_n}. To see that such a $\bar{\Lambda}$ exists, recall that $\Lambda_{z_n} = \left[\int_0^\infty \left(1 - G_{z_n} \left(t \right) \right) dt \right]^{-1}$ and that for all $n > 1$, $\int_0^\infty \left(1 - G_{z_n} \left(t \right) \right) dt = \int_0^{z_n} \left(1 - \frac{G(t)}{G(z_n)} \right) dt > \int_0^{z_1} \left(1 - \frac{G(t)}{G(z_1)} \right) dt$. Thus, it is enough to pick $\bar{\Lambda} > \left[\int_0^{z_1} \left(1 - \frac{G(t)}{G(z_1)} \right) dt \right]^{-1}$.

If $\int_0^\infty \left(1 - G \left(t \right) \right) t^k dt < \infty$ then $\bar{\Lambda} \int_0^\infty \left(1 - G \left(t \right) \right) t^k dt < \infty$ and all the conditions of the theorem are satisfied. It follows that $\lim_{n \to \infty} \int_0^\infty \left(1 - G_{z_n} \left(t \right) \right) t^k dt = \int_0^\infty \left(1 - G \left(t \right) \right) t^k dt$ and $\Lambda \int_0^\infty \sum_{k=1}^{K} a_k \frac{t^k}{k} \left(1 - G \left(t \right) \right) dt = \lim_{z \to \infty} \Lambda_z \int_0^\infty \sum_{k=1}^{K} a_k \frac{t^k}{k} \left(1 - G_z \left(t \right) \right) dt.$ $\qquad \square$