Economic Quarterly—Volume 96, Number 4—Fourth Quarter 2010—Pages 317–337

Monetary Policy and Global Equilibria in a Production Economy

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Macroeconomic models that are applied to the study of monetary policy often exhibit multiple equilibria.[1] Prior to the mid-1990s, applied monetary theory typically modeled monetary policy in terms of a rule for the money supply, and it was well understood that multiple equilibria often arose under constant money supply policies. Starting in the mid-1990s, applied work shifted to modeling monetary policy in terms of interest rate rules. This was mainly because of the accumulating observations that central banks in fact operated with interest rate targets rather than money supply targets. A particular class of interest rate rules—so called "active Taylor rules," featuring a strong response of the policy interest rate to inflation—attracted special attention. In linearized models these policy rules were shown to guarantee a locally unique nonexplosive equilibrium. Benhabib, Schmitt-Grohé, and Uribe looked beyond the local dynamics in a series of articles (e.g., 2001a, 2001b, 2002), and showed that active Taylor rules could in fact lead to multiple equilibria. Whereas local analysis ignored the zero bound on nominal interest rates, global analysis showed that the zero bound implied the existence of a second steady-state equilibrium, with low inflation and a low nominal interest rate. This second steady state proved to be the "destination" for paths that had appeared explosive in the local analysis. Benhabib, Schmitt-Grohé, and Uribe's results attracted much attention in the academic literature because the prevailing wisdom had held that active Taylor

The views in this paper are those of the authors and do not represent the views of the Federal Reserve Bank of Richmond, the Federal Reserve Board of Governors, or the Federal Reserve System. For helpful comments, the authors thank Huberto Ennis, Brian Gaines, Andreas Hornstein, and Thomas Lubik. E-mails: tim.hursey@rich.frb.org; alexander.wolman@rich.frb.org.

[1] Michener and Ravikumar (1998) provide a taxonomy of multiple equilibria in monetary models that predates the recent sticky-price literature.

rules generated a *unique* equilibrium. More recently, the persistence of low inflation and low nominal interest rates has brought attention to Benhabib, Schmitt-Grohé, and Uribe's work in policy circles. Most notably, Bullard (2010) argued that monetary policy in the United States could unintentionally be leading the economy to a steady state in which inflation is below its target.

This article provides an introduction to Benhabib, Schmitt-Grohé, and Uribe's work on multiple equilibria under active Taylor rules, using two simple models. While the type of results presented here is not new, the specific modeling framework—Rotemberg price setting in discrete time—is new, and it fits neatly into the frameworks typically used for applied monetary policy analysis. Furthermore, we provide computer programs in the open source software R to replicate all the results in the article. The programs are available at www.richmondfed.org/research/economists/bios/wolman_bio.cfm.

Section 1 places the topic of this article in historical perspective. Section 2 shows the existence of multiple equilibria in a reduced-form model consisting only of an active Taylor rule and a Fisher equation, assuming that the real interest rate is exogenous and fixed. Section 3 describes the discrete-time Rotemberg pricing model to be used in the remainder of the article. Steady-state equilibria and local dynamics are described in Section 4, and global dynamics are described in Section 5. Section 6 concludes.

1. HISTORICAL CONTEXT

Multiple equilibria is a common theme in monetary economics, and has been at least since the work of Brock (1975). On the theory side, there has been a steady stream of work on multiple equilibria since the 1970s. In contrast, emphasis on multiple equilibria in applied monetary policy research has fluctuated as new theoretical results have appeared, the tools of analysis have evolved, and economic circumstances have changed. The immediate explanation for why the theoretical results described in this article have attracted attention in policy circles—10 years after those results first appeared—involves economic circumstances, namely the existence of low inflation and near-zero nominal interest rates in the United States. There is a longer history, however, that also involves the ascent of interest rate feedback rules and linearized New Keynesian models, and the accompanying focus on active Taylor rules as a descriptive and prescriptive guide to central bank behavior.

Beginning with Bernanke and Blinder (1992), quantitative research on monetary policy in the United States rapidly shifted from modeling monetary policy as controlling the money supply to modeling monetary policy as controlling interest rates.[2] At around the same time, Henderson and McKibbin

[2] Bernanke and Blinder were not the first to suggest modeling monetary policy in terms of interest rates. See for example McCallum (1983).

(1993) and Taylor (1993) influentially proposed particular rules for the conduct of monetary policy. These rules involved the policy rate (federal funds rate in the United States) being set as a linear function of a small number of endogenous variables, typically including inflation and some measure of real activity. Henderson and McKibbin focused on the normative aspects of interest rate rules, whereas Taylor also argued that what would become known as the "Taylor rule" actually provided a reasonable description of short-term interest rates in the United States from 1986–1992.

Just as Taylor rules were attracting more attention, another shift was occurring in the nature of quantitative research on monetary policy. Bernanke and Blinder's 1992 article had used vector autoregressions (VARs) for its empirical analysis and, in their policy analysis, Henderson and McKibbin employed linear rational expectations models with some rule-of-thumb behavior. These two approaches—VARs and linear rational expectations models—had become standard in applied monetary economics for empirical analysis and policy analysis, respectively. Beginning with Yun (1996), King and Wolman (1996), and Woodford (1997), however, the tide shifted toward what Goodfriend and King (1997) called New Neoclassical Synthesis (NNS) models. NNS models represented a melding of real business cycle (RBC) methodology—dynamic general equilibrium—with nominal rigidities and other market imperfections. Nominal rigidities made the NNS models appealing frameworks for studying monetary policy, and the RBC methodology meant that it was straightforward to model the behavior of monetary policy as following a Taylor-style rule.

While NNS models, like RBC models, were fundamentally nonlinear, they were typically studied using linear approximation. In linearized NNS models (as with their predecessors, the linear rational expectations models), the question of existence and uniqueness of equilibrium generally was presumed to be identical to the question of whether the model possessed unique stable local dynamics in the neighborhood of the steady state around which one linearized.[3] In turn, the nature of the local dynamics depended on the properties of the interest rate rule. Although specific conditions can vary across models, the results in Leeper (1991) and Kerr and King (1996) were the basis for a useful rule of thumb in many monetary models: Taylor-style interest rate rules were consistent with unique stable local dynamics only if the coefficient on inflation was greater than one; a coefficient less than one would be consistent with a multiplicity of stable local dynamics. Taylor rules with a coefficient greater than one became known as active Taylor rules, and the rule of thumb

[3] For example, see Blanchard and Kahn (1980) or King and Watson (1998). In many economic models, explosive paths for some variables are inconsistent with equilibrium. For example, explosive paths for the capital stock can be inconsistent with a transversality condition (in nontechnical terms, consumers would be leaving money on the table), and explosive paths for real money balances can violate the requirement of a nonnegative price level. See Obstfeld and Rogoff (1983) for a discussion of these issues.

that active Taylor rules guaranteed a unique equilibrium became known as the Taylor principle.[4] Passive Taylor rules, in contrast, are Taylor rules with a coefficient on inflation less than one.

Some intuition for the Taylor principle comes from the much earlier work of Sargent and Wallace (1975) and McCallum (1981). Sargent and Wallace showed that if the nominal interest rate is held fixed by the central bank, then in many models expectations of future inflation will be pinned down, but the current price level is left indeterminate. McCallum followed up by showing that if the nominal interest rate responds to some nominal variable it is also possible to pin down the price level. The Taylor principle states that multiplicity can occur if the nominal interest rate does not respond strongly enough to inflation, consistent with the message of Sargent and Wallace and McCallum.

With widespread understanding of the Taylor principle came empirical applications by Clarida, Gali, and Gertler (2000) and Lubik and Schorfheide (2004). These authors argued that (i) violation of the Taylor principle could help explain the macroeconomic instability of the 1970s, and (ii) a shift in policy so that the Taylor principle did hold could help explain the subsequent stability after 1982. Although this work brought multiple equilibria into the mainstream of applied research on monetary policy, it proceeded under the assumption that the local linear dynamics gave an accurate picture of the nature of equilibrium. These articles also helped to cement the idea that the Taylor principle characterized "good" monetary policy, because the Taylor principle would guarantee that inflation stayed on target.

Beginning with their 2001a article, Benhabib, Schmitt-Grohé, and Uribe (BSU) showed that when there is a lower bound on nominal interest rates, the local dynamics can be misleading about the uniqueness of equilibrium when monetary policy is described by an active Taylor rule. The details of BSU's argument will become clear below. The rough intuition is as follows. Arguments for (local) uniqueness of equilibrium with active Taylor rules posit that without shocks, the model has a unique equilibrium at the inflation rate targeted by the interest rate rule. Any other candidate solutions to the model equations would have the inflation rate exploding to plus or minus infinity, or oscillating explosively. But many of these explosive paths would violate the lower bound on the nominal interest rate. When that bound is imposed and the model is studied nonlinearly, it becomes clear that (i) there is a second steady-state equilibrium at a lower inflation rate, and (ii) there are many

[4] Note that Leeper (1991) emphasizes that an active rule guarantees uniqueness only in conjunction with an assumption about fiscal policy, specifically that fiscal policy takes care of balancing the government budget. We maintain that assumption here. Benhabib, Schmitt-Grohé, and Uribe (2002) discuss the implication of alternative assumptions about fiscal policy for multiple equilibria induced by the zero bound on nominal interest rates.

non-steady-state equilibria in which the inflation rate converges to the low-inflation steady state in the long run.

Initially, while the articles by BSU were widely cited, they did not attract much attention in policy circles. This is somewhat surprising because the articles were showing that a policy advocated in large part because it was believed to deliver a unique equilibrium actually delivered multiple equilibria in some models! Furthermore, a rule that violated the Taylor principle—a passive rule—would actually be consistent with keeping inflation close to its targeted value, even though there could be multiple equilibria with this property. Recently however, the results in BSU have attracted substantial attention in policy circles. The simultaneous occurrence of low inflation and low nominal interest rates in the United States is suggestive of some of the equilibria identified by BSU, so it is natural to wonder whether we are experiencing outcomes associated with those global equilibria. Policymakers care about this because the global equilibria involve average inflation below its intended level.

2. A SIMPLE FRAMEWORK WITH ONLY NOMINAL VARIABLES

As a simple framework for communicating some of the key ideas in BSU, this section works through a two-equation model of the nominal interest rate and inflation. That minimal structure is sufficient to illustrate the potential for the local and global dynamics to diverge when monetary policy is given by an active Taylor rule.

Assume that the real interest rate is exogenous and fixed, $r_t = r$, whereas the nominal interest rate (R_t) and the inflation rate (π_t) are endogenous.[5] Expectations are rational. The model consists of a Fisher equation relating the short-term nominal interest rate to the short-term real interest rate and expected inflation,

$$R_t = r E_t \pi_{t+1}, \tag{1}$$

and a rule specifying how the central bank sets the nominal interest rate—in this case as a function only of the current inflation rate, with an inflation target of π^*:

$$R_t = 1 + \left(R^* - 1\right)\left(\pi_t/\pi^*\right)^{\gamma}, \tag{2}$$

where

$$R^* = r\pi^*; \tag{3}$$

[5] Throughout the article, interest rates and inflation rates are measured in gross terms—that is, a 4 percent nominal interest rate would be written as $R_t = 1.04$.

that is, the targeted nominal interest rate is the one that is implied by the steady-state Fisher equation when inflation is equal to its target.

The interest rate rule in (2) may look unfamiliar relative to standard linear Taylor rules. We use the nonlinear rule because it will simplify the analysis in the second part of the article.[6] Furthermore, the linear approximation to the rule in (2) around $\{R^*, \pi^*\}$ is

$$R_t - R^* = \gamma \left(\frac{R^* - 1}{\pi^*} \right) \left(\pi_t - \pi^* \right), \tag{4}$$

a simple inflation-only Taylor rule in which the coefficient on inflation is $\gamma \left(R^* - 1 \right) / \pi^*$, and we assume that $\gamma \left(R^* - 1 \right) / \pi^* > r > 1$. The standard local-linear approach around the point $\{R^*, \pi^*\}$ involves combining the linearized Taylor rule (4) with the linearized Fisher equation ($R_t - R^* = (R^*/\pi^*) E_t (\pi_{t+1} - \pi^*)$), which yields an expectational difference equation in inflation:

$$E_t \left(\pi_{t+1} - \pi^* \right) = \gamma \left(\frac{R^* - 1}{R^*} \right) \left(\pi_t - \pi^* \right).$$

For simplicity, assume perfect foresight—that is, the future is known with certainty, so that $E_t (\pi_{t+1} - \pi^*)$ can be replaced with $\pi_{t+1} - \pi^*$. Perfect foresight is clearly an unrealistic assumption, but it is a convenient one for illustrating the difference between local and global dynamics. With perfect foresight, we have

$$\left(\pi_{t+1} - \pi^* \right) = \gamma \left(\frac{R^* - 1}{R^*} \right) \left(\pi_t - \pi^* \right). \tag{5}$$

By assumption the coefficient on $\pi_t - \pi^*$ is greater than one—the rule obeys the Taylor principle. Consequently, we can show that there is a unique nonexplosive equilibrium. Constant inflation at the targeted steady-state level ($\pi_t = \pi^*$) is clearly an equilibrium because it represents a solution to the difference equation (5). If inflation in period t were equal to any number other than π^*, inflation would have to follow an explosive path going forward because the coefficient on current inflation is greater than one. Any such explosive path would be ruled out as an equilibrium by assumption in the standard local-linear approach.[7]

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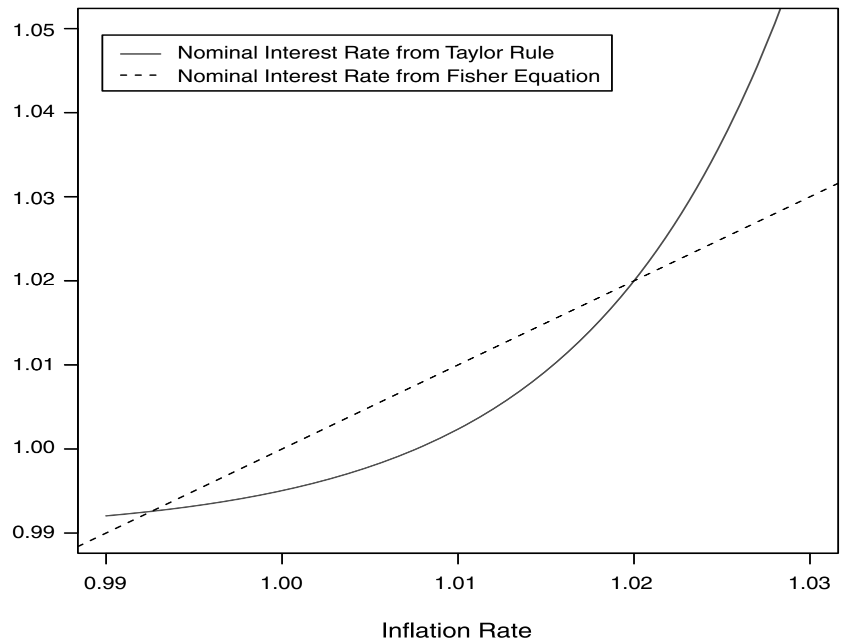
[6] Imposing the zero bound on an otherwise linear rule creates a nondifferentiability, making computation more difficult.

[7] Since the model here is itself ad-hoc, we cannot complain about ruling out explosive paths as equilibria by assumption. Depending on the particular model, explosive paths up or down may or may not be equilibria—see footnote 3. What is important here is that the ad-hoc model we wrote down is nonlinear, and the nonlinear analysis yields different conclusions about equilibrium than the linear analysis.

Figure 1 Steady-State Equilibria



Steady-State Equilibria

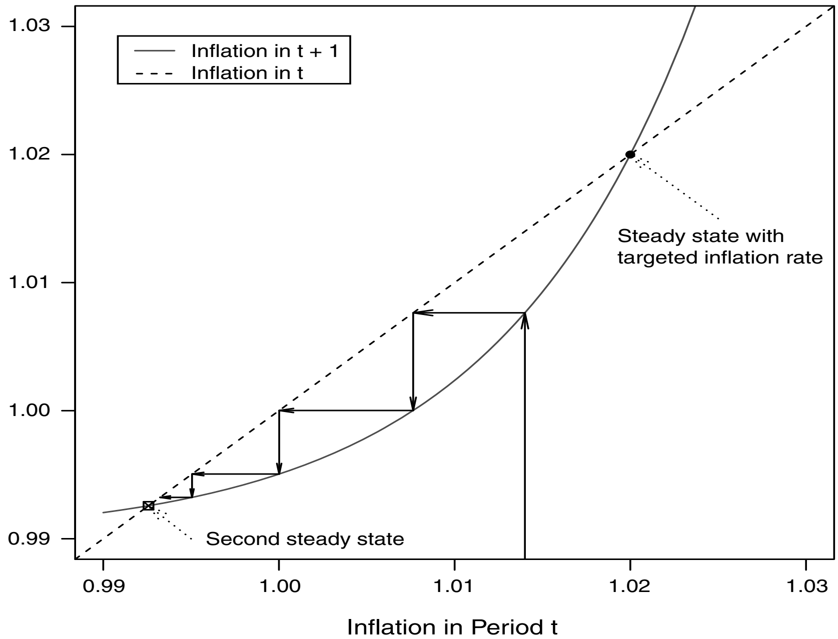
It is obvious that $\{R^*, \pi^*\}$ represents a steady-state solution to the Fisher and Taylor equations ([1] and [2]). Less obviously, there is also a second steady-state solution with a lower inflation rate and a lower nominal interest rate. To see this, combine the steady-state Fisher and Taylor equations into a single equation in π:

$$\pi = r^{-1}\left(1 + \left(R^* - 1\right)\left(\pi/\pi^*\right)^{\gamma}\right). \tag{6}$$

Figure 1 displays a plot of the right-hand side of (6) (essentially the Taylor rule) against the 45-degree line—which is also the left-hand side, or the Fisher equation. The two intersections of the right-hand side and left-hand side represent the two steady-state equilibria. The targeted inflation rate is 2 percent, and the other steady state involves slight deflation.

The specific Taylor rule we chose for this example never allows the nominal interest rate to hit the zero bound. Alternatively, if we had chosen a typical linear Taylor rule ($R_t = \max\{R^* + f(\pi_t - \pi^*), 0\}$), there would be a kink in the steady-state Taylor curve at $\pi = 1/r$, and the second steady state would be at $\pi = \pi^* - (1/f)R^*$. BSU (2001a) and Bullard (2010) contain pictures of the analogues to Figure 1 implied by several different interest rate rules that

Figure 2 Example of a Non-Steady-State Equilibrium



all satisfy the Taylor principle at the targeted steady state, and all imply the existence of a second steady state with lower inflation.

Example of a Non-Steady-State Equilibrium

The fact that there are two steady-state equilibria suggests that there may also be equilibria in which inflation and nominal interest rates fluctuate. Returning now to the nonlinear model, by combining the Fisher equation (1) and the interest rate rule (2) and imposing perfect foresight, we have a first-order difference equation for the inflation rate:

$$\pi_{t+1} = r^{-1}\left(1 + \left(R^{*} - 1\right)\left(\pi_t/\pi^{*}\right)^{\gamma}\right). \tag{7}$$

This is the nonlinear analogue of (5). In contrast to the linearized model, we can show that there is a continuum of nonexplosive equilibria.[8] In Figure 2 we plot the right-hand side of (7): It is an identical curve to the solid line in

[8] Note the sensitivity of this result to whether current or (expected) future inflation is the argument in the policy rule. If the policy rule responds to π_{t+1} instead of π_t, then the same two steady-state equilibria exist; but the system is entirely static and, under perfect foresight, the two steady-state equilibria are also the only two equilibrium values for inflation in any period. The

Figure 1. The dotted line is the 45-degree line, which is also the left-hand side of (7). The intersections between the two lines are the steady states and, starting with any initial inflation rate below the targeted steady state, we can trace an equilibrium path using the solid line and the 45-degree line. For example, from an initial inflation rate of 1.014, the vertical solid lines with arrows pointing down indicate the successive values of inflation going forward. Generalizing from this example, the figure shows that all perfect foresight equilibria except for the targeted steady state converge to the nontargeted steady state. In contrast, the conventional local linear approach applied to the targeted steady state would conclude that the targeted steady state was the only equilibrium—other solutions are locally explosive and would be ruled out by assumption. Figure 2 conveys the essence of the literature that began with BSU (2001a): Local analysis suggests a unique equilibrium, whereas global analysis reveals that many solutions ruled out as explosive instead lead to a second steady-state equilibrium.

Because the qualitative results involving a second steady state and multiple equilibria will carry over into the model with an endogenous real interest rate and endogenous output, it is interesting to discuss the economics behind these results. In a neighborhood of the targeted steady state, the interest rate rule responds to an upward (downward) deviation of inflation from target by moving the interest rate upward (downward) more than proportionally. This sets off a locally explosive chain: The Fisher equation (1) dictates that an increase in the current nominal interest rate must correspond to a higher future inflation rate, which then is met with a further increase in next period's interest rate, etc. One notable aspect of this process is that there is no sense in which a higher nominal interest rate represents "tighter" monetary policy. The model has only nominal variables, and a higher nominal interest rate *must* correspond to higher expected inflation. In contrast, the Taylor principle is often thought of as ensuring that an increase in inflation is met with a monetary tightening, as represented by a higher nominal interest rate. In models with real effects of monetary policy—such as the one discussed below—an increase in the nominal interest rate does not have to correspond to higher expected inflation. However, we have learned from the two-equation model that this association of higher interest rates with tight monetary policy is not an inherent ingredient in the local uniqueness and global multiplicity associated with the Taylor principle.[9]

"economy" can bounce arbitrarily between those two values in a deterministic way. There may also be rational expectations equilibria with stochastic fluctuations.

[9] See Cochrane (2011) for a similar argument.

3. A MODEL WITH REAL VARIABLES AND MONETARY NONNEUTRALITY

The model above taught us that the Fisher equation together with a Taylor rule that responds strongly to inflation can lead to multiple steady states and other equilibria because of the lower bound on nominal interest rates. However, the only endogenous variables in that model are nominal variables. One of the simplest ways to endogenize real variables and introduce real effects of monetary policy is with a version of the Rotemberg (1982) model, which has quadratic costs of nominal price adjustment. In this model, there is a representative household that takes all prices and aggregate quantities as given, and chooses how much to consume and how much to work. There is a continuum of monopolistically competitive firms that face convex costs of adjusting their nominal prices, and there is a monetary authority that sets the short-term nominal interest rate according to a time-invariant feedback rule.

The representative household has preferences over consumption (c_t) and (disutility of) labor (h_t) given by

$$\sum_{t=0}^{\infty} \beta^t \left(\ln\left(c_t\right) - \chi h_t\right). \tag{8}$$

There is a competitive labor market in which the real wage is w_t per unit of time. The consumption good is a composite of a continuum of differentiated products ($c_t(z)$), each of which are produced under monopolistic competition:

$$c_t = \left(\int_0^1 c_t\left(z\right)^{\frac{\varepsilon-1}{\varepsilon}} dz\right)^{\frac{\varepsilon}{\varepsilon-1}}. \tag{9}$$

Households own the firms. An individual household's budget constraint is

$$c_t + R_t^{-1} B_t / P_t = w_t h_t + B_{t-1}/P_t + \Pi_t / P_t, \tag{10}$$

where Π_t represents nominal dividends from firms, P_t is the price of the composite good, and B_t is the quantity of one-period nominal discount bonds. As above, R_t is the gross nominal interest rate. The household's intratemporal first-order conditions representing optimal choice of labor input and consumption are given by

$$\lambda_t w_t = \chi, \tag{11}$$

and

$$\lambda_t = 1/c_t, \tag{12}$$

and the intertemporal first-order condition representing optimal choice of bondholdings is given by

$$\frac{\lambda_t}{P_t} R_t^{-1} = \beta \cdot \frac{\lambda_{t+1}}{P_{t+1}}. \tag{13}$$

In these equations, the variable λ_t is the Lagrange multiplier on the budget constraint for period t—it can also be thought of as the marginal utility of an additional unit of consumption at time t. Note that the intertemporal first-order condition (13) corresponds to the Fisher equation from the first model, with the real interest rate now endogenous and given by

$$r_t = \beta^{-1}\frac{c_{t+1}}{c_t}.$$

Firms face a cost (ξ_t) in terms of final goods of changing the nominal price of the good they produce (z):

$$\xi_t(z) = \frac{\theta}{2}\left(\frac{P_t(z)}{P_{t-1}(z)} - 1\right)^2. \tag{14}$$

Because goods are produced both for consumption and for accomplishing price adjustment, the market-clearing condition is

$$y_t = c_t + \frac{\theta}{2}(\pi_t - 1)^2, \tag{15}$$

where y_t denotes total output of the composite good, π_t denotes the gross inflation rate (P_t/P_{t-1}), and we have imposed symmetry across firms, meaning that all firms choose the same price.

An individual firm chooses its price each period to maximize the expected present value of profits, where profits in any single period are given by revenue minus costs of production minus costs of price adjustment. The demand curve facing each firm is $y_t(z) = (P_t(z)/P_t)^{-\varepsilon} y_t$, so the profit maximization problem for firm z is

$$\max_{P_{t+j}(z)} \sum_{j=0}^{\infty} \beta^j \left(\frac{\lambda_{t+j}}{\lambda_t}\right)\left[\frac{P_{t+j}(z)}{P_{t+j}}\left(\frac{P_{t+j}(z)}{P_{t+j}}\right)^{-\varepsilon} y_{t+j}\right.$$
$$\left. -w_{t+j}\left(\frac{P_{t+j}(z)}{P_{t+j}}\right)^{-\varepsilon} y_{t+j} - \frac{\theta}{2}\left(\frac{P_{t+j}(z)}{P_{t+j-1}(z)} - 1\right)^2\right].$$

The first term in the square brackets is the real revenue a firm earns charging a price $P_{t+j}(z)$ in period $t+j$; it sells $\left(P_{t+j}(z)/P_{t+j}\right)^{-\varepsilon} y_{t+j}$ units of goods for relative price $P_{t+j}(z)/P_{t+j}$. The second term in the square brackets (in the second line of the expression) is the real costs the firm incurs in period $t+j$, number of goods sold multiplied by average cost, which is equal to marginal cost and to the real wage because labor productivity is constant and equal to one. Finally, the third term in the square brackets is the real cost of adjusting the nominal price from $P_{t+j-1}(z)$ to $P_{t+j}(z)$. Note that the price chosen in any period shows up only in two periods of the infinite sum. Thus, the part of the objective function relevant for the choice of a price in period t

is

$$\frac{P_t(z)}{P_t}\left(\frac{P_t(z)}{P_t}\right)^{-\varepsilon} y_t - w_t\left(\frac{P_t(z)}{P_t}\right)^{-\varepsilon} y_t$$
$$-\frac{\theta}{2}\left(\frac{P_t(z)}{P_{t-1}(z)}-1\right)^2 - \beta\left(\frac{\lambda_{t+1}}{\lambda_t}\right)\frac{\theta}{2}\left(\frac{P_{t+1}(z)}{P_t(z)}-1\right)^2.$$

The first-order condition is

$$(1-\varepsilon)\frac{1}{P_t}\left(\frac{P_t(z)}{P_t}\right)^{-\varepsilon} y_t + \varepsilon w_t \frac{1}{P_t}\left(\frac{P_t(z)}{P_t}\right)^{-\varepsilon-1} y_t$$
$$-\theta\frac{1}{P_{t-1}(z)}\left(\frac{P_t(z)}{P_{t-1}(z)}-1\right) + \beta\left(\frac{\lambda_{t+1}}{\lambda_t}\right)\theta\frac{P_{t+1}(z)}{P_t(z)^2}\left(\frac{P_{t+1}(z)}{P_t(z)}-1\right) = 0.$$

If we multiply both sides by P_t and impose symmetry—that is, assume that all firms choose the same price in any given period, the expression simplifies to

$$(1-\varepsilon)\, y_t + \varepsilon w_t y_t$$
$$-\theta\pi_t(\pi_t-1) + \beta\left(\frac{\lambda_{t+1}}{\lambda_t}\right)\theta\pi_{t+1}(\pi_{t+1}-1) = 0.$$

Using the goods market clearing condition (15) and the household's optimality conditions, the previous equation simplifies to a form that we will refer to as the New Keynesian Phillips Curve:[10]

$$\begin{aligned}(\pi_t-1)\,\pi_t &= \left(\frac{c_t}{\theta}+\frac{(\pi_t-1)^2}{2}\right)(1-\varepsilon+\chi\varepsilon c_t)\\ &\quad +\beta E_t\left(\frac{c_t}{c_{t+1}}(\pi_{t+1}-1)\,\pi_{t+1}\right),\end{aligned} \tag{16}$$

where π_t is the gross inflation rate.

Finally, monetary policy is given by a nominal interest rate rule similar to what was used in the two-equation model, with the one difference that the interest rate responds to expected future inflation instead of to current inflation:

$$R_t = 1 + \left(\pi^*/\beta - 1\right)\left(\pi_{t+1}/\pi^*\right)^{\gamma}. \tag{17}$$

Recall that in the two-equation model, using a policy rule identical to (17) would render the model entirely static, whereas the rule that responds to current inflation introduces dynamics. In the current model, optimal pricing already introduces dynamics, so we choose to use the future-inflation version of the policy rule.[11] Combining the policy rule with the household's intertemporal

[10] We should note that the term "New Keynesian Phillips Curve" typically refers to the linearized version of (16).

[11] Note that with current inflation in the policy rule, the steady states do not change and it would be possible to study dynamic equilibria in the same way we do here—tentative results

first-order condition (13), using the definition of the inflation rate to eliminate the price level, and using the household's intratemporal first-order condition (12) to eliminate λ, we have

$$\left(\frac{c_t}{\pi_{t+1}c_{t+1}}\right)^{-1} = \beta\left(1 + \left(\pi^*/\beta - 1\right)\left(\pi_{t+1}/\pi^*\right)^{\gamma}\right). \qquad (18)$$

The model has now been reduced to two nonlinear difference equations (16) and (18) in the variables c_t, π_t, c_{t+1}, and π_{t+1}.

4. LOCAL DYNAMICS AROUND STEADY-STATE EQUILIBRIA

As with the ad-hoc model in Section 2, there are two steady-state equilibria. That there are two steady-state equilibrium inflation rates is immediately apparent from (18)—in a steady state it is identical to (6). One of the steady states has inflation equal to the targeted inflation rate π^*, and the other steady state has a lower inflation rate.[12] The steady-state levels of consumption are determined by (16).

To study dynamic equilibria, we follow the same steps as in the two-equation model, beginning with the linearized model and then moving on to the exact nonlinear model. The two dynamic equations (16) and (18) can be represented as

$$\begin{bmatrix} F\left(c_t, c_{t+1}, \pi_t, \pi_{t+1}\right) \\ G\left(c_t, c_{t+1}, \pi_t, \pi_{t+1}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where

$$\begin{gathered} F\left(c_t, c_{t+1}, \pi_t, \pi_{t+1}\right) = \\ \left(\pi_t - 1\right)\pi_t - \left(\frac{c_t}{\theta} + \frac{\left(\pi_t - 1\right)^2}{2}\right)\left(1 - \varepsilon + \chi\varepsilon c_t\right) \\ -\beta\left(\frac{c_t}{c_{t+1}}\left(\pi_{t+1} - 1\right)\pi_{t+1}\right) \end{gathered}$$

$$G\left(c_t, c_{t+1}, \pi_{t+1}\right) = \pi_{t+1}c_{t+1} - \beta c_t\left(1 + \left(\pi^*/\beta - 1\right)\left(\pi_{t+1}/\pi^*\right)^{\gamma}\right).$$

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suggest that qualitatively similar results apply with current inflation in the policy rule. Our approach in this article is positive rather than normative. For a policymaker choosing a rule, whether multiple equilibria arise would be one important consideration in that choice.

[12] This statement relies again on γ being sufficiently large. In contrast, for low enough γ such that $R'\left(\pi^*\right) < 1$, the second steady state will involve inflation higher than π^*.

Table 1 Parameter Values

β	0.99
ε	6
θ	17.5
χ	5
γ	90
π^*	1.005

Linearizing around the steady state with the targeted inflation rate (denoted $[c^*, \pi^*]$) yields

$$\begin{bmatrix} F_2(c^*, c^*, \pi^*, \pi^*) & F_4(c^*, c^*, \pi^*, \pi^*) \\ G_2(c^*, c^*, \pi^*) & G_3(c^*, c^*, \pi^*) \end{bmatrix} \begin{pmatrix} c_{t+1} - c \\ \pi_{t+1} - \pi \end{pmatrix} \approx$$
$$-\begin{bmatrix} F_1(c^*, c^*, \pi^*, \pi^*) & F_3(c^*, c^*, \pi^*, \pi^*) \\ G_1(c^*, c^*, \pi^*) & 0 \end{bmatrix} \begin{pmatrix} c_t - c \\ \pi_t - \pi \end{pmatrix}, \qquad (19)$$

where $H_j(\mathbf{s})$ denotes the j^{th} partial derivative of the generic function $H()$, evaluated at $\mathbf{s}$.

The existence and uniqueness of a nonexplosive equilibrium in the linearized model depends on the eigenvalues of the Jacobian matrix J, given by

$$J = -\begin{bmatrix} F_2(.) & F_4(.) \\ G_2(.) & G_3(.) \end{bmatrix}^{-1} \begin{bmatrix} F_1(.) & F_3(.) \\ G_1(.) & 0 \end{bmatrix}.$$

Neither c_t nor π_t are predetermined variables, so the condition for a unique nonexplosive equilibrium is that both eigenvalues of J be less than one in absolute value. Because we are not able to provide a general proof of the parameter conditions under which equilibrium exists and is unique, we turn to a numerical example, which we will stay with for the rest of the article.[13] Table 1 contains the parameters for that example; they are chosen to be consistent with a 2 percent annual inflation target (the model is a quarterly model), a 4 percent real interest rate, a markup of 20 percent, and a coefficient in the Taylor rule of 1.33 when the Taylor rule is linearized around the targeted steady state. In addition, our choice of θ implies that price adjustment costs are less than $\frac{2}{10}$ percent of output.

At the targeted steady state, the local (nonexplosive) dynamics are unique, in a trivial sense. The Jacobian's eigenvalues are $0.99771321 \pm 0.12791602i$, which means that both eigenvalues have absolute value 1.0059. Local to the

[13] If the targeted inflation rate were zero ($\pi^* = 1$) then it would be straightforward to characterize uniqueness conditions analytically—this is the standard New Keynesian Phillips Curve. With a nonzero inflation target there are price-adjustment costs incurred in steady state, and the analysis is less straightforward.

targeted steady state, the fact that both eigenvalues have absolute value greater than one and are imaginary means that any solution to the difference equation system (19) other than the steady state itself oscillates explosively. In the linearized model the local dynamics *are* the global dynamics, so the only nonexplosive solution is the targeted steady state itself.

Suppose instead that we linearize around the low-inflation steady state. There the Jacobian's eigenvalues are 1.1291231 and 0.89509305. This eigenvalue configuration, with one explosive root and one stable root (less than one), means that there is a saddlepath: Given an initial value for c (or an initial value for π), there is a unique initial value for π (or for c) such that the economy will converge from that point to the steady state with low inflation. If either inflation or consumption were predetermined variables, then this saddlepath would describe the unique equilibrium at any point in time. Because neither variable is predetermined, the saddlepath represents one dimension of equilibrium indeterminacy at any point in time. That is, any value of c (or π) is consistent with equilibrium in period t, but as was stated above, once that value of c (or π) has been selected, the associated value of π (or c) is pinned down, as is the entire subsequent equilibrium path.[14]

The conventional linearization approach to studying NNS models, as followed, for example, by King and Wolman (1996), involves implicitly ignoring the steady state with low inflation. In that approach it is presumed that the only relevant steady state is the targeted one. From the same kind of reasoning used in the discussion following (5), the explosiveness of paths local to the targeted steady state means there is a unique nonexplosive equilibrium, the steady state itself. One can then proceed to study the properties of the model when subjected to shocks, for example to productivity or monetary policy. However, the fact that there are two steady states suggests that it may be revealing to investigate the global dynamics. Furthermore, if one extrapolates the local dynamics around the two steady states, it leads to the conjecture that paths that explode locally from the targeted steady state may in fact end up as stable paths converging at the low-inflation steady state. This is indeed what we will find in studying the global dynamics.

5. GLOBAL DYNAMICS

Studying the model's global dynamics means analyzing the nonlinear equations ([18] and [16]). We will combine the nonlinear equations with information about the local dynamics to trace out the global stable manifold of the low-inflation steady state. The global stable manifold is the set of inflation and

[14] Because we are dealing here with perfect foresight paths, the discussion of period t really should apply only to an *initial* period, prior to which the perfect foresight assumption does not apply. After that initial period the equilibrium outcomes are unique.

consumption combinations such that if inflation and consumption begin in that set, there is an equilibrium path that leads in the long run to the low-inflation steady state. While this approach may not yield a comprehensive description of the perfect foresight equilibria, it will provide a coherent picture of how the two steady states relate to the dynamic behavior of consumption and inflation.[15] We will find that the local saddlepath can be understood as part of a path (the global stable manifold) that begins arbitrarily close to the targeted steady state and cycles around that steady state with greater and greater amplitude before converging monotonically to the low-inflation steady state.

From Local to Global

Before plunging into the global dynamics, it may be helpful to take stock of our knowledge. There are two steady-state equilibria, one with the targeted inflation rate (π^*) and one with a lower inflation rate (π_l). The levels of consumption in the two steady states are c^* and c_l. Local to the targeted steady state, all dynamic paths oscillate explosively. Local to the low inflation steady state many paths explode and one path converges to that steady state. To go further, we will combine the forward dynamics local to the low inflation steady state with the nonlinear *backward dynamics*. This approach will allow us to compute the global stable manifold of the low-inflation steady state. Since all paths diverge around the targeted steady state, no analogous approach can be applied there.

As described above, the local dynamics around $\{c_l, \pi_l\}$ involve a unique path in $\{c, \pi\}$ space that converges to the steady state. If we begin with a point on that path, very close to the low-inflation steady state, and then iterate the nonlinear system *backward*, we can trace out the global dynamics associated with the saddlepath—the global stable manifold. We now describe this process algorithmically.

1. To find a point on the local saddlepath of the low-inflation steady state, follow the approach described in Blanchard and Kahn (1980). First, decompose the Jacobian matrix J into its Jordan form: $J = P\Lambda P^{-1}$, where Λ is a diagonal 2×2 matrix whose diagonal elements are the eigenvalues of J, and where P is a 2×2 matrix whose columns are the eigenvectors of J. Next, rewrite the system in terms of canonical variables $x_{1,t}$ and $x_{2,t}$, which are linear combinations of c_t and π_t : $\left[x_{1,t} \;\; x_{2,t}\right]' = P\,[c_t - c_l \;\; \pi_t - \pi_l]'$. The system is
$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}. \tag{20}$$

[15] While we have not proved that the global stable manifold contains all perfect foresight equilibria, we conjecture this to be the case.

Note that at the steady state c_l, π_l, we have $x_{1,l} = x_{2,l} = 0$. Recall that one of the roots (λ_1, λ_2) is greater than one. Without loss of generality, assume that $\lambda_1 > 1$. Any point on the local saddlepath must have $x_{1,t} = 0$, because $x_{1,t+j} = \lambda_1 x_{1,t+j-1}$, and if $x_{1,t} \neq 0$ then $x_{1,t+j}$ could not approach 0 as $j \to \infty$. Select one such point within an ε ball of the low-inflation steady state and call that point $\{c_T, \pi_T\}$. Set $t = T$.

2. From (18) we have

$$c_{t-1} = \frac{c_t}{\beta}\left(\frac{\pi_t}{1 + (\pi^*/\beta - 1)(\pi_t/\pi^*)^\gamma}\right).$$

3. Compute π_{t-1} by solving (16):

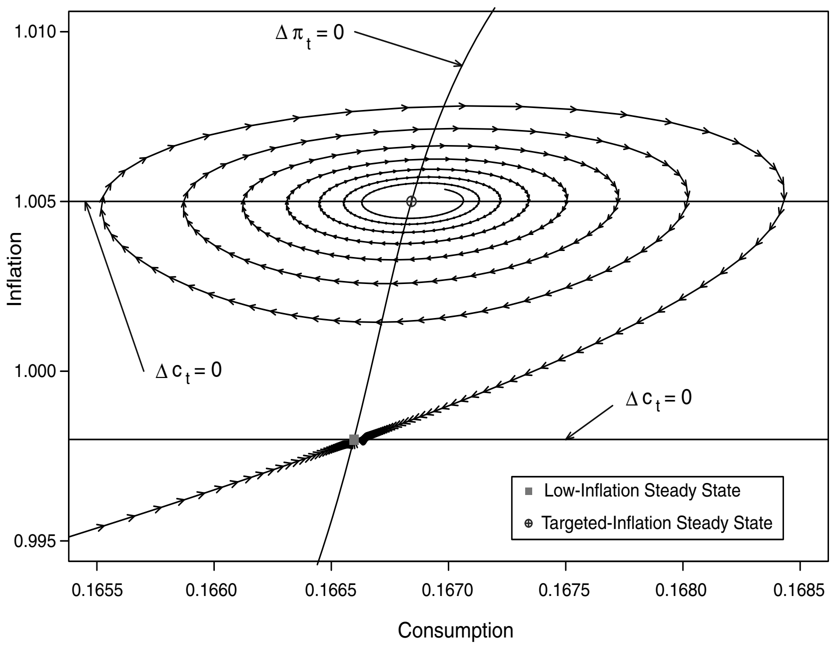
$$\left(1 - \frac{1}{2}(1 - \varepsilon(1 - \chi c_{t-1}))\right)\pi_{t-1}^2 - \varepsilon(1 - \chi c_{t-1})\pi_{t-1} - \left(\frac{c_{t-1}}{\theta}(1 - \varepsilon(1 - \chi c_{t-1})) + \beta\left(\frac{c_{t-1}}{c_t}(\pi_t - 1)\pi_t\right)\right) = 0. \quad (21)$$

With c_{t-1}, c_t, and π_t all known, (21) is a quadratic equation in π_{t-1}. The presence of two solutions is rooted in the properties of the firm's profit-maximization problem—while there is a unique profit-maximizing price, there are multiple solutions to the first-order condition. Only the positive root of the quadratic is consistent with the firm maximizing profits—the negative root typically implies a negative gross inflation rate, which would imply a negative price level.

4. Set $t = t - 1$, return to step 2.

Figure 3 describes the results of iterating backward for 450 periods in steps 2 through 4. The figure is in c, π space. It plots the two steady states and the global stable manifold of the low-inflation steady state, constructed as just described. The arrows represent forward movement in time, as opposed to the backward movement that characterizes the algorithm. The algorithm starts at a point close to the low-inflation steady state and goes backward in time. The figure shows that the only path that converges to a steady-state equilibrium initially involves spirals around the targeted steady state and ends with monotonic convergence to the low-inflation steady state. The figure provides us with a unified understanding of the local results around the two steady states. From the local dynamics we learn that all paths local to the targeted steady state oscillate explosively. From Figure 3, we see that *one* of those paths is not globally explosive, instead converging at the low-inflation steady state. This path is what we refer to as the global stable manifold.

Figure 3 Global Stable Manifold of Low-Inflation Steady State



6. CONCLUSION

Since late 2008, both inflation and nominal interest rates have been extremely low in the United States. These facts have focused attention on ideas motivated by the theory in BSU (2001a, 2001b, 2002): An active Taylor rule, together with a moderate inflation target, could have the unintended consequence of leading the economy to undesirably low inflation with a near-zero nominal interest rate. The article by St. Louis Federal Reserve Bank President James Bullard (2010) represents the leading example of this attention.

The aim of this article was to provide an accessible introduction to the ideas in BSU (2001a). Much of the literature in this area uses models that are either set in continuous time or that assume prices are flexible. In contrast, the model in this article is set in discrete time and has sticky prices. Discrete time reduces mathematical tractability, but makes it easy to compute specific solutions; in addition, the quantitative literature on monetary policy overwhelmingly uses discrete time models. Sticky prices are also a central element in the applied monetary policy literature. In adapting BSU's analysis to a discrete-time framework with sticky prices, we have seen that the general conclusions of their work also apply to the specific example we have analyzed. First, with an active Taylor rule, the presence of a lower bound on the nominal interest rate leads to the presence of two steady states, one at the targeted inflation rate and one at a lower inflation rate. Second, the targeted steady state, which is a unique equilibrium according to the conventional local analysis, instead is the source for a global stable manifold of the low-inflation steady-state equilibrium.

In closing we will offer some caveats regarding using the kind of analysis in this article to interpret current economic outcomes. It is tempting to conclude from Figure 3 that the low-inflation steady state is "more likely" because it does possess a stable manifold while the targeted steady state does not. However, the model only tells us what equilibria exist, not how likely they are to occur. It is also tempting to conclude from this work that policy may be unwittingly leading the economy to the unintended steady state. However, the theoretical analysis is based on perfect information about the model and the equilibrium by all agents. It is interesting to think about situations where policymakers and private decisionmakers do not understand the structure of the economy, but that is not the situation analyzed here. Finally, we should stress that before using this kind of framework for quantitative analysis, it would be desirable to enrich the model to incorporate capital accumulation. The behavior of the capital stock plays a key role in interest rate determination, and at this point it is an open question whether the kind of dynamics described here carry over to models with capital accumulation.

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